Max Brückner’s Wunderkammer of Paper Polyhedra

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Abstract

In 1900 the German mathematician Max Brückner published a book with photographs of 146 amazing paper polyhedron models. While containing little that was cutting-edge mathematically and not produced as fine art, the photographs have had an enormous influence on mathematical art ever since. The artist M.C. Escher was particularly influential in spreading Brückner's ideas. I argue that the import of the book can best be understood by seeing it as a *Wunderkammer*—a cabinet of curiosities—that excited wonder in the reader. This paper explores Brückner's work and its legacy from that perspective.

Introduction and History

Max Brückner (1860–1934) was a German mathematician who received his Ph.D. at Leipzig University in 1886 under the supervision of the renowned Felix Klein. Not a university professor, he taught first at a grammar school and then a gymnasium (an academic high school). Brückner is best known for his 1900 book *Vielecke und Vielflache: Theorie und Geschichte (Polygons and Polyhedra: Theory and History),* which summarized much of what was then known about polyhedra. It is illustrated with hundreds of engraved images and ten full sheets of photographic plates illustrating 146 spectacular paper models neatly arrayed on shelves. Figures 1–3 show Brückner's Tables IX, VIII, and X respectively [24, 3].

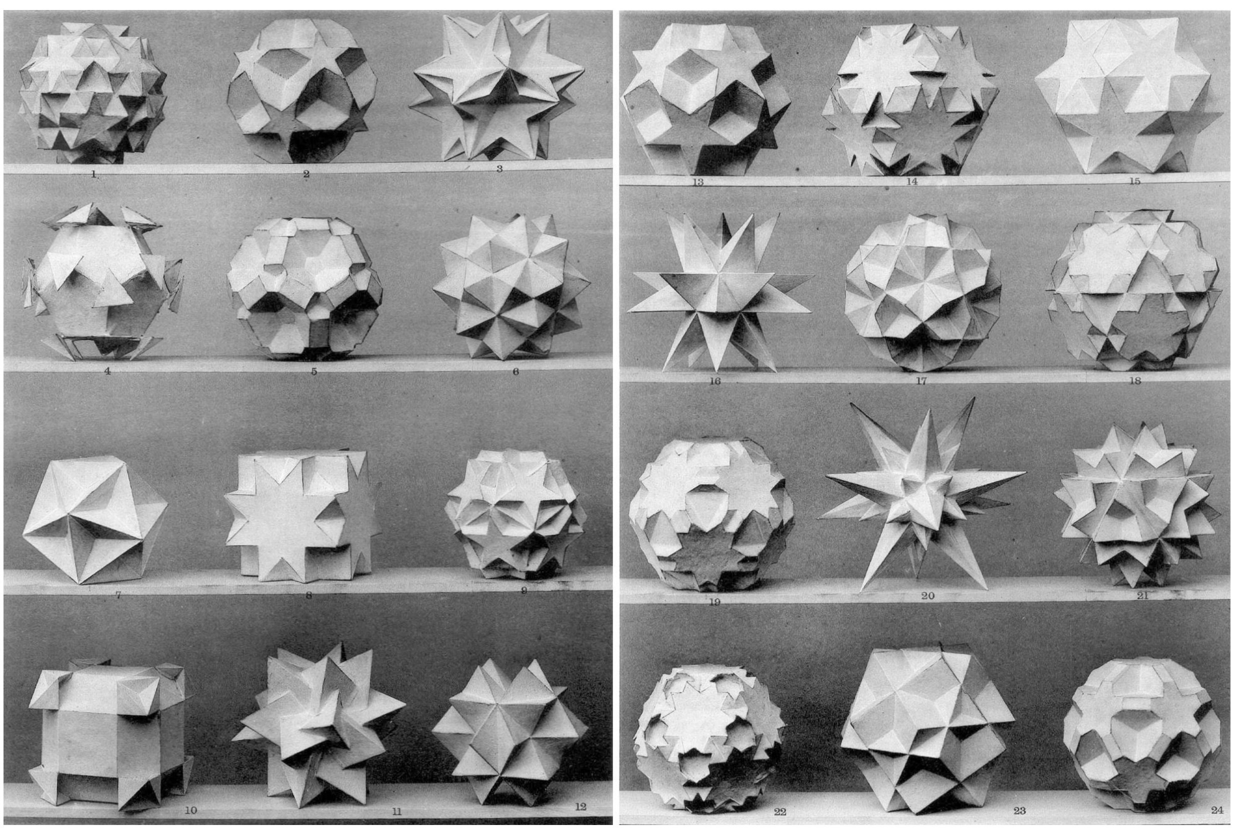

Figure 1: *Table IX from Brückner (two full page leafs in the original). n. 23 is a compound of three cubes; n. 6 is five octahedra; n. 11 is five tetrahedra; n. 3 is ten tetrahedra; n. 17 and n. 20 are stellations of the icosahedron; n. 7 and n. 16 are non-convex regular polyhedra; n. 2, 8, 9, 13, 15, 19, and 21 are (almost) uniform polyhedra.*

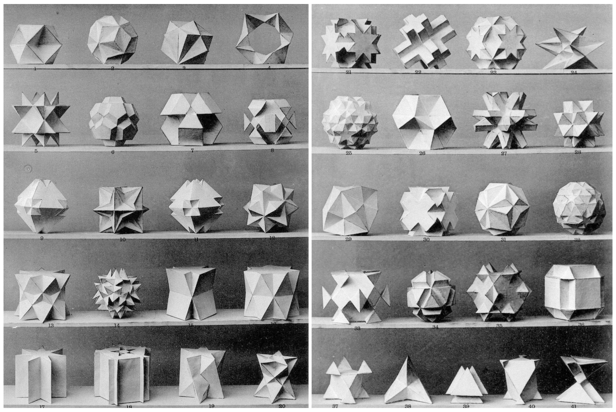

Figure 2: *Table VIII from Brückner. n. 12 is three octahedra, n. 3 is two cubes, both in Escher's "Stars."*

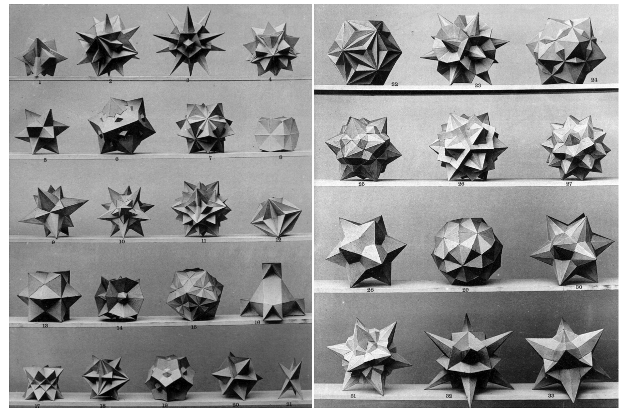

Figure 3: *Table X from Brückner. n. 13 is the first stellation of the rhombic dodecahedron.*

Vielecke und Vielflache has repeatedly been regarded as historically significant. At the 1984 *Shaping Space* conference on polyhedra, the geometer Joseph Malkevitch gave a talk "Milestones in the History of Polyhedra" listing the book as item #17 (chronologically) out of two dozen all-time important discoveries and publications [14]. The 1997 book *Beyond the Cube,* about polyhedra in architecture, contains a history chapter on "Polyhedra from Pythagoras to Alexander Graham Bell" with a section singling out "Max Brückner and his Paper Model Collection" [26]. The Bodleian Libraries of Oxford University highlights *Vielecke und Vielflache* as part of its 2019 exhibition "Thinking 3D from Leonardo to the Present," which tells "the story of the development of three-dimensional communication over the past 500 years" [2, 11].

Since ancient times, polyhedra have been associated with a sense of structure and rationality, while providing an accessible gateway to the deep beauty of mathematical thinking. Authors may present the subject at various levels, aimed for their target audience. Brückner's tome comprises 222 pages of technical mathematics and formulas summarizing much of the polyhedral knowledge of the time, so clearly was not aimed at general readers. But it attracted attention because it was the first major work that used photographs to communicate the beauty of mathematics. In 1900, photographic plates in printed books were a relatively recent medium that conveyed a feeling of modernity and a sense of exact scientific truth. The ten large photos ultimately influenced a huge audience via later mathematicians and artists.

In the foreword to *Vielecke und Vielflache*, Brückner mentions that he went to great pains over several years in making the paper models shown in the book. He gives credit to the photography firm which took the pictures and invites readers to come visit the model collection in person. It is difficult now, after 120 years, to gauge how readers of the time might have understood the images. (I haven't found any contemporary book reviews or commentary.) Did research-level mathematicians of the time see the work as important? Could non-mathematicians make geometric sense of the images? Did educators find pedagogical value in the material? Would anyone associate the forms with fine art in any way? What was the route by which the book became historically significant?

Current readers are often enchanted by the illustrations without any significant understanding of what they are. A quick internet search will reveal how the photos are periodically rediscovered and recirculated on social media. However, a careful reading of the associated commentary shows that online readers are not aware of the geometric questions to which Brückner was providing answers. My sense is that modern viewers are largely impressed with (a) the sheer intricacy of the structures, (b) a sense of wonder that they are samples from an extensive but completely alien mathematical world, and (c) the fact that anyone would patiently create such complex structures from such familiar materials as paper, scissors, and glue. In brief, they form a "cabinet of curiosities" or "Wunderkammer"—a collection which evokes awe and sparks superficial curiosity without significant understanding.

Since the seventeenth century, cabinets of curiosities have amazed visitors with a mix of unusual animal or plant specimens, ethnographic artifacts from distant cultures, ornate devices such as globes or astrolabes, unidentified fossils, extraordinary mineral samples, fabulous objects like unicorn horns, freakish religious relics, claimed antiquities, or pretty much any eye-catching rarity. All were indiscriminately amassed into collections intended to excite wonder in the viewer. In later centuries they evolved into organized, curated museums, but traditional cabinets were famously eclectic [15].

I will return below to the reasons why I see Brückner's collection as a Wunderkammer, but first let me close this history by mentioning that although Brückner's photographs live on, there is no record of what happened to the original paper polyhedra. His collection apparently managed to survive World War I, as he donated 200 models to Heidelberg University a few years before his death. But it is hard to imagine them lasting past World War II and the University has no documentation as to their ultimate fate.

Mathematics

I see three important points that should be understood about the mathematics of *Vielecke und Vielflache*: (1) typical viewers clicking through the images have no understanding of what the models are intended to communicate, (2) the professional math world that did understand wasn't very interested, and (3) the presentation is a mess and misses many things. Let us take these up in order:

It is difficult for a casual viewer to fully comprehend nonconvex polyhedra, because images only show the outer portions of what must be understood as intricate arrangements of interpenetrating polygons. Consider one of the easiest of Brückner's models to decode: How long might it take a viewer to parse the relatively simple Table IX, n. 23 (Figure 1) as three concentric interpenetrating cubes? The structure can be understood if one imagines three superimposed copies of a unit cube centered at the origin, then rotate the first 45 degrees about the X axis, the second 45 degrees about the Y axis, and the third 45 degrees about the Z axis. Thus arranged, the cube faces pass through each other in a symmetric manner with only small facets of each square face externally visible. Most of each face must be imagined as planes passing through the interior, dissecting the volume into small cells.

After visualizing that, it is still a challenge to understand how Table VIII, n. 12 shows three similarly rotated regular octahedra or how Table IX, n. 6 is a more complex symmetric arrangement of five regular octahedra. Other compound models shown in the plates are five tetrahedra, ten tetrahedra, and five cubes. But more complex models with other types of components, non-regular faces, less familiar rotation angles, or fundamentally different construction principles each take serious study to appreciate. I know from teaching this material at the university level that even college students with physical models or interactive computer-visualization software have difficulties coming to grips with such intricate interpenetrating structures. I'm certain that most casual viewers see little or none of this.

Professional geometers who read the book or were otherwise familiar with polyhedra, compounds, and stellations would have understood the purpose of these paper models. But I don't think many cared. *Vielecke und Vielflache* is merely a descriptive text. There are many geometric facts and historical references, but the book ignores the important lessons of abstract algebra developed in the second half of the nineteenth century. Group theory became the unifying language for describing symmetry, but is totally absent from the book. Dr. Brückner certainly knew much modern mathematics. His Ph.D. dissertation concerned complex variables and conformal mappings. His advisor, Klein, was most famous not for the one-sided bottle but for his 1872 "Erlangen program" that classified geometries in terms of symmetry groups of transformations, yet Brückner ignored group theory and its application here.

In the 20th century, H.S.M. Coxeter's approach to polyhedra gave great insight into many of same topics from the modern perspective of group theory and geometric transformations. It is difficult to gauge how influential a publication is to a professional research community, but one indication of Brückner's not having many academic followers is that Coxeter's 1960 text *Regular Polytopes* has 25 times as many citations on Google Scholar as *Vielecke und Vielflache.* It should also be mentioned that Klein's pedagogical 1908 book *Elementary Mathematics from an Advanced Standpoint* discusses polyhedra and the role of transformation groups in geometry, but does not mention the magnum opus of his student Brückner, which presumably lacked an "advanced standpoint."

Another of the book's weaknesses is that what is presented is disorganized and incomplete. Brückner's photographs include ten stellations of the icosahedron, several of which he was the first to construct (e.g., the “complete” or “final” stellation of Table XI, n. 14), but miss many others. Some of the remaining ones were later discovered by A.H. Wheeler in 1924 and a much expanded set was presented by Coxeter et al. in 1938 [6]. Likewise, only some stellations of the rhombic dodecahedron and the rhombic triacontahedron are included. While dozens of uniform polyhedra appear in the book, a complete enumeration had to wait for systematic studies by Coxeter and others in the 1930's, eventually published in 1953 [5]. Similarly, the reader is shown an assortment of uniform polyhedral compounds, but a systematic investigation and complete enumeration first appeared in a 1976 paper by John Skilling [22]. The book also discusses isohedra while missing many examples. In a detailed analysis of isohedra and the mathematical weaknesses of *Vielecke und Vielflache,* Branko Grünbaum [10] demonstrates that “Brückner’s presentation is completely ad hoc, with no particular guiding ideas and no clear classification or description principles; moreover, it is very incomplete and misses some of the most interesting polyhedra of the types it purports to enumerate.”

I do not fault Brückner for missing many examples. Discovering them was the hard work of many mathematicians over the following decades and in some cases computer techniques were needed for proving completeness. But it is important to understand that his approach was more like collecting a menagerie than a modern mathematician's structured investigation based on systematic principles. For this reason and because of its “old school” obliviousness to group theory, I believe most twentieth century professional mathematicians found little value in *Vielecke und Vielflache* beyond a cabinet of curiosities.

Physical Models in Education

From the late 1800s through the mid 1900s, largely due to the influence of Klein, physical models were highly regarded in teaching mathematics. Klein reinvented math education and under his influence several German companies sold paper, wire, wood, and plaster models that were used by schools and universities internationally to tangibly demonstrate mathematical structures. For example, in addition to algebraic surfaces, crystal structures, and various mechanisms, Walther Dyck's 1892 *Katalog* describes a variety of polyhedral models ranging from basic Platonic and Archimedean solids, compounds, and non-convex polyhedra, up through projection models of regular four-dimensional polytopes [7]. Similar catalogs by Brill and Schilling continued to offer such models into the 1930's [21]. In the US, analogous models were sold through the catalog of Richard P. Baker [13, 23]. Prominent universities in Europe and the US had sizeable collections of mathematical models and instruments both for display and for classroom use, some of which are still on exhibit [1, 27].

In this context, anyone would have seen Brückner's paper models as having pedagogical value for teaching the more advanced geometric ideas that they illustrate. However, there was no significant interest in these topics. These complex 3D structures were too advanced for the high school curriculum and of little interest to the university-level professional mathematics community that had moved on from 3D Euclidean geometry to newer areas of research. No doubt Brückner had also made paper models of the simpler Platonic solids, Archimedean solids, and other well-known polyhedra that might have found a place in K-12 educational circles, but considered them too familiar for the expensive photographic section of *Vielecke und Vielflache*. All the photographed models are non-convex.

I see a close American parallel to Brückner in the high school teacher Albert Harry Wheeler (1873–1950), who discovered some of the stellations of the icosahedron mentioned above. He was known in the early twentieth century for promoting polyhedra-making in education, but attracted little interest from academic mathematicians. Part of the reason was that Wheeler did not understand the language of group theory and so was merely tolerated as a well-intentioned outsider by professional mathematicians [17].

It is worth contrasting Brückner's models with the paper polyhedral models made by Alicia Boole Stott around 1890 [16]. Stott's models illustrate 3D cross-sections of the regular 4D polytopes described in an academic paper she published in 1900. They were recognized as important by the mathematical community and are still on display at Cambridge University in England and the University of Groningen in The Netherlands. While not as visually impressive as Brückner's, Stott's models illustrate topics in higher-dimensional geometry, one of the areas to which research-level mathematical interest had moved.

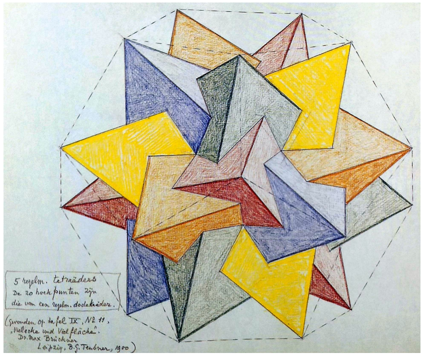

Figure 4: *Escher, five tetrahedra.*

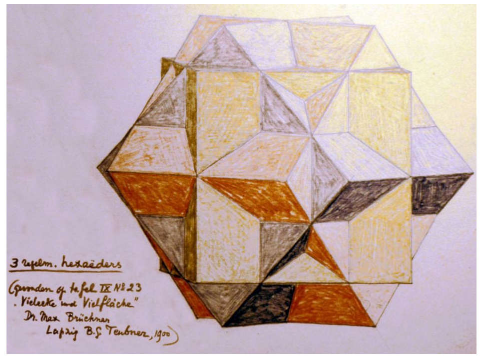

Figure 5: *Escher, three cubes.*

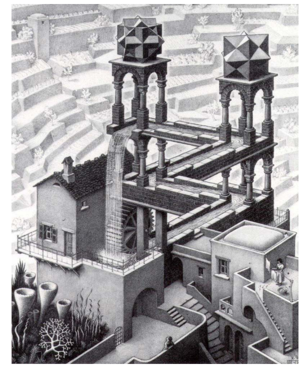

Figure 6: *Escher, Waterfall.*

Brückner's work must be seen in the context of an ongoing schism in mathematicans' styles—a war between purely algebraic techniques and the power of visual imagination. Perhaps Joseph-Louis Lagrange started it in his 1801 *Analytical Mechanics*, by supplanting Isaac Newton's diagram-based vector methods and boasting "No figures will be found in this work. The methods I present require neither constructions nor geometrical or mechanical arguments, but solely algebraic operations..." Leading the opposing camp, Klein was a strong proponent of images and physical models for developing mathematical intuition. In this battle, *Vielecke und Vielflache* seems to present an ineluctable argument for Klein's view: even something as simple as a compound of three familiar cubes could not be fully understood without the physical models and their photographic images. Although there is no doubt that model makers such as Brückner and his followers have won the hearts and minds of the public, cautious mathematicians argue that images can be misleading and a focus on notable examples might distract from important generalizations. In the 20th century, a highly respected group of mathematicians working under the *nom de plume* "Bourbaki" wrote an influential set of texts widely seen as systematizing and unifying all of pure mathematics while expressly avoiding any images. The weaknesses noted above in Brückner's text can be cited as support for this formalist perspective. No doubt, the correct approach (as with many dichotomies) is to pursue an informed synthesis of the two extremes.

Art and M.C. Escher

Brückner painstakingly made his models by hand and their complexity, accuracy, and variety set a high bar for all later model makers, but were they seen in their time as fine art? No, I'm sure they weren't. In 1900, sculpture was still always representational and paper was not seen as an artistic medium suitable for sculpture. The modern viewer may see Brückner's "star bodies" as abstract art, but before the twentieth century and non-representational sculpture, the contemporary reader would only have seen geometric models, presumably associated with simpler ones seen elsewhere for education. Each image might be studied at length as a visual puzzle with its own internal logic to marvel at and decode, and any sensitive viewer would have to appreciate the craftsmanship involved, but no one would consider them fine art *per se*. So it is curious that *Vielecke und Vielflache* has had its greatest impact via the world of art.

The route from Brückner's mathematical models to twentieth century fine art will seem less peculiar if we first review two interesting antecedents in which pedagogical models in mathematical collections influenced artists [27, 9]. One occurred when the surrealist Max Ernst encountered plaster models of algebraic surfaces (of types found in the Dyck, Brill, and Schiller catalogs) on display at the Institute Henri Poincaré in Paris. He brought Man Ray to make photographs of them, which then influenced other surrealists. Ernst incorporated prints of mathematical surfaces into his collages and Ray also made paintings inspired by their sensuous forms. The second school of artists to be influenced by mathematical models was the constructivists. The brothers Naum Gabo and Antoine Pevsner encountered surface models strung with ruled lines (probably in university collections) which influenced them to create sculptures and paintings clearly reminiscent of the lined texture. Later, the British sculptors Henry Moore and Barbara Hepworth developed their own styles of sculpture enriched by this technique of strung surfaces. They were influenced, respectively, by string models at the London Science Museum and "hidden away in a cupboard" in Oxford [26]. In all these cases, the artists discovered how to see ideas of structure, form, beauty, or exoticness in the models originally made for educational purposes, which they adapted and incorporated into their own purely artistic creations. I can think of no greater success for a cabinet of curiosities.

In an analogous manner, the Dutch graphic artist M.C. Escher (1898–1972) studied Brückner's figures in great detail, made paper and cardboard models, and incorporated many polyhedral structures into his own work [8, 18, 20]. (Escher's half-brother Berend was a geology professor with a special interest in crystallography and is known to have provided technical references [20], so is a likely source of the Brückner book.) We know M.C. Escher studied *Vielecke und Vielflache* specifically, because he made an undated colored-pencil sketch of the compound of five tetrahedra with a handwritten note specifying that he learned of it in Table IX, n. 11 of Brückner (Figure 4). It is interesting to observe that although the drawing has the same handedness, is not from exactly the same viewpoint as Brückner's photo, so Escher did not merely trace the image from the plate. He understood the 3D structure well from making his own cardboard model, which still survives [8, 18]. From that, he drew the perspective image of Figure 4 from a new point of view (centered exactly on a vertex) and rendered it in five colors to bring out the important idea (harder to see in the original black and white photo) that it is a compound of five simple tetrahedral forms. In 1958 Escher also carved a 15 cm diameter floral sculpture from maple based on this structure, evolving it into a symmetric organic centerpiece.

Escher also made an undated colored-pencil sketch of the compound of three cubes, with a note that he learned of it in Table IX, n. 23 of Brückner (Figure 5). Again, it is from a different viewpoint than the photo in the book. The use of color lucidly emphasizes the three individual overlapping cubes. I am told he also made a cardboard model of this compound [19].

Many other Escher works starting in the late 1940's are based on polyhedral structures that Brückner presented. *Double Planetoid* (1949) is based on the "stella octangula," which is shown as an engraving in Table VII, n. 20. *Order and Chaos* (1950) and *Gravitation* (1952) each feature a small stellated dodecahedron, shown in Table X, n. 5. The 1961 lithograph *Waterfall* (Figure 6) features geometric curiosities atop its two impossible towers: The left one holds the compound of three cubes which he sketched from Table IX, n. 23. On the right is the first stellation of the rhombic dodecahedron, undoubtedly seen in Brückner's Table X, n. 13. While many Escher works feature forms found in *Vielecke und Vielflache,* one must be singled out as almost an homage to Brückner, because it contains so many images from the book. The 1948 wood engraving *Stars* (Figure 7) features, among dozens of floating polyhedra, two chameleons inhabiting a cage-like compound of three octahedra, found in Brückner's Table VIII, n. 12. Escher displays it with open faces, clearly influenced by the Pacioli/Leonardo style that makes the interior structure so apparent. To its lower right is a solid-faced version of the same form, akin to Brückner's paper model, and below it is an open-faced version of a compound of two cubes with a common 3-fold axis, seen in Table VIII, n. 3.

Escher's artwork demonstrated to an enormous world-wide audience how fine art can be enriched by mathematical foundations. Many later artists took on the implied challenge of incorporating polyhedral structures into their own work. An internet search for "polyhedra art" or "polyhedra sculpture" will produce many thousands of examples. There isn't space here to fully explore the wide variety of ways that polyhedra have inspired both 2D and 3D artwork, so I will just mention three examples I like of artists working very close to the tradition of Brückner's paper models. These are all constructions by passionate model makers—not educators like Brückner who would have a pedagogical purpose, but geometry aficionados who put enormous care into their work for aesthetic reasons.

The artist Ulrich Mikloweit makes paper models with openings that reveal the interior structure of multilayered polyhedra, using colour to emphasize the facets common to larger faces. His lovely riff on the compound of five tetrahedra (Figure 8) makes the interior accessible so the large triangles become evident. The geometry constrains the boundaries, but allows Mikloweit the artistic freedom to choose the curves of the window shapes. The great snub dodecicosidodecahedron (Figure 9) is a typical example of

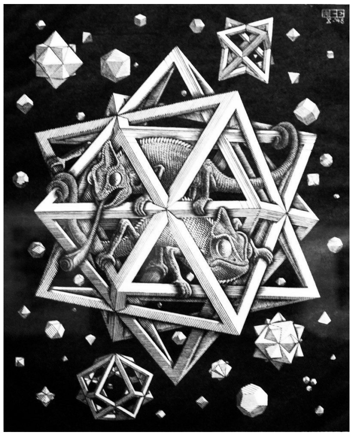

Figure 7: *Escher, Stars.*

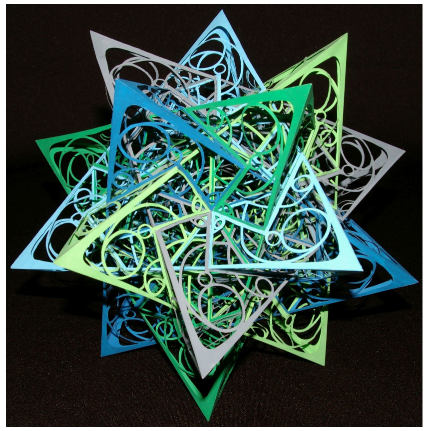

Figure 8: *Five tetrahedra paper model, Ulrich Mikloweit. (25 cm)*

Figure 9: *Great Snub Dodecicosidodecahedron, Ulrich Mikloweit. (40 cm)*

his more complex work, with large triangles and pentagrams connecting edge-to-edge throughout the interior. It gives a much richer sense of structural complexity than the photos in *Vielecke und Vielflache* can convey. A paper model of the compound of two great retrosnub icosidodecahedra (Figure 10) crisply constructed by Marcel Tünnissen took "around 222 hours" to make. The small inverted retrosnub icosicosidodecahedron created in wood by Tom Lechner (Figure 11) required four months of work with a table saw. This eight-foot tall uniform polyhedron is an amazing example of the legacy of Brückner's Wunderkammer: beautiful works now made not to teach mathematics, but as fine art.

Artists and model makers seeking to understand and extend Brückner's work now have many resources, including powerful software such as "Great Stella" by Robert Webb [28] and a series of how-to books by Magnus Wenninger. Intensely coloured card stock allows for vibrant models. Modern high-resolution photography lets constructors around the world share details lost in Brückner's small photos.

Wunderkammern

As a mathematical cabinet of curiosities, Brückner's photos provide us with a rich source of visual pleasure and structural inspiration. The models exemplify a mathematical world that only a few specialists fully appreciate, presented with a level of construction expertise that only the most patient can achieve. Although the mathematical academy was largely unimpressed, for over a century the images have excited perceptive artists and the popular imagination with wonder and amazement. Viewers likely think "What could be the story here that someone would spend so much effort to create these things?"

The world needs more such mathematical Wunderkammern. Sometimes books may serve this role: I love browsing through the old German model catalogs for inspiration and I recommend Steinhaus's *Mathematical Snapshots* as a classic sampler that can be analogously viewed [25]. In 1906 Brückner published a follow-on book with nine new photographic plates that can be similarly studied [4]. However the experience of seeing mathematical models in person has a much greater impact than images. In my lifetime, the closest thing I have experienced to what Brückner's studio must have been like was the model room at the Shaping Space conference at Smith College in 1984 (Figure 12). After Brückner's collection, it was no doubt the largest physical assemblage ever of similar paper models. Of course, now we can easily 3D print complex polyhedra to illustrate their structure [12], but knowing that a paper model is constructed by hand with many patient hours of care adds greatly to its perceived value.

Brückner's legacy lies in his demonstrating the power of images of exotic paper polyhedra models to amaze the viewer. His Wunderkammer displays intricate 3D specimens imported from strange mathematical lands. We are invited to appreciate both the mathematical elegance and the visual beauty of these ornate relics. The careful observer may glean deep insight into extraordinary structures while discovering that there is a strange culture of model fabricators willing to dedicate countless hours to the rituals of polyhedral construction. So amazing is this collection that even great artists like Escher have

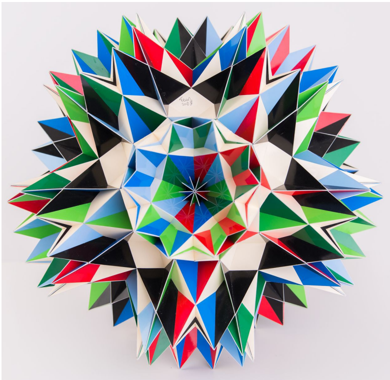

Figure 10: *Paper compound polyhedron by Marcel Tünnissen 50cm.*

Figure 11: *Wood polyhedron sculpture by Tom Lechner, 2.5 m.*

Figure 12: *Display room at the 1984 Shaping Space Conference at Smith College, with H.S.M. Coxeter and the author appreciating the paper polyhedra.*

found inspiration among its treasures. And as with many of the old jumbled cabinets, later curators like Coxeter and Grünbaum have revisited it to re-organize it into a more systematic museum. I invite the reader to delve into online photos and enjoy the exotic wonders of Brückner's collection.

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