Equilibrium Tides

Along with surface gravity waves, the predictable rise and fall of sea level at the coast are our most familiar periodic motions. These tides do not represent a new class of waves, but rather are the manifestation of strong forcing at precisely repeating periods. Gravitational tide-producing forces originating with the moon and the sun occur at periods of about twice per day (**semidiurnal**). The ocean response to these forces is in the form of *barotropic surface gravity waves*. (Interaction of the barotropic tides with topography can then also generate *baroclinic internal gravity waves*, as we have seen.) Since the periods are relatively long compared to the rotation period, the Coriolis force has a role, and the tides propagate as Poincare (inertia-gravity) waves and Kelvin waves.

Observations of tides

The methods of observing tides have evolved from the simple and straightforward local measurement to the use of satellite sensors:

- *Classical method* -- rod with scale -- height recorded hourly by visual observations.

- *Float type recorders*. Stilling well (to reduce effects of waves and wind) with a float on a lever arm. Arm connected to analog recording device like a chart recorder.

- *Pressure sensors*. Pressure recorded at a fixed depth (say 10 m) is converted to an equivalent sea level using the hydrostatic equation.

- *Radar altimetry*. Measures variations in distance from an orbiting satellite to the surface of the ocean and use satellite altitude to infer sea level.

The altimeter has fostered a surge of interest in tides because sea level in the deep ocean is now measured globally. A network of tide gauges at island stations was the primary tool for tidal analysis previously and this network has been used in conjunction with the altimeter to improve the radar's calibration, as well as to improve the tide estimates. Currently estimates of the tidal components are made using a combination of numerical tide models and observations.

Tidal variations in sea level are periodic (repeating with a fixed period) and, with sufficiently long time-series of tide measurements at a given point, the tides can be predicted with great accuracy. Kelvin's method of empirical tide prediction uses harmonic analysis to decompose a long time series into sinusoidal components of various frequencies. These components are called **partial tides** or **tidal constituents**. For example, if we want to find a semidiurnal tidal component due to the moon (called the M_2 tide), we would use a model of the tidal function as

$$\eta(t) = M_2 \cos(\omega_{M2} \, t + \phi_{M2})$$

and then fit this function (using a least-squares method) to our available data of sea level to obtain the amplitude, M_2, and phase, φ, of the lunar semidiurnal tide.

To understand the phase of the tide, if we assume that noon is the time reference locally, and the phase is 45 degrees, then to find when high tide occurs we write
$$\omega_{M2} \, t + \pi/4 = 2\pi$$
since high tide will occur at phase of 2π. We know that the period of the M2 tide is 12.42 hours so we solve for the time using
$$t = 7\pi/4/2/\pi * 12.42 = 10.9 hours$$
so the next high tide will occur at about 11 in the evening.

Another example is, if the tidal amplitude is 1 m, at 2pm, the height will be
$$\eta = 1\cos(2\pi/12.42 * 2 + \pi/4) = -0.22m$$

Since the tidal variation in any one place is more complex than this, we add additional components that represent the other frequencies of interest. For example, to obtain the lunar semi-diurnal and the solar diurnal components, we would use

$$\eta(t) = M_2 \cos(\omega_{M2} \, t + \phi_{M2}) + S_1 \cos(\omega_{S1} \, t + \phi_{S1})$$

Because some of these components may be similar (e.g., the semi-diurnal tides M_2 and S_2) a relatively long time series of sea level observations is required to distinguish the amplitudes and phases. Decomposing the tidal signal into its components is known as ***harmonic analysis***.

In most locations, the four largest amplitude tidal components turn out to be:

Symbol	Name	period
M_2	Principal lunar	12.42 h
K_1	Luni-solar diurnal	23.93
S_2	Principal solar	12.00
O_1	Principal lunar diurnal	25.82

A list of other components can be found in Knauss table 10.1. Particularly important is the fortnightly (2 week) tide, often written M_f. See Figure 10.15 in Knauss for plots of partial tides. Tides in different locations are classified based on the predominant frequency of the tide using a function called the **form ratio**

$$F = \frac{K_1 + O_1}{M_2 + S_2}$$

which measures the relative strength of the diurnal and semi-diurnal tides.

F > 3	Diurnal	1 High, 1 Low per day
0.25 < F < 3	Mixed	2 Highs, 2 Lows per day, but of different strength
F < 0.25	Semidiurnal	2 Highs, 2 Lows per day, similar strength.

The following figure is from Pond and Pickard and shows time series of tidal height at different locations.
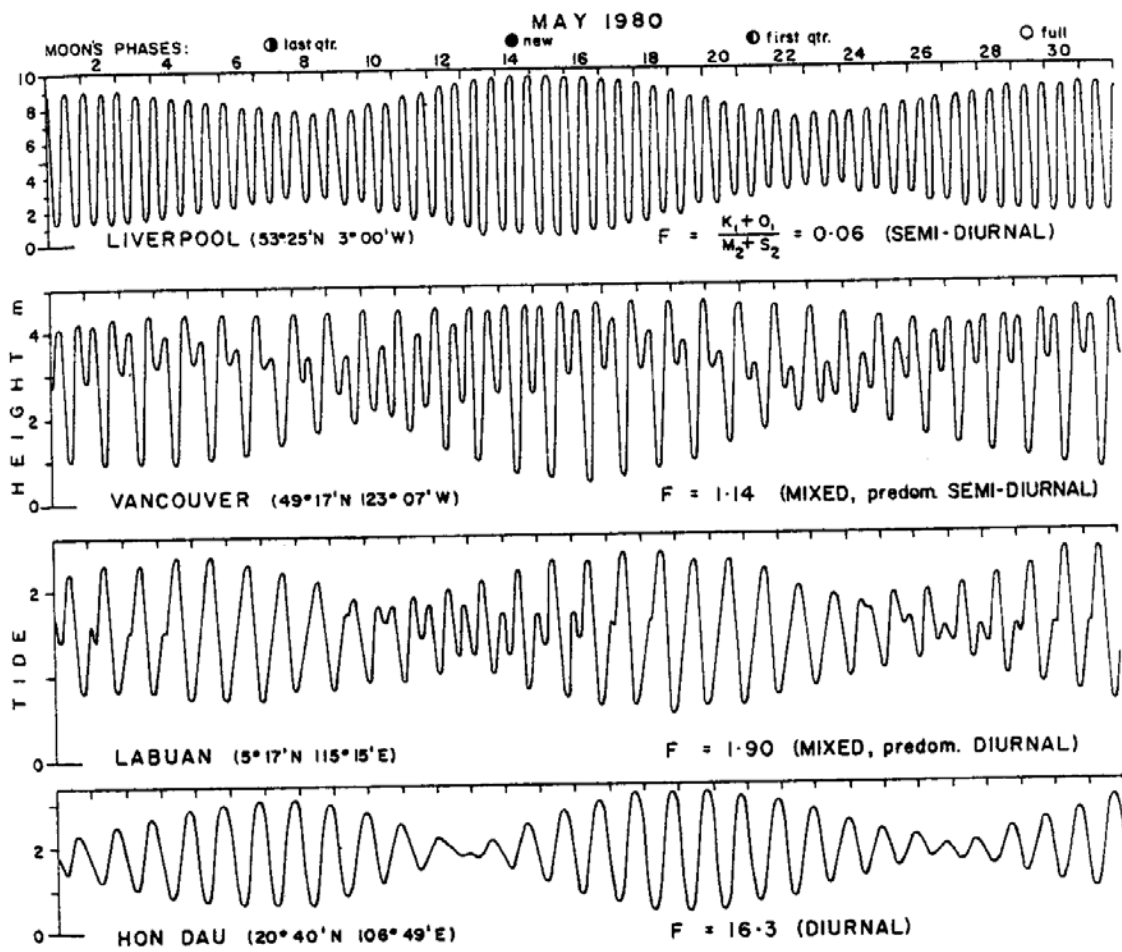

FIG. 13.1   *Tide curves for May 1980 (data from Admiralty Tide Tables) showing four types in terms of the "form ratio"* $F = (K_1 + O_1)/(M_2 + S_2)$ *of major diurnal to semi-diurnal constituents.*
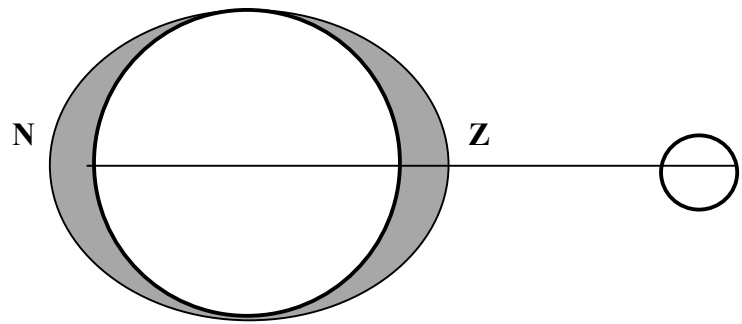
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Tides

So far, we have considered four forces that act on the sea water to cause it to move. These are: pressure gradient force, gravity of the earth, the Coriolis force, and friction. We have talked about different situations in which one or more of the four is more prominent over the other, and different types of motions that thus arise: geostrophic current, Ekman transport, surface and internal waves, inertial oscillation.

We now introduce a fifth force: gravity of sun and moon. This causes the phenomenon we know very well as the tide. Our final topic of the course is tides, and important physical phenomena associated with or controlled by tides.

Tidal forcing

To understand how tides are generated, let's first just consider the effect of the moon. Imagine that we could freeze the motion of the Earth and the moon and let the ocean come into *equilibrium* with the gravitational field. At the point on the Earth directly under the moon **Z**, we can see intuitively that the water would be drawn towards the moon by its gravitational attraction. What is not so obvious is that fluid on the far side of the Earth **N** would also be drawn toward the moon, creating a *second bulge.* Now imagine that the Earth is rotating on its axis. If the system could stay in equilibrium, the two bulges would propagate around the Earth, giving two highs and two lows for each rotation (day), hence, the semi-diurnal tide.

Let's assume that this motion of the bulge could keep up with the forcing. This leads to the **equilibrium theory of tides,** or **direct tides**. There are two forces, gravity and centripetal force, that together produce the **tide-producing force**.

The Earth and moon rotate around the center of mass **C** of the combined system once every 27.3 days. The center of mass of this system is nearly at the center of the Earth because the mass of the Earth is so much larger than the mass of the moon. A **centripetal force** is needed to maintain the rotation of the Earth-moon system and that force is provided by the *gravitational* attraction between the two masses. [Note that the Earth-moon rotation is independent of the Earth's rotation on its axis.]

The **gravitational force** varies according to the familiar inverse square law
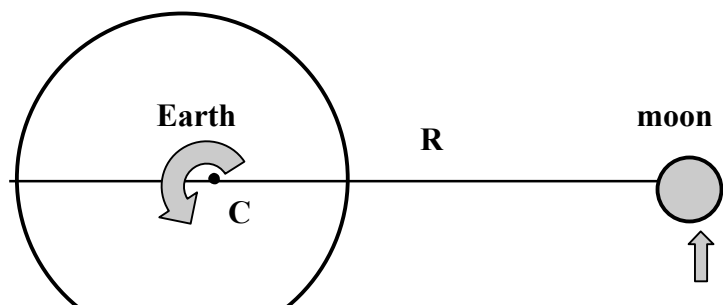
$$F_G = \frac{GMm}{r^2}$$

M = mass of Earth
m = mass of moon
r = distance between a point on the Earth and the center of the moon
G = universal gravitational constant

The gravitational force on a fluid parcel (per unit mass) is

$$F_G = \frac{Gm}{r^2} \quad (21.1)$$

and is directed *towards the moon*.

The distance from the Earth's center to the moon **R** is determined by the requirement that gravity supply the centripetal force. So we know that the magnitude of the centripetal force is the average gravitational force above. However, the gravitational force on individual water parcels may be either larger or smaller than the average force.

Why does the tide happen twice a day?

I'm sure you have wondered about this question. After all, we are usually told only that tides are due to gravity of sun and moon; so it's natural to think tides would happen only once a day, like the sunrise or the moonrise. In fact, the statement "tides are caused by the gravity of the sun and the moon" is inadequate in an important way, and understanding this point holds the key to the understanding of the phenomenon. Tides are caused due to the gravity of the sun and the moon *and because they vary in their strength from one side of the earth to the other.*

The argument for this is rather subtle, so I beg for your patience. To understand this, consider a hypothetical earth-moon system in which each member does not rotate but they revolve around each other. Here, rotation is relative to the rest of the universe: relative to the positions of distant stars, as it were. And note that I said that the earth and the moon revolve around each other. Although it's usual to think of the moon as
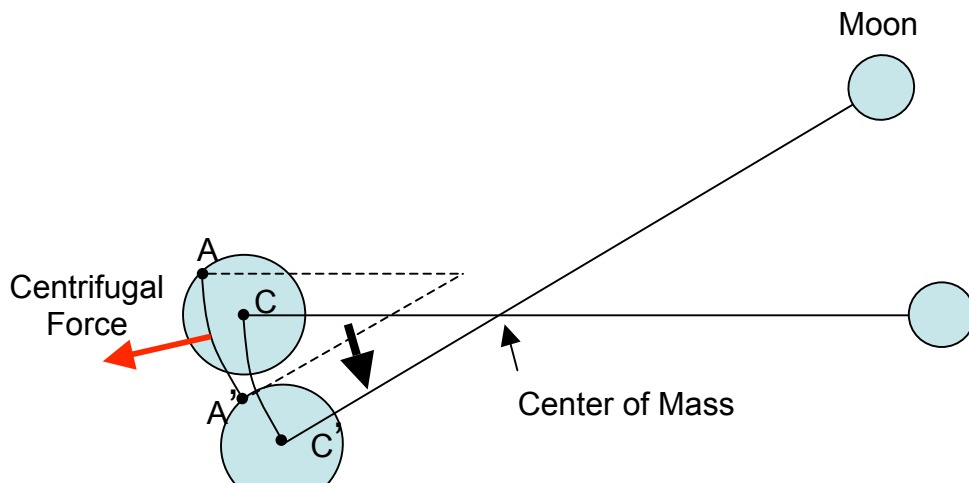
revolving around the earth, this is because the earth has more mass, and thus moves less.

At the center of the earth, there is an exact balance between the gravitational attraction of the moon and the centrifugal force of earth's revolution. The former is given by Newton's law of gravitation, in which the force is proportional to the product to the masses of the earth and the moon, and is inversely proportional to the distance between the two.

The latter is given by the mass of the earth times the centrifugal acceleration, given by the square or the angular rate of rotation times the radius of revolution. A balance of the same nature applies to the moon, and from this one can deduce that the radius of revolution for the earth is the distance to the moon divided by the ratio of the moon's mass and the total mass of the earth-moon system. (Consult any standard textbook on classical mechanics that describe Newton's theory of gravitation for details.)

Since this hypothetical earth does not rotate, each point on the earth executes the same circular motion the center of the earth executes, and thus experiences the same centrifugal force as a result. This centrifugal force is balanced by the moon's gravitational attraction. But the moon's gravity is not the same everywhere on the earth. For one thing, it is oriented slightly differently at different points because of the parallax effect; but more importantly, it is slightly stronger on the side of the earth facing the moon than the other side, because of the inverse-square law. Because of this, there is a slight imbalance between the centrifugal force of revolution and the moon's gravity everywhere on the earth except at the center. The solid earth can even out such imbalance via internal elasticity, but mobile objects such as the ocean can be set in motion by this force imbalance.

Now let's add rotations to this. The earth's rotation carries with it its own centrifugal

Earth
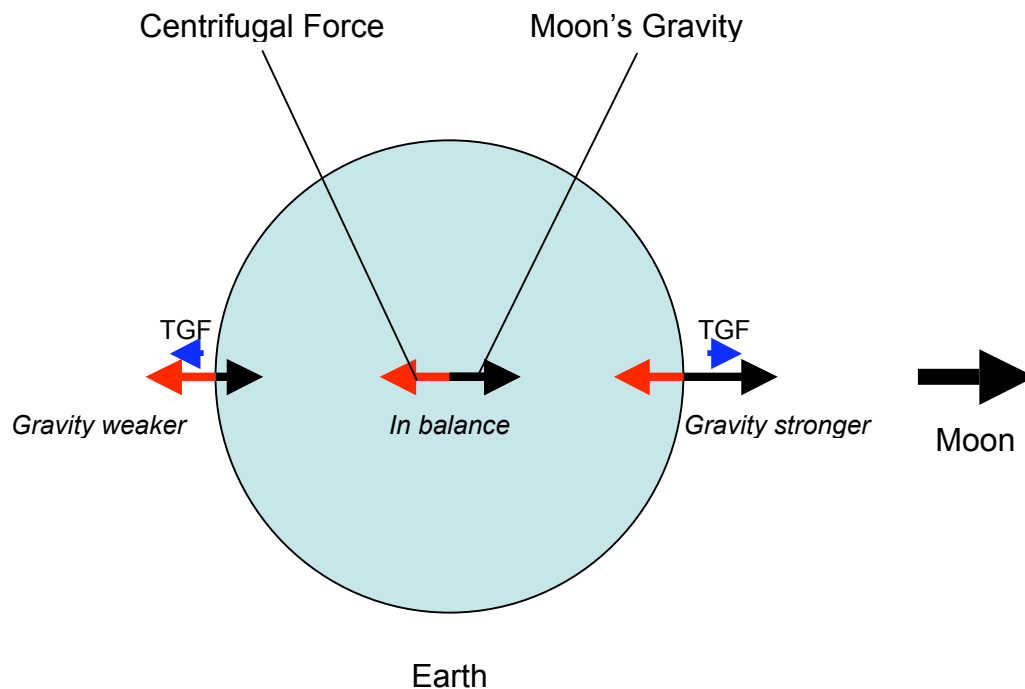
force, but this is balanced by gravity (in fact, what we normally think of gravity is gravity minus the centrifugal force) so we can focus on the forces associated with revolution. When we are on the side closer to the moon, the moon's gravity slightly outweighs the centrifugal force of revolution, and the imbalance is a force directed towards the moon. When we are on the other side, the centrifugal force outweighs the moon's gravity, and the imbalance is away from the moon. On both occasions though, the imbalance is *away from the center of the earth*, or *upward*. So, as we go round with the earth once each day, we experience a *twice-daily upward force*, and this is the force that's responsible for the tidal phenomenon.
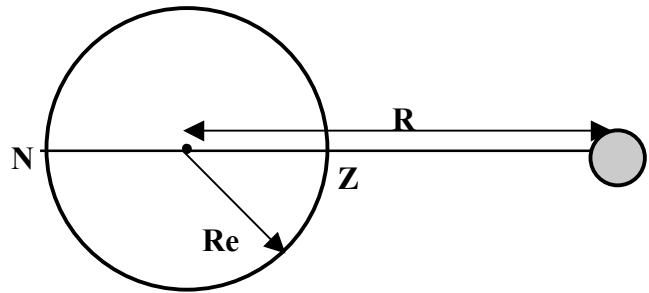
The actual imbalance force, called the *tide-generating force* (TGF) has horizontal components (called *tractive force*) as well as the vertical component, and is distributed as shown below (this is a hypothetical case exaggerated to show the direction of TGF). Because it is difficult to generate vertical movement of water in the ocean, tractive forces are much more important for the generation of tides than the vertical component of TGF. That the moon's gravity is stronger on one side of the earth than the other is crucial to the argument we just made. In fact, physicists generalize from this and call any deformational force that results from gravity varying from one point of an object to another "tidal". Such forces are of great astronomical importance, for example in the dynamics of binary stars (pairs of stars that orbit each other closely). Tidal forces were responsible for the breakup of the comet Shoemaker-Levy 9 before it collided with Jupiter in 1994.
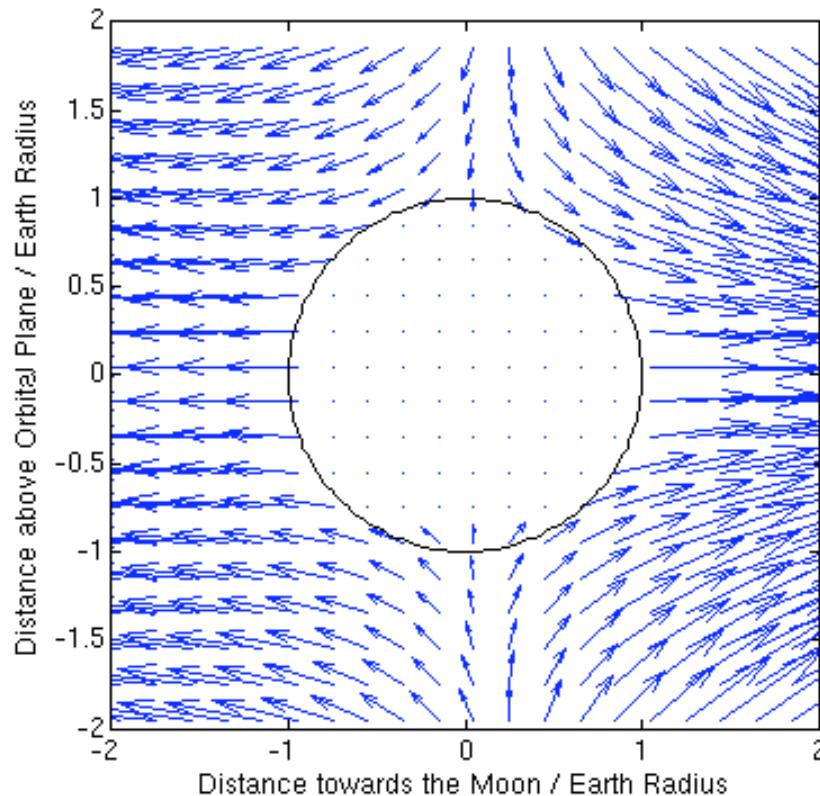
This is also the reason why oceanographers have been discouraging the use of words "tidal waves" to describe the *tsunami*. For tsunamis are not caused tidally, that is due to differential gravity; they are, rather, caused by seafloor movements such as submarine landslides and tectonic movements.

Magnitude of tidal forces on earth

How strong is the tidal force on the earth? The above argument shows that the maximum tidal force on earth would be the maximum difference in the moon's gravitational pull between a point on earth and the earth center. We can estimate this from Newton's inverse square law of gravity. For an object of unit mass, the moon's gravitational pull will be given by GM_M/d^2, where G is the constant of gravitation (6.673×10^{-11} m^3 kg^{-1} s^{-2}), M_M the mass of the moon (7.36×10^{22} kilograms), and d the distance to the moon. The average earth-moon distance is 3.78×10^8 meters while the radius of the earth is 6.38×10^6 meters; so the moon's gravity at the center of the earth is 3.44×10^{-5} m s^{-2} per unit mass, while at the point on the surface directly below the moon it is 3.56×10^{-5} m s^{-2}(this is the sublunary or zenith point, the point opposite if the nadir or antipodes) The difference is 1.2×10^{-6} m s^{-2} – this is the maximum acceleration we can expect from lunar tidal forces on earth. Compared with the earth's own gravitational acceleration – 9.8 m s^{-2} – this is minuscule, and indeed we humans don't feel the sway of the tide! The ocean, however, represents the largest movable mass on the earth; and tidal forces act globally and throughout its depth. For example, when integrated over the average depth of the ocean of four kilometers, this becomes 4.8 Newtons per square meter – an order of magnitude greater than typical wind stress.

The sun also contributes to the tides by its gravity. The sun is far more massive than the moon (1.98892×10^{30} kg), but it's also much further away (1.5×10^{11} m) and its gravitational effect is weaker. Still, the above estimate applied to the sun would show the sun's tide-generating force to be about 5.5×10^{-7} m s^{-2}, or 45 percent of the lunar tide.

Semi-diurnal and diurnal tides

Tides that occur twice a day are characterized as *semi-diurnal*. The magnitude of each tidal occurrence is not the same; typically, one is larger than the other. Thus we talk of a *major* flood and a *minor* flood, a major ebb and a minor ebb, with respect to tidal currents. This can be understood in terms of another, *diurnal*, tide that occurs once a day added on top of the semi-diurnal tide. So after all, there is a once-a-day component to the tides!

We've just discussed how the twice-daily tidal force arises from the moon's gravity being stronger on one side of the earth than the other. So how does the diurnal tide arise? For one thing, the amount of imbalance is not exactly the same (the imbalance on the side closer to the moon is slightly larger). But more important is the effect of the earth's rotational axis being inclined to the orbital plane, or *obliquity*.

If you are on the Arctic Circle, as depicted below, then when you are furthest away from the moon your distance to the moon is the same as the earth's center. So you experience no force imbalance; at other times the force imbalance you experience is always directed towards the moon. This means your imbalance force goes only through once a day cycle. While this effect is the easiest to understand near the poles, the *magnitude* of diurnal variation is greater at lower latitudes because the distance to the moon varies by a greater amount there.
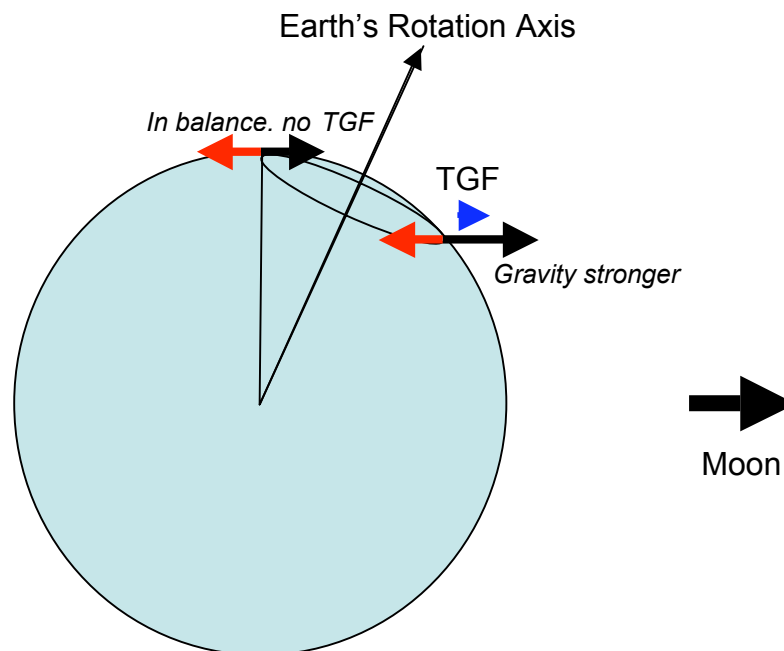
The actual relative importance of semi-diurnal and diurnal tides are determined by the ocean's depth and geometry as well as the distribution of tide-generating forces. The figure below (from the web site http://www.physicalgeography.net/fundamentals/8r.html) shows which type of tide predominates along coastlines of the world ocean. When both types are important, the tide is characterized as "mixed"; we will give a precise definition

of the term, when we discuss constituent description of
tides.
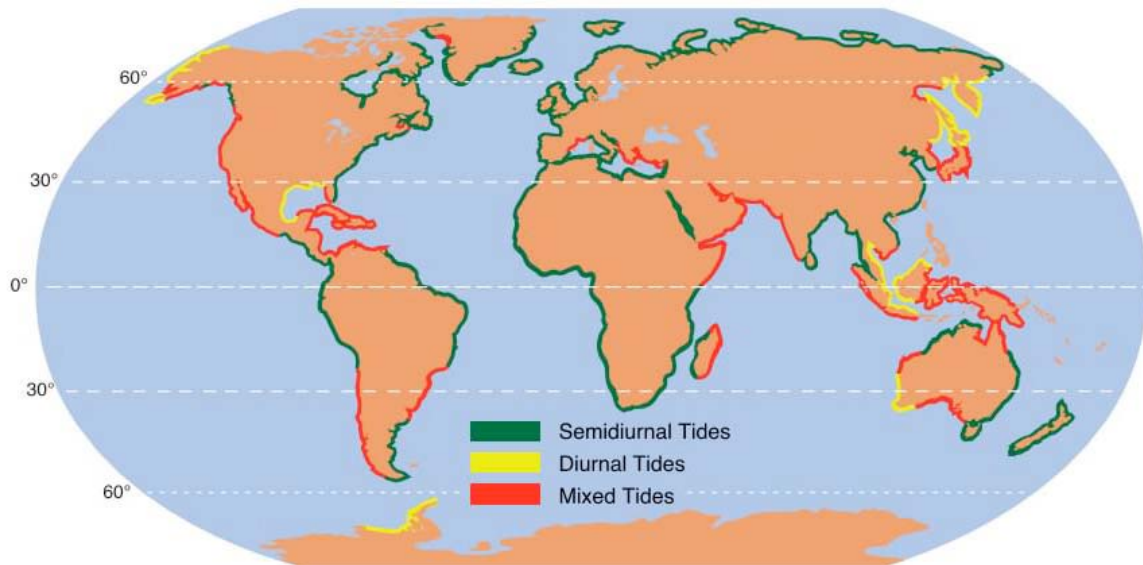
Equilibrium response.

The ocean does *not* produce tides as a direct response to the vertical forces at the bulges.
The tidal force is only about 1 ten millionth the size of the gravitational force owing to
the Earth's gravity. It is the horizontal component of the tidal force that produces the
tidal ellipsoid, causing fluid to converge (and bulge) at the sublunar and antipodal points
and move away from the poles, causing a contraction there.

The projection of the tidal force onto the horizontal direction is called the **tractive force**
(see Knauss, Fig. 10.11). This force causes an acceleration of water towards the sublunar
and antipodal points, building up water until the pressure gradient force from the bulging
sea surface exactly balances the tractive force field.

In the equilibrium model, as the Earth rotates under this tidal force, each point on the
Earth would experience 2 highs and 2 lows per day, that is, the equilibrium tide is
semidiurnal. We define the *lunar day* as the period (24 h 50 m) between times when the
moon is above the same point on the Earth. The semidiurnal lunar tide (the M_2 tide) has
a period of 12 h 25 min. The period of the semidiurnal solar tide is 12 h (S2 tide).
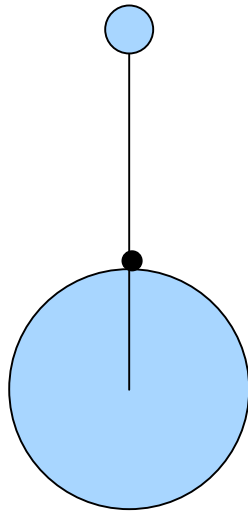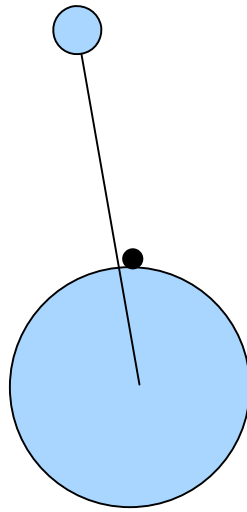
T = 0, moon overhead   **T = 24 hours**   **T = 24 hr 50 min, moon overhead**
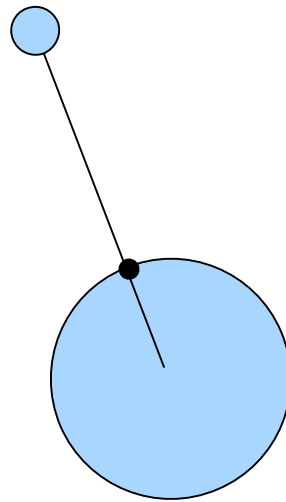
The **declination of the moon** (the angle the moon makes with the Earth's equatorial plane, up to 28°) varies over a cycle with a period of 27.2 days. Lunar declination causes uneven highs and lows and introduces a diurnal component to the tides, that is, to produce the kind of signal seen at a fixed point on the Earth, you would need to add diurnal and semidiurnal sine waves. The effect is greatest at a location under the moon at maximum declination, ± 28°. Thus we expect the largest diurnal tides near in the tropics ("tropic tides"). We expect predominantly semidiurnal tides at the equator and at high latitudes.
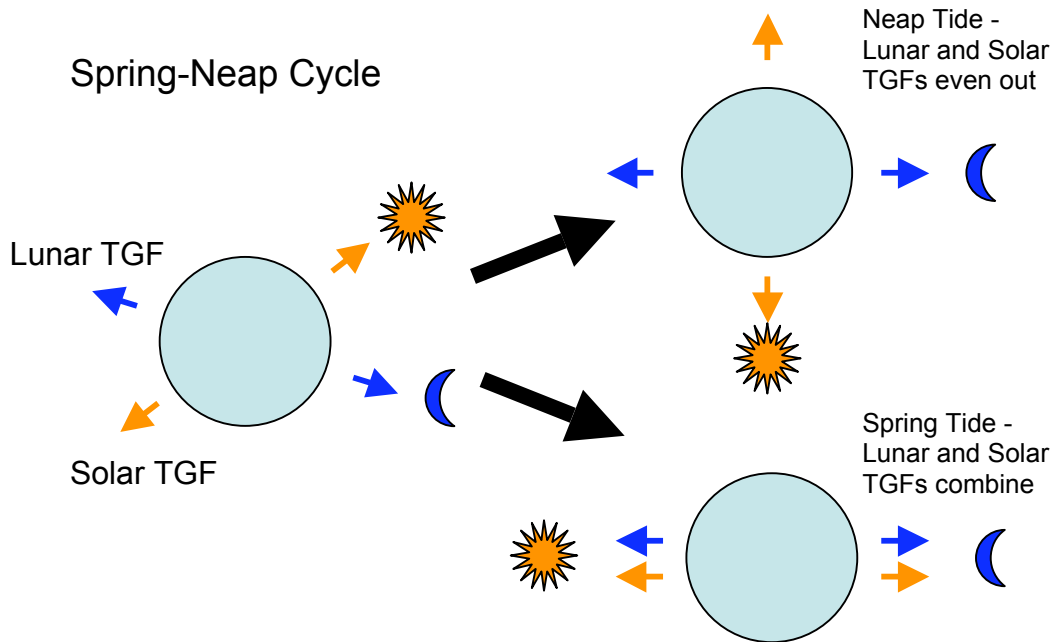
The periods of the diurnal tides are not exactly 24 hours. To understand the periods of the tides, we define a speed with which the phase progresses. For instance, for the solar semi-diurnal tide with period 12 hours, it moves 360 degrees. That means that it moves 30 degrees per hour. To find the diurnal tides, we take the speed of the rotation of the Earth about its axis, and then add to it the speed with which the Earth rotates around the sun. The rotation of the Earth about its axis has speed 15 (that is 360 degrees in 24 hours), while the rotation around the Earth has speed 360/365.25/24=0.0411 degrees per hour. There are two diurnal tides, the K1 and the P1 that have speeds 15+/-0.0411. This gives periods of 23.93 hours and 24.07 hours respectively. Likewise, the lunar diurnal tide has speed given by half that of the M2 tide (14.49 degrees per hour) and then add or subtract from that 360/27.32/24=0.549 degrees per hour. This gives a period of 25.82 hours and   23.93 hours respectively. Thus we have three primary diurnal tides.

Variability in Tides

In addition to repeating from day to day, tides show variability in strength over longer time scales. The most noticeable of this is the *fortnightly cycle,* in which tidal strength go

round a cycle of strong-to-weak-to-strong in about two weeks. The Earth and moon are rotating about their center of mass (period = 27.3 days) and the combined system is rotating around the sun. The cycle of the moon position relative to sun is 29.5 days long and results in the modulation of tidal amplitude that we call **spring** and **neap tides**.

The best-known source of the fortnightly tidal variability is the *spring-neap cycle* due to constructive and destructive interference of solar and lunar tides. When the sun and the

Spring-Neap Cycle

Lunar TGF

Solar TGF

Neap Tide - Lunar and Solar TGFs even out

Spring Tide - Lunar and Solar TGFs combine

moon are on the same side of the earth (new moon) or on the opposite sides (full moon), the tide-generating forces of the sun and moon combine, and give rise to tides of greater amplitudes, or *spring tides*. When the moon is ninety degrees away from the sun in the sky (first or last quarter), the tide-generating forces tend to even out, and result in weaker, or *neap*, tides. Since the period of the moon's revolution relative to the sun is 29.5 days, the spring-neap cycle occurs with a period half of that, or approximately two weeks. Note that this is once again a demonstration that it is the difference in the gravitational force on one side of the earth from the other that is responsible for the tides; for if tides were due to a simple pull by the moon and the sun, then one might expect the largest tides at new moon when the sun's and the moon's gravities combine, and the weakest tides at full moon when they oppose.

A less known cause, often more important when diurnal tides dominante, is the *lunar declination cycle*. The earth's rotation axis is always (at least over human time scales) pointing in the same direction: the north pole more or less in the direction of Polaris, and the south pole, towards an obscure star called Sigma Octans. As the moon goes around the earth, the diection of the moon relative to the earth's rotation axis (called *lunar declination*) changes. The picture we drew when we discussed the diurnal tide depicted the situation when the axis is most aligned with the moon. On the other hand, if the moon is ninety degrees from that position – perpendicularly above the page – then the effect of obliquity is the least strong, and the diurnal tide almost disappears. The lunar declination cycle has a similar to but slightly shorter period (half the moon's sidereal revolution

period of 27.3 days) than the spring-neap cycle. Its amplitude effect on the diurnal tide is much greater than that of the spring-neap cycle on the semi-diurnal tide. In common parlance however, it's usual to refer to any large tide that occurs every two weeks as "spring", and weak tide as "neap".

Further variability comes from *solar declination cycle,* which arises in the same way but with respect to the sun and has the period of half a year. The solar diurnal tide is the greatest at the solstices, when the earth's rotation axis is the most oriented towards the sun. You will be familiar with this if you live in the Puget Sound area and are engaged in beachcombing, because the lowest low tides occur in June and December. There are also variability that comes from varying distances between the sun and the earth, and the earth and the moon, at different points on their respective orbits. These introduce monthly and annual components to the tidal variability. The oscillation of the lunar distance to the Earth has a period of 27.5546 days and this gives the monthly tide. There are also very long period tides associated with changes in the moon's orbit around the Earth. The lunar orbit repeats in a 18.61 year cycle. The lunar phase repeats with a 19 year cycle, and there is an eclipse cycle if 18.03 years.

Lunar Declination Cycle

Pole

Rotation axis oriented 90° to the moon - minimum diurnal tide

Rotation axis oriented to the moon - maximum diurnal tide

Viewed from above the orbital plane