Performance of Quicksort

We will count the number C(n) of comparisons performed by quicksort in sorting an array of size n.

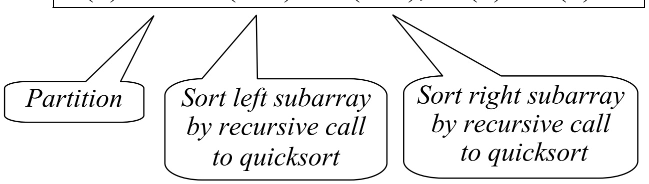
We have seen that *partition*() performs n comparisons (possibly $n-1$ or $n+1$, depending on the implementation).

> In fact, $n-1$ is the lower bound on the number of comparisons that any partitioning algorithm can perform.
>
> The reason is that every element other than the pivot must be compared to the pivot; otherwise we have no way of knowing whether it goes left or right of the pivot.

So our recurrence for C(n) is:

$$\boxed{\mathrm{C}(n) = n + \mathrm{C}(k-1) + \mathrm{C}(n-k), \quad \mathrm{C}(0) = \mathrm{C}(1) = 0}$$

(k = final position of pivot element)



A bad case (actually the worst case): At every step, *partition*() splits the array as unequally as possible ($k = 1$ or $k = n$).

> Then our recurrence becomes
>
> $$\mathrm{C}(n) = n + \mathrm{C}(n-1), \quad \mathrm{C}(0) = \mathrm{C}(1) = 0$$
>
> This is easy to solve.

$$\begin{aligned}
\mathrm{C}(n) &= n + \mathrm{C}(n-1) \\
&= n + n-1 + \mathrm{C}(n-2) \\
&= n + n-1 + n-2 + \mathrm{C}(n-3) \\
&= n + n-1 + n-2 + \ldots + 3 + 2 + \overset{0}{\cancel{\mathrm{C}(1)}} \\
&= (n + n-1 + n-2 + \ldots + 3 + 2 + 1) - 1 \\
&= n(n+1)/2 \ - 1 \\
&\approx n^2/2
\end{aligned}$$

> This is terrible. It is no better than simple quadratic time algorithms like straight insertion sort.

A good case (actually the best case): At every step, *partition*() splits the array as equally as possible ($k = (n+1)/2$; the left and right subarrays each have size $(n-1)/2$)).

> This is possible at every step only if $n = 2^k - 1$ for some k. However, it is always possible to split nearly equally. The recurrence becomes
>
> $$\mathrm{C}(n) = n + 2\mathrm{C}((n-1)/2), \quad \mathrm{C}(0) = \mathrm{C}(1) = 0,$$
>
> which we approximate by
>
> $$\mathrm{C}(n) = n + 2\mathrm{C}(n/2), \quad \mathrm{C}(1) = 0$$
>
> This is the same as the recurrence for mergesort, except that the right side has n in place of $n-1$. The solution is essentially the same as for mergesort:
>
> $$\mathrm{C}(n) = n\lg(n).$$
>
> This is excellent — essentially as good mergesort, and essentially as good as any comparison sorting algorithm can be.

The expected case: Here we assume either (i) the array to be partitioned is randomly ordered, or (ii) the pivot element is selected from a random position in the array.

In either case, the pivot element will be a random element of the array to be partitioned. That is, for $k = 1, 2, ..., n$, the probability that the pivot element is the k^{th} largest element of the array is $1/n$. (Recall that, if the pivot element is the k^{th} largest element of the array, it ends up after partitioning in position k.)

In the recurrence

$$C(n) = n + C(k-1) + C(n-k), \quad C(0) = C(1) = 0,$$

all values of k are equally likely. We must average over all k.

$$\begin{aligned} C(n) &= (1/n)\textstyle\sum_{k=1}^{n} \big(n + C(k-1) + C(n-k)\big), \quad C(0) = C(1) = 0, \\ &= n + (1/n)\textstyle\sum_{k=1}^{n} C(k-1) + (1/n)\sum_{k=1}^{n} C(n-k) \end{aligned}$$

Note: $\sum_{k=1}^{n} C(k-1) = \sum_{i=0}^{n-1} C(i)$, by substituting $i = k-1$.
$\sum_{k=1}^{n} C(n-k) = \sum_{i=0}^{n-1} C(i)$, by substituting $i = n-k$.

So our recurrence becomes

$$C(n) = n + (2/n)\textstyle\sum_{i=0}^{n-1} C(i), \quad \text{or}$$

$$n\,C(n) = n^2 + 2\textstyle\sum_{i=0}^{n-1} C(i)$$

Writing down the same recurrence with $n-1$ replacing n, we get

$$(n-1)\,C(n-1) = (n-1)^2 + 2\textstyle\sum_{i=0}^{n-2} C(i).$$

Subtracting this recurrence from the one above it gives

$$n\,C(n) - (n-1)\,C(n-1) = n^2 - (n-1)^2 + 2\,C(n-1), \quad \text{or}$$

$$n\,C(n) = (n+1)\,C(n-1) + 2n-1$$

Dividing by $n(n+1)$ gives

$$C(n)/(n+1) = C(n-1)/n + (2n-1)/(n(n+1)).$$

To a very good approximation,

$$C(n)/(n+1) = C(n-1)/n + 2/n.$$

Now if let $D(n) = C(n)/(n+1)$, then the recurrence becomes

$$D(n) = D(n-1) + 2/n, \quad D(1) = 0.$$

This is easy to solve:

$$\begin{aligned} D(n) &= D(n-1) + 2/n \\ &= D(n-2) + 2/(n-1) + 2/n \\ &= D(n-3) + 2/(n-2) + 2/(n-1) + 2/n \\ &= \overset{0}{\cancel{D(1)}} + 2/2 + 2/3 + \ldots + 2/(n-2) + 2/(n-1) + 2/n \\ &= 2\ln(n) - 2 \\ &\approx 2\ln(n) \\ &= 2\ln(2)\lg(n) \\ &\approx 1.39\lg(n) \end{aligned}$$

So $C(n) = (n+1)\,D(n) \approx 1.39\,(n+1)\,\lg(n)$, or $\boxed{C(n) \approx 1.39\,n\lg(n)}$

The expected case for quicksort is fairly close to the best case (only 39% more comparisons) and nothing like the worst case.

In most (not all) tests, quicksort turns out to be a bit faster than mergesort.

Quicksort performs 39% more comparisons than mergesort, but much less movement (copying) of array elements.

We saw that, in the expected case, quicksort performs one exchange for every six comparisons, or about $1.39\,n\lg(n)/6 \approx 0.23\,n\lg(n)$ exchanges.

A slightly different partitioning algorithm performs one move (copy) for each three comparisons, or about $0.46\,n\lg(n)$ moves.

By contrast, the version of mergesort given in class performs $2\,n\lg(n)$ moves, although this can be reduced to $n\lg(n)$ moves — still more than twice as many as quicksort is likely to perform.

With a randomized version of quicksort (pivot element chosen randomly), the standard deviation in the number of comparisons is also small.

The probability of performing substantially more than $1.39\,n\lg(n)$ comparisons is extremely low.

Quicksort is not stable, since it exchanges nonadjacent elements.

If stability is not required, quicksort provides a very attractive alternative to mergesort.

Quicksort is likely to run a bit faster than mergesort — perhaps 1.2 to 1.4 times as fast.

Quicksort requires less memory than mergesort.

A good implementation of quicksort is probably easier to code than a good implementation of mergesort.