

# Generating Function of Some k-Fibonacci and k-Lucas Sequences

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**Abstract** – The k-Fibonacci and the k-Lucas numbers are particular cases of the different generalizations of the classical Fibonacci and Lucas numbers made by different authors. In this paper, first of all, we present the k-Fibonacci and the k-Lucas numbers and we remember some of the properties that we will need throughout this article. Then, we study the relationship between the product of two k-Fibonacci or k-Lucas numbers with subscripts in linear form and the k-Lucas numbers. We thus enter the main part of the paper and find the generating function of some k-Fibonacci and k-Lucas numbers, that can be used later studies. It is interesting to note that because the definition of the k-Fibonacci and the k-Lucas numbers is based on the same recurrence relation, we find that generating functions are similar for both types of numbers. As for the denominators, there is only difference in some sign, while the numerators are different because the initial conditions of both sequences are different. By last, we give examples of application of the preceding formulas to find the generation function of some new sequences.

**Keywords** – k-Fibonacci and k-Lucas numbers, Binet Identity, Generating Function, Convolution.

## I. INTRODUCTION

One of the more studied sequences is the Fibonacci sequence [1, 2, 3, 4] and it has been generalized in many ways [5]. Here, we use the following one-parameter generalization of the Fibonacci sequence [6, 7].

### 1.1 Definition 1

For any integer number  $k \geq 1$ , the k-Fibonacci sequence, say  $\{F_{k,n}\}_{n \in \mathbb{N}}$ , is defined recurrently as  $F_{k,n+1} = k F_{k,n} + F_{k,n-1}$  for  $n \geq 1$  with initial conditions  $F_{k,0} = 0$ ,  $F_{k,1} = 1$

First few k-Fibonacci numbers are  $\{0, 1, k, k^2 + 1, k^3 + 2k, \dots\}$ . Note for  $k = 1$  the classical Fibonacci sequence is obtained and for  $k = 2$ , it is the Pell sequence:

$$F = \{0, 1, 1, 2, 3, 5, 8, \dots\}, \text{ A000045 in OEIS [8].}$$

$$P = \{0, 1, 2, 5, 12, 29, 70, \dots\}, \text{ A000129}$$

Characteristic equation from the definition is  $r^2 = kr + 1$  whose solutions are  $\sigma_{1,2} = \frac{k \pm \sqrt{k^2 + 4}}{2}$  that verify the following properties:

$$\sigma_1 \sigma_2 = -1, \sigma_1 + \sigma_2 = k, \sigma_1 - \sigma_2 = \sqrt{k^2 + 4}, \sigma^2 = k\sigma + 1, \sigma_1 > 0, \sigma_2 < 0.$$

Generating function of the k-Fibonacci numbers is  $f(k,x) = \frac{x}{1 - kx - x^2}$ . For the properties of the k-Fibonacci

numbers see [6, 7]. We can find any k-Fibonacci number by mean of the Binet Identity [9]  $F_{k,n} = \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2}$ . Finally,

$$F_{k,-n} = (-1)^{n+1} F_{k,n}$$

### 1.2 Definition 2

For any integer number  $k \geq 1$ , the  $k$ -Lucas sequence, say  $\{L_{k,n}\}_{n \in \mathbb{N}}$ , is defined recurrently as  $L_{k,n+1} = k L_{k,n} + L_{k,n-1}$  for  $n \geq 1$  and initial conditions  $L_{k,0} = 2$ ,  $L_{k,1} = k$ , [10].

For  $k = 1$  the classical Lucas sequence is obtained and for  $k = 2$ , it is the Lucas-Pell sequence. Generating function of the  $k$ -Lucas numbers is  $l(k,x) = \frac{2-kx}{1-kx-x^2}$ . Binet Identity for the  $k$ -Lucas numbers is  $L_{k,n} = \sigma_1^n + \sigma_2^n$ .

The  $k$ -Lucas numbers are related to the  $k$ -Fibonacci numbers by the relation  $L_{k,n} = F_{k,n-1} + F_{k,n+1}$ . Moreover,  $L_{k,-n} = (-1)^n L_{k,n}$  in [11] the following formulas are proven. If  $r \in \mathbb{N} - \{0\}$ ,

$$\sum_{i=0}^m L_{k,ri+p} = \frac{L_{k,r(m+1)+p} - (-1)^r L_{k,rm+p} + (-1)^p L_{k,r-p} - L_{k,p}}{L_{k,r} - (-1)^r - 1} \quad (1)$$

$$\sum_{i=0}^m (-1)^i L_{k,ri+p} = \frac{(-1)^m L_{k,r(m+1)+p} + (-1)^{r+m} L_{k,rm+p} + (-1)^m L_{k,r-p} + L_{k,p}}{L_{k,r} + (-1)^r + 1} \quad (2)$$

In particular,  $\sum_{j=0}^n L_{k,j} = \frac{1}{k} (L_{k,n+1} + L_{k,n} + k - 2)$

## II. PRODUCT OF TWO $k$ -FIBONACCI AND $k$ -LUCAS NUMBERS

Before we prove a lemma that we need to find the sum of the products of two  $k$ -Fibonacci numbers with subscripts in linear form.

### 2.1 Lemma (Product of two $k$ -Fibonacci Numbers)

Let  $p, q$  be integer numbers. Product of two  $k$ -Fibonacci numbers is

$$F_{k,p} F_{k,q} = \frac{1}{k^2 + 4} (L_{k,p+q} - (-1)^q L_{k,p-q}) \quad (3)$$

*Proof.*

Applying the Binet Identity, and taking into account  $\sigma_1 \sigma_2 = -1$ ,

$$\begin{aligned} F_{k,p} F_{k,q} &= \frac{1}{k^2 + 4} (\sigma_1^p - \sigma_2^p) (\sigma_1^q - \sigma_2^q) = \frac{1}{k^2 + 4} (\sigma_1^{p+q} + \sigma_2^{p+q} - (\sigma_1^p \sigma_2^q + \sigma_1^q \sigma_2^p)) \\ &= \frac{1}{k^2 + 4} (L_{k,p+q} - (\sigma_1^{p-q+q} \sigma_2^q + \sigma_1^q \sigma_2^{p-q+q})) = \frac{1}{k^2 + 4} (L_{k,p+q} - (-1)^q (\sigma_1^{p-q} + \sigma_2^{p-q})) \\ &= \frac{1}{k^2 + 4} (L_{k,p+q} - (-1)^q L_{k,p-q}) \end{aligned}$$

### 2.2 Relationship between the $k$ -Fibonacci and the $k$ -Lucas Numbers

If in the equation (3) it is  $q = 1$ , we obtain the formula that relates the  $k$ -Fibonacci and the  $k$ -Lucas numbers:

$$F_{k,p} = \frac{1}{k^2 + 4} (L_{k,p+1} + L_{k,p-1}) \quad (4)$$

### 2.3 Sum of the Products of two $k$ -Fibonacci Numbers with Subscripts in Linear form

If in the equation (3) it is  $p = ai + r$  and  $q = bi + s$ , then

$$F_{k,ai+r} F_{k,bi+s} = \frac{1}{k^2 + 4} \left( L_{k,(a+b)i+(r+s)} - (-1)^{bi+s} L_{k,(a-b)i+(r-s)} \right) \quad (5)$$

From this equation (5), the sum of the products of two k-Fibonacci numbers with subscripts in linear form is

$$\sum_{i=0}^n F_{k,ai+r} F_{k,bi+s} = \frac{1}{k^2 + 4} \left( \sum_{i=0}^n L_{k,(a+b)i+(r+s)} - (-1)^s \sum_{i=0}^n L_{k,(a-b)i+(r-s)} \right) \quad (6)$$

where the sums are calculated by means of the formulas (1) and (2), doing  $m = a + b$ ,  $p = r + s$ , or  $m = a - b$ ,  $p = r - s$ .

Find the form of this formula lacks interest and it is much more practical to impose conditions to the numerical values involved in it.

#### 2.4 Theorem: Sum of the Squares of the k-Fibonacci Numbers

The sum of the squares of the k-Fibonacci numbers is

$$\sum_{j=0}^n F_{k,ai}^2 = \frac{1}{k^2 + 4} \left( \frac{L_{k,2a(n+1)} - L_{k,2an}}{L_{k,2a} - 2} - (-1)^{an} - ((-1)^a + 1)n \right) \quad (7)$$

We will divide the proof in two parts according “a” is odd or even. In the formula (3), if both subscripts are equal, then

$$F_{k,p}^2 = \frac{1}{k^2 + 4} \left( L_{k,2p} - 2(-1)^p \right) \quad (8)$$

In the formula (6), let us suppose  $b = a, s = r = 0$ . Then,

(1) If “a” is odd ( $a = 2p + 1$ ), the sum of the squares is

$$\sum_{i=0}^n F_{k,(2p+1)i}^2 = \frac{1}{k^2 + 4} \left( \frac{L_{k,2(2p+1)(n+1)} - L_{k,2(2p+1)n}}{L_{k,2(2p+1)} - 2} - (-1)^n \right)$$

*Proof.*

If we apply the formulas (8), (6) and (1),

$$\begin{aligned} \sum_{i=0}^n F_{k,(2p+1)i}^2 &= \frac{1}{k^2 + 4} \left( \sum_{i=0}^n L_{k,2(2p+1)i} - \sum_{i=0}^n (-1)^{(2p+1)i} 2 \right) \\ &= \frac{1}{k^2 + 4} \left( \frac{L_{k,2(2p+1)(n+1)} - L_{k,2(2p+1)n} + L_{k,2(2p+1)} - L_{k,0}}{L_{k,2(2p+1)} - 2} - 2 \frac{(-1)^{(2p+1)n} + 1}{2} \right) \\ &= \frac{1}{k^2 + 4} \left( \frac{L_{k,2(2p+1)(n+1)} - L_{k,2(2p+1)n}}{L_{k,2(2p+1)} - 2} - (-1)^n \right) \end{aligned}$$

(2) Similarly, if “a” is Even ( $a = 2p$ ),

$$\sum_{i=0}^n F_{k,2pi}^2 = \frac{1}{k^2 + 4} \left( \frac{L_{k,p(n+1)} - L_{k,4pn}}{L_{k,p} - 2} - 2n - 1 \right)$$

Joining both formulas in one we obtain the formula (7).

As particular cases

$$\sum_{i=0}^n F_{k,i}^2 = \frac{1}{k^2 + 4} \left( \frac{L_{k,2n+1}}{k} - (-1)^n \right)$$

$$\sum_{i=0}^n F_{k,2i}^2 = \frac{1}{k^2 + 4} \left( \frac{L_{k,4(n+1)} - L_{k,4n}}{L_{k,4} - 2} - 2n - 1 \right) \rightarrow \sum_{i=0}^n F_{k,2i}^2 = \frac{1}{k^2 + 4} \left( \frac{F_{k,4n+2}}{k^2} - 2n - 1 \right)$$

$$\sum_{i=0}^n F_{k,3i}^2 = \frac{1}{k^2 + 4} \left( \frac{L_{k,6(n+1)} - L_{k,6n}}{L_{k,6} - 2} - (-1)^n \right)$$

$$\sum_{i=0}^n F_{k,4i}^2 = \frac{1}{k^2 + 4} \left( \frac{L_{k,8(n+1)} - L_{k,8n}}{L_{k,8} - 2} - 2n - 1 \right) \rightarrow \sum_{i=0}^n F_{k,4i}^2 = \frac{1}{k^2 + 4} \left( \frac{F_{k,8n+2}}{k^2(k^2 + 2)} - 2n - 1 \right)$$

If  $a = r = 1$  and  $b = a$ , the equation (6) is

$$\sum_{i=0}^n F_{k,ai+1}^2 = \frac{1}{k^2 + 4} \left( \frac{L_{k,2a(n+1)+2} + L_{k,2an+2} + L_{k,2a-2} - L_{k,2}}{L_{k,2a} - 2} + 2 \sum_{i=0}^n (-1)^{ai} \right)$$

Simply apply again formulas (8), (6) and (1).

In particular, if  $a = 2$ , the sum of the squares of the odd k-Fibonacci numbers, taking into account the equation (4), is

$$\sum_{i=0}^n F_{k,2i+1}^2 = \frac{1}{k^2 + 4} \left( \frac{L_{k,4n+6} - L_{k,4n+2}}{k(k^2 + 4)} + 2n + 1 \right) = \frac{1}{k^2 + 4} \left( \frac{F_{k,4n+4}}{k} + 2(n+1) \right)$$

### 2.5 Sum of the Products of two Consecutive k-Fibonacci Numbers

If  $b = a = 1, r = 0, s = 1$ , the equation (6) becomes

$$\begin{aligned} \sum_{j=0}^n F_{k,i} F_{k,i+1} &= \frac{1}{k^2 + 4} \left( \sum_{i=0}^n L_{k,2i+1} + \sum_{i=0}^n (-1)^i L_{k,-1} \right) = \frac{1}{k^2 + 4} \left( \frac{L_{k,2(n+1)+1} - L_{k,2n+1} - L_{k,1} - L_{k,1}}{L_{k,2} - 2} + L_{k,-1} \frac{(-1)^n + 1}{2} \right) \\ &= \frac{1}{k^2 + 4} \left( \frac{k L_{k,2n+2} - 2k}{k^2} - k \eta(n) \right) \rightarrow \sum_{i=0}^n F_{k,i} F_{k,i+1} = \frac{1}{k^2 + 4} \left( \frac{L_{k,2n+2} - 2}{k} - k \eta(n) \right) \text{ with } \eta(n) = \frac{(-1)^n + 1}{2} \end{aligned}$$

### 2.6 Sum of the Products of Two k-Lucas Numbers

First we will prove

$$L_{k,p} L_{k,q} = L_{k,p+q} + (-1)^q L_{k,p-q} \tag{9}$$

*Proof.*

Taking into account the Binet Identity  $L_{k,r} = \sigma_1^r + \sigma_2^r$  and  $\sigma_1 \cdot \sigma_2 = -1$ , it is

$$\begin{aligned} L_{k,p}L_{k,q} &= (\sigma_1^p + \sigma_2^p)(\sigma_1^q + \sigma_2^q) = (\sigma_1^{p+q} + \sigma_2^{p+q}) + (\sigma_1^p\sigma_2^q + \sigma_1^q\sigma_2^p) = L_{k,p+q} + (\sigma_1^{p-q+q}\sigma_2^q + \sigma_1^q\sigma_2^{q-p+p}) \\ &= L_{k,p+q} + (-1)^q (\sigma_1^{p-q} + \sigma_2^{p-q}) = L_{p+q} + (-1)^q L_{k,p-q} \end{aligned}$$

Then, the sum of the products of two k-Lucas numbers is similar to the formula (6) without the coefficient  $\frac{1}{k^2+4}$  and changing  $-(-1)^s$  by  $+(-1)^s$  :

$$\sum_{i=0}^n L_{k,ai+r} L_{k,bi+s} = \sum_{i=0}^n L_{k,(a+b)i+(r+s)} + (-1)^s \sum_{i=0}^n (-1)^{bi} L_{k,(a-b)i+(r-s)}$$

In particular

$$\sum_{i=0}^n L_{k,i} L_{k,i+1} = \frac{L_{k,2n+2} - 2}{k} + k \eta(n)$$

$$\sum_{i=0}^n L_{k,i}^2 = \frac{L_{k,2n+1}}{k} + 2 + (-1)^n$$

$$\sum_{i=0}^n L_{k,2i}^2 = \frac{L_{k,4n+4} - L_{k,4n}}{k^2(k^2+4)} + 2n + 3 = \frac{F_{k,4n+2}}{k} + 2n + 3$$

$$\sum_{i=0}^n L_{k,2i+1}^2 = \frac{L_{k,4n+6} - L_{k,4n+2}}{k^2(k^2+4)} - 2n - 2 = \frac{F_{k,4n+4}}{k} - 2n - 2$$

### III. GENERATING FUNCTION OF SOME K-FIBONACCI AND K-LUCAS NUMBERS

From definition of k-Fibonacci numbers it is easy to prove

$$F_{k,p} = (k^2 + 2)F_{k,p-2} - F_{k,p-4} \tag{10}$$

#### 3.1 Theorem: Generating Function of the even k-Fibonacci Numbers

Generating function of the even k-Fibonacci numbers  $\{F_{k,2n}\}$  is  $f_e(x) = \frac{kx}{1 - (k^2 + 2)x + x^2}$

*Proof.*

Taking into account the recurrence relation (10)

$$\begin{aligned} f_e(x) &= F_{k,0} + F_{k,2}x + F_{k,4}x^2 + F_{k,6}x^3 + \dots \\ (k^2 + 2)x f_e(x) &= (k^2 + 2)F_{k,0}x + (k^2 + 2)F_{k,2}x^2 + (k^2 + 2)F_{k,4}x^3 + \dots \\ x^2 f_e(x) &= F_{k,0}x^2 + F_{k,2}x^3 + \dots \end{aligned}$$

$$f_e(x)(1 - (k^2 + 2)x + x^2) = \frac{F_{k,0} + (F_{k,2} - (k^2 - 2)F_{k,0})x}{1 - (k^2 + 2)x + x^2} \rightarrow f_e(x) = \frac{kx}{1 - (k^2 + 2)x + x^2}$$

From the equation (3),  $F_{k,j}F_{k,j+1} = \frac{1}{k^2 + 4} (L_{k,2j+1} - (-1)^j k)$ . So, and taking into account Theorem 4, the generating function of the sequence  $\{F_{k,n}F_{k,n+1}\}$  is  $ff(k,x) = \frac{1}{k^2 + 4} \left( \frac{k(1+x)}{1 - (k^2 + 2)x + x^2} - \frac{k}{1+x} \right) = \frac{kx}{1 - (k^2 + 1)(x + x^2) + x^3}$

From the equation (9),  $L_{k,j}L_{k,j+1} = L_{k,2j+1} + (-1)^j k$ . Then, the generating function of the sequence  $\{L_{k,n}L_{k,n+1}\}$  is  $ll(k,x) = \frac{k(2 - k^2x + 2x^2)}{1 - (k^2 + 1)(x + x^2) + x^3}$

Finally, the generating function of the sequence  $\{F_{k,n}^2\}$  is  $f2(k,x) = \frac{x - x^2}{1 - (k^2 + 1)(x + x^2) + x^3}$

### 3.2 Generating Function of the Sequences $\{L_{k,n-2i}\}$ and $\{(-1)^i L_{k,n-2i}\}$

Special cases are the finite sequences  $\{L_{k,n-2i}\}$  and  $\{F_{k,n-2i}\}$  whose generating functions we will find next.

Let  $p(k,x)$  the generating function of the sequence  $\{L_{k,n-2i}\} = \{L_{k,n}, L_{k,n-2}, L_{k,n-4}, \dots\}$ , finalizing in  $L_{k,1}$  or  $L_{k,0}$  according to "n" is odd or even, respectively. Then, taking into account the formula (10)

$$\begin{aligned} p(k,x) &= L_{k,n} + L_{k,n-2}x + L_{k,n-4}x^2 + L_{k,n-6}x^3 + \dots \\ (k^2 + 2)x p(k,x) &= (k^2 + 2)L_{k,n}x + (k^2 + 2)L_{k,n-2}x^2 + (k^2 + 2)L_{k,n-4}x^3 + \dots \\ x^2 p(k,x) &= L_{k,n}x^2 + L_{k,n-2}x^3 + \dots \end{aligned}$$

$$\begin{aligned} (1 - (k^2 + 2)x + x^2) p(k,x) &= L_{k,n} + (L_{k,n-2} - (k^2 + 2)L_{k,n})x \rightarrow \\ p(k,x) &= \frac{L_{k,n} + (L_{k,n-2} - (k^2 + 2)L_{k,n})x}{1 - (k^2 + 2)x + x^2} \rightarrow p(k,x) = \frac{L_{k,n} - L_{k,n+2}x}{1 - (k^2 + 2)x + x^2} \end{aligned}$$

Similarly, we can prove the generating function of the alternated sequence  $\{(-1)^i L_{k,n-2i}\}$  is

$$pa(k,x) = \frac{L_{k,n} + L_{k,n+2}x}{1 + (k^2 + 2)x + x^2}$$

We can use this method to find the generating function of some special k-Fibonacci and k-Lucas sequences. Next we present some of them:

- a)  $\{F_{k,2n}\} \mapsto \frac{kx}{1 - (k^2 + 2)x + x^2}$
- b)  $\{F_{k,2n+1}\} \mapsto \frac{1-x}{1 - (k^2 + 2)x + x^2}$
- c)  $\{L_{k,2n}\} \mapsto \frac{2 - (k^2 + 2)x}{1 - (k^2 + 2)x + x^2}$
- d)  $\{L_{k,2n+1}\} \mapsto \frac{k(1+x)}{1 - (k^2 + 2)x + x^2}$
- e)  $\{(-1)^n F_{k,2n}\} \mapsto \frac{-kx}{1 + (k^2 + 2)x + x^2}$

$$f) \{(-1)^n F_{k,2n+1}\} \mapsto \frac{1-x}{1+(k^2+2)x+x^2}$$

$$g) \{(-1)^n L_{k,2n}\} \mapsto \frac{2+(k^2+2)x}{1+(k^2+2)x+x^2}$$

$$h) \{(-1)^n L_{k,2n+1}\} \mapsto \frac{k(1+x)}{1+(k^2+2)x+x^2}$$

$$i) \{F_{k,n}F_{k,n+1}\} \mapsto \frac{kx}{1-(k^2+1)(x+x^2)+x^3}$$

$$j) \{L_{k,n}L_{k,n+1}\} \mapsto \frac{k(2-k^2x+2x^2)}{1-(k^2+1)(x+x^2)+x^3}$$

$$k) \{L_{k,n-2i}\} \mapsto \frac{L_{k,n-2i}}{1-(k^2+2)x+x^2}$$

### 3.3 Applications

These formulas can be used to find the generating functions of the sequences of k-numbers whose general terms can be expressed as a linear expression of k-Fibonacci or k-Lucas numbers.

#### Example 1.

Generating function of the sequence  $\{F_{k,n}^2\}$

From the equation (3),  $F_{k,n}^2 = \frac{1}{k^2+4}(L_{k,2n} - (-1)^n 2)$ . Then, its generating function is a combination of the generating function of  $L_{k,2n}$  and that of  $\{(-1)^n\}$ :

$$g.f.(\{F_{k,n}^2\}) = \frac{1}{k^2+4}(g.f.(\{L_{k,2n}\}) - 2g.f.(\{(-1)^n\})) = \frac{1}{k^2+4}\left(\frac{2-(k^2+2)x}{1-(k^2+2)x+x^2} - 2\frac{1}{1+x}\right) = \frac{1}{k^2+4} \frac{(k^2+4)(x-x^2)}{1-(k^2+1)(x+x^2)+x^3}$$

$$\{F_{k,n}^2\} \mapsto \frac{x-x^2}{1-(k^2+1)(x+x^2)+x^3}$$

It is interesting to note the denominator of this generating function shows the recurrence relation of the squares of the k-Fibonacci numbers:  $F_{k,n}^2 = (k^2+1)(F_{k,n-1}^2 + F_{k,n-2}^2) - F_{k,n-3}^2$

#### Example 2:

Generating function of the sequence  $\{L_{k,n}^2\}$

From the equation (9),  $L_{k,n}^2 = L_{k,2n} + (-1)^n 2$  and using the formula © of the page 9, the generating function of the  $L_{k,n}^2$  numbers is  $l2(k,n) = \frac{2-(k^2+2)x}{1-(k^2+2)x+x^2} + \frac{2}{1+x} = \frac{4-(3k^2+4)x-k^2x^2}{1-(k^2+1)(x+x^2)+x^3}$

#### Recurrence Relations [12].

On the other hand, the denominator of these generating functions shows the recurrence relation for both sequences

nces is  $M_{k,n} = (k^2 + 1)(M_{k,n-1} + M_{k,n-2}) - M_{k,n-3}$  with initial conditions  $M_{k,0} = F_{k,0}^2 = 0$ ,  $M_{k,1} = F_{k,1}^2 = 1$ ,  $M_{k,2} = F_{k,2}^2 = k^2$  and  $M_{k,0} = L_{k,0}^2 = 4$ ,  $M_{k,1} = L_{k,1}^2 = k^2$ ,  $M_{k,2} = L_{k,2}^2 = (k^2 + 2)^2$ , respectively.

#### IV. CONCLUSIONS

In this article we have used two different ways to find the generating function of the k-numbers.

In the first place, we have used the general method that consists of following the indications shown in the law of recurrence of the different types of k-numbers. We have used this method to find the generating function of even numbers in section 3.1 from formula (10).

Second, to find the generating function of some more complex numbers, we have transformed them into simpler ones and later found the generating function of these new numbers. So we have done in Example 1, to find the generating function of the squares of the k-Fibonacci numbers.

Consequently, this paper can serve as a basis for other researchers to find the generating function of some more complex expression k-numbers.

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