THE EVOLUTION OF

LARGE CARDINAL AXIOMS IN SET THEORY

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There are more things in heaven
and earth, Horatio,
Than are dreamt of in your philosophy.
—Hamlet, I.v.166-7.

Large cardinals have been around for a long time. Indeed, the last fifteen or so years bear witness to a tremendous amount of set theoretical research invested in their study. However, a casual foray into this extensive domain can be a bewildering experience: The proliferation and fragmentation of concepts and terminology can leave one awed by the entrancing beauty of the free flow of ideas, but also, with some distrust and unease about the apparently quixotic irrelevance of it all to foundational studies. An attempt is made in this paper to present a unifying point of view concerning the development of the theory and applications of large cardinals, as part of the mainstream of the axiomatic study of set theory. In doing this, we hope to show that concepts which may at first seem arbitrary and disconnected actually fit into a natural and coherent scheme, and that there is an abiding inner logic in the synthesis of new large cardinal axioms.

The paper will be expository, organized around the flow of ideas in the historical development of the subject. We have had to cope with the difficult problem of compartmentalizing and presenting in linear sequence the development of a theory with thematic interrelationships of high genus. By tracing principal themes, each time picking up the beginning threads but emphasizing the growing interconnections, we hope to guide the reader successfully through the labyrinth. On the large scale, our first four chapters deal with Statement and Development, and our last three with Integration and Application. Each chapter is subdivided into several sections, which usually can be read independently with the help of various notational referrals to previous sections.

We hope to be reasonably complete and precise in presenting concepts and their known interconnections. Sometimes, however, when detailed proofs have appeared elsewhere, rather than break the exposition we shall content ourselves with outlining the main ideas, and giving sufficient references to the extant literature. In particular, there will be a natural selection in favor of presenting unpublished, and also, shorter, illustrative proofs. We believe that a paper of this sort which encompasses recent developments in the theory of large cardinals is somewhat overdue, and we will count ourselves successful if we have been able to convey to the reader even some of the enthusiasm and spirit of adventure which we ourselves have felt, delving into this subject.

Our set theoretical notation is standard, and the following litany should take care of any possible variations: The letters $\alpha, \beta, \gamma, \ldots$ denote ordinals, whereas $\kappa, \lambda, \mu, \ldots$ are reserved for infinite cardinals. V_α is the collection of sets of rank $< \alpha$, often understood as the relational system $\langle V_\alpha, \epsilon \rangle$. In fact, the short-hand $X \prec Y$ is often used for $\langle X, \epsilon \rangle \prec \langle Y, \epsilon \rangle$. If x is a set, $|x|$ is its cardinality, $P(x)$ is its power set, and $P_\kappa x = \{y \in P(x) \mid |y| < \kappa\}$. $\lambda^{<\kappa} = \bigcup\{\lambda^\alpha \mid \alpha < \kappa\}$, and hence is the cardinality of $P_\kappa \lambda$. If f is a function, then $f''x = \{f(y) \mid y \in x\}$ is the image of x under f, and $f|x = f \cap (x \times V)$ is the restriction of f to x. ${}^y x$ denotes the collection of functions: $x \to y$, so that λ^κ is the cardinality of ${}^\kappa\lambda$. It is implicit in the notation $\phi(v_1, \ldots, v_n)$ for formulas that the free variables of ϕ are among the $v_1, \ldots, v_n$, so that in particular, $\mathfrak{A} \models \phi[a_1, \ldots, a_n]$ means that ϕ is satisfied in $\mathfrak{A}$ with the variable assignment taking v_i to $a_i \in$ domain of $\mathfrak{A}$. OR denotes the class of ordinals, AC is the Axiom of Choice, and GCH is the Generalized Coninuum Hypothesis. Finally, the marks $\vdash \dashv$ bracket a proof, signalling a break in the flow of exposition.

If I is a set, then an (ultra)filter F <u>over</u> I is a (maximal) filter <u>on</u> the Boolean algebra $P(I)$. (The preceding sentence(s) exemplifies the distinction between over and on.) Throughout this paper, <u>all filters F will be assumed to be non-principal</u>, i.e. $\bigcap F = \emptyset$. F is uniform iff whenever $X \in F$, $|X| = |I|$. F is κ-complete iff whenever $T \subseteq F$ and $|T| < \kappa$, then $\bigcap T \in F$.

ZF denotes Zermelo-Frankel set theory, ZFC denotes Zermelo-Frankel set theory plus the Axiom of Choice, and GB denotes Gödel-Bernays set theory. <u>We work in ZFC</u>, unless it is made explicit that we are discussing a situation where the Axiom of Choice fails. If θ is a formula or term, $(\theta)^M$ denotes the usual relativization of θ to the class M. An inner model is a transitive ϵ-model of ZF containing all the ordinals. There is a formula $\mathrm{Inn}(\cdot)$ such that for any axiom θ of ZF, $\vdash_{ZF} \mathrm{Inn}(M) \to (\theta)^M$. As first introduced by Lévy[1960], if X is a class, $L[X]$ denotes the inner model relatively constructible from X: $L[X] = \bigcup_\alpha L_\alpha[X]$, where $L_{\alpha+1}[X] = \{a \subseteq L_\alpha[X] \mid a$ is first-order definable in $\langle L_\alpha[X], \epsilon, X \cap L_\alpha[X] \rangle$ from parameters in $L_\alpha[X]\}$, and $L_\gamma[X] = \bigcup_{\alpha<\gamma} L_\alpha[X]$ for limit ordinals γ. It is well-known that there is a sentence σ provable in ZF so that whenever $\langle A, \epsilon \rangle$ is a transitive (possibly set) model of σ with $X \cap A \in A$, then for any ordinal $\alpha \in A$, $(L_\alpha[X \cap A])^A = L_\alpha[X \cap A] = L_\alpha[X]$. In particular, this is true for A an inner model.

Concerning our forcing formalism, we often find it convenient to take V as a relative term for the ground model and to construct extensions $V[G]$. The forcing relation with conditions in a partial order ("the notion of forcing") will be used in preference to Boolean-valued models, although we avoid deciding whether $p \le q$ is to mean p is stronger or weaker than q by various equivocations. A notion of forcing is λ-closed iff whenever $\langle p_\alpha \mid \alpha < \gamma \rangle$ is a sequence of increasing conditions with $\gamma \le \lambda$, there is a condition stronger than every p_α. The variation $<\lambda$-closed has the obvious meaning. In the forcing language developed in V for a particular notion of forcing, $\check{x}$ is the name for $x \in V$, and $\underset{\sim}{G}$ is the name for the generic object. Thus, for instance, $p \Vdash \check{p} \in \underset{\sim}{G}$. The $\check{x}$ notation will often be suppressed, especially for x an ordinal. Finally, if τ is a defined term in set theory, $\dot{\tau}$ is its corresponding name in the forcing language. For the basic precepts about forcing, which we shall take for granted, consult Solovay[1970], Shoenfield[1971], or Bell[1977].

The citation of primary sources will be by author(s) and usually by the year of publication [in square brackets], referring to the bibliography. This may be historically misleading, however, as the results often appear in publication several years after they were first proved, depending on the sluggish pace of the publication process or the indolence of the author. Those primary sources which have not yet appeared in print will be cited by the year of their first manuscript appearance (in parentheses), referring to the bibliography.

There are several good secondary sources which deal with the classical theory of large cardinals, to which we refer for details on our first two chapters: Drake [1974] (set theoretical and closest to our approach), Chang-Keisler[1973] (from the point of view of model theory), and Comfort-Negrepontis[1974] (with a preoccupation with ultrafilters). Boos[1975] contains a great deal of information about the topics we discuss in our first four chapters as well as §20, but its bristling technical detail may be more appreciated by the advanced reader. Finally, the fine expository work of Devlin is quite relevant to many of our sections: [1975] for §§3,4,5; [1973] for §6; and [1973a] for §§7,9,10,19,20,21.

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§0. Introduction

It has often been remarked that the ZFC axioms for set theory have a persuasive, almost inevitable, nature. Indeed, they formally present, and solicit justification in, a clear picture of the universe of sets as $V = \bigcup_\alpha V_\alpha$, the cumulative hierarchy. Thus, the universe is to be the totality obtained by iterating the power set operation through the transfinite. The Power Set Axiom is widely recognized as a very powerful axiom. Its assertion easily implies its own independence: the class of hereditarily countable sets is then a set, and a model of all the axioms except the Power Set Axiom itself. However, the presence of the Axiom of Replacement also plays an essential role in the proceedings, dictating as it does that the cumulation of sets must proceed through the transfinite in a real sense: without it, it is already sufficient to stop at $V_{\omega+\omega}$. Thus, the Power Set Axiom provides the width, and Replacement, the requisite length.

Regarding Replacement in this way as a strong postulate of closure on the extent of the class of ordinals, we can consider various closure operations like the processes of Mahlo as natural generalizations. However, iterations of closure operations (or sequences like $T_0 = ZFC, \ldots, T_n, T_{n+1} = T_n + Con(T_n), \ldots$) are intuitively r.e., and so the feeling quickly emerges that such processes must be incomplete in axiomatizing what is "true" about sets. Hence, in order to get at qualitatively stronger axiom systems about the nature of the cumulative hierarchy, distinctly new ideas must be introduced: we must switch from universal to existential (and hence "theological") postulates. This is not at all far-fetched; some existential axiom must be introduced to insure that the set theoretical universe is not empty to begin with, and in fact the Axiom of Infinity has this attribute. It is in similar spirit to this axiom that we introduce large cardinals.

Like the Power Set Axiom, the Axiom of Infinity has the notable characteristic that its assertion implies its own independence (this time, consider the class of hereditarily finite sets). Large cardinals will also have this property, as well as establishing further new consistency statements. Thus, our theological intentions are fulfilled even at the level of arithmetical statements; this consequence of Gödel's Second Incompleteness Theorem is by now quite familiar, and in fact by Matijasevič's result new Diophantine equations become solvable.

What is a large cardinal? The following is a reasonable, working meta-definition: Let us say that a property is of large cardinal character if it has the following consequences: (a) the existence of a cardinal with a property which (at least in some inner model of ZFC) renders it essentially "larger" than cardinals with weaker properties (in the sense that it is a fixed point of reasonable thinning procedures, like Mahlo's, beginning from these cardinals), and (b) a discernible new strength in set theory, as for example in the emergence of new combinatorial properties. Let us say that the assertion of a large cardinal property is a strong axiom of infinity.

The adaptation of strong axioms of infinity is thus a theological venture, involving basic questions of belief concerning what is true about the universe. However, one can alternately construe work in the theory of large cardinals as formal mathematics, that is to say the investigation of those formal implications provable in first-order logic. This being the case, there is no denying the value of work in this area, especially in view of relative consistency results. There is here a pleasing analogy: In order for a true believer to really know Mount Everest, he must slowly and painfully trudge up its forbidding side, climbing the rocks amid the snow and the slush, with his confidence waning and his skepticism growing as to the possibility of ever scaling the height. But in these days of great forward leaps in technology, why not get into a helicopter, fly up to the summit, and quickly survey the rarefied realm--all while having a nice cup of tea?

In tracing the development of the theory of large cardinals, we will emphasize the interplay of three major themes. The first of these is the role of those abstract motivating principles which have led to the formulation of large cardinal properties. We can isolate, and thereby designate, four such principles:

(i) Generalization. To the hereditarily finite sets, ω seems a tremendously large cardinal; indeed, we have already remarked on the strength and necessity of the existential postulation of ω. The comparative size of ω can be formally described by several properties, and the attribution of these properties to uncountable cardinals yield similar points of strong closure. After all, it would seem rather accidental if ω can be characterized by these properties. We mean by generalization such a process of reasonable induction from familiar situations to higher orders, with the concommitant confidence in the recurring richness of the cumulative hierarchy.

(ii) Reflection. The ordinary Reflection Principle in set theory says that any particular statement true of V is already true at some initial segment V_α. This invites generalizations to cardinals endowed with similar downward reflection properties, thus rendering them strong closure points. In another approach (see Reinhardt[1974]), what is involved is a formulation of various reflection properties the class OR intuitively ought to have, the antithetical realization that OR ought to be essentially indescribable in set theory, and thus the synthesis in the conclusion that there must already be some cardinal at which these properties obtain. Note that this in itself is a reflection argument.

(iii) Resemblance. This is closely related to (i) and (ii). Because of reflection considerations, and, generally speaking, because the cumulative hierarchy is neutrally defined by iterations of the power set operation, it is reasonable to suppose that there are $< V_\alpha , \varepsilon >$'s which resemble each other. Such considerations lead naturally to elementary embeddings and indiscernibility arguments, the stuff of which large cardinals are made.

(iv) Restriction. Much of on-going mathematics is involved with weakening known assertions to gain more information, to discern the essential properties involved in order to sharpen implications. Such structural considerations have not only brought the landscape of large cardinals into sharper focus, but have led to the isolation of new principles, like the existence of the set of integers $0^\#$.

The second of the major themes which we discuss is the efficacy and esthetics of concepts, particularly in the context of relative consistency results. One of the foundational results of proof theory is Gentzen's characterization of the strength of Peano Arithmetic as induction up to the ordinal ε_0. Many ordinals have since been established as giving precise measures to the consistency of various subsystems of analysis. In analogy, large cardinals via the method of forcing turn out to be the natural measures of the consistency strength of $ZFC + \Phi$ for various statements Φ in the language of set theory. In many cases, actual equiconsistency results are known between the existence of some large cardinals and natural combinatorial principles concerning the real numbers, or such small cardinals as ω_1 and ω_2. In other cases, the only known way to prove the consistency of the statement at issue is to assume the consistency of the existence of some large cardinal, and the prevalent view insists on the possibility of being able to prove the converse. Of course, the analogy to the proof-theoretic ordinals is not very exact, since those ordinals correspond precisely to the suprema of heights of proof trees with bottom node falsity. However, the known large cardinals fit into a linear order via consistency strength, and so provide an abiding esthetics in neatly categorizing statements about sets.

The third of the major themes which will be discussed is the fruitfulness of the methods introduced in the context of large cardinal theory in leading to new, "standard" theorems of ZFC. Psychologically, work in the theory of large cardinals might result in an intuitive picture of V as rich and populated by a multitude of distinctive entities, and in such a fertile landscape, glimmerings of new interconnections are bound to emerge, indicating possible new applications of known methods and ultimately leading to new theorems.

I. EARLY RESULTS

1. Beginnings

We pick up the beginning threads of the subject in its early history. At about the same time that Zermelo introduced his initial axiomatization of set theory, Hausdorff[1908] had already isolated the notion of a weakly inaccessible cardinal (as a regular fixed point of the $\aleph$ function). It is now well known that such cardinals κ cannot be proved in ZFC to exist (as $L_\kappa \models$ ZFC), but in the initial preoccupation with the study of cardinal functions in general, it is difficult to determine whether ontological commitment was an important issue.

Further fixed points now ensue: define κ is 0-weakly hyperinaccessible iff κ is weakly inaccessible; κ is $(\alpha+1)$-weakly hyperinaccessible iff κ is a regular limit of α-weakly hyperinaccessibles; and for limit γ, κ is γ-weakly hyperinaccessible iff κ is α-weakly hyperinaccessible for all $\alpha < \gamma$. We can now diagonalize, i.e. take those κ which are κ-weakly hyperinaccessible and consider the regular limit of these, and so forth. This is a typical case of an intuitively r.e. process which can be applied to any given class of cardinals to yield larger cardinals. The interest in such processes quickly fades as we consider significantly new concepts, and one was soon forthcoming in the present case:

Mahlo[1911][1912][1913] discovered a new process, using for the first time the concepts of closed unbounded, and stationary subsets of a regular cardinal (the term "Mahlo" is sometimes used for "stationary", for example by Jensen). For a regular uncountable cardinal κ, $R_\kappa = \{ \alpha < \kappa \mid \alpha \text{ is regular} \}$ cannot, of course, be closed unbounded. But we can ask for next best thing: κ is weakly Mahlo iff R_κ is stationary in κ. The weakly Mahlo cardinals result after one application on the class of regular cardinals of Mahlo's Operation: $M(X) = \{ \kappa \in X \mid X \cap \kappa \text{ is stationary in } \kappa \}$. We can iterate this operation to get weakly Mahlo cardinals of higher order. Of course, the hierarchy of weakly hyperinaccessibles can be regarded as resulting from iterating the operation $L(X) = \{ \kappa \in X \mid X \cap \kappa \text{ is unbounded in } \kappa \}$, also starting from the class of regular cardinals. The operation L is much weaker than M: if κ is weakly Mahlo, then κ is κ-weakly hyperinaccessible. Thus, we have a typical case of a new concept outstripping r.e. processes from below. There have been several generalizations of Mahlo's Operation, some quite recently: see Fodor[1966], Gaifman[1967a] and Glöede[1973].

The closed unbounded subsets of a regular uncountable cardinal κ are very important, being the natural copies of the order type κ, and generating a κ-complete filter. From the contemporary point of view, it is a significant, useful, and easy-to-prove fact that given a structure $< \kappa, \epsilon, R >$ where $R \subseteq \kappa$, the set $\{ \alpha \mid < \alpha, \epsilon, R \cap \alpha > \prec < \kappa, \epsilon, R > \}$ is closed unbounded. Using this,

we can get a modern characterization of weakly Mahlo cardinals by a reflection principle, in analogy with regular cardinals, as follows: an uncountable cardinal κ is regular (weakly Mahlo, respectively) iff for any $R \subseteq \kappa$, there is a $\lambda < \kappa$ (and λ is regular, respectively) so that $< \lambda, \epsilon, R \cap \lambda > \prec < \kappa, \epsilon, R >$.

Considerations involving the width of the set theoretical universe came into play when Sierpinski-Tarski[1930] introduced the notion of a strongly inaccessible cardinal, as a regular fixed point of the $\beth$ function. Thus, a strongly inaccessible cardinal is a regular uncountable cardinal κ which is a strong limit: if $\alpha < \kappa$, then $2^\alpha < \kappa$. (Unless distinctions are being emphasized, the present-day habit of using inaccessible to mean strongly inaccessible will be adopted throughout this paper.) The limit and Mahlo processes can be started on the class of inaccessible cardinals, yielding the strongly hyperinaccessible and strongly Mahlo cardinals, and in fact the term Mahlo nowadays usually means strongly Mahlo.

Of course, the classes of weakly and strongly inaccessible cardinals coincide under the GCH, but without any such assumption, closure considerations involving the power set operation need strong inaccessibility. For example, if κ is strongly inaccessible, then $V_\kappa \models$ ZFC. Note that this property does not characterize strong inaccessibility: ZFC being a first-order theory, it is straightforward to see by a Löwenheim-Skolem argument that if there is any V_α which models ZFC at all, the least such α must have cofinality ω. However, we can go on to a second-order conservative extension: $V_\kappa \models$ GB iff κ is strongly inaccessible. The "classes" here are to be members of $V_{\kappa+1}$; the point here is that κ must be regular for arbitary "classes" to satisfy the Replacement Schema. Width considerations also come into play in the following reflection characterizations, since V_α replaces α in the "weak" versions: κ is inaccessible (Mahlo, respectively) iff for any $R \subseteq V_\kappa$, there is a $\lambda < \kappa$ (and λ is inaccessible, respectively) so that $< V_\lambda, \epsilon, R \cap V_\lambda > \prec < V_\kappa, \epsilon, R >$. (Now, $\{ \alpha \mid < V_\alpha, \epsilon, R \cap V_\alpha > \prec < V_\kappa, \epsilon, R > \}$ is closed unbounded in κ.)

Inaccessibility generalizes, of course, a (first-order) property of ω. Postulating the existence of many inaccessibles amounts to imposing a superstructure on V comprised of initial segments $V_{\kappa+1}$ where sufficient closure properties obtain to yield natural models of GB. This is a first step to extending the axioms of ZFC; the universe is to be rich enough so that closure points exist for basic set theoretic operations from below.

In a significant paper introducing several new ideas, Ulam[1930] considered the abstract measure problem: is there an infinite set X and a measure on all the subsets of X, i.e. a function $\mu: P(X) \to [0,1]$ so that:

(i) $\mu(X) = 1$ and $\mu(\{x\}) = 0$ for any $x \in X$.

(ii) μ is countably additive, i.e. if $\{ X_n \mid n \in \omega \} \subseteq P(X)$ are disjoint, then $\mu(\bigcup_n X_n) = \sum_n \mu(X_n)$.

Of course, Lebesgue measure on the unit interval satisfies (i) and (ii), but is not total: with the Axiom of Choice (or even the existence of a (non-principal) ultrafilter over ω) there are non-Lebesgue measurable sets. Carrying such a μ is only a property of the cardinal κ of X , and it was quickly noticed that if κ is the least cardinal with this property, then any measure $\mu\colon P(\kappa) \to [0,1]$ is in fact κ-additive, i.e. for any $\gamma < \kappa$, if $\{ X_\alpha \mid \alpha < \gamma \} \subseteq P(\kappa)$ are disjoint, then $\mu(\bigcup_\alpha X_\alpha) = \sum_\alpha \mu(X_\alpha)$:

$\vdash$ Suppose otherwise. Then $\gamma > \omega$, and since only countably many α's are such that $\mu(X_\alpha) > 0$, we can suppose by throwing these away through countable additivity that in fact every $\mu(X_\alpha) = 0$, yet $\mu(\bigcup_\alpha X_\alpha) = r > 0$. It is now easy to show that $\rho\colon P(\gamma) \to [0,1]$ defined by $\rho(Y) = (1/r)\cdot\mu(\bigcup\{ X_\alpha \mid \alpha \in Y \})$ is a measure, contradicting the leastness of κ . $\dashv$

Since the ontological commitment is the same, we consider only the existence of those κ so that there are κ-additive measures μ on $P(\kappa)$. It is easy to see that κ must consequently be regular and uncountable. Say that $A \subseteq \kappa$ is an atom for μ iff $\mu(A) > 0$ yet whenever $B \subseteq A$, $\mu(B) = \mu(A)$ or $\mu(B) = 0$. The following dichotomy was noticed:

(a) μ has no atoms. Then κ is nowadays called real-valued measurable. Ulam showed that such a κ is weakly inaccessible (see §11), and also that $\kappa \le 2^\omega$:

$\vdash$ First, a technical lemma stated without proof: as μ is not atomic, for any $Y \subseteq \kappa$, there are disjoint $Z, \bar{Z} \subseteq Y$ so that $Z \cup \bar{Z} = Y$ and $\mu(Z) = \mu(\bar{Z}) = \frac{1}{2}\mu(Y)$. Now define positive measure sets for the nodes of the binary tree: Let $X_{<>} = \kappa$, and if X_s has already been defined for $s \in {}^n2$, some n , let $X_{s^\frown<0>}, X_{s^\frown<1>}$ be disjoint with their union X_s , so that $\mu(X_{s^\frown<0>}) = \mu(X_{s^\frown<1>}) = \frac{1}{2}\mu(X_s)$. For $f \in {}^\omega2$, if $T_f = \bigcap_n X_{f|n}$, then $\mu(T_f) = 0$. However, $\bigcup\{ T_f \mid f \in {}^\omega2 \} = \kappa$. Thus, κ is not $(2^\omega)^+$-additive, and so $\kappa \le 2^\omega$. $\dashv$

(b) μ has an atom A. Then define $\rho\colon P(A) \to [0,1]$ by $\rho(Y) = \frac{\mu(Y)}{\mu(A)}$. Then ρ is a κ-additive measure on $P(A)$. Since $\mu(A) > 0$, $|A| = \kappa$. Thus, via a bijection of A and κ we can conclude that there is a two-valued measure on $P(\kappa)$. A cardinal κ with such a measure is called measurable.

We refer to sections §11 and §24 for Solovay's basic results concerning real-valued measurability, and for now proceed to discuss (two-valued) measurability. Ulam went on to show that a measurable cardinal is (strongly) inaccessible. Nowadays, of course, much stronger results are known, but the converse question of whether the least inaccessible can be measurable turned out to be an open question for about thirty years. After the Fall, we may smile at the naiveté of Adam and Eve, but this is a typical case of where new ideas were necessary—and these were to emerge, perhaps surprisingly, from linguistic considerations (see §3).

If μ is a two-valued, κ-additive measure on $P(\kappa)$, then $U = \{ X \subseteq \kappa \mid \mu(X) = 1 \}$ is a (non-principal) ultrafilter over κ , which is easily seen to be κ-complete, i.e. whenever $\gamma < \kappa$ and $\{ X_\alpha \mid \alpha < \gamma \} \subseteq U$, then $\bigcap_{\alpha<\gamma} X_\alpha \in U$. Since by the Boolean Prime Ideal Principle there are (non-principal) ultrafilters over ω (which of course are "ω-complete"), the assertion that there is a measurable cardinal is a direct generalization. The finite intersection property is automatically preserved when taking the union of increasing chains of filters, so that just a maximal principle is needed to get ultrafilters over ω . But κ-completeness for $\kappa > \omega$ is not similarly preserved, so that the existence of a κ-complete ultrafilter over κ must be directly postulated. We mention here that Tarski[1939] approached the concept of measurability via the route of Boolean algebras and κ-complete ideals—just the dual notion.

The measurability of κ is a third-order existential statement about V_κ , and gives an intuitive feel to the cardinal κ as a higher order of actual infinity above the elements of V_κ . It turns out that several theorems in various mathematical fields are known to hold only in the ZFC model V_κ where κ is the least measurable cardinal. (See for example Hewitt[1948] in functional analysis, Fuchs[1970] in abelian group theory, and Reyes[1972] for a recent result in model theory. For further references to results involving measurable cardinals in standard mathematical fields, see page 272 of Keisler-Tarski[1964].) This confluence is an interesting phenomenon in the study of infinite sets; the world V_κ is well-behaved in a sense, and to it κ is an infinity of a higher order. It remained for Scott to use the measurability hypothesis in a positive way, showing that the presence of a measurable cardinal definitely leads to a richer structure on the cumulative hierarchy.

§2. Scott's Work

Though Gödel[1938] introduced the inner model L , the class of constructible sets, and thereby proved the consistency of AC and the GCH relative to ZF, he only considered L a device. His skepticism about the truth of the Axiom of Constructibility (the assertion $V = L$) rested in part in his distrust of the Continuum Hypothesis itself (see Gödel[1947]). $V = L$ is quite a strong axiom in being able to decide several combinatorial questions, but to Gödel's mind, in a direction against his intuition—an attitude couched firmly in neo-Platonism. In the present, post-Cohen era it is the nuts and bolts of the technology of forcing to show the relative consistency with ZF of $V \ne L$, but it was Scott[1961] who first realized the possibility, in showing that: If there is a measurable cardinal, then $V \ne L$. Let us be aware of the difference: while forcing can be construed as a possible world semantics in which actual realization of a generic extension is a transcendence to a next universe, Scott's result depends on a direct existential (and hence theological) postulation leading to a result about the present universe.

The usefulness of the ultraproduct construction in model theory was just becoming understood at the time, when Scott struck on the idea of taking the ultrapower of V by a κ-complete ultrafilter U over κ .

We quickly chart the landmarks of a path well-known today: $V^\kappa/\mathcal{U}$ is well-founded (as per its derived membership relation) as $\mathcal{U}$ is at least ω_1-complete. Since a strategem (known as Scott's trick) is available for construing each ultrapower equivalence class as a set, by Mostowski's collapsing lemma there is a unique isomorphism $M_\mathcal{U} \cong V^\kappa/\mathcal{U}$, where $M_\mathcal{U}$ is a transitive class. $M_\mathcal{U}$ is thus an inner model of set theory. The canonical embedding of V into the ultrapower induces an elementary embedding $j_\mathcal{U}$ into $M_\mathcal{U}$, which we write schematically,

$$j_\mathcal{U}: V \to M_\mathcal{U} \cong V^\kappa/\mathcal{U}.$$

This notational convention for analogous situations will be adopted in this paper; for $f: \kappa \to V$, by $[f]_\mathcal{U}$ we denote the element of $M_\mathcal{U}$ to which the ultrapower equivalence class of f corresponds, under the transitizing isomorphism. (The subscript $\mathcal{U}$ will often be dropped, when clear from the context.) Now note several facts: by κ-completeness, $\alpha < \kappa$ implies $j_\mathcal{U}(\alpha) = \alpha$, while $\kappa \leq [id]_\mathcal{U} < j_\mathcal{U}(\kappa)$, where $id: \kappa \to \kappa$ is the identity function.

We can go on to establish Scott's theorem as follows: Assume κ is the least measurable cardinal, and, by way of contradiction, that $V = L$. Since $M_\mathcal{U}$ is then an inner model which by elementarity also satisfies the Axiom of Constructibility, $M_\mathcal{U} = L = V$. But also by elementarity $j_\mathcal{U}(\kappa)$ is supposed to be the least measurable cardinal, yet $j_\mathcal{U}(\kappa) > \kappa$, a contradiction. (In fact, a stronger mechanism is known: without assuming that κ is the least measurable or that $V = L$, it can be established that $\mathcal{U} \notin M_\mathcal{U}$. Now $V = L$ would still mean that $M_\mathcal{U} = L$, contradicting $\mathcal{U} \in V - M_\mathcal{U}$. See §8, 8.7.)

It was soon noticed that Scott's construction has a direct converse: <u>if there is an elementary embedding $j: V \to M$ where M is an inner model and j is not the identity, then there is a measurable cardinal</u>:

$\vdash$ Let κ be the <u>critical point</u> of j, i.e. the least ordinal moved by j (there must be such an ordinal—it is the least rank of any set moved by j). Define $\mathcal{U}$ by

$$X \in \mathcal{U} \text{ iff } X \subseteq \kappa \ \& \ \kappa \in j(X).$$

It turns out that $\mathcal{U}$ is a κ-complete ultrafilter over κ, and hence κ is a measurable cardinal. $\dashv$

It may be surprising that non-trivial ultrafilters can result from elementary embeddings; here, κ is not in the range of j and so can be considered a "generic" element which generates $\mathcal{U}$. Thus, the really structural characterization of measurable cardinals in set theory emerged: they exactly correspond to elementary embeddings. An old concept dons a new guise, and this insight partially set the stage for the further use of model theoretic techniques in set theory.

As an interesting sidelight, we prove in passing a fact that will be useful in future sections. <u>Closure Lemma: if κ is measurable, $\mathcal{U}$ is a κ-complete ultrafilter over κ, and $j: V \to M \cong V^\kappa/\mathcal{U}$, then ${}^\kappa M \subseteq M$, i.e. M is closed under arbitrary sequences of length κ. In particular, $V_{\kappa+1} \subseteq M$.</u>

$\vdash$ Suppose $\{[f_\alpha] \mid \alpha < \kappa\} \subseteq M$. We must find a $g: \kappa \to V$ so that $[g] = \langle [f_\alpha] \mid \alpha < \kappa \rangle$. Let $h: \kappa \to V$ so that $[h] = \kappa$. For each $\xi < \kappa$, let $g(\xi)$ be with domain $= h(\xi)$, so that $g(\xi)(\alpha) = f_\alpha(\xi)$. Then by Łoś' Theorem, $[g]$ is a function with domain $[h] = \kappa$, and for each $\alpha < \kappa$, $[g](\alpha) = [f_\alpha]$. $\dashv$

Scott also isolated the concept of a <u>normal</u> filter. If $\mathcal{U}$ is a κ-complete ultrafilter over κ, then $\kappa \leq [id]_\mathcal{U}$, as was stated before, where $id: \kappa \to \kappa$ is the identity function. Equality here was desirable, and Scott proceeded as follows: Let $f: \kappa \to \kappa$ be such that $\kappa = [f]_\mathcal{U}$. Define an object $f_*(\mathcal{U})$ by:

$$X \in f_*(\mathcal{U}) \ \text{iff} \ X \subseteq \kappa \ \& \ f^{-1}(X) \in \mathcal{U}.$$

Then $f_*(\mathcal{U})$ is also a κ-complete ultrafilter over κ, with the additional property that $\kappa = [id]_{f_*(\mathcal{U})}$. This property is equivalent to asserting that $f_*(\mathcal{U})$ is normal, in the sense of the following definition:

For any λ-complete filter F over a cardinal λ, F is <u>normal</u> <u>iff</u> one (and hence both) of the following equivalent conditions hold:

(a) Whenever $\{X_\alpha \mid \alpha < \lambda\} \subseteq F$, then $\Delta_\alpha X_\alpha = \{\beta < \lambda \mid \alpha < \beta \to \beta \in X_\alpha\} \in F$. (Notice that for uniform F, this subsumes λ-completeness.)

(b) Whenever $f: \lambda \to \lambda$ is regressive on a set of F-positive measure, i.e. $Y = \{\beta < \lambda \mid f(\beta) < \beta\}$ has F-positive measure, then there is an $\alpha < \lambda$ so that $\{\beta \in Y \mid f(\beta) = \alpha\}$ has F-positive measure. (A set $A \subseteq \lambda$ has <u>F-positive measure</u> <u>iff</u> $F \cup \{A\}$ generates a (proper) filter, i.e. $\lambda - A \notin F$.)

Of course, for an ultrafilter vis-à-vis (b), having positive measure is the same as being in the ultrafilter. Scott saw the utility of the normality condition in the context of elementary embeddings. Let $\mathcal{N}$ be a normal, κ-complete ultrafilter over κ. Since $V_{\kappa+1} \subseteq M_\mathcal{N}$ by the Closure Lemma, we have $M_\mathcal{N} \models \kappa$ is inaccessible. But $\kappa = [id]_\mathcal{N}$ by normality, so it follows from Łoś' Theorem that $\{\alpha < \kappa \mid \alpha$ is inaccessible$\} \in \mathcal{N}$. Thus, the least inaccessible cardinal cannot be measurable. Tarski was the first to prove this fact using ideas of Hanf (see §3), but Scott's conclusion is stronger. (Even stronger results about normal ultrafilters are possible and will be cited. Note the apparently counter-intuitive fact that though an ultrafilter is usually thought to consist of large sets, the sparse set $\{\alpha < \kappa \mid \alpha$ is inaccessible$\} \in \mathcal{N}$. This is illustrative of the strong reflection phenomena occurring at measurable cardinals.) Scott also showed that if $\{\alpha < \kappa \mid 2^\alpha = \alpha^+\} \in \mathcal{N}$, then $2^\kappa = \kappa^+$, the prototype of several similar results about powers of cardinals, culminating in Silver's theorem (see the end of §13, and §29).

With the notion of normality made explicit, it was in a sense a rediscovery that some work on regressive functions had already been done, and that Fodor[1956] in fact had shown that <u>the closed unbounded filter $\mathcal{C}_\lambda$ on any regular uncountable cardinal λ is normal</u>. The stationary subsets of λ are just the $\mathcal{C}_\lambda$-positive measure sets, and the result is usually stated via definition (b) of normality as follows: whenever $f\colon \lambda \to \lambda$ is regressive on a stationary subset of λ, it is constant on a stationary subset. The following is a proof (made simpler than Foder's original proof by hindsight) via definition (a) of normality:

⊢ We want to show that if $\{C_\alpha \mid \alpha < \lambda\}$ are closed unbounded sets, then so is $\Delta_\alpha C_\alpha$. That it is closed is immediate. To show that it is unbounded, let $\beta_0 < \lambda$. Inductively define β_n for $n > 0$ as follows: given β_k, let $\beta_{k+1} > \beta_k$ and $\beta_{k+1} \in \bigcap\{C_\alpha \mid \alpha < \beta_k\}$. This is possible since this last set is closed unbounded. Let $\beta = \sup \beta_n$. Then $\beta < \lambda$ as λ is regular, and $\beta \in \Delta_\alpha C_\alpha$. ⊣

In a real sense, what distinguishes regular uncountable cardinals from all others is the fact that they carry an easily defined normal filter. The closed unbounded filter over a regular uncountable cardinal is contained in every other normal filter, and the theory of several large cardinals will have as basic features some naturally defined normal filters. In the presence of the Axiom of Choice, the closed unbounded filter is not an ultrafilter. But we shall see in later sections that it can get quite close to being ultra, and that with the Axiom of Determinacy (§28) the closed unbounded filter over ω_1 is ultra, and hence a normal ultrafilter over a measurable cardinal. This is quite an interesting phenomenon: the ostensibly strong, existential postulation of measurability is connected up with an easily defined filter over a small set. In this connection, see also §9 and §22.

§3. Compactness Properties of Languages

We must now backtrack a bit to pick up the development of ideas in a different direction, ideas which in fact led to the first modern results about large cardinals. Just before Scott's work, Hanf considered the problem of when analogues of the Compactness Theorem of first-order logic holds for infinitary languages $L_{\kappa\kappa}$. In general, $L_{\lambda\mu}$ is a direct generalization of the usual finitary language, which allows conjunctions of lengths $< \lambda$ and quantification blocks (homogeneous, in that no alternations of quantifiers is permitted within a block) of lengths $< \mu$. In the present context, it will only be necessary to consider possible additions of finitary function and relation symbols.

(See Karp[1965] or Dickmann[1974] for elaboration on the languages $L_{\lambda\mu}$. The study of generalized languages and logics has developed beyond this first, perhaps naive, generalization, with more attention paid to structural considerations. Substantial results like the Barwise Compactness Theorem and Lindstrom's Theorem showed that there were real secrets to be unearthed, and the present preoccupation with generalized quantifiers is a promising venture. In this context, $L_{\omega_1\omega}$ has an interesting theory (see Keisler[1971]), and is thought nowadays to be enough for adequately exhibiting the mathematically significant properties of the various $L_{\lambda\mu}$'s .)

A collection of sentences of $L_{\kappa\kappa}$ is satisfiable iff it has a model under the

natural interpretation of infinitary conjunction and quantification; and is κ-satisfiable iff every subcollection of cardinality $< \kappa$ is satisfiable. We now define for $\kappa > \omega$: κ is weakly compact iff whenever Σ is a κ-satisfiable collection of sentences of $L_{\kappa\kappa}$ using at most κ non-logical symbols, then Σ is satisfiable. (Actually, Hanf had the more stringent hypothesis $|\Sigma| \leq \kappa$. The present formulation is more convenient since it implies the inaccessibility of κ; see Drake [1974], page 300, or Devlin[1975]). κ is strongly compact iff whenever Σ is a κ-satisfiable collection of sentences of $L_{\kappa\kappa}$ (with any number of non-logical symbols), then Σ is satisfiable. (The higher order analogue is extendibility; see §16.)

Since $L_{\omega\omega}$ is just the usual finitary language, it is clear that these concepts are direct generalizations of the usual Compactness Theorem. Hanf[1964] was able to show that weakly compact cardinals are very Mahlo, and then Tarski[1962] showed that a measurable cardinal is weakly compact, thus providing the first proof that the least inaccessible is not measurable. Strongly compact cardinals were quickly seen to be measurable. (These results will become clear shortly.) It seems remarkable that metamathematical concepts, first studied with no thought to size considerations, should lead to large cardinal properties.

It turned out that weak compactness has many diverse characterizations, which is good evidence for the naturalness and efficacy of the concept. One is already given here, and for others see §4, §5, and the full treatment Devlin[1975]. κ is weakly compact iff (Keisler) whenever $R \subseteq V_\kappa$, there is a transitive X so that $\kappa \in X$ and an $S \subseteq X$, so that $\langle V_\kappa, \epsilon, R\rangle \prec \langle X, \epsilon, S\rangle$:

⊢ First, assume that κ is weakly compact and that $R \subseteq V_\kappa$. The following is a generalization of the use of the usual Compactness Theorem to get proper extensions of models. From (our formulation of) weak compactness one can prove that κ is inaccessible. Thus, $|V_\kappa| = \kappa$. Using new constants c_x for $x \in V_\kappa$, let Σ be the $L_{\kappa\kappa}$ theory of $\langle V_\kappa, \epsilon, R, x\rangle_{x \in V_\kappa}$ together with sentences "c is an ordinal" and "$c \neq c_x$" for every $x \in V_\kappa$. Σ is then κ-satisfiable, and so by weak compactness it is satisfiable. Now well-foundedness is expressible already in $L_{\omega_1\omega_1}$, and since Σ has a member saying ϵ is well-founded, any structure satisfying Σ is well-founded with respect to its "membership" relation. By Mostowski's collapsing lemma, let $\langle X, \epsilon, S, \bar{x}, \alpha\rangle_{x \in V_\kappa}$ be the transitive isomorph of such a structure. Σ has sentences of $L_{\kappa\kappa}$ stating exactly the members of each $x \in V_\kappa$, so by induction on rank, $x \in V_\kappa$ implies $x = \bar{x}$. Clearly, α is an ordinal $\geq \kappa$, so that $\kappa \in X$. Hence, the reduct $\langle X, \epsilon, S\rangle$ satisfies all the requirements.

For the converse, first show by straightforward means that the inaccessibility of κ is implied by the Keisler property. Then, note that for any inaccessible λ, the following two Löwenheim-Skolem-type theorems for $L_{\lambda\lambda}$ are easy to prove: (i) If σ is a satisfiable sentence of $L_{\lambda\lambda}$, then it has a model of cardinality $< \lambda$. (ii) If Σ is a satisfiable collection of sentences of $L_{\lambda\lambda}$ of cardinality $\leq \lambda$, then it has a model of cardinality $\leq \lambda$.

Suppose now that Σ is a κ-satisfiable collection of sentences of $L_{\kappa\kappa}$, using at most κ non-logical symbols. By the inaccessibility of κ , $|\Sigma| \leq \kappa$, so let $\Sigma = \{\sigma_\alpha \mid \alpha < \kappa\}$ and $R = \{\langle\alpha,\sigma_\alpha\rangle \mid \alpha < \kappa\}$. We construe Σ as a subset of V_κ . Using (i) above, it is easy to see that $\langle V_\kappa, \varepsilon, R\rangle \models \forall\alpha(\{\sigma_\beta \mid \beta < \alpha\}$ is satisfiable). By the Keisler property, let $\langle V_\kappa, \varepsilon, R\rangle \prec \langle X, \varepsilon, S\rangle$ where X is transitive and $\kappa \in X$. Thus by elementarity, $\langle X, \varepsilon, S\rangle \models \{\sigma_\alpha \mid \alpha < \kappa\}$ is satisfiable. But $\langle X, \varepsilon\rangle$ is a model of ZFC, and so by (ii) above, $\langle X, \varepsilon, S\rangle \models$ "$\{\sigma_\alpha \mid \alpha < \kappa\}$ has a model A with universe $\subseteq V_\kappa$". Finally, V_κ is closed under arbitrary $<\kappa$-sequences and $V_\kappa \subseteq X$, so that satisfaction for A is absolute between X and V , i.e. A really models $\{\sigma_\alpha \mid \alpha < \kappa\} = \Sigma$. $\dashv$

Strong compactness also has several characterizations; the following algebraic one was seen early on: κ is strongly compact iff for any set I , every κ-complete filter over I can be extended to a κ-complete ultrafilter over I :

$\vdash$ First, assume that κ is strongly compact and that F is a κ-complete filter over I . Just like getting a measurable cardinal from an elementary embedding, we want to get a "generic" element to generate the desired ultrafilter. So, using constants c_x for $x \subseteq I$, let Σ be the $L_{\kappa\kappa}$ theory of $\langle I, \varepsilon, x\rangle_{x\subseteq I}$ together with the sentences "$c \in c_x$" for every $x \in F$, where c is a new constant. Σ is κ-satisfiable as F is κ-complete. By strong compactness, let A model Σ . Now define U by: $x \in U$ iff $x \subseteq I$ & $A \models c \in c_x$. U will then be an ultrafilter extending F , and Σ has $L_{\kappa\kappa}$ sentences which assure that U is κ-complete.

Conversely, let $\Sigma = \{\sigma_\alpha \mid \alpha < \lambda\}$ be a κ-satisfiable collection of sentences of $L_{\kappa\kappa}$. We generalize the ultraproduct proof of the usual Compactness Theorem. Recall that $P_\kappa\lambda = \{s \subseteq \lambda \mid |s| < \kappa\}$. For any $s \in P_\kappa\lambda$ let A_s model $\bigwedge_{\alpha\in s} \sigma_\alpha$. Let U be any κ-complete ultrafilter over $P_\kappa\lambda$, extending the κ-complete filter generated by the sets $\{\{s \in P_\kappa\lambda \mid \alpha \in s\} \mid \alpha < \lambda\}$. Consider the ultraproduct $A = \Pi A_s/U$. Then for any $\alpha < \lambda$, $\{s \mid A_s \models \sigma_\alpha\} \supseteq \{s \mid \alpha \in s\} \in U$, so that $A \models \sigma_\alpha$ by Łoś' Theorem. $\dashv$

As an immediate corollary, we have: a strongly compact cardinal is measurable. The fact that such proofs as the preceding one directly lift known techniques from $L_{\omega\omega}$ seems appropriate. It is to be emphasized that strong compactness of κ is a global property of κ affecting all higher orders of the cumulative hierarchy (see §15 for Solovay's result on the GCH above a strongly compact cardinal). An appropriate weakening to a local property of the preceding characterization of strong compactness exists for weak compactness (see the ultrafilter properties in Devlin [1975]), and is quite natural in the context of Tarski's earlier work on ideals in (representable) Boolean algebras. It is an interesting latter-day fact that measurability can also be recast as a sort of compactness property (see Chang-Keisler [1973], page 198): κ is measurable iff whenever Σ_α for $\alpha < \kappa$ are satisfiable collections of sentences of $L_{\kappa\kappa}$ (with any number of non-logical symbols) so that

$\alpha < \beta < \kappa$ implies $\Sigma_\alpha \subseteq \Sigma_\beta$, then $\bigcup_\alpha \Sigma_\alpha$ is satisfiable.

Much of what might be called the classical theory of large cardinals was summed up in the long paper Keisler-Tarski [1964]. This paper presented many results about the extent of the classes C_0, C_1, C_2 of cardinals which are, respectively, not weakly compact, not measurable, and not strongly compact. Nowadays, we tend to think that weak compactness is a relatively weak concept, and that there are many more interesting train stops on the way to measurability. To the set theorist of today, flipping through the pages of Keisler-Tarski is an interesting experience, as he repeatedly sees theorems which he himself would state in dual or contrapositive form. The many theorems with hypotheses of form "if C_i contains all the cardinals" are perhaps suggestive of a point of view that these theorems are partial results toward ultimately establishing these hypotheses true outright.

Of course, the sky looks highly unapproachable from the ground. But in the current climate of numerous relative consistency results and the Axiom of Determinacy, we are not afraid to get into our mental helicopters and, Icarus-like, soar unashamedly in speculative altitudes implicit in forms of statements like "if there is a measurable cardinal, then . . .". If the sun begins to melt the wax on our wings, we can always don the formalist parachute by saying that these are interesting implications of ZFC. It may be historically interesting to trace this change of attitude from the days of Keisler-Tarski; does it have any connection to the switch from prime ideal to the dual notion, ultrafilter, in model theory? Thinking big sometimes has its advantages.

Once the concept of strong compactness became known, a nice global generalization of Scott's result was soon found. We assume the reader's familiarity with the notion of relative constructibility; for any set x, L[x] is the smallest inner model M of ZF so that $x \cap M \in M$, and for any such M , $L[x]^- = (L[x])^M = (L[x \cap M])^M$. Vopěnka-Hrbáček [1966] proved the following result: if there is a strongly compact cardinal κ , then $V \neq L[x]$ for any set x :

$\vdash$ Suppose to the contrary that $V = L[x]$, where without loss of generality we can assume x is transitive and $|x| = \lambda \geq \kappa$. Using the previous characterization of strong compactness we can easily get a uniform κ-complete ultrafilter U over λ^+ by extending the (κ-complete) filter $F = \{Y \subseteq \lambda^+ \mid |\lambda^+ - Y| \leq \lambda\}$. Let $j: V \to M \cong V^{\lambda^+}/U$.

Now look at the substructure of V^{λ^+}/U consisting of equivalence classes of those $f: \lambda^+ \to V$ so that $|\text{Range}(f)| \leq \lambda$. Call this substructure V^{λ^+}/U^- . It is true that Łoś' Theorem holds for this restricted ultrapower (the induction step to the existential quantifier must be checked), and so in particular the natural embedding $V \to V^{\lambda^+}/U^-$ is also elementary. V^{λ^+}/U^- is well-founded with respect to its "membership" relation, so let N be its transitive isomorph. Set $k: V \to N \cong V^{\lambda^+}/U^-$, and for $f: \lambda^+ \to V$ so that $|\text{Range}(f)| \leq \lambda$, let $[f]^-$ denote the element of N

corresponding to the equivalence class of f in V^{λ^+}/U^-. Define $i: N \to M$ by $i([f]^-) = [f]$. The following will then be true:

(i) i is elementary, and $j = i \cdot k$.

(ii) $i(\alpha) = \alpha$ for every $\alpha < k(\lambda^+)$, and $i(k(x)) = k(x)$. (One can prove inductively that: for any f so that the transitive closure of Range(f) has cardinality $\leq \lambda$, $[f]^- = [f]$.)

(iii) $k(\lambda^+) = \sup\{k(\alpha) \mid \alpha < \lambda^+\} \leq [id] < j(\lambda^+)$, where $id: \lambda^+ \to \lambda^+$ is the identity function.

We now get a contradiction as follows: Since $V = L[x]$, by absoluteness of relative constructibility $M = L[k(x)] = N$, as $i(k(x)) = k(x)$. Now in M, $j(\lambda^+)$ is the successor of $j(\lambda)$, and in N, $k(\lambda^+)$ is the successor of $k(\lambda)$. So, as $k(\lambda) = i(k(\lambda)) = j(\lambda)$ by (ii), $M = N$ implies $k(\lambda^+) = j(\lambda^+)$. This contradicts (iii). ⊣

§4. Indescribability

Linguistic generalization in a different direction, expressibility with higher type variables rather than in infinitely long formulas, turned out to yield a classification scheme which provides an efficient means of comparing the sizes of various large cardinals. In the brief note Hanf-Scott [1961] was introduced the idea of looking at higher order reflection properties that hold for $< V_\kappa, \varepsilon, R >$ where $R \subseteq V_\kappa$.

As usual, let Π^m_n (Σ^m_n, respectively) be the class of formulas (in the (finitary) language with ε and higher type variables) which in prenex form has at most n alternating blocks of quantifiers of (m+1)th order variables, starting with the universal (existential, respectively) quantifier. Now define (for $m,n > 0$) κ is Π^m_n-indescribable iff whenever ϕ is Π^m_n in one free second-order variable and $R \subseteq V_\kappa$, then $< V_\kappa, \varepsilon, R > \models \phi(R)$ implies that there is an $\alpha < \kappa$ so that $< V_\alpha, \varepsilon, R \cap V_\alpha > \models \phi(R \cap V_\alpha)$. Define Σ^m_n-indescribable analogously. Because there is sufficient coding apparatus in V_κ for κ a cardinal, this definition is equivalent to one where R is replaced by any finite number of relations on V_κ, some possibly k-ary for $1 \leq k < \omega$. As the intention is for higher order variables to have the standard interpretations, Π^m_n-indescribability could also have been defined in terms of satisfiability of just first-order formulas in $V_{\kappa+m}$ reflecting down to a $V_{\alpha+m}$. One can also pursue indescribability via transfinite types (see Drake [1974]).

Many of the large cardinals κ that have been investigated are defined via some higher order property for $< V_\kappa, \varepsilon >$. The point of the indescribable cardinals is that they are characterized by some degree of uncharacterizability. The degree of closure at some height V_κ is to be measured by the linguistic complexity of formulas which reflect. This is a direct generalization of the ordinary Reflection Principle; the new added feature of considering higher order statements is possible

because aggrandizement in the cumulative hierarchy continues beyond V_κ.

The first thing to see about the definitions is that Σ^1_{n+1}-indescribability is equivalent to Π^1_n-indescribability, since we can use the $R \subseteq V_\kappa$ for (second-order) existential instantiation. Similarly, Σ^1_1-indescribability just becomes "first-order" indescribability, and so a straightforward argument like for the model theoretic characterization of inaccessibility in §1 shows that: κ is Σ^1_1-indescribable iff κ is inaccessible. What about Π^1_1-indescribability? Hanf and Scott found that it provides a nice characterization: κ is Π^1_1-indescribable iff κ is weakly compact:

⊢ First, suppose κ is Π^1_1-indescribable. Then κ is inaccessible by the comment on first-order indescribability. We intend to show that κ has the Keisler property for weak compactness. By the Löwenheim-Skolem Theorem, $< V_\kappa, \varepsilon, R > \prec < X, \varepsilon, S >$ for some transitive X with $\kappa \in X$ and $S \subseteq X$ just in case there is such an X with the additional property that $|X| = |V_\kappa| = \kappa$. Thus, assuming the Keisler property did not hold for some $R \subseteq V_\kappa$, then $< V_\kappa, \varepsilon, R, \kappa > \models$ $(\forall X \forall E \forall S \forall J \forall \gamma$(J is not an isomorphism of $< X, E, S, \gamma>$ to a transitive elementary extension of the universe, which contains κ, so that S is to interpret R)). When properly formalized, this becomes a Π^1_1 statement, say $\phi(R,\kappa)$.

By another Löwenheim-Skolem argument, for any $\alpha < \kappa$ there is an $< X_\alpha, \varepsilon, S_\alpha, \gamma_\alpha > \prec < V_\kappa, \varepsilon, R, \kappa >$ so that X_α is transitive, $V_\alpha \cup \{\alpha\} \subseteq X_\alpha$, and $|V_\alpha| = |X_\alpha|$. Now by a comment in §1, $C = \{\alpha < \kappa \mid < V_\alpha, \varepsilon, R \cap V_\alpha, \alpha > \prec < V_\kappa, \varepsilon, R, \kappa > \}$ is closed unbounded in κ. Observe that for any $\alpha \in C$, $< V_\alpha, \varepsilon, R \cap V_\alpha, \alpha > \prec < X_\alpha, \varepsilon, S_\alpha, \gamma_\alpha >$, so that we can conclude $<V_\alpha, \varepsilon, R\cap V_\alpha, \alpha> \models \neg\phi(R \cap V_\alpha, \alpha)$.

Let $\psi(C)$ be a (first-order) statement saying that C is unbounded. Thus, $< V_\kappa, \varepsilon, R, \kappa, C > \models \psi(C) \;\&\; \phi(R,\kappa)$. By Π^1_1-indescribability, there is an $\alpha < \kappa$ so that $< V_\alpha, \varepsilon, R\cap V_\alpha, \alpha, C \cap \alpha > \models \psi(C\cap\alpha) \;\&\; \phi(R \cap V_\alpha, \alpha)$. However, the first clause insures that $\alpha \in C$ as C is closed, and so the second clause contradicts the conclusion of the previous paragraph.

For the converse, assume $< V_\kappa, \varepsilon, R > \models \phi(R)$, where ϕ is Π^1_1. By the Keisler property of weak compactness, $< V_\kappa, \varepsilon, R > \prec < X, \varepsilon, S >$ for some transitive X with $\kappa \in X$ and $S \subseteq X$. Since Π^1_1 formulas are preserved under restriction, we have $< X, \varepsilon, S > \models (\exists\alpha(<V_\alpha, \varepsilon, S\cap V_\alpha> \models \phi(S \cap V_\alpha)))$, since κ is such an α and $S \cap V_\kappa = R$ (note that $\kappa \in X$ implies $V_\kappa \in X$, as X is a transitive model of ZFC extending V_κ). However, this sentence is first-order, so that $< V_\kappa, \varepsilon, R > \models (\exists\alpha(<V_\alpha, \varepsilon, R\cap V_\alpha> \models \phi(R\cap V_\alpha)))$ by elementarity. This means (since for any $\alpha < \kappa$, all subsets of V_α are in V_κ) that for some $\alpha < \kappa$, $< V_\alpha, \varepsilon, R \cap V_\alpha > \models \phi(R \cap V_\alpha)$, which was to be proved. ⊣

Finally, Hanf-Scott [1961] also stated the following result: a measurable cardinal is Π^2_1-indescribable:

⊢ Suppose that $\langle V_\kappa, \epsilon, R\rangle \models \forall X\phi(R)$, where X is a third-order variable and ϕ has at most second-order quantifiers. Let U be a κ-complete ultrafilter over κ, and $j: V \to M \approx V^\kappa/U$. By the Closure Lemma of §2, $V_{\kappa+1} \subseteq M$, so that second-order statements about $\langle V_\kappa, \epsilon\rangle$ are absolute between V and M. But the third-order universal quantifier is preserved under restriction, so that $M \models (\langle V_\kappa, \epsilon, R\rangle \models \forall X\phi(R))$, i.e. $M \models (\exists\alpha<j(\kappa)(\langle V_\alpha, \epsilon, j(R)\cap V_\alpha\rangle \models \forall X\phi(j(R)\cap V_\alpha))$, as κ is such an α and $j(R)\cap V_\kappa = R$. Thus by elementarity, there is an $\alpha < \kappa$ so that $\langle V_\alpha, \epsilon, R\cap V_\alpha\rangle \models \forall X\phi(R\cap V_\alpha)$. ⊣

Notice that this is the best possible, since there is a Σ^2_1 sentence characterizing the least measurable cardinal. Nowadays, it is known that Π^2_1-indescribability by no means characterizes measurability: If U is a normal ultrafilter over a measurable cardinal κ, then $\{\alpha < \kappa \mid \alpha$ is Π^m_n-indescribable for every $m,n\} \in U$:

⊢ Let $j: V \to M \approx V^\kappa/U$. To get the result it suffices to show that $M \models$ "κ is Π^m_n-indescribable for every m,n", for then we can apply normality and Łoś' Theorem. The following argument works with very little restriction on the precise nature of ϕ: Suppose $M \models (\langle V_\kappa, \epsilon, R\rangle \models \phi(R))$. Thus, $M \models (\exists\alpha<j(\kappa)(\langle V_\alpha, \epsilon, j(R)\cap V_\alpha\rangle \models \phi(j(R)\cap V_\alpha))$, as κ is such an α and $j(R)\cap V_\kappa = R$. By elementarity, there is an $\alpha < \kappa$ so that $\langle V_\alpha, \epsilon, R\cap V_\alpha\rangle \models \phi(R\cap V_\alpha)$, which is the same in M as in V because $V_\kappa^M = V_\kappa$. But then, we are done. ⊣

Thus, for example, though the least measurable cardinal can be described easily enough by a third-order existential statement, there are many cardinals below it which are highly indescribable. In particular, the presence of one measurable cardinal in the universe renders the theory of the hierarchy of indescribable cardinals highly non-trivial.

Unlike measurable cardinals, we should point out that the indescribable cardinals are compatible with $V = L$: If κ is Π^m_n-indescribable, then (κ is Π^m_n-indescribable)^L. Similarly for Σ^m_n. In particular, if κ is weakly compact, then κ is weakly compact in L. See Devlin[1975] for a proof. The point is that for κ inaccessible, $(V_\kappa)^L = L_\kappa$, and, for example, for taking care of the case $m = 1$, that there is a Σ^1_1 formula $\phi(\cdot)$ so that whenever κ is inaccessible and $X \subseteq L_\kappa$, $X \in L$ iff $V_\kappa \models \phi(X)$. In other words, relatively simple formulas describe constructibility, and these can be used to prove the relativizations to L.

Once the stage has been set, it is not unreasonable that there should be direct analogues between degrees of indescribability and Kleene's Arithmetical Hierarchy. Indeed, the following enumeration theorem should have a familiar ring (Lévy([1971]; see also Devlin[1975]): For any $m,n > 0$ there is a Π^m_n formula $\chi_{mn}(\cdot,\cdot)$ so that for any Π^m_n formula $\phi(\cdot)$, there is an integer k so that for any limit α and $R \subseteq V_\alpha$,

$$\langle V_\alpha, \epsilon, R\rangle \models \phi(R) \text{ iff } \langle V_\alpha, \epsilon, R\rangle \models \chi_{mn}(k,R).$$

(Analogously for Σ^m_n.)

The proof, of course, uses the satisfaction predicate. We have the following corollaries:

(a) For any $n > 0$ there is a Π^1_{n+1} sentence Γ so that $\langle V_\kappa, \epsilon\rangle \models \Gamma$ iff κ is Π^1_n-indescribable.

(b) If $m > 1$, for any $n > 0$ there is a Π^m_n (Σ^m_n respectively) sentence Γ so that $\langle V_\kappa, \epsilon\rangle \models \Gamma$ iff κ is Σ^m_n- (Π^m_n- respectively) indescribable.

⊢ For (a), let Γ be: $\forall R\forall k(\chi_{1n}(k,R) \to \exists\alpha(\alpha$ is a limit and $\langle V_\alpha, \epsilon, R\cap V_\alpha\rangle \models \chi_{1n}(k,R\cap V_\alpha)))$. Here, χ_{1n} is as in the previous theorem. Because of the $\forall R$, and as χ_{1n} is Π^1_n and stands to the left of the implication sign, Γ is Π^1_{n+1}. (b) is like (a), but using χ_{mn} instead; the second order $\forall R$ now gets subsumed in the proceedings, as $m > 1$. ⊣

Let us make some further definitions: π^m_n = the least Π^m_n-indescribable cardinal, and σ^m_n = the least Σ^m_n-indescribable cardinal. (In these and similar cases, it is to be implicit that any theorem using this notation is actually an implication of the type "if a Π^m_n-indescribable cardinal exists, then . . ." Thus, we do not worry over such vacuities as the baldness of the present king of France!) Since we have already remarked that Σ^1_{n+1}-indescribability is equivalent to Π^1_n-indescribability, it is clear from (a) and (b) above that:

σ^1_1 = least inaccessible

$\pi^1_1 = \sigma^1_2 < \pi^1_2 = \sigma^1_3 < \ . \ . \ .$

$\sigma^m_n \neq \pi^m_n$,

$\sigma^m_n < \sigma^m_{n+1}$, π^m_{n+1} and

$\pi^m_n < \sigma^m_{n+1}$, π^m_{n+1} for any $m,n > 0$.

It is still not known (in ZFC alone) whether $\sigma^m_n < \pi^m_n$ or vice versa. In this connection we describe an interesting historical development. A potentially very fruitful spin-off of the study of large cardinals which has not been fully investigated is the consideration of their various analogues in simplified contexts, such as second-order arithmetic or admissible set theory. (See for example Harrington [1974] for the notion of a recursively Mahlo ordinal.) The theory of inductive definitions provided a typical context:

Richter[1971], Aczel-Richter[1972][1974] developed connections between analogues of large cardinals and closure ordinals of inductive definitions. For example, if $|\Gamma|$ denotes the supremum of closure ordinals of (not necessarily monotone) inductive operations on $P(\omega)$ in the class Γ, it was seen that $|\Pi^1_1|$ can be characterized by a reflection property very much like Π^1_1-indescribability. It was also realized that $|\Pi^1_1| \neq |\Sigma^1_1|$, but it remained for Aanderaa[1973] to show among other things that $|\Pi^1_1| < |\Sigma^1_1|$. Finally, Moschovakis noticed that Aanderaa's method shows: If $V = L$, then $\sigma^m_n < \pi^m_n$ for any $m,n > 0$. See Devlin[1975] for a proof; the assumption $V = L$ unfortunately seems essential to lift the effective aspects of Aanderaa's proof. Thus we have come full circle: the indescribable cardinals are complemented by natural analogues, and their study leads again to a result about indescribable cardinals. Can the assumption $V = L$ be eliminated from the above result? An independence result is perhaps unlikely, but still possible.

In emphasizing the concept of normality for filters in §2, we were in part foreshadowing the interesting phenomenon that many large cardinals turn out to carry naturally defined normal filters over them. This becomes quite an asset in the structural study of these cardinals, and provides new insights into similarities with measurable cardinals and their normal ultrafilters. Lévy[1971] first discovered this phenomenon, in the context of the indescribable cardinals:

Let us first generalize a definition: an $X \subseteq \kappa$ is called Π^m_n-indescribable iff whenever $\phi(\cdot)$ is a Π^m_n formula and $R \subseteq V_\kappa$, then $\langle V_\kappa, \in, R\rangle \models \phi(R)$ implies that there is an $\alpha \in X$ so that $\langle V_\alpha, \in, R\cap V_\alpha\rangle \models \phi(R\cap V_\alpha)$. We then have: If κ is Π^m_n-indescribable, then $\mathcal{F}^m_n = \{ X \subseteq \kappa \mid \kappa - X$ is not Π^m_n-indescribable $\}$ is a κ-complete normal filter over κ. (Similarly for Σ^m_n.) See Lévy[1971] or Baumgartner[1975] for a proof; in Lévy's terminology, the members of $\mathcal{F}^m_n$ are called Π^m_n-enforceable.

The Π^m_n-indescribable sets are the positive measure sets with respect to $\mathcal{F}^m_n$, so they are analogous to the stationary sets for the closed unbounded filter. Note that for any Π^m_n formula $\phi(\cdot)$ and any $R \subseteq V_\kappa$, the set $\{ \alpha < \kappa \mid \langle V_\alpha, \in, R\cap V_\alpha\rangle \models \phi(R\cap V_\alpha) \}$ must be in $\mathcal{F}^m_n$. Also, any such set must be stationary since $\mathcal{F}^m_n$ is normal (and hence must extend the closed unbounded filter). The following is now a consequence of these facts, and (a), (b) above: If κ is Π^m_{n+1}-indescribable, then both $\{ \alpha < \kappa \mid \alpha$ is Π^m_n-indescribable $\}$ and $\{ \alpha < \kappa \mid \alpha$ is Σ^m_n-indescribable $\}$ are in $\mathcal{F}^m_{n+1}$ and hence are stationary in κ. (Similarly for Σ^m_{n+1}, when $m > 1$.)

Strictly speaking, we do not need the $\mathcal{F}^m_n$'s to show the mere stationariness of these sets, but the stronger statements exhibited have an intrinsic interest, and typifies how often in hierarchies of large cardinals, a cardinal at one level defines for itself its transcendent largeness as compared to cardinals of lower levels.

II. PARTITION PROPERTIES

§5. The Properties $\alpha \to (\beta)^n_\delta$, and Trees

We now take up yet another stream which will flow into our developing framework. Erdös-Rado[1952][1956], Erdös-Hajnal[1958] (see also the surveys Erdös-Hajnal-Rado [1965], and Erdös-Hajnal[1971]) developed a theory of combinatorics in set theory based on a partition calculus. For our present context, it is interesting that (as pointed out in Erdös-Hajnal[1962]) the 1958 paper had already contained enough results to deduce Tarski's result that the least inaccessible is not measurable.

We first define the partition symbol in this calculus for directly generalizing Ramsey's famous theorem. Recall that if X is a set of ordinals then $[X]^\gamma = \{ Y \subseteq X \mid Y$ has order type $\gamma \}$. Then

$$\alpha \to (\beta)^\gamma_\delta$$

means that whenever $f\colon [\alpha]^\gamma \to \delta$, there is an $H \in [\alpha]^\beta$ homogeneous for f, i.e. $|f''[H]^\gamma| = 1$. The idea behind this somewhat arcane symbolism is that the relation is preserved upon making the ordinal on the left larger, or making any of the ordinals on the right smaller. Ramsey's theorem states that $\omega \to (\omega)^m_n$ for any positive integers m, n. As we are presently assuming the Axiom of Choice, and since it is known in this case that $\alpha \to (\omega)^\omega_2$ does not hold for any α, we shall for the moment restrict ourselves to the consideration of the partition symbol only for $\gamma < \omega$ (see §28 for partitions of infinite tuples).

Of their many results, what has become known as the Erdös-Rado Theorem is the statement that for any κ,

$$(\beth_n(\kappa))^+ \to (\kappa^+)^{n+1}_\kappa ,$$

where we define $\beth_0(\kappa) = \kappa$ and $\beth_{n+1}(\kappa) = 2^{\beth_n(\kappa)}$. (See below for a proof.) This is known to be the best possible (in the sense that one cannot lower the left or raise anything on the right), and shows in particular that for any λ, n, δ, there is a κ so that $\kappa \to (\lambda)^n_\delta$. The possibility of a fixed point $\kappa = \lambda$ had already been voiced in Erdös-Tarski[1943], and in fact it was discovered in the late fifties that $\kappa \to (\kappa)^2_2$ is equivalent to the weak compactness of κ (see below).

What lies at the heart of these matters soon emerged as a property of cardinals involving trees (called "ramification systems" in early papers). Let us quickly review the terminology: A tree is a partially ordered set $\langle T, <_T\rangle$ so that the $<_t$-predecessors of any element are well-ordered by $<_T$. The level α of T consists of those $t \in T$ so that the $<_T$-predecessors of t under $<_T$ has order type α. The height of T is the least α so that T has no elements in level α. A branch is a subset of T which is a linearly ordered initial segment under $<_T$. Finally, an α-branch of T is a branch of order type α under $<_T$. The connection between trees and partition relations is isolated in the following key

Lemma: If $n \geq 2$ and $f\colon [\kappa]^n \to \lambda$, then there is a tree ordering $<_f$ with domain κ so that:

(i) If $\xi <_f \delta$, then $\xi < \delta$.

(ii) If $\xi_1 <_f \cdots <_f \xi_{n-1} <_f \delta <_f \eta$, then $f(\{\xi_1,\ldots,\xi_{n-1},\delta\}) = f(\{\xi_1,\ldots,\xi_{n-1},\eta\})$.

(iii) For each infinite α, the level α of the tree $\langle\kappa,<_f\rangle$ has cardinality at most $\lambda^{|\alpha|}$. For each $m \in \omega$, the level m has cardinality at most λ^{ω}.

The proof of this lemma involves building the desired tree by straightforward induction. (Set $0 <_f 1 <_f \cdots <_f n$. In general, if $\alpha < \kappa$ and all $\beta < \alpha$ have been taken care of, choose *one* maximal branch b of the tree already constructed, with the property that $\beta_1 <_f \cdots <_f \beta_n$ in b implies $f(\{\beta_1,\ldots,\beta_{n-1},\beta_n\}) = f(\{\beta_1,\ldots,\beta_{n-1},\alpha\})$, and designate $\beta <_f \alpha$ for each $\beta \in b$. In the case that no such branch exists, put α into level $n-1$.) To show the usefulness of this lemma, we outline from it the proof of the Erdös-Rado Theorem stated above:

$\vdash$ Proceed by induction on n; the case $n = 0$ follows from the regularity of κ^+. Suppose now that $(\beth_n(\kappa))^+ \to (\kappa^+)^{n+1}_{\kappa}$ is already known, and assume $f\colon [(\beth_{n+1}(\kappa))^+]^{n+2} \to \kappa$. Let $<_f$ be a tree on $(\beth_{n+1}(\kappa))^+$ as in the lemma, for this f.

We claim that the level $\beth_n(\kappa)^+$ of the tree is not empty (and this is precisely where the $\beth$-type numbers over κ come into play). If this were *not* the case, by (iii) of the lemma we would have $\beth_{n+1}(\kappa)^+ = |T| \leq \bigcup\{\kappa^{|\alpha|} \mid \alpha < \beth_n(\kappa)^+\} \leq 2^{\beth_n(\kappa)} < \beth_{n+1}(\kappa)^+$, a contradiction.

Thus, there is an η at level $\beth_n(\kappa)^+$ of the tree. Consider the branch $C = \{\xi \mid \xi <_f \eta\}$. By (ii) of the lemma C is a *prehomogeneous* set for f, i.e. for $\xi_1 <_f \cdots <_f \xi_{n+1} <_f \delta$ all in C, $f(\{\xi_1,\ldots,\xi_{n+1},\delta\})$ only depends on $\langle\xi_1,\ldots,\xi_{n+1}\rangle$, and so $g\colon [C]^{n+1} \to \kappa$ defined by

$$g(\{\xi_1,\ldots,\xi_{n+1}\}) = f(\{\xi_1,\ldots,\xi_{n+1},\delta\}) \quad \text{for any } \delta \in C \text{ greater than all the } \xi_i\text{'s}$$

is well-defined. Now the inductive hypothesis can be applied to g to extract a subset of C of order type κ^+, homogeneous for the original f. $\dashv$

As this proof shows, the existence of large homogeneous sets can be related to the existence of long branches in trees. Ramsey's theorem can be proved in this fashion, and the basic phenomenon involved here can be isolated as follows: A *κ-tree* is a tree of height κ with each level of cardinality $< \kappa$. The *tree property* for κ is the assertion that every κ-tree has a κ-branch.

The well-known König's Lemma is just the statement that ω has the tree property. A classical result of Aronszajn is that ω_1 does not have the tree property, and so a counterexample to the tree property for κ in general has become known as a *κ-Aronszajn tree*. If κ is singular, it is easy to see that there is a κ-Aronszajn

tree. Specker[1951] remarked that Aronszajn's construction can be extended to show generally that whenever $2^{<\kappa} = \kappa$ and κ is regular, there is a κ^+-Aronszajn tree. However, little is known about the existence of κ^+-Aronszajn trees for κ singular. That the tree property for κ may not imply the inaccessibility of κ was first noticed by Silver[1966] when he showed that: *If κ is real-valued measurable, then κ has the tree property*. For more on trees, see the expository Jech[1971]. We discuss in §21 an actual equi-consistency result involving the tree property for ω_2.

Combining inaccessibility with the tree property leads to the following characterizations of weak compactness, known by the early sixties: *The following are equivalent:*

(i) κ is weakly compact.

(ii) κ is inaccessible and has the tree property.

(iii) $\kappa \to (\kappa)^n_\lambda$ for every $n < \omega$, $\lambda < \kappa$.

(iv) $\kappa \to (\kappa)^2_2$.

$\vdash$ *(i) → (ii).* To establish the tree property, let $\langle T,<_T\rangle$ be a κ-tree. To each $t \in T$ associate a propositional letter P_t, and consider the collection of $L_{\kappa\kappa}$ (in fact $L_{\kappa\omega}$) sentences consisting of: $\sigma_\alpha = \bigvee\{P_t \mid t \text{ is at level } \alpha\}$ for each $\alpha < \kappa$, and $\neg(P_t \,\&\, P_{\bar t})$ for each pair $t,\bar t$ in T which are not $<_T$-comparable. Since T is a tree of height κ, this collection of sentences is κ-satisfiable. Hence by weak compactness, it is satisfiable, say by a model $\mathfrak{A}$. Clearly, $\{t \in T \mid \mathfrak{A} \models P_t\}$ then constitutes a κ-branch through T.

(ii) → (iii). The proof (for all $\lambda < \kappa$, by induction on m) of this assertion is just like the proof of the Erdös-Rado Theorem given above; the tree constructed in the lemma is a κ-tree by the inaccessibility of κ, and the tree property assures that there is a κ-branch, and hence a homogeneous set of size κ.

(iii) → (iv). is trivial.

(iv) → (i). This seems to involve a long trudge back, suitable only for the robust and careful reader. (It seems incumbent upon us to give this proof, since we presented proofs of all other directions!) We first show *(iv) → (*)*, where (*) says: *Whenever $\langle L,<_L\rangle$ is a total ordering of cardinality $\geq \kappa$, there is either an increasing or decreasing $<_L$-sequence of order type κ.* To establish this, let $e\colon \kappa \to L$ be an injection and let $f\colon [\kappa]^2 \to 2$ be defined by $f(\{\alpha,\beta\}) = 0$ *iff* $\alpha < \beta$ & $e(\alpha) <_L e(\beta)$. Any homogeneous set of cardinality κ then corresponds to a $<_L$-sequence of the desired sort.

We next establish that *(*) implies (ii)*. *Firstly*, κ is regular since otherwise let $\kappa = \bigcup_{\delta<\lambda} X_\delta$ where $\lambda < \kappa$ and each X_δ has cardinality $< \kappa$. Define $\prec$ on κ by $\alpha \prec \beta$ *iff* either $\alpha < \beta$ and they are in the same X_δ; or $\alpha \in X_\gamma$, $\beta \in X_\eta$, and $\gamma > \eta$. Then $\prec$ is a total order on κ with no increasing or decreasing suborder of type κ, a contradiction of (*). *Secondly*, κ is inaccessible, for otherwise let $\lambda < \kappa$ be least so that $\kappa \leq 2^\lambda$. Let $\ll$ be the lexigraphic ordering of ${}^\lambda 2$, i.e. $f \ll g$ *iff* when $\alpha < \lambda$ is the least so that $f(\alpha) \neq g(\alpha)$, then

$f(\alpha) < g(\alpha)$. Then, the collection of functions in ${}^{\lambda}2$ which are eventually zero is dense in $<<$, and this collection has cardinality $\bigcup\{2^{\nu} \mid \nu < \lambda\} < \kappa$ by choice of λ . Thus, $<<$ cannot have an increasing or decreasing suborder of order type κ , again a contradiction of (*) .

Finally, κ has the tree property, for suppose that $\langle T, <_T \rangle$ is a κ-tree. If $t \in T$ is at a level α and $\beta < \alpha$, denote by $\pi_\beta(t)$ the predecessor of t at level β . It is straightforward to define a total ordering $\prec$ of T which extends $<_T$ so that if t_1 and t_2 are at levels $> \beta$, then $t_1 \prec t_2$ implies $\pi_\beta(t_1) \prec \pi_\beta(t_2)$ or $\pi_\beta(t_1) = \pi_\beta(t_2)$. (To do this, simply order each level "left to right" inductively up through T .) Applying (*), let $S = \langle s_\xi \mid \xi < \kappa \rangle$ be a $\prec$-increasing sequence of type κ . (S could not have been $\prec$-decreasing, as each level of T has cardinality $< \kappa$ and κ is regular; for the same reasons S has elements in arbitrarily high levels of T .) For $\alpha < \kappa$, let η_α be so that whenever $\eta_\alpha < \xi < \kappa$, s_ξ is at a level $\geq \alpha$. By the properties of $\prec$, note that $\langle \pi_\alpha(s_\xi) \mid \eta_\alpha < \xi < \kappa \rangle$ is non-$\prec$-decreasing in ξ , and hence eventually constant. Let t_α be this constant value. Then $\langle t_\alpha \mid \alpha < \kappa \rangle$ is a branch through T (as any two t_α's have a common successor in T). We have thus established the tree property for κ , and hence (*) $\rightarrow$ (ii).

We now complete the proof by showing that (ii) implies the Keisler property for weak compactness. So, suppose $R \subseteq V_\kappa$. We want to find an extension $\langle V_\kappa, \epsilon, R \rangle \prec \langle X, \epsilon, S \rangle$ so that X is transitive with $\kappa \in X$, and $S \subseteq X$. Let W be a well-ordering of V_κ . By the inaccessibility of κ we have (as remarked before) that $C = \{\alpha < \kappa \mid \langle V_\alpha, \epsilon, R \cap V_\alpha, W \cap V_\alpha \rangle \prec \langle V_\kappa, \epsilon, R, W \rangle\}$ is closed unbounded in κ . Let $\langle \alpha_\xi \mid \xi < \kappa \rangle$ be the ascending enumeration of C .

We will now define a tree $\langle T, <_T \rangle$. First, for $\alpha_\xi < \beta < \kappa$, let $H(\xi,\beta)$ be the Skolem Hull in $\langle V_\kappa, \epsilon, R, W \rangle$ of $V_{\alpha_\xi} \cup \{\beta\}$, so that $\langle V_{\alpha_\xi}, \epsilon, R \cap V_{\alpha_\xi}, W \cap V_{\alpha_\xi} \rangle \prec H(\xi,\beta)$. As we have augmented the proceedings with the well-ordering W , we can consider $H(\xi,\beta)$ to be well-defined. Say that $H(\xi,\beta) \sim H(\bar{\xi},\bar{\beta})$ iff $\xi = \bar{\xi}$ and there is isomorphism between the two structures fixing V_{α_ξ} and sending β to $\bar{\beta}$. Clearly, $\sim$ is an equivalence relation, and so let $[H(\xi,\beta)]$ denote the equivalence class of $H(\xi,\beta)$. We define the elements of our tree T to be the $[H(\xi,\beta)]$'s . Finally, set $[H(\xi,\beta)] <_T [H(\bar{\xi},\bar{\beta})]$ iff $\xi < \bar{\xi}$, $\beta \leq \bar{\beta}$, and $H(\xi,\beta) \sim$ the Skolem Hull in $H(\bar{\xi},\bar{\beta})$ of $V_{\alpha_\xi} \cup \{\bar{\beta}\}$. (Again, Skolem Hulls can be considered unique because of W .)

That $\langle T, <_T \rangle$ is indeed a tree is easy to see; note that the ξth level of T is $\{[H(\xi,\beta)] \mid \beta < \kappa\}$. That $\langle T, <_T \rangle$ is actually a κ-tree follows from the inaccessibility of κ and the fact that there are at most $2^{|V_{\alpha_\xi}|} < \kappa$ ways (up to isomorphism) to build Skolem Hulls over V_{α_ξ} with one free variable to be interpreted as some $\beta > \alpha_\xi$.

Thus, by the tree property for κ let $\{[H(\xi,\beta_\xi)] \mid \xi < \kappa\}$ be a κ-branch through T . By definition of $<_T$, whenever $\xi \leq \eta < \kappa$ there is an elementary embedding $i_{\xi\eta}: H(\xi,\beta_\xi) \rightarrow H(\eta,\beta_\eta)$ that fixes V_{α_ξ} , so that $i_{\xi\eta}(\beta_\xi) = \beta_\eta$. From the construction, it can be seen that $\xi \leq \eta \leq \rho < \kappa$ implies $i_{\xi\rho} = i_{\eta\rho} \cdot i_{\xi\eta}$. Thus, we can form the direct limit, and it is straightforward to see that it is well-founded, and hence that its transitive isomorph $\langle X, \epsilon, S, \bar{W} \rangle$ is an elementary extension of $\langle V_\kappa, \epsilon, R, W \rangle$. Finally, note that the β_ξ's get identified together to correspond to an ordinal $\beta \in X$ with $\beta \geq \kappa$. As X is transitive, we have $\kappa \in X$, and laying the good warrior W to rest, the reduct $\langle X, \epsilon, S \rangle$ is as required. The proof is (finally!) complete. ⊣

Perhaps some technical remarks concerning the preceding proof are in order. The proof of (ii) $\rightarrow$ (iii) deployed on ω establishes Ramsey's theorem, and thus, weak compactness can be seen as a generalization of a property of ω in this rather precise way. It is interesting that a more direct, combinatorial way of showing (iv) $\rightarrow$ (iii) is not known. Finally, in the process of establishing (iv) $\rightarrow$ (i) we also isolated a principle (*) equivalent to weak compactness and interesting in its own right.

§6. The Properties $\alpha \rightarrow (\beta)^{<\omega}_\delta$, and Rowbottom and Jonsson Cardinals

It remained for partition relations of a stronger character to invite the infusion of model theoretic techniques even more directly into the study of large cardinals. Recall that $[\kappa]^{<\omega} = \bigcup_{n\in\omega} [\kappa]^n$. Erdös, Hajnal, and Rado also considered the following symbol in their partition calculus:

$$\alpha \rightarrow (\beta)^{<\omega}_\delta$$

means that whenever $f: [\alpha]^{<\omega} \rightarrow \delta$, there is an $H \in [\alpha]^\beta$ homogeneous for f , i.e. for every n , $|f"[H]^n| = 1$. For any ordinal α , the Erdös cardinal $\kappa(\alpha)$ is the least cardinal λ so that $\lambda \rightarrow (\alpha)^{<\omega}_2$. Finally, a Ramsey cardinal is a fixed point of the sequence of Erdös cardinals, i.e. a cardinal λ so that $\lambda \rightarrow (\lambda)^{<\omega}_2$. Thus, a Ramsey cardinal is weakly compact. (The appellation "Ramsey" suggesting a generalization is perhaps inappropriate, since it can be shown that $\omega \rightarrow (\omega)^{<\omega}_2$ is false—however, the term has stuck.)

As with σ_n^m and π_n^m , it is to be implicit in the use of the term $\kappa(\alpha)$ that we are considering a situation in which there is some λ so that $\lambda \rightarrow (\alpha)^{<\omega}_2$. Erdös-Hajnal[1958] showed by direct combinatorial arguments that $\omega \leq \alpha < \beta$ implies $\kappa(\alpha)$ is regular and $\kappa(\alpha) < \kappa(\beta)$, and that for α a limit ordinal, $\kappa(\alpha)$ is inaccessible. (Silver[1966] later showed that for α a limit ordinal, in fact $\kappa(\alpha) \rightarrow (\alpha)^{<\omega}_\mu$ for every $\mu < \kappa(\alpha)$, showing model theory to good advantage in proving a combinatorial result. For these and more results, see Drake[1974] or Baumgartner-Galvin(1977) ,§1). What brought this hierarchy into the greater scheme of things was their further result that a measurable cardinal is Ramsey. This result is nowadays conceptually seen as a corollary to the following important strengthening, due to Rowbottom[1964]: If κ is measurable and U is a normal ultrafilter over κ , then whenever $f: [\kappa]^{<\omega} \rightarrow \lambda$ where $\lambda < \kappa$, there is a set $X \in U$ homogeneous for f:

⊢ Given such an f, let $f_n = f|[\kappa]^n$. If for each n we could find sets $X_n \in U$ homogeneous for f_n, then $X = \bigcap_n X_n \in U$ would be as required. Thus, it suffices to establish the following for every $n \in \omega$: whenever $g: [\kappa]^n \to \lambda$ and $\lambda < \kappa$, there is a set $H \in U$ homogeneous for g.

Proceed by induction on n. The case $n = 1$ is immediate by the κ-completeness of U. So, assume the statement is true for n, and consider $g: [\kappa]^{n+1} \to \lambda$ where $\lambda < \kappa$. Define related partitions $g_\alpha: [\kappa]^n \to \lambda$ for each $\alpha < \kappa$ as follows:

$$g_\alpha(s) = \begin{cases} g(\{\alpha\} \cup s) & \text{if } \alpha < \bigcap s , \\ 0 & \text{otherwise.} \end{cases}$$

By inductive hypothesis, let $H_\alpha \in U$ be homogeneous for g_α. Notice that by κ-completeness and $\lambda < \kappa$, there is exactly one $\gamma < \lambda$ so that $T = \{ \alpha < \kappa \mid g_\alpha``[H_\alpha]^n = \{\gamma\} \} \in U$. Finally, set $H = T \cap \Delta_{\alpha<\kappa} H_\alpha$. Then $H \in U$ by normality, and we claim that $g``[H]^{n+1} = \{\gamma\}$, so that H is a homogeneous set for g, as desired: Suppose $t \in [H]^{n+1}$, written $t = \{\alpha\} \cup s$ where $\alpha < \bigcap s$. Then $g(t) = g_\alpha(s) = \gamma$, as $s \in [H_\alpha]^n$ and $\alpha \in T$. ⊣

We remark that a typical indescribability argument can be used to show that the least Ramsey cardinal is not measurable: If κ is measurable and U is a normal ultrafilter over κ, then $\{ \alpha < \kappa \mid \alpha$ is Ramsey $\} \in U$.

⊢ Let $j: V \to M \approx V^\kappa/U$. As κ is Ramsey and $V_{\kappa+1} \subseteq M$ by the Closure Lemma in §2, it can easily be seen that $M \models \kappa$ is Ramsey. The conclusion now follows as usual by normality: use $\kappa = [\mathrm{id}]$ and Łoś' Theorem. ⊣

We now describe Rowbottom's basic discovery of the equivalence of certain partition relations to model theoretic principles, which set the stage for further investigations into applications of model theory in set theory in the work of Silver. The ultimate isolation of the set of integers $0^\#$ in the context of Silver's results on L and large cardinals is a landmark achievement, but Rowbottom the precursor was not just a voice in the wilderness; his influence was felt throughout, and the concepts he isolated were later seen to be closely intertwined with a problem of Jonsson in universal algebra (see below).

First, some model theoretic definitions. A structure $\mathfrak{A} = \langle A, R, \ldots \rangle$, where $R \subseteq A$, appropriate for a countable language with a distinguished unary predicate, is said to be type $\langle\kappa,\lambda\rangle$ iff $|A| = \kappa$ and $|R| = \lambda$. We say $\langle\kappa,\lambda\rangle \twoheadrightarrow \langle\bar\kappa,\bar\lambda\rangle$ iff whenever a structure $\mathfrak{A}$ has type $\langle\kappa,\lambda\rangle$, there is a $\mathfrak{B} \prec \mathfrak{A}$ of type $\langle\bar\kappa,\bar\lambda\rangle$. (We use the double arrow $\twoheadrightarrow$ here to distinguish the present concept from the (better known) concept $\langle\kappa,\lambda\rangle \to \langle\bar\kappa,\bar\lambda\rangle$, defined analogously but with "$\mathfrak{B} \prec \mathfrak{A}$" replaced by the weaker "$\mathfrak{B} \equiv \mathfrak{A}$".) Variations on this symbolism should be self-explanatory. We can now state a version of Rowbottom's result:

Let $\kappa > \lambda > \bar\lambda > \omega$ and $\kappa \geq \bar\kappa > \bar\lambda$. The following are then equivalent:

(a) $\langle\kappa,\lambda\rangle \twoheadrightarrow \langle\bar\kappa,\bar\lambda\rangle$.

(b) Whenever $f: [\kappa]^{<\omega} \to \lambda$, there is an $X \subseteq \kappa$ so that $|X| = \bar\kappa$ and $|f``[X]^{<\omega}| < \bar\lambda$.

⊢ First, assume (a) and suppose that $f: [\kappa]^{<\omega} \to \lambda$. For each $n \in \omega$, set $f_n: {}^n\kappa \to \lambda$ by $f_n(\langle\xi_1,\ldots,\xi_n\rangle) = f(\{\xi_1,\ldots,\xi_n\})$. Consider the structure $\mathfrak{A} = \langle\kappa, \lambda, f_n\rangle_{n\in\omega}$. By hypothesis, there is an $X \subseteq \kappa$ so that $|X| = \bar\kappa$ and $|X \cap \lambda| < \bar\lambda$, and $\langle X, X\cap\lambda, f_n|^nX\rangle_{n\in\omega} \prec \mathfrak{A}$. Clearly, X is as desired, establishing (b).

Conversely, assume (b) and suppose that $\mathfrak{A} = \langle\kappa, \lambda, \ldots\rangle$ is any structure of type $\langle\kappa,\lambda\rangle$. The basic idea is to use the combinatorial hypothesis (b) by Skolemizing $\mathfrak{A}$. So, let $\{h_n \mid n \in \omega\}$ be a complete set (closed under functional composition) of Skolem functions for $\mathfrak{A}$. If h_n is $k(n)$-ary, by augmenting the list with at most $k(n)^{k(n)}$ new functions, we can assume that $h_n(\langle\xi_1,\ldots,\xi_{k(n)}\rangle)$ only depends on $\{\xi_1,\ldots,\xi_{k(n)}\}$. We can further assume, by possibly introducing trivial functions and trivially altering domains, that $h_n: [\kappa]^n \to \kappa$. Define $f: [\kappa]^{<\omega} \to \lambda$ as follows:

$$f(s) = \begin{cases} h_n(s) & \text{if } n = |s| \text{ and } h_n(s) < \lambda , \\ 0 & \text{otherwise .} \end{cases}$$

By hypothesis (b), there is an $X \subseteq \kappa$ so that $|X| = \bar\kappa$ and $|f``[X]^{<\omega}| < \bar\lambda$. Let $B = \{h_n(\{\xi_1,\ldots,\xi_n\}) \mid \xi_1,\ldots,\xi_n \in X\}$. Then $|B| = \bar\kappa$ and generates a structure $\langle B, B\cap\lambda, \ldots\rangle \prec \mathfrak{A}$. Also, as $B\cap\lambda \subseteq f``[X]^{<\omega}$ we have $|B\cap\lambda| < \bar\lambda$, and the proof is complete. ⊣

In light of this result, the conspicuous feature of a Ramsey cardinal that concerned Rowbottom was the following, now given his name: an uncountable cardinal κ is υ-Rowbottom iff whenever $f: [\kappa]^{<\omega} \to \lambda$ where $\lambda < \kappa$, there is an $X \subseteq \kappa$ so that $|X| = \kappa$ and $|f``[X]^{<\omega}| < \upsilon$. κ is just called Rowbottom iff it is ω_1-Rowbottom. With the reminder that Silver[1966] showed that if κ is Ramsey, then $\kappa \to (\kappa)^{<\omega}_\mu$ for every $\mu < \kappa$, we have immediately that a Ramsey cardinal is Rowbottom (the converse is not true, as a Rowbottom cardinal may not even be regular; see below). With his model theoretic characterization in hand, Rowbottom went on to sharpen considerably Scott's original result on the incompatibility of $V = L$ with with some large cardinal axioms. We state the following as a typicality: If there is a Rowbottom cardinal, then (i) there are only countably many constructible subsets of ω, and (ii) ω_1 is inaccessible in L. This result will soon be subsumed (see §7, §10); we now go on to consider a related concept.

In a real sense, model theory can be considered the amplification of universal algebra through the infusion of the syntactical methods of logic. It seems appropriate that Rowbottom's model theoretic interpretation of partition relations turns out to be closely connected to a well-known problem in universal algebra. By an algebra we simply mean a structure $\mathfrak{A} = \langle A, f_n\rangle_{n\in\omega}$ where each f_n is a finitary operation on A into A. (Allowing infinitary functions is an interesting possibility, and leads to another tale of large cardinals; see §17.) A Jonsson algebra is an algebra without a proper subalgebra of the same cardinality. Finally, a Jonsson cardinal is a cardinal κ such that there is no Jonsson algebra of cardinal κ. Note that we are only concerned with a very gross property of algebras; the domains A might as well be cardinals. Jonsson's Problem is: are there any Jonsson cardinals?

Through Skolemization, one sees that κ is Jonsson iff any structure of cardinality κ (for a countable language) has a proper elementary substructure of cardinality κ. Note that ω is not Jonsson, and that a straightforward argument shows that if κ is ν-Rowbottom for some $\nu < \kappa$, then κ is Jonsson. Indeed, an analogous argument to Rowbottom's (above) reduces the question to one of partitions: κ is Jonsson iff whenever $f\colon [\kappa]^{<\omega} \to \kappa$, there is an $X \subseteq \kappa$ so that $|X| = \kappa$ and $f``[X]^{<\omega} \neq \kappa$.

For illustrative purposes we prove the following facts: (i) If κ is not Jonsson, neither is κ^+. (ii) The least Jonsson cardinal is either weakly inaccessible, or has cofinality ω.

⊢ For (i), by hypothesis and the previous result, whenever $\kappa \leq \alpha < \kappa^+$ let $f_\alpha\colon [\alpha]^{<\omega} \to \alpha$ be such that for $X \subseteq \alpha$ with $|X| = \kappa$, $f_\alpha``[X]^{<\omega} = \alpha$. Define $g\colon [\kappa^+]^{<\omega} \to \kappa^+$ by $g(s) = f_\alpha(s - \{\alpha\})$ where $\alpha = \max(s)$. Then g establishes that κ^+ is not Jonsson.

For (ii), suppose κ is the least Jonsson cardinal. Then (i) shows that κ is a limit cardinal, so assume to the contrary that $\omega < cf(\kappa) = \lambda < \kappa$. Let $\{\mu_\alpha \mid \alpha < \lambda\}$ be a closed set of cardinals cofinal in κ, with $\mu_0 > \lambda$. For each $\alpha < \lambda$, let $f_\alpha\colon [\mu_\alpha]^{<\omega} \to \mu_\alpha$ witness that μ_α is not Jonsson, i.e. whenever $X \subseteq \mu_\alpha$ with $|X| = \mu_\alpha$ we have $f_\alpha``[X]^{<\omega} = \mu_\alpha$. Define $f\colon [\kappa]^{<\kappa} \to \kappa$ by:

$$f(s) = \begin{cases} f_\alpha(s - \{\alpha\}) & \text{if } s < \mu_\alpha, \text{ where } \alpha = \min(s), \\ 0 & \text{otherwise.} \end{cases}$$

Also, let $g\colon [\lambda]^{<\omega} \to \lambda$ witness that λ is not Jonsson, and let $h\colon \kappa \to \lambda$ be defined by $h(\beta) =$ the unique α so that $\mu_\alpha \leq \beta < \mu_{\alpha+1}$.

Consider $\mathfrak{A} = \langle \kappa, f, g, h, \{\lambda\} \rangle$. The claim is that $\mathfrak{A}$ is a Jonsson algebra, which would yield the required contradiction. So, suppose $X \subseteq \kappa$ with $|X| = \kappa$, and X is the domain of a subalgebra of $\mathfrak{A}$. As $|X| = \kappa$, by the use of h we have $|X \cap \lambda| = \lambda$. Thus, by the use of g we have $X \cap \lambda = \lambda$. Assume now that $\beta < \kappa$ is arbitrary. Let $\alpha_0 < \lambda$ so that $\beta < \mu_{\alpha_0}$, and by induction let $\alpha_n \leq \alpha_{n+1} < \lambda$ so that $|X \cap \mu_{\alpha_{n+1}}| \geq \mu_{\alpha_n}$. Finally, as we are assuming $\lambda = cf(\kappa) > \omega$, we have $\bar{\alpha} = \sup \alpha_n < \lambda$. As the μ_α's are a closed set of ordinals, $|X \cap \mu_{\bar{\alpha}}| = \mu_{\bar{\alpha}}$. Finally since $f_{\bar{\alpha}}``((X \cap \mu_{\bar{\alpha}}) - \lambda)^{<\omega} = \mu_{\bar{\alpha}}$, by definition of f and the fact that $\bar{\alpha} \in \lambda = X \cap \lambda$, we have $\beta \in X$. Since the choice of $\beta < \kappa$ was arbitrary, we have shown $X = \kappa$, as desired. ⊣

Concerning (i) above, Erdős-Hajnal-Rado[1965] showed that if $2^\kappa = \kappa^+$, then κ^+ cannot be Jonsson. However, without the GCH it is not even known that a successor cardinal cannot be Jonsson. Recently, Shelah[1977] generalized the construction of Erdős-Hajnal-Rado to show that there is a ω_1-Jonsson group (a Jonsson algebra of cardinality ω_1 which is a group), answering questions of Kurosh and Mackenzie.

We have already remarked that any Rowbottom cardinal is Jonsson. Kleinberg [1973a](1973) established the following fact: Con(ZFC & there is a Rowbottom cardinal) iff Con(ZFC & there is a Jonsson cardinal). He first showed that any Jonsson cardinal κ has the Rowbottom-type property that for some $\nu < \kappa$, $\langle \kappa, \nu \rangle \nrightarrow \langle \kappa, \langle \nu \rangle$; that if κ were the least Jonsson cardinal, κ is ν-Rowbottom for such a ν; and then went on to prove the full result by forcing. This is certainly an interesting result: it measures exactly the difficulty of Jonsson's problem in universal algebra via an ostensibly more stringent property that arose through mainstream large cardinal considerations.

None of the results thus far disallow the possibility that ω_ω can be Jonsson. Indeed, this is a thriving open question. Prikry[1970] had established (via Prikry forcing; see §23) that: Con(ZFC & there is a measurable cardinal) implies Con(ZFC & there exists a Rowbottom cardinal with cofinality ω). So, it is hoped that this situation can be transferred down to ω_ω by forcing, for example by some variant of techniques used recently by Magidor in connection with the Singular Cardinals Problem (see §29). Jonsson and Rowbottom-type cardinals are the first cardinals that we have come across which may not in themselves be very large. However, their existence will have drastic consequences for L, and imply the existence of many large cardinals (as $0^\#$ then exists; see §7, §10), and thus they heartily qualify as properties of large cardinal character.

A 1974 result of Silver has a significant bearing on these matters. He established the following theorem: Set $\kappa = \omega_\omega$. If $2^\omega < \kappa$, and κ is Jonsson, then κ is measurable in an inner model. This result shows that, with one power set cardinality restriction, producing a small, Rowbottom-type singular cardinal essentially necessitates having started with actual measurability. Though we shall be jumping ahead of ourselves, we sketch a proof of Silver's result for the experts, a proof which becomes more natural in the context of iterated ultrapower techniques, discussed later in this paper.

⊢ First of all, a Kleinberg result cited in the penultimate paragraph assures that, since κ must be the least Jonsson cardinal, κ must be ω_n-Rowbottom for some n. As we are assuming that $2^\omega < \kappa$, we can take $\omega_n \geq 2^\omega$. Our ultimate goal will be to get a sequence of cardinals $\langle \kappa_i \mid i < \omega \rangle$ cofinal in κ so that if F is the filter over κ generated by this sequence (i.e. $X \in F$ iff $X \subseteq \kappa$ & $\exists j \forall i > j (\kappa_i \in X)$), then in $L[F]$, the universe relatively constructed from F, $F \cap L[F]$ is a normal κ-complete ultrafilter.

We first need to make a comment on structures with many functions: If $\kappa \subseteq A$ and $\mathfrak{A} = \langle A, f_\alpha \rangle_{\alpha \in \omega_n}$ is a structure (with the f_α's finitary functions), then there is a $\mathfrak{B} = \langle B, \ldots \rangle \prec \mathfrak{A}$ so that $|B \cap \kappa| = \kappa$ and $|B \cap \omega_{n+1}| \leq \omega_n$. To show this, we can certainly assume that among the f_α's are a complete set of Skolem functions, and that the f_α's are closed under functional composition. Now define $g\colon [A]^{<\omega} \to \omega_{n+1}$ by: $g(s) = \sup\{f_\alpha(s) \mid f_\alpha(s) < \omega_{n+1}$ & $\alpha < \omega_n\}$. As κ is ω_n-Rowbottom, let $\bar{B} \subseteq \kappa$ so that $|\bar{B}| = \kappa$ and $|g"[\bar{B}]^{<\omega}| < \omega_n$. Set $B = \bigcup\{f_\alpha"[\bar{B}]^{<\omega} \mid \alpha < \omega_n\}$. Then clearly B is the domain of an elementary substructure of $\mathfrak{A}$, and we have $B \cap \omega_{n+1} \subseteq \sup(g"[\bar{B}]^{<\omega}) < \omega_{n+1}$. This is as was required.

We now say that a structure of type $\mathfrak{A} = \langle A, f_\alpha \rangle_{\alpha<\omega_n}$ with $\kappa \subseteq A$ <u>admits</u> a sequence of cardinals $\langle \kappa_i \mid i \in \omega \rangle$ cofinal in κ <u>iff</u> there is a $\mathfrak{B} \prec \mathfrak{A}$ so that: κ_{i+1} is the κ_ith ordinal of $\mathfrak{B}$ for every $i \in \omega$. The claim is: <u>(*) There is a sequence admitted by every structure of the above type</u>. Well, suppose not. Then for every ω-sequence s of cardinals cofinal in κ, there is a structure $\langle A^s, f^s_\alpha \rangle_{\alpha<\omega_n}$ which does not admit s. Without loss of generality, we can assume $A^s = \kappa+\kappa$, using the second set of κ ordinals to code anything necessary beyond the first κ ordinals of A^s. Fix a coding $g: \kappa\times\kappa \to \kappa$ where $g(\beta,\cdot)$ is a bijection $\beta \leftrightarrow |\beta|$. Now set $\mathfrak{A} = \langle A, \epsilon, f^s_\alpha, g \rangle_{\alpha<\omega_n, s}$, where $A \supseteq \kappa+\kappa$ and $\langle A, \epsilon \rangle$ models enough of ZF to talk about cardinality. Thus, for $\beta < \kappa$, $\mathfrak{A} \models$ "β is a cardinal" just in case β really is a cardinal.

$\mathfrak{A}$ is a structure with at most ω_n functions (as there are at most $2^\omega \leq \omega_n$ ω-sequences s of cardinals cofinal in κ; this is where we need the cardinality assumption). Thus, by the remark of the penultimate paragraph, let $\mathfrak{B} = \langle B, \ldots \rangle \prec \mathfrak{A}$ be so that $|B \cap \kappa| = \kappa$ and $|B \cap \omega_{n+1}| \leq \omega_n$. Define inductively: $\bar{\kappa}_0 = \omega_{n+1}$ and in general $\bar{\kappa}_{i+1} = \bar{\kappa}_i$th element of $B \cap \kappa$ ($= \bar{\kappa}_i$th element of any A^s). Thus, $\bar{\kappa}_0 < \bar{\kappa}_1 < \ldots$ If we can show all the $\bar{\kappa}_i$'s to be cardinals, then we can get a contradiction of the choice of $\langle A^{\bar{s}}, f^{\bar{s}}_\alpha \rangle_{\alpha<\omega_n}$, where $\bar{s} = \langle \bar{\kappa}_i \mid i \in \omega \rangle$, as this structure is encoded in $\mathfrak{A}$. To this end, let $\pi: \mathfrak{T} \cong \mathfrak{B} \prec \mathfrak{A}$ be the transitization. As $\bar{\kappa}_0$ is a cardinal, $\mathfrak{T} \models$ "$\bar{\kappa}_0$ is a cardinal", i.e. $\mathfrak{A} \models$ "$\pi(\bar{\kappa}_0) = \bar{\kappa}_1$ is a cardinal" so that by the aforementioned absoluteness, $\bar{\kappa}_1$ really is a cardinal. Now repeat this argument to show $\bar{\kappa}_2$ a cardinal, and so forth! This establishes (*).

Let us now fix a $\langle \kappa_i \mid i \in \omega \rangle$ satisfying (*), and, as anticipated above, let F be the filter over κ generated by this sequence. We conclude by showing that: <u>$L[F] \models$ "$\bar{F} = F \cap L[F]$ is a normal (κ-complete) ultrafilter over κ"</u>. For instance, suppose $\bar{F}$ were not ultra in $L[F]$, and let X be the least subset of κ (in the canonical well-ordering of $L[F]$) so that $X \notin \bar{F}$ and $(\kappa - X) \notin \bar{F}$. Notice that $X \in L_{\kappa^+}[F]$ by the standard proof of $L[F] \models$ "$2^\kappa = \kappa^+$". We can suppose <u>mutatis mutandis</u> that $\kappa_0 \in X$. By (*), let $B \prec L_{\kappa^+}[F]$ be so that $|B \cap \kappa| = \kappa$ and κ_{i+1} is the κ_ith element of $B \cap \kappa$ for every $i \in \omega$. Let $\pi: T \cong B$ be the transitization; thus, $\pi(\kappa_i) = \kappa_{i+1}$ for every i. But then for $Y \subseteq \kappa$ in the structure T, $Y \in F$ <u>iff</u> $\pi(Y) \in F$, as π preserves final segments of $\langle \kappa_i \mid i \in \omega \rangle$. Hence, by the relativized condensation lemma, $T = L_\delta[F]$ for some δ, i.e. $\pi: L_\delta[F] \to L_{\kappa^+}[F]$ is elementary. As X was <u>definable</u>, $X \in L_\delta[F]$ and $\pi(X) = X$. But $\kappa_0 \in X$ and hence $\kappa_1 = \pi(\kappa_0) \in \pi(X) = X$. We can now repeat to establish that $\kappa_2 \in X$, and so forth! This shows $X \in \bar{F}$, yielding a contradiction. Analogous arguments finish the proof, e.g. if $\bar{F}$ were not normal in $L[F]$, let $f \in L[F]$ be the least regressive function counterexample, assume $f(\kappa_j) = \gamma < \kappa_j$

and then proceed to show $f(\kappa_i) = \gamma$ for every $i \in \omega$ with $j \leq i$. ⊣

For the sake of thematic unity, we have strayed chronologically ahead with this proof; let us now return to 1966, and the story of Silver's Hauptsatz for L.

§7. Indiscernibles and the Story of $0^\#$

Coming quickly on the scene following Rowbottom, it was Silver who realized that Ramsey and Erdös cardinals were intimately bound up with the notion of a <u>set of indiscernibles for a structure</u>. Rowbottom had seen the drastic consequences for L of the model theoretic consequences of strong partition relations, but Silver considerably sharpened these results by demonstrating a trivialization of the generation of L, via indiscernibles from a fixed blueprint of formulas to be true of their ascending sequences. Thus, in the presence of a suitably large cardinal in the universe, many strong results about the uniform generation of L now follow from this intrinsic structural characterization, and L takes on the character of a very thin inner model indeed, bare ruined choirs appended to the slender life-giving spine which is the class of ordinals. That blueprint, after some refinement, has been designated via Gödelization to be the (unique) set of integers $0^\#$. It will become clear that the existence of $0^\#$ is a large cardinal axiom, and the isolation of this concept is certainly a significant achievement, adding force to the importance of set theory in foundational studies, as the source of genuinely new principles.

In addressing themselves to the problem of getting models of theories with a large number of automorphisms, Ehrenfeucht-Mostowski[1956] introduced the notion of indiscernibility, and brought Ramsey's theorem into model theoretic prominence. If $\mathfrak{A}$ is a structure and X is a subset of the domain of $\mathfrak{A}$, linearly ordered by $<$ (not necessarily a relation in $\mathfrak{A}$), then <u>X (with $<$) is a set of indiscernibles for $\mathfrak{A}$ iff</u> whenever $x_1 < \ldots < x_n$ and $y_1 < \ldots < y_n$ in X, then

$$\mathfrak{A} \models \phi[x_1,\ldots,x_n] \quad \underline{\text{iff}} \quad \mathfrak{A} \models \phi[y_1,\ldots,y_n]$$

for every formula $\phi(v_1,\ldots,v_n)$ in the language for $\mathfrak{A}$. The following is the basic result of Ehrenfeucht-Mostowski[1956], the basis of so much to come later: <u>Suppose T is a theory with infinite models, and $\langle X, < \rangle$ is a linearly ordered set. Then there is a model $\mathfrak{A}$ of T so that X is contained in its domain and X is a set of indiscernibles for $\mathfrak{A}$.</u>

⊢ Expand the language by introducing new constants c_x for $x \in X$, and consider the theory $\bar{T} = T \cup \{c_x \neq c_y \mid x \neq y \text{ in } X\} \cup \{\phi(c_{x_1},\ldots,c_{x_n}) \leftrightarrow \phi(c_{y_1},\ldots,c_{y_n}) \mid \phi(v_1,\ldots,v_n)$ is a formula of the language of T and $x_1 < \ldots < x_n$ and $y_1 < \ldots < y_n$ in $X\}$. It certainly suffices to show that $\bar{T}$ is consistent, which by the Compactness Theorem amounts to showing that every finite subset of $\bar{T}$ is satisfiable.

So, assume $\Gamma \subseteq \bar{T}$ is finite. Let $\mathfrak{B}$ be an infinite model of T, and $A = \{a_i \mid i \in \omega\}$ be a countably infinite subset of the domain of $\mathfrak{B}$. Let $n \in \omega$ be the number of new constants appearing among the members of Γ, and for $k \leq n$ let

f_k with domain $[A]^k$ be defined by:

$$f_k(\{a_{i_1},\dots,a_{i_k}\}) = \{\phi(v_1,\dots,v_k) \mid \mathfrak{A} \models \phi[a_{i_1},\dots,a_{i_k}] \text{ and } \phi(c_{x_1},\dots,c_{x_k}) \leftrightarrow \phi(c_{y_1},\dots,c_{y_k}) \in \Gamma\}$$

when $i_1 < \dots < i_k$. Since Γ is finite, the range of f_k is finite, and so we can apply Ramsey's theorem n times, successively whittling down A to get an infinite $B \subseteq A$ homogeneous for each f_k.

It is now straightforward to see that $\mathfrak{A}$ satisfies Γ, with any n elements of B assigned to the new constants appearing in Γ in corresponding ascending order. Thus, Γ is satisfiable, and we are done. ⊣

Thus, partition relations first entered model theory; notice that the proof gives a method of passing from a given model to a new model by a typical compactness argument, and so little information can be culled about the new model. Silver saw that strong partition relations can be characterized by sets of indiscernibles already existing in given structures:

<u>7.0. Theorem</u>: $\kappa \to (\alpha)_2^{<\omega}$ <u>iff</u> whenever $\mathfrak{A}$ is a structure for a countable language and κ is a subset of the domain of $\mathfrak{A}$, then there is a set of indiscernibles $X \subseteq \kappa$ for $\mathfrak{A}$, of order type α.

⊢ Let $\langle \phi_n \mid n \in \omega \rangle$ enumerate the formulas of the language, where without loss of generality ϕ_n has at most the variables $v_1,\dots,v_n$ free. Define $f\colon [\kappa]^{<\omega} \to 2$ by $f(\{\alpha_1,\dots,\alpha_n\}) = 0$ <u>iff</u> $\alpha_1 < \dots < \alpha_n$ and $\mathfrak{A} \models \phi_n[\alpha_1,\dots,\alpha_n]$. Then, as before, a set homogeneous for f would be a set of indiscernibles for $\mathfrak{A}$.

Conversely, suppose $f\colon [\kappa]^{<\omega} \to 2$ and X is a set of indiscernibles for the structure $\langle \kappa, \in, f \mid [\kappa]^n \rangle_{n\in\omega}$. Then X is homogeneous for f. ⊣

We now cast Silver's work into a series of lemmata ultimately leading to $0^\#$. Note that for ease of exposition, we are doffing some of our informality and numbering our definitions and results. Silver first stated his results in general model theoretic terms, and then derived the particular consequences for L, but we shall content ourselves with working in the sharpened context of L directly.

We assume in what follows that languages are automatically augmented by a <u>complete set of Skolem terms</u> closed under functional composition, and that any theory in the language has been augmented with sentences giving the usual role of these terms. Thus, all Skolem Hulls will be <u>well-defined</u>. By $\mathcal{L} \cup \{c_n \mid n \in \omega\}$ we mean as usual the expansion of the language $\mathcal{L}$ by the addition of new constant symbols $\{c_n \mid n \in \omega\}$.

<u>7.1. Definition</u>: An <u>Ehrenfeucht-Mostowski (EM) blueprint</u> is a complete consistent theory in a countable language of form $\mathcal{L} \cup \{c_n \mid n \in \omega\}$ so that T includes ZF + V=L together with sentences asserting that the c_n's are an increasing sequence of indiscernible ordinals. Thus, T includes "$\phi(c_{i_1},\dots,c_{i_n}) \leftrightarrow \phi(c_{j_1},\dots,c_{j_n})$" for every formula $\phi(v_1,\dots,v_n)$ of $\mathcal{L}$ and $i_1 < \dots < i_n$, $j_1 < \dots < j_n$; as well as "c_0 is an ordinal" and "$c_0 < c_1$". Hence, to every model M (for a countable language $\mathcal{L}$) of ZF + V=L and infinite set of indiscernibles $X \subseteq \mathrm{OR}^M$, there corresponds a unique EM blueprint T (in $\mathcal{L} \cup \{c_n \mid n \in \omega\}$) so that

$$\phi(c_1,\dots,c_n) \in T \text{ \underline{iff} } M \models \phi[x_1,\dots,x_n] \text{ for any (all) } x_1 < \dots < x_n \text{ in } X.$$

<u>7.2. Definition</u>: If T is an EM blueprint and $\langle X, < \rangle$ is an infinite, linearly ordered set, let $M(T,X)$ be the unique (up to isomorphism) model M of T such that:

(i) $\langle X, < \rangle$ is order-isomorphically embedded in OR^M.

(ii) For any $x_1 < \dots < x_n$ in X, $M \models \phi[x_1,\dots,x_n]$ <u>iff</u> $\phi(c_1,\dots,c_n) \in T$.

(iii) M = Skolem Hull in M of (the isomorphic copy of) X. (Recall that we are assuming that a fixed complete set of Skolem terms is available, and so we mean here the well-defined Skolem Hull with respect to these.)

This definition is justified since (the proof of) the Ehrenfeucht-Mostowski theorem above yields a model with (i) and (ii), and then (iii) renders $M(T,X)$ unique. Note that for any element a in the domain of $M(T,X)$, $a = t(x_1,\dots,x_n)$ for $x_1 \dots x_n$ in X and t a Skolem term; we shall use this sort of description with this understanding, throughout. The following is immediate:

<u>7.3. Lemma</u>: Suppose T is an EM blueprint, X_1 and X_2 are both infinite, linearly ordered sets, and $i\colon X_1 \to X_2$ is an order-preserving injection. Then i extends uniquely to an elementary embedding $\bar{i}\colon M(T,X_1) \to M(T,X_2)$.

When $M(T,X)$ is well-founded, we henceforth <u>identify</u> it with its transitive isomorph, which must be of form L_δ. Note that in this case, by 7.3. we might as well assume that X is an infinite ordinal with the natural ordering. Those EM blueprints T so that $M(T,\alpha)$ is well-founded for every infinite α naturally interest us:

<u>7.4. Lemma</u>: The following are equivalent for an EM blueprint T:

(i) <u>Property I</u>: for every infinite $\alpha < \omega_1$, $M(T,\alpha)$ is well-founded.

(ii) for every infinite α, $M(T,\alpha)$ is well-founded.

⊢ To prove the non-trivial direction, assume that $M = M(T,\alpha)$ is ill-founded for some α. Let $\langle x_n \mid n \in \omega \rangle$ be elements of M so that $x_{n+1} \in_M x_n$ for every n. Each $x_n = t_n(s_n)$ for some finite set s_n of indiscernibles and t_n a Skolem term. Let $S = \bigcup\{s_n \mid n \in \omega\}$, and set N = the Skolem Hull in M of S. Then N is ill-founded, yet clearly $N = M(T,\beta)$ where $\beta < \omega_1$ is the order type of S, contradicting (i). ⊣

<u>7.5. Definition</u>: If $M(T,\alpha)$ is well-founded (and hence by our convention of form L_δ for some δ), let $\Gamma^{T,\alpha} = \{\gamma_\xi^{T,\alpha} \mid \xi < \alpha\}$ be <u>the</u> set of generating indiscernibles for $M(T,\alpha)$.

With the reduction in 7.4., we can now call upon a partition property strong enough to produce an EM blueprint which yields well-founded models. Though the existence of a Ramsey cardinal will do, we shall be more parsimonious and use the much smaller cardinal $\kappa(\omega_1)$, which we remind the reader is the least λ so that $\lambda \to (\omega_1)^{<\omega}_2$. Let $f_\delta\colon [\delta]^{<\omega} \to 2$ for $\delta < \kappa(\omega_1)$ be counterexamples to $\delta \to (\omega_1)^{<\omega}_2$, and consider the structure:

<u>7.6.</u> $\mathfrak{A} = \langle L_{\kappa(\omega_1)}, \varepsilon, \{\langle n,s,\delta\rangle \mid \delta < \kappa(\omega_1) \ \& \ |s| = n \ \& \ f_\delta(s) = 0\}\rangle$

(Incorporating the f_δ's into the structure is a technical device that will come in handy later.) By 7.0., there is a set of indiscernibles for $\mathfrak{A}$ of order type ω_1. Henceforth, fix such a set of indiscernibles X, and let T_0 be the corresponding EM blueprint.

<u>7.7. Lemma</u>: T_0 has Property I.

$\vdash$ Given any $\beta < \omega_1$, it is clear that the Skolem Hull in $\mathfrak{A}$ of the first β elements of our fixed set X of indiscernibles is isomorphic to $M(T_0,\beta)$, and hence $M(T_0,\beta)$ is certainly well-founded. $\dashv$

Note that the existence of T_0 already yields Rowbottom's conclusion:

<u>7.8. Lemma</u>: If there is an EM blueprint T with Property I, then there are only countably many constructible subsets of ω.

$\vdash$ Let α be any ordinal $\geq \omega_1$. Then $M(T,\alpha) = L_\delta$ for some $\delta \geq \omega_1$. Thus by Gödel's proof of the GCH in L, $P(\omega) \cap L = P(\omega) \cap M(T,\alpha)$.

Now if x is any subset of ω in $M(T,\alpha)$, x has form $t(\gamma^{T,\alpha}_{\xi_1},\ldots,\gamma^{T,\alpha}_{\xi_n})$. But each $i \in \omega$ is definable, and so by indiscernibility, $i \in t(\gamma^{T,\alpha}_{\xi_1},\ldots,\gamma^{T,\alpha}_{\xi_n})$ <u>iff</u> $i \in t(\gamma^{T,\alpha}_1,\ldots,\gamma^{T,\alpha}_n)$. Hence, $x = t(\gamma^{T,\alpha}_1,\ldots,\gamma^{T,\alpha}_n)$. There are only countably many such forms, so we can hence conclude that $P(\omega) \cap L$ is countable. $\dashv$

Thus, the existence of $\kappa(\omega_1)$ is already a very strong property, though it is a weakening of the Ramsey property in a different direction from Rowbottomness. Silver[1970] showed that in the hierarchy of Erdös cardinals, $\kappa(\omega_1)$ is the exact breaking point of L:

<u>7.9. Theorem</u>: If $\kappa \to (\alpha)^{<\omega}_2$ and $\alpha < \omega_1^L$, then $(\kappa \to (\alpha)^{<\omega}_2)^L$.

$\vdash$ Suppose $f\colon [\kappa]^{<\omega} \to 2$, with $f \in L$. We want an $X \in L$ of order type α, homogeneous for f. Since $\alpha < \omega_1^L$, let $g\colon \omega \to \alpha$ with $g \in L$. Set $D = \{d \mid d$ is an order-preserving injection: $g``n \to \kappa$ for some n, whose range is homogeneous for $f\}$, and define a partial ordering on D by $d \mathrel{\hat{<}} \bar{d}$ <u>iff</u> d extends $\bar{d}$. As $g \in L$ and $([\kappa]^{<\omega})^L = [\kappa]^{<\omega}$, we have $D \in L$. It is straightforward to see that the following is a <u>theorem of ZFC</u>: $\hat{<}$ is ill-founded <u>iff</u> there is an X of order type α, homogeneous for f.

It thus follows from $\kappa \to (\alpha)^{<\omega}_2$ that $\hat{<}$ is ill-founded in V, i.e. there is no order-preserving injection from $\hat{<}$ into the ordinals. But then, there cannot be any such map in L either, so $\hat{<}$ is ill-founded in L. Finally, the relativization of the above stated theorem of ZFC to L now yields the desired result. $\dashv$

It turns out that there are even further sharpenings in the search for the exact breaking point of L in the gap between $\sup\{\kappa(\alpha) \mid \alpha < \omega_1\}$ and $\kappa(\omega_1)$. Baumgartner-Galvin(1977) define a generalized version of Erdös cardinals which is sensitive to the possible EM blueprints they produce, and show that assumptions weaker than $\kappa(\omega_1)$ in their cosmology still yield $0^\#$. Roughly, the starting point of their work is the remark that $\kappa(\omega_1)$ yields sets of indiscernibles of type ω_1, whereas 7.4. only required sets of indiscernibles of all types $< \omega_1$, as long as they cohere with the same blueprint.

Well, let us get back to T_0, and go on with the story of $0^\#$. We gradually unveil further properties of EM blueprints, and show that these properties have interesting consequences.

<u>7.10. Lemma</u>: T_0 has <u>Property II</u>: For any term t, "$t(c_1,\ldots,c_n) \in OR \to t(c_1,\ldots,c_n) < c_{n+1}$" is in T_0.

$\vdash$ Suppose on the contrary that "$t(c_1,\ldots,c_n) \in OR \ \& \ c_{n+1} \leq t(c_1,\ldots,c_n)$" is in T_0. Let $x_1 < \ldots < x_n$ be the first n members of our fixed set X of indiscernibles for $\mathfrak{A}$. Set $Y = X - \{x_1,\ldots,x_n\}$ and $\delta = t(x_1,\ldots,x_n) < \kappa(\omega_1)$. It follows from our assumption that $Y \subseteq \delta+1$. However, as $f_{\delta+1}$ was encoded in $\mathfrak{A}$, it follows that Y is homogeneous for $f_{\delta+1}$. As Y has order type ω_1, this clearly contradicts the choice of $f_{\delta+1}$. $\dashv$

<u>7.11. Lemma:</u> If an EM blueprint T has Properties I & II, then whenever α is a limit ordinal, $\Gamma^{T,\alpha} = \{\gamma^{T,\alpha}_\xi \mid \xi < \alpha\}$ is cofinal in $M(T,\alpha)$.

$\vdash$ Immediate. $\dashv$

<u>7.12. Lemma</u>: T_0 has <u>Property III</u>: For any term t, "$t(c_1,\ldots,c_n,c_{n+1},\ldots,c_{n+m}) \leq c_n \to t(c_1,\ldots,c_n,c_{i_1},\ldots,c_{i_m}) = t(c_1,\ldots,c_n,c_{j_1},\ldots,c_{j_m})$" is in T_0, for any $n < i_1 < \ldots < i_m$ and $n < j_1 < \ldots < j_m$.

$\vdash$ For our mutual convenience in this and similar proofs, let $\vec{x}$ denote a strictly increasing sequence, and let $\vec{x} < \vec{y}$ mean that every element in $\vec{x}$ is less than every element in $\vec{y}$.

Assume now that $\vec{x} \in {}^nX$ and $\vec{y} \in {}^mX$, $\vec{x} < \vec{y}$, and $t(\vec{x},\vec{y}) \leq x_n$, the last element of x. We must show that $t(\vec{x},\vec{a}) = t(\vec{x},\vec{b})$ for any $\vec{a},\vec{b}$ in ${}^mX$ so that $\vec{x} < \vec{a},\vec{b}$. A typical argument shows that it suffices in fact to establish this for $\vec{a},\vec{b}$ in ${}^mX$ so that $\vec{x} < \vec{a} < \vec{b}$ (for then, given <u>arbitrary</u> $\vec{e},\vec{f} > \vec{x}$, let $\vec{g} \in {}^mX$ so that $\vec{e},\vec{f} < \vec{g}$; by indiscernibility, $t(\vec{x},\vec{e}) = t(\vec{x},\vec{g}) = t(\vec{x},\vec{f})$). Thus, argue by contradiction and suppose $\vec{x} < \vec{a} < \vec{b}$, yet $t(\vec{x},\vec{a}) \neq t(\vec{x},\vec{b})$. There are two cases to consider:

(i) $t(\vec{x},\vec{a}) > t(\vec{x},\vec{b})$. Then any sequence $\vec{x} < \vec{b}_1 < \vec{b}_2 < \vec{b}_3 \ldots$ with each $\vec{b}_i \in {}^mX$ generates by indiscernibility an infinite descending sequence $t(\vec{x},\vec{b}_1) > t(\vec{x},\vec{b}_2) > \ldots$ of ordinals, a contradiction.

(ii) $t(\vec{x},\vec{a}) < t(\vec{x},\vec{b})$. Then if $\langle \vec{b}_\alpha \mid \alpha < \omega_1\rangle$ is a sequence of elements of ${}^mX$ so that $\alpha < \beta < \omega_1$ implies $\vec{x} < \vec{b}_\alpha < \vec{b}_\beta$, then clearly $\{t(\vec{x},\vec{b}_\alpha) \mid \alpha < \omega_1\}$

is an increasing set of indiscernibles for $\mathfrak{A}$, with $t(\vec{x},\vec{b}_\alpha) \le x_n$ for every $\alpha < \omega_1$. This clearly contradicts the choice of f_δ for any $x_n < \delta < \kappa(\omega_1)$, as before in 7.10. ⊣

<u>7.13. Lemma</u>: If an EM blueprint T has Properties I-III, then:

(a) For any α, $|\gamma_\xi^{T,\alpha}| = |\xi|$ for every $\omega \le \xi < \alpha$.

(b) For any uncountable cardinal λ, $M(T,\lambda) = L_\lambda$.

⊢ We drop the superscripts on the indiscernibles in what follows.

For (a), consider an arbitrary $\beta < \gamma_\xi$. Then $\beta = t(\gamma_{\delta_1},\ldots,\gamma_{\delta_n},\gamma_{\zeta_1},\ldots,\gamma_{\zeta_m})$ with the indiscernibles in ascending order and $\delta_n \le \xi < \zeta_1$. By Property III, we have $\beta = t(\gamma_{\delta_1},\ldots,\gamma_{\delta_n},\gamma_{\xi+1},\ldots,\gamma_{\xi+m})$. Thus there are at most $|\xi| = |\{\gamma_\delta | \delta \le \xi\}|^\omega$ ordinals altogether, below γ_ξ, by a counting argument using such Skolem forms.

For (b), note that by Property II, $\{\gamma_\xi | \xi < \lambda\}$ is cofinal in $M(T,\lambda)$. Also, since λ is a cardinal, we have that for any $\xi < \lambda$, $\gamma_\xi < |\xi|^+ \le \lambda$, by (i). Thus, $M(T,\lambda) \subseteq L_\lambda$. But for any infinite α, it is obvious that $M(T,\alpha) \supseteq L_\alpha$, and so we are done. ⊣

<u>7.14. Corollary</u>: With the hypotheses of 7.13., whenever $\omega < \lambda < \mu$ are cardinals, then $L_\lambda \prec L_\mu \prec L$, and in particular $L_\lambda \models$ ZFC.

We now specializae a bit to capture a fourth property. Let $\bar{X}$ be a set of indiscernibles of order type ω_1 for our $\mathfrak{A}$ (see 7.6.), <u>with the extra proviso that its ωth element is the least possible</u> among all such sets of indiscernibles. Let T_1 be the corresponding blueprint; this T_1 surely has Properties I-III.

<u>7.15. Lemma</u>: T_1 has <u>Property IV</u>: For any term t, "$t(c_1,\ldots,c_n,c_{n+1},\ldots,c_{n+m}) < c_{n+1} \rightarrow t(c_1,\ldots,c_n,c_{i_1},\ldots,c_{i_m}) = t(c_1,\ldots,c_n,c_{j_1},\ldots,c_{j_m})$" is in T_1 for any $n < i_1 <\ldots< i_m$ and $n < j_1 <\ldots< j_m$.

⊢ Let $\vec{x}$ be the first n elements of our $\bar{X}$. We suppose that T is such that $t(\vec{x},\vec{a})$ < the first element of $\vec{a}$ for any $\vec{a} > \vec{x}$, with $\vec{a} \in {}^m\bar{X}$. Argue by contradiction and assume that $\vec{x} < \vec{a} < \vec{b}$, yet $t(\vec{x},\vec{a}) \ne t(\vec{x},\vec{b})$. (As in 7.12. it is only necessary to consider those $\vec{a},\vec{b} > x$ so that $\vec{x} < \vec{a} < \vec{b}$.) There are again two cases to consider:

(i) $t(\vec{x},\vec{a}) > t(\vec{x},\vec{b})$. Then as in 7.12., we get an infinite descending sequence of ordinals and hence a contradiction.

(ii) $t(\vec{x},\vec{a}) < t(\vec{x},\vec{b})$. In this case, let $\vec{b}_0$ = the first m elements of $\bar{X}$ after $\vec{x}$, b_1 = the next m elements, and so forth; then by the assumption of (ii), $\{t(\vec{x},\vec{b}_\alpha) | \alpha < \omega_1\}$ is an increasing sequence. It is easy to see that this set in fact constitutes a set of indiscernibles of order type ω_1 for $\mathfrak{A}$. However, by our construction the first element y of $\vec{b}_\omega$ is the ωth element of $\bar{X}$, and $t(\vec{x},\vec{b}_\omega) < y$ by our underlying assumption about t. This is a contradiction of the choice of the set $\bar{X}$ of indiscernibles as having the least ωth element. ⊣

<u>7.16. Lemma</u>: If an EM blueprint T has Properties I-IV, then for all infinite α,

(i) $\Gamma^{T,\alpha}$ is a closed set of ordinals.

(ii) $\lambda \subset \Gamma^{T,\alpha}$ for every cardinal $\lambda < \alpha$.

⊢ Again, we drop the superscripts from our indiscernibles. To establish (i), suppose to the contrary that ζ is a limit, yet $\rho = \sup\{\gamma_\xi | \xi < \zeta\} < \gamma_\zeta$. We write $\rho = t(\gamma_{\xi_1},\ldots,\gamma_{\xi_n},\gamma_{\zeta_1},\ldots,\gamma_{\zeta_m})$ with the indiscernibles in ascending order and $\xi_n < \zeta \le \zeta_1$. But then, $\rho = t(\gamma_{\xi_1},\ldots,\gamma_{\xi_n},\gamma_{\xi_n+1},\ldots,\gamma_{\xi_n+m}) < \gamma_{\xi_n+m+1} < \rho$. (Here, the equality is by Property IV, the first inequality by Property II, and the last inequality by definition of ρ.) This is a contradiction.

(ii) follows from (i), 7.11., and 7.13. ⊣

<u>7.17 Lemma</u>: If EM blueprints T and $\bar{T}$ both have Properties I-IV, then $T = \bar{T}$.

⊢ For any regular uncountable λ, $\Gamma^{T,\lambda}$ and $\Gamma^{\bar{T},\lambda}$ are closed unbounded in λ by 7.11. and 7.16. Thus, $C = \Gamma^{T,\lambda} \cap \Gamma^{\bar{T},\lambda}$ is infinite (in fact, closed unbounded in λ). It is now clear that $T = \bar{T}$: Given any formula $\phi(c_1,\ldots,c_n)$, let $\zeta_1 <\ldots< \zeta_n$ be any n elements of C. Then $\phi(c_1,\ldots,c_n) \in T$ <u>iff</u> $L_\lambda \models \phi[\zeta_1,\ldots,\zeta_n]$ <u>iff</u> $\phi(c_1,\ldots,c_n) \in \bar{T}$. (Note that by 7.13.(b), we have $M(T,\lambda) = L_\lambda = M(\bar{T},\lambda)$.) ⊣

Hence, there is at most one EM blueprint satisfying I-IV. Perhaps for this reason, the collections of sentences specified in II-IV have been called "remarkable" by some. We keep the reader in suspense for just a bit more, before naming our unique blueprint.

<u>7.18. Lemma</u>: If T is an EM blueprint with Properties I-IV, and $\alpha < \beta$ with α a limit ordinal, then the Skolem Hull of $\{\gamma_\xi^{T,\beta} | \xi < \alpha\}$ in $M(T,\beta)$ is precisely $L_{(\gamma_\alpha^{T,\beta})}$.

⊢ Let N be the stated Skolem Hull. It suffices to show that $OR^N = \gamma_\alpha^{T,\beta}$. Again, we drop the superscripts T,β from our indiscernibles.

First, if δ is an ordinal of N, then for some $\xi_1 <\ldots< \xi_n < \alpha$, we have $\delta = t(\gamma_{\xi_1},\ldots,\gamma_{\xi_n}) < \gamma_{\xi_n+1} < \gamma_\alpha$ by Property II and as α is a limit.

Conversely, if $\rho < \gamma_\alpha$, then $\rho = \bar{t}(\gamma_{\zeta_1},\ldots,\gamma_{\zeta_n},\gamma_{\tau_1},\ldots,\gamma_{\tau_m})$ with indiscernibles in ascending order and $\zeta_n < \alpha \le \tau_1$. We then have by Property IV that $\rho = \bar{t}(\gamma_{\zeta_1},\ldots,\gamma_{\zeta_n},\gamma_{\zeta_n+1},\ldots,\gamma_{\zeta_n+m})$, and surely this is in N as α is a limit ordinal greater than ζ_n. ⊣

<u>7.19. Corollary</u>: With the hypotheses of 7.18., we have:

(i) $M(T,\alpha) = L_{(\gamma_\alpha^{T,\beta})}$.

(ii) $\gamma_\xi^{T,\beta} = \gamma_\xi^{T,\alpha}$ for every $\xi < \alpha$.

⊢ (i) is immediate. For (ii), note that $\{\gamma_\xi^{T,\beta} | \xi < \alpha\}$ is the generating set of indiscernibles for (the transitized) $M(T,\alpha) = L_{(\gamma_\alpha^{T,\beta})}$. ⊣

The climax is now upon us:

7.20. Definition: Assuming that it exists, the unique EM blueprint T satisfying I-IV is designated $0^\#$. For every ξ , we define $\gamma_\xi = \gamma_\xi^{0^\#,\alpha}$ for any $\alpha > \xi$, α a limit ordinal (by 7.19(ii) this is well defined).

7.21. Theorem(Summary): If there is a λ so that $\lambda \to (\omega_1)_2^{<\omega}$, then there exists exactly one EM blueprint satisfying I-IV. This blueprint is called $0^\#$, and to it corresponds a canonical, closed unbounded class of indiscernibles $\{\gamma_\xi \mid \xi \in OR\}$ for L containing every uncountable cardinal, so that for $\alpha < \beta$ both limits, L_{γ_α} is the Skolem Hull of $\{\gamma_\xi \mid \xi < \alpha\}$ and $L_{\gamma_\alpha} \prec L_{\gamma_\beta} \prec L$.

If we assume some fixed recursive Gödelization of our language so that, in particular, a theory becomes identified with a set of integers, then note that "X is an EM blueprint with Properties I-IV" is a Π^1_2 predicate on $P(\omega)$:
⊢ The "remarkable" conditions II-IV are just arithmetical, and we can state I as: $\forall R \subseteq \omega\times\omega(R$ is a well-ordering $\to M(X,R)$ is well-founded). $M(X,R)$ is, of course, to be encoded as a structure on ω , and to say a relation is well-founded or well-rdered is Π^1_1 . Hence, our statement is Π^1_2 . ⊣

Thus, $0^\#$ as a set of integers is the unique solution of a Π^1_2 predicate Φ . Hence, $0^\#$ is a Δ^1_3 set of integers :

$$n \in 0^\# \text{ iff } \exists X(\Phi(X) \;\&\; n \in X)$$
$$\text{iff } \forall X(\Phi(X) \to n \in X) .$$

$0^\#$ is obviously not constructible, and since Shoenfield's Absoluteness Lemma implies that all Σ^1_2 sets of integers are constructible, $0^\#$ has the least possible complexity of a non-constructible set of integers. Strictly speaking, one does not have to assume a large cardinal axiom to obtain the existence of a non-constructible Δ^1_3 set of integers. Jensen-Solovay[1970] used a method, now known as "almost disjoint" forcing, to get the consistency of the existence of such a set, relative to just ZFC. (Solovay was also deeply involved in the formulation of $0^\#$; see his [1967]. Almost disjoint forcing turns out to be a powerful tool for getting small sets to code a great deal of information; see Harrington[1973](and later work) for applications to descriptive set theory, Miller[1977] for applications to the theory of Boolean algebras, as well as Jensen(1975) which we shall mention again shortly.)

In the presence of $0^\#$, the extent of the trivialization of L can be further emphasized beyond 7.21. by pointing out that the predicate "$L \models \phi(\vec{x})$" is definable in ZFC by reflection, since for any limit α , $L_\alpha \prec L$. Also, it is clear that $\{n \mid n$ is the Gödel number of a sentence σ and $L_{\gamma_\omega} \models \sigma\}$ is recursive in $0^\#$, and hence is Δ^1_3 . Finally, note that if $x \subseteq \omega$ is constructible, then $x = t(\gamma_{\xi_1},\ldots,\gamma_{\xi_n})$ say, and so $i \in x$ iff "$i \in t(c_1,\ldots,c_n)$" is in $0^\#$, i.e. every constructible subset of ω is recursive in $0^\#$, and hence is Δ^1_3. (For the

recursion theorists: as our Gödelization is recursive, the function $f(i) =$ "$i \in t(c_1,\ldots,c_n)$" is recursive and so any constructible subset of ω is in fact many-one reducible to $0^\#$.)

Also, it is not unexpected that the presence of $0^\#$ should lead to consequences of large cardinal character:

7.22. Theorem: If $0^\#$ exists, then:

(a) There is an elementary embedding: $L \to L$ (which is not the identity).

(b) Each γ_ξ is Π^m_n-indescribable in L for every $m,n \in \omega$.

(c) For every $\alpha \geq \omega$, $|P(\alpha) \cap L| = |\alpha|$.

⊢ For (a), note that the shifting map $j(\gamma_\xi) = \gamma_{\xi+1}$ for every ξ extends uniquely via the Skolemization to an elementary embedding $\bar{j}: L \to L$.

For (b), it suffices by indiscernibility to show that γ_0 is Π^m_n-indescribable for every $m,n \in \omega$. Take the $\bar{j}$ given in the proof of (a); it is not hard to see that γ_0 must be the first ordinal moved by $\bar{j}$. The result now follows by an argument of the sort in §4.

For (c), observe first that for any ξ , $|\gamma_\xi| = |\gamma_{\xi+1}|$, and that $|\gamma_0| = \omega$. In particular, we need only consider $\alpha \geq \gamma_0$. Then, since the γ_ξ's are a closed class of ordinals, let η be the unique ordinal such that $\gamma_\eta \leq \alpha < \gamma_{\eta+1}$. Then $|P(\alpha) \cap L| = |(\alpha^+)^L| < \gamma_{\eta+1}$, as $\gamma_{\eta+1}$ is inaccessible in L by (b) . Thus, $|P(\alpha) \cap L| \leq |\gamma_{\eta+1}| = |\gamma_\eta| = |\alpha|$. ⊣

Note that (c) above subsumes 7.8. Also, (a) implies $0^\#$ exists; see §10.

7.23. Theorem: If $0^\#$ exists, then there are Cohen, random, etc., generic reals over L . Also, no (set) notion of forcing over L yields $0^\#$.

⊢ The first statement follows from $|P(\omega) \cap L| = \omega$. For the second, suppose that $P \in L$ were a notion of forcing with G a P-generic filter over L . Since P is a set, there is a λ so that P has the λ^+-c.c. in L , i.e. $(\lambda^+)^L$ is a cardinal in $L[G]$. If $0^\# \in L[G]$, this would contradict 7.22.(c), since $|(\lambda^+)^L| = |P(\lambda) \cap L| = |\lambda|$. ⊣

With sufficiently strong large cardinal assumptions, we can relativize the construction of $0^\#$ to $L[x]$, the universe relatively constructible from x . Hence, $^\#$ can be construed as an operation on sets, $0^\#$ being appropriate for $L = L[0]$. To be more precise, suppose $x \subseteq \lambda$. It turns out that we need a language with names $\underline{\alpha}$ for every $\alpha \leq \lambda$ (so that we can preserve x and λ by requiring for any Skolem term t that "$t(c_1,\ldots,c_n) < \underline{\lambda} \to t(c_1,\ldots,c_n) = \underline{\alpha}$" is in the blueprint, for some $\alpha < \lambda$.) This necessitates having available a cardinal κ so that $\kappa \to (\omega_1)_{2^\lambda}^{<\omega}$ at least, to be able to prove a version of 7.0. for structures for languages of cardinality $\leq \lambda$. All further details now go through, and we get a blueprint $x^\#$. Traditionally, $x^\#$ for $x \subseteq \omega$ hold the main interest for set theorists; note that in this case, the language need only be countable, so we have:

7.24. Theorem: If there is a λ so that $\lambda \to (\omega_1)_2^{<\omega}$, then $x^\#$ exists for every $x \subseteq \omega$.

The assumption that $x^\#$ exists for every $x \subseteq \omega$ provides a rich landscape for the study of degrees of constructibility: confining ourselves to subsets of ω, we write $x \leq_c y$ to mean $x \in L[y]$, and $<_c$ and $=_c$ are to have the appropriate derived meanings. $=_c$ is then an equivalence relation, and its equivalence classes are called degrees of constructibility. There is a minimum degree, the degree of any constructible set, and the degrees form an upper semi-lattice. Since $x <_c x^\#$, $^\#$ can be regarded as a "jump" operation on degrees. Thus, every x is the start of an infinite chain. Paris, by looking at the relative density of indiscernibles, was able to prove: if $y \leq_c x < y^\#$, then $x^\# =_c y^\#$. (The analogous statement for Turing degrees is false.) Call an $x \subseteq \omega$ L-generic iff x is in some generic extension of L via a (set) notion of forcing. Thus, 7.23. asserts that $0^\#$ is not L-generic. Much of the work on degrees and $^\#$ (by Kechris and Harrington, for instance) was motivated by various conjectures formulated by Solovay about $0^\#$ being in some sense the $\leq_c$-least non-L-generic set. Then Jensen (1975) addressed himself to these matters and showed in fact that if $0^\#$ exists, then there is a non-L-generic $x <_c 0^\#$. Actually, his main result was that if $0^\#$ is not in the universe, there is a (proper class) notion of forcing so that any generic extension by it satisfies "$V = L[a]$" for some $a \subseteq \omega$. This is an impressive "coding of the universe in a real", which employs variations of the almost disjoint forcing alluded to earlier, and preserves many properties of the ground model (for instance, $0^\#$ does not appear in the extension, and all cofinalities are preserved).

An even more impressive result is Jensen's Covering Theorem: If $0^\#$ does not exist, then whenever X is an uncountable set of ordinals, there is a $Y \in L$ so that $X \subseteq Y$ and $|X| = |Y|$. This is a deep structure theorem about the close relationship between V and L in the absence of $0^\#$, and shows in particular that much of the uniform behavior of cardinalities and cofinalities in L lifts to V. See §29 for a more detailed discussion.

All in all, the study of the principle "$0^\#$ exists" has led to profound insights on the nature of sets. The initial isolation of this principle was an important achievement in set theory and foundational studies in general, and Jensen's recent work has transformed it into a focal point for the study of basic structural principles about the set theoretical universe. We are eons away from Cantor's Garden of Eden; would not he himself have marvelled at the subtle structure of the universe he had fathered, now after a sea-change, even more rich and strange?

We close this section with the statement of a beautiful theorem; it is a list of equivalences, much deeper than the confluence seen at weak compactness. In the theorem, (A) → (C) is 7.22.(a), and (C) → (A) is established in §10; (A) → (D) is obvious, and (D) → (A) is a corollary to Jensen's Covering Theorem (see §29);

and finally, see §27 for (A) → (B). Let us now gaze—silent:

7.25. Theorem: The following are equivalent:

(A) $0^\#$ exists.

(B) (lightface) Π_1^1-Determinacy.

(C) There is an elementary embedding $j: L \to L$ (which is not the identity).

(D) ω_ω is regular in L.

III. ASPECTS OF MEASURABILITY

§8. Iterated Ultrapowers

In the late sixties, the concept of measurability was further elaborated in terms of iterated ultrapowers, inner models, and the approach from saturated ideals. The sections of this chapter deal with the somewhat technical development of these themes.

About the same time that Rowbottom and Silver were deriving their beautiful results about L in the presence of strong partition properties that follow from measurability, Gaifman (see [1967] and the final exposition [1974]) was proving similar results from measurability through a method of iterating ultrapowers. He developed a general theory of "self-extension" operators which is also relevant to §17, but in the context of measurability it was Kunen who in a tour de force refined Gaifman's method to provide an elegant and powerful format for reproving Silver's results as well as deriving deep structure theorems about inner models of measurability.

Let us start, inauspiciously perhaps, by considering how to take products of ultrafilters:

<u>8.1. Definitions</u>: If U is an ultrafilter over I and V over J, then $U \times V$ over $I \times J$ is defined by: $X \in U \times V$ <u>iff</u> $X \subseteq I \times J$ & $\{i \mid \{j \mid \langle i,j\rangle \in X\} \in V\} \in U$. U^n is defined inductively: $U^1 = U$ and $U^{n+1} = U \times U^n$.

It is not hard to see that $U \times V$ is indeed an ultrafilter. Suppose now that U and V are ω_1-complete. Then following Scott, we can define a transitive class $N_1 \cong V^I/U$ and an elementary embedding $j: V \to N_1$. Note that $j(V)$ is an ultrafilter over $j(J)$ in N_1, so that within N_1 we can construct an ultrapower N_2 using $j(V)$. Tracing through the definitions, it turns out that N_2 is isomorphic to $V^{I \times J}/U \times V$. (A note of warning: $V^{I \times J}/U \times V$ is also isomorphic to $(V^J/V)^I/U$, i.e. iterated from the outside in the model theoretic sense, and in the reverse order. Keisler took this route; see Chang–Keisler[1973],§ 6.5. We stick with the approach of viewing successive iterations from within.) The main interest lies in the case $U = V$, and this then amounts to considering $V^{{}^2I}/U^2$, and for the n-fold iteration, to considering $V^{{}^nI}/U^n$.

Gaifman showed how to extend the process through the transfinite, roughly by taking direct limits over finite supports. At first, one may not think that anything new would come out of this, but Gaifman grasped the uniform definability of the process, and saw that the system of directed embeddings can yield well-founded direct limits.

Realizing the significance of Gaifman's idea, Kunen formulated the minimal conditions that still permit the construction of iterated ultrapowers of a transitive model M. He noticed that one need only have an ultrafilter U on the Boolean

algebra $P(\kappa) \cap M$, and U need not be a member of M as long as an essential iterability condition was satisfied. In particular, this showed that in M one need not assume actual measurability, and opened the door to new results on $0^{\#}$ and saturated ideals. We refer to Kunen[1970] for an efficient, though somewhat Spartan, exposition, and content ourselves in this section with presenting the special case when $U \in M$; the modifications for the general case are discussed in §12.

We have seen how taking finite ultrapowers amounts to taking one ultrapower by a product ultrafilter; in what follows, the idea will be to similarly view infinite iterations, using finite support. The advantage lies in the fact that we shall still be working in the initial model M rather than in successive ultrapowers, and will be able to cull more information from the consequent uniformity of construction.

<u>8.2. Definitions</u>: Let U be an arbitrary ultrafilter over a cardinal κ, and α any ordinal > 0.

(i) $f: {}^\alpha\kappa \to V$ has <u>finite support</u> y <u>iff</u> $y \subseteq \alpha$ is a finite set so that $f(s)$ depends only on $s|y$, i.e. $f(s) = f(t)$ whenever $s|y = t|y$.

(ii) $X \subseteq {}^\alpha\kappa$ has <u>finite support</u> y <u>iff</u> its characteristic function has finite support y: $y \subseteq \alpha$ is a finite set so that $s|y = t|y$ implies $s \in X$ <u>iff</u> $t \in X$.

(iii) If $X \subseteq {}^\alpha\kappa$ has finite support $y = \{\alpha_1,\ldots,\alpha_n\}$ with $\alpha_1 < \ldots < \alpha_n$, set $X_y = \{\langle s(\alpha_1),\ldots,s(\alpha_n)\rangle \mid s \in X\}$.

(iv) If $f: {}^\alpha\kappa \to V$ has finite support $y = \{\alpha_1,\ldots,\alpha_n\}$ with $\alpha_1 < \ldots < \alpha_n$, set $f_y: {}^n\kappa \to V$ by $f_y(\langle s(\alpha_1),\ldots,s(\alpha_n)\rangle) = f(s)$. (This may be confusing, but is well-defined by support considerations; any n-tuple from κ can be written as $\langle s(\alpha_1),\ldots,s(\alpha_n)\rangle$ for some $s \in {}^\alpha\kappa$.)

(v) $Fn_\alpha(\kappa) = \{f \mid f: {}^\alpha\kappa \to V$ with some finite support $\}$.

(vi) $P_\alpha(\kappa) = \{X \mid X \subseteq {}^\alpha\kappa$ with some finite support $\}$.

(vii) $U_\alpha = \{X \in P_\alpha(\kappa) \mid X$ has a finite support y with $|y| = n$, so that $X_y \in U^n\}$.

$P_\alpha(\kappa)$ is a field of sets, i.e. it is closed under finite union, finite intersection, and complementation relative to ${}^\alpha\kappa$. Some sweat must go into showing that if $X \in P_\alpha(\kappa)$ has finite supports y and z where $|y| = m$ and $|z| = n$, then $X_y \in U^m$ <u>iff</u> $X_z \in U^n$. Using this fact, we can conclude that <u>U_α is an ultrafilter on the field of sets $P_\alpha(\kappa)$</u>. Note that $U_n = U^n$ for $0 < n < \omega$.

Observe that for any formula ϕ in n free variables and $f_1,\ldots,f_n \in Fn_\alpha(\kappa)$, we have $\{s \in {}^\alpha\kappa \mid \phi(f_1(s),\ldots,f_n(s))\} \in P_\alpha(\kappa)$. Hence, we can take an ultrapower of V via U_α if we <u>only</u> use functions: ${}^\alpha\kappa \to V$ in $Fn_\alpha(\kappa)$. An appropriate version of Łoś' Theorem goes through, and so we can make the following general definitions:

<u>8.3. Definitions</u>: Let M be an inner model of ZFC and in M, suppose U is an ultrafilter over a cardinal κ. (Thus, κ is not necessarily a cardinal in V; this is somewhat of an abuse of our notational conventions.)

(a) Carry out the following construction <u>completely within</u> M: define U_α as before, and take the ultrapower of M with it, i.e. set $N_\alpha = \{[f]_\alpha \mid f \in Fn_\alpha(\kappa)\}$

and E_α the usual lifting of the membership relation (mod U_α) . Here, $[f]_\alpha$ denotes the usual equivalence class of f (mod U_α), construed as a set in M , by Scott's trick of taking only members of minimal rank. Then $\mathrm{Ult}_\alpha(M,U) = \langle N_\alpha, E_\alpha\rangle$ is <u>the αth iterated ultrapower of M by U</u> . Henceforth, whenever $\mathrm{Ult}_\alpha(M,U)$ is well-founded, we identify N_α with its transitive isomorph, and E_α with ε . By convention, we take $\mathrm{Ult}_0(M,U) = \langle M,\varepsilon\rangle = \langle N_0,E_0\rangle$.

(b) For $0 < \alpha \leq \beta$, define $i_{\alpha\beta}\colon N_\alpha \to N_\beta$ by $i_{\alpha\beta}([f]_\alpha) = [\langle f(s|\alpha) \mid s \in {}^\beta\kappa\rangle]_\beta$. Set $i_{0\alpha}\colon M \to N_\alpha$ as usual by $i_{0\alpha}(x) = ([\langle x \mid s \in {}^\alpha\kappa\rangle]_\alpha)$.

The following are now easy consequences of the structure theory:

<u>8.4. Lemma</u>: Whenever $0 \leq \alpha < \beta < \gamma$, $i_{\alpha\beta}$ is an elementary embedding, and $i_{\alpha\gamma} = i_{\beta\gamma} \circ i_{\alpha\beta}$. For limit $\delta > 0$, N_δ is isomorphic to the direct limit of the system $\langle N_{\alpha\beta}, i_{\alpha\beta} \mid \alpha \leq \beta < \delta\rangle$.

Finally, we would like to relate all this apparatus developed with the product-type ultrafilters U_α to the original idea of successive iterations. The following result is straightforward, but takes some work to trace through the definitions.

<u>8.5. Theorem</u>: In the situation of 8.3., if N_α is well-founded, then for any β there is an isomorphism $e_{\alpha\beta}\colon N_{\alpha+\beta} \to \mathrm{Ult}_\beta(N_\alpha, i_{0\alpha}(U))$ so that the following diagram commutes:

$$\begin{array}{ccc} & \xrightarrow{\;i_{0\beta}^{(\alpha)}\;} & \mathrm{Ult}_\beta(N_\alpha, i_{0\alpha}(U)) \\ N_\alpha & & \uparrow\ e_{\alpha\beta} \\ & \xrightarrow[\;i_{\alpha,\alpha+\beta}\;]{} & N_{\alpha+\beta} \end{array}$$

(Here, $i_{0\beta}^{(\alpha)}$ is the elementary embedding defined within N_α corresponding to $i_{0\alpha}(U)$ which in N_α is an ultrafilter over $i_{0\alpha}(\kappa)$.)

It naturally transpires that those ultrafilters which yield well-founded iterated ultrapowers become particularly interesting. The following lemma is perhaps expected; its proof illustrates in particular the conceptual usefulness of having developed the U_α in the initial model M .

<u>8.6. Lemma</u>: If in the inner model M , U is a ω_1-complete ultrafilter over a cardinal κ , then all the corresponding N_α are well-founded.

$\vdash$ The following argument is carried out entirely in M , to show that $M \models N_\alpha$ is well-founded. This will suffice, as M itself is an inner model.

Suppose to the contrary that $[f_{n+1}]_\alpha\, E_\alpha\, [f_n]_\alpha$ for every $n \in \omega$. Thus, $X_n = \{s \in {}^\alpha\kappa \mid f_{n+1}(s) \in f_n(s)\} \in U_\alpha$ for every n . The idea is to build a function $t \in \bigcap_n X_n$. Then $\ldots f_2(t) \in f_1(t) \in f_0(t)$ would be an infinite descending chain, and we would have before us a contradiction.

Let us fix n for the present paragraph, so that we do not have to worry about appending subscripts in the discussion: If X_n has support $y = \{\alpha_1,\ldots,\alpha_m\}$ listed

in ascending order, then $Z = \{\langle s(\alpha_1),\ldots,s(\alpha_m)\rangle \mid s \in X_n\} \in U^m$. We thus want our t to have the property that $\langle t(\alpha_1),\ldots,t(\alpha_m)\rangle \in Z$. In general, how can one "inductively" obtain an element of Z ? Well, in the non-trivial case $m > 1$, we have $Z \in U^m$ <u>iff</u> $\{\alpha \mid \{x \mid \langle\alpha\rangle^\frown x \in Z\} \in U^{m-1}\} \in U$ by the definition of the m-fold product. So, we can first choose such an α . Then $\{x \mid \langle\alpha\rangle^\frown x \in Z\} \in U^{m-1}$, and presuming $m-1 > 1$, we have $\{\beta \mid \{y \mid \langle\alpha,\beta\rangle^\frown y \in Z\} \in U^{m-2}\} \in U$. Thus, we can proceed to choose such a β , and so forth. After m steps, we will indeed have an m-tuple in Z , having made certain along the way that we can always keep choosing the next element from a set in U , which of course would be non-empty.

Now the procedure for constructing our t becomes clear(er): we can use the ω_1-completeness of U to perform the above procedure simultaneously for every n . Suppose that $t|\beta$ has already been defined, having made sure inductively for any typical n that, in the notation of the previous paragraph: if $\beta \cap y = \{\alpha_1,\ldots,\alpha_k\}$ where $k \leq m$, then $\langle t(\alpha_1),\ldots,t(\alpha_k)\rangle$ was chosen as in the previous paragraph. Then we can choose $t(\beta)$ as a member of an appropriate set in U . The point is that β might be in the support of infinitely many of the X_n's—but this will not cause a difficulty as we can take a countable intersection of the sets in U from which we need simultaneously to choose an element to keep the induction going. $\dashv$

We now specialize to the case which holds the main interest: <u>For the rest of this section, assume M is an inner model of ZFC, and in M , U is a κ-complete ultrafilter over a measurable cardinal κ</u> . The previous lemma assures that the corresponding N_α are all well-founded. Also, by elementarity, $N_\alpha \models i_{0\alpha}(U)$ is an $i_{0\alpha}(\kappa)$-complete ultrafilter over $i_{0\alpha}(\kappa)$. We would now like to state a theorem giving more structural information, but we need a preliminary lemma (first established by Scott) on further aspects of the initial ultrapower construction given in §2.

<u>8.7. Lemma</u>: Suppose V is a λ-complete ultrafilter over a measurable cardinal λ , and $j\colon V \to N \approx V^\lambda/V$. Then

(i) $2^\lambda \leq (2^\lambda)^N < j(\lambda) < (2^\lambda)^+$.

(ii) $V \notin N$.

$\vdash$ For (i), note first that if $[f] < j(\lambda)$, then we can take $f \in {}^\lambda\lambda$, so that $j(\lambda)$ = order type of $\{[f] \mid f \in {}^\lambda\lambda\}$. Thus, $j(\lambda) < (2^\lambda)^+$. Also, by the Closure Lemma of §2 , we have $P(\lambda) \subseteq N$, so that $2^\lambda \leq (2^\lambda)^N$. Finally, by elementarity $j(\lambda)$ is measurable and hence inaccessible <u>in N</u>, so surely $\lambda < j(\lambda)$ implies $(2^\lambda)^N < j(\lambda)$.

For (ii), assume to the contrary that $V \in N$. Then as ${}^\lambda\lambda \subseteq N$ by the Closure Lemma, we can carry out in N the bounding of $j(\lambda)$ from above as in (i) to show that $j(\lambda) < ((2^\lambda)^+)^N$. This contradicts the inaccessibility of $j(\lambda)$ in N . $\dashv$

Now we can state:

8.8 Theorem: If $\alpha < \beta$, then:

(i) $\xi < i_{0\alpha}(\kappa)$ implies $i_{\alpha\beta}(\xi) = \xi$.

(ii) $i_{\alpha\beta}(i_{0\alpha}(\kappa)) = i_{0\beta}(\kappa) > i_{0\alpha}(\kappa)$.

(iii) $P(i_{0\alpha}(\kappa)) \cap N_\alpha = P(i_{0\alpha}(\kappa)) \cap N_\beta$.

(iv) $i_{0\alpha}(U) \notin N_\beta$.

(v) $N_\alpha \supseteq N_\beta$ and $N_\alpha \neq N_\beta$.

(vi) If β is a limit, $i_{0\beta}(\kappa) = \sup\{i_{0\gamma}(\kappa) \mid \gamma < \beta\}$.

$\vdash$ By 8.5. we can construe N_β as an iterated ultrapower of N_α by $i_{0\alpha}(U)$. Thus if $\beta = \alpha+1$, then (i), (ii), (iii), and (iv) are just (the relativized versions of) facts about ultrapowers corresponding to measurable cardinals, where in particular (iii) follows from the Closure Lemma of §2 and (iv) is 8.7.(ii). For $\beta > \alpha+1$ just proceed by induction, using 8.5. at successor stages and the direct limit characterization of 8.4. at limit stages.

For (v), note that $N_\alpha \supseteq N_\beta$ follows from 8.5. as the construction of N_β can be considered carried out entirely within N_α; that equality cannot hold is evinced from (iv).

Finally, for (vi) if $\xi < i_{0\beta}(\kappa)$, then $\xi = i_{\gamma\beta}(\eta)$ for some $\eta < i_{0\gamma}(\kappa)$ by the nature of direct limit. Since $i_{\gamma\beta}(\eta) = \eta$ by (i), we have $\xi < i_{0\gamma}(\kappa)$. $\dashv$

All this is already quite interesting; we next want some information on how ordinals are moved:

8.9. Theorem: With cardinalities and cofinalities taken either in M or V:

(i) $i_{0\alpha}(\gamma) < (|\gamma|^{\kappa}\cdot|\alpha|)^+$.

(ii) If λ is a cardinal $> 2^\kappa$, then $i_{0\lambda}(\kappa) = \lambda$.

(iii) If δ is a limit ordinal so that $cf(\delta) > \kappa$, then $i_{0\gamma}(\delta) = \sup\{i_{0\gamma}(\xi) \mid \xi < \delta\}$.

(iv) If λ is a strong limit cardinal of cofinality $> \kappa$ then whenever $\gamma < \lambda$, $i_{0\gamma}(\lambda) = \lambda$.

$\vdash$ (i) is proved by a counting argument like for 8.7.(i): If $[f]_\alpha < i_{0\alpha}(\gamma)$ then we can take $f: {}^{\alpha}\kappa \to \gamma$. Each finite $y \subseteq \alpha$ can be the support of at most $|\gamma|^\kappa$ such functions, and so there are at most $|\gamma|^{\kappa}\cdot|\alpha|$ such functions in $Fn_\alpha(\kappa)$. Thus the successor of this cardinal strictly bounds the order type $i_{0\alpha}(\gamma)$.

To establish (ii) note that by (i) and 8.8.(vi), $i_{0\lambda}(\kappa) = \sup\{i_{0\alpha}(\kappa) \mid \alpha < \lambda\} \leq \sup\{(2^{\kappa}\cdot|\alpha|)^+ \mid \alpha < \lambda\} \leq \lambda$. But clearly for any ordinal α we must have $i_{0\alpha}(\kappa) \geq \alpha$, as the $i_{0\alpha}(\kappa)$'s form a strictly increasing sequence of ordinals by 8.8.(ii). Hence, (ii) follows.

For (iii), note that if $[f]_\gamma < i_{0\gamma}(\delta)$, then we can take $f: {}^{\gamma}\kappa \to \delta$, and $|Range(f)| \leq \kappa$ by looking at its finite support. Hence $f: {}^{\gamma}\kappa \to \xi$ for some $\xi < \delta$ by $cf(\delta) > \kappa$, and the result follows.

Finally, for (iv) observe that by (i) and (iii), $\lambda \leq i_{0\gamma}(\lambda) = \sup\{i_{0\gamma}(\xi) \mid \xi<\lambda\} \leq \sup\{(|\xi|^{\kappa}\cdot|\gamma|)^+ \mid \xi < \lambda\} \leq \lambda$, and hence equality obtains throughout. $\dashv$

The point of (iv) is merely that we have arbitrarily large cardinals λ such that λ is fixed by any $i_{0\gamma}$ for $\gamma < \lambda$. We could have gotten away with weaker hypotheses on λ, but there was no need. The next section will reveal how the technique of iterated ultrapowers was used by Kunen to prove fundamental results about inner models of measurability.

§9. Inner models of measurability

One of the main plausibility arguments for measurable cardinals is that they have natural inner models. Indeed, if U is a normal ultrafilter over a measurable cardinal κ, let $L[U]$ be the usual universe relatively constructible from U. If we set $\bar{U} = U \cap L[U] \in L[U]$, it is easy to see that $L[U] \models \bar{U}$ is a normal ultrafilter over κ. As we shall see below, Kunen established that $L[U] \subseteq M$ for any inner model M in which κ is measurable; in particular, this characterization only depends on κ and not on U.

We thus have a canonical inner model which plays a role for the theory (ZF & κ is a measurable cardinal) directly analogous to the role L plays for ZF. That the uniform generation and combinatorial clarity of L is also mirrored to a substantial extent in $L[U]$ is evidenced by such gross structure results as $L[U] \models$ GCH (see below), as well as work of Solovay and more recently Dodd-Jensen[1977] on the fine structure of $L[U]$. Remembering that measurability entails the existence of a plethora of indescribables, Ramsey cardinals, and so forth, $L[U]$ is still a rather complicated class, and so we have a detailed vision of a rich but uniform world.

The model $L[U]$ was not hard to find; the first hint of its uniformity was the result of Silver[1971] that $L[U] \models$ GCH. Of course, the GCH does not generally hold in arbitrary $L[X]$, since for example X could be coding ω_2 subsets of ω. Silver saw that measurability allows model theoretic arguments which push through the proof for $L[U]$. It is historically interesting that Jensen had also established the consistency of the GCH with the existence of a measurable cardinal, by a forcing argument which anticipates Silver Forcing (see §25).

The proof of the GCH in $L[U]$ devolves into two cases. If $\alpha \geq \kappa$, a version of Gödel's original argument for L works to establish in fact that $P(\alpha) \cap L[U] \subseteq L_{\alpha^+}[U]$, since a Condensation Lemma holds. Thus, the proof relativized to $L[U]$ shows within $L[U]$ that: $2^\alpha \leq |L_{\alpha^+}[U]| \leq \alpha^+$. If $\alpha < \kappa$, the problem is that many new subsets of α may first appear at rather high levels $< \kappa^+$. For example, the impish set of integers $0^\#$ is in $L_{\kappa+\omega+1}[U] - L_{\kappa+\omega}[U]$. However, Silver saw that an indiscernibility argument establishes that whenever $x \subseteq \alpha < \kappa$ with $x \in L_{\delta+1}[U] - L_\delta[U]$, $|P(\alpha) \cap L_{\delta+1}[U]| \leq |\alpha|$. The argument then carried out within $L[U]$ shows that: there are at most $|\alpha|$ subsets of α constructed before any new subset of α is constructed, and hence $2^\alpha = \alpha^+$.

There is also a way to prove Silver's result using iterated ultrapowers, which has the advantage of allowing a conceptual generalization to show that the GCH holds in the models of Mitchell[1974]. Silver[1971b] also showed that there is a Δ^1_3-well-ordering of the reals in L. Δ^1_3 is the best possible here, in view of Shoenfield's Absoluteness Lemma and the existence of non-constructible reals in $L[U]$.

Why the normality of U? Well, it actually came into play already in Silver's proof of the GCH below κ in $L[U]$, essentially in the use of Rowbottom's partition theorem. The penetrating analysis of $L[U]$ by iterated ultrapowers in Kunen[1970] revealed much deeper phenomena concerning the canonical nature of normality, and a surprising connection with that most basic of all normal filters, the closed unbounded filter. We need some definitions to encapsulate an important situation.

Definition: A κ-model is an inner model M in which there is a normal ultrafilter U over a measurable cardinal κ so that $M = L[U]$. Such a $U \in M$ is called a constructing ultrafilter for M. Finally, a filter F over an ordinal ρ is strong iff $L[F]$ is a ρ-model with constructing ultrafilter $F \cap L[F]$.

It is important to remember that although there may be a κ-model, κ need not actually be measurable (in V), nor indeed even be a cardinal. Thus, we shift the emphasis to the relativized universe. Typically, a constructing ultrafilter for a κ-model will generate a filter F in V, but F as viewed in V may no longer retain any properties of large cardinal character.

The following important lemma is the beginning of the harvest from the seeds sown in §8.

Lemma: Suppose that M is a κ-model with constructing ultrafilter U. Then if α is a limit ordinal > 0 and the corresponding αth iterated ultrapower N_α is well-founded, $X \in i_{0\alpha}(U)$ iff $X \in P(i_{0\alpha}(\kappa)) \cap N_\alpha$ & $\exists\beta<\alpha(\{i_{0\gamma}(\kappa) \mid \beta \leq \gamma < \alpha\} \subseteq X)$.

$\vdash$ Note first that if $\gamma < \alpha$, then N_γ is also well-founded as $i_{\gamma\alpha}$ is an elementary embedding. For such γ, by elementarity $N_\gamma \models i_{0\gamma}(U)$ is a normal ultrafilter over $i_{0\gamma}(\kappa)$. In particular, if $X \in P(i_{0\gamma}(\kappa)) \cap N_\gamma$, we have $X \in i_{0\gamma}(U)$ iff $\{\delta \mid \delta \in X\} \in i_{0\gamma}(U)$ iff $i_{0\gamma}(\kappa) \in i_{\gamma,\gamma+1}(X)$. This last equivalence follows from 8.5. and Łoś' Theorem since $N_{\gamma+1} = \mathrm{Ult}_1(N_\gamma, i_{0\gamma}(U))$, remembering that we identify well-founded ultrapowers with their transitive isomorphs.

We are now able to prove the desired equivalence. Suppose first that $X \in i_{0\alpha}(U)$. Then by the direct limit characterization, $X = i_{\beta\alpha}(Y)$ for some $\beta < \alpha$ and $Y \in i_{0\beta}(U)$. Then for any γ such that $\beta \leq \gamma < \alpha$, we have $i_{\beta\gamma}(Y) \in i_{0\gamma}(U)$. Thus, $i_{0\gamma}(\kappa) \in i_{\beta,\gamma+1}(Y)$ by the previous paragraph. But clearly $i_{\beta,\gamma+1}(Y) \subseteq X$, as every member of $i_{\beta,\gamma+1}(Y)$ is fixed by $i_{\gamma+1,\alpha}$ by 8.8.(i). We have thus proved one direction.

Conversely, suppose for some $\beta < \alpha$ that $\{i_{0\gamma}(\kappa) \mid \beta \leq \gamma < \alpha\} \subseteq X$. Let $i_{\gamma\alpha}(Y) = X$ where we can assume $\beta \leq \gamma$. But then $i_{0\gamma}(\kappa) \in X \doteq i_{\gamma\alpha}(Y)$ and since $i_{\gamma+1,\alpha}$ fixes $i_{0\gamma}(\kappa)$ by 8.8.(i), we have $i_{0\gamma}(\kappa) \in i_{\gamma,\gamma+1}(Y)$ by elementarity. Thus, by the penultimate paragraph $Y \in i_{0\gamma}(U)$ and so $X = i_{\gamma\alpha}(Y) \in i_{0\alpha}(U)$. $\dashv$

This lemma says that $i_{0\alpha}(U)$ for limit $\alpha > 0$ is externally generated by a "generic" sequence, which is comprised of just the iterates of κ. This nicely generalizes the fact that normality means κ "generically" generates U. It is quite crucial that the generating sequence is closed unbounded in $i_{0\alpha}(\kappa)$:

Strong Filters Theorem: Suppose that M is a κ-model with constructing ultrafilter U, and ρ is a cardinal in M greater than $(\kappa^+)^M$. If (the real) $\mathrm{cf}(\rho) > \omega$ and F is the closed unbounded filter over ρ, then F is a strong filter, with: $L[F] = \mathrm{Ult}_\rho(M,U)$, $i_{0\rho}(\kappa) = \rho$, and $i_{0\rho}(U) = F \cap L[F]$.

$\vdash$ That $i_{0\rho}(\kappa) = \rho$ follows from 8.9.(ii), noting that $(\kappa^+)^M = (2^\kappa)^M$. It is immediate from the Lemma that $i_{0\rho}(U) \subseteq F \cap \mathrm{Ult}_\rho(M,U)$, as $\{i_{0\gamma}(\kappa) \mid \gamma < \rho\}$ is a closed unbounded subset of ρ. Thus, equality must hold, as $i_{0\rho}(U)$ is an ultrafilter on $P(\rho) \cap \mathrm{Ult}_\rho(M,U)$. We then have: $\mathrm{Ult}_\rho(M,U) = L[i_{0\rho}(U)] = L[F \cap \mathrm{Ult}_\rho(M,U)] = L[F]$. From this also follows that $i_{0\rho}(U) = F \cap L[F]$. $\dashv$

With this theorem, Kunen established an unexpectedly close connection between measurability and the closed unbounded filter. That such an easily defined filter when relativized to an inner model can actually witness measurability was a profound insight. We now see that normality was, after all, an intrinsic aspect of Ulam's concept of measurability.

The assumption $\mathrm{cf}(\rho) > \omega$ in the theorem is not sacrosanct; it was only used so that we could call upon the closed unbounded filter over ρ. To bring in the case $\mathrm{cf}(\rho) = \omega$, at least for a limit cardinal ρ, define the cardinal filter over ρ as the set $\{X \subseteq \rho \mid \exists\eta<\rho\forall\gamma(\eta \leq \gamma = |\gamma| < \rho \text{ implies } \gamma \in X)\}$. Then the above theorem also holds if "(the real) $\mathrm{cf}(\rho) > \omega$" is replaced by "$\rho$ a (real) limit cardinal", and "closed unbounded filter" is replaced by "cardinal filter". (The only nicety to notice is that now, a final segment of cardinals $< \rho$ appear among the $i_{0\gamma}(\kappa)$'s for $\gamma < \rho$, by 8.9.(ii).) This version can be viewed as the ideological source of Silver's proof given at the end of §6. It is only a minor irony of this paper that Silver's proof was presented first!

We are now in a position to present the major results on the canonicity of $L[U]$. First, a preliminary

Definability Lemma: Let M be a κ-model with constructing ultrafilter U. Suppose that S is any set of ordinals all greater than κ so that S has order type $\geq (\kappa^+)^M$, and θ is a cardinal greater than every member of S. Then every element of $P(\kappa) \cap M$ is definable in $L_\theta[U]$ from a finite subset of $\kappa \cup S \cup \{U\}$.

$\vdash$ Let A be the collection of sets so definable. Since $L_\theta[U]$ has a well-ordering definable from U, $A \prec L_\theta[U]$. Let $i: A \cong T$ be the transitizing isomorphism. Since $\kappa \subseteq A$, i is the identity on κ and so $i(x) = x$ for any $x \in P(\kappa) \cap A$. Thus, $i(U) = U \cap T$ and so $T = L_\delta[U]$ for some δ. Since S has order type $\geq (\kappa^+)^M$, we have $\delta \geq (\kappa^+)^M$. But then, by what we mentioned about the proof of the GCH in $L[U]$, we have that $P(\kappa) \cap M \subseteq L_\delta[U]$. Thus, $P(\kappa) \cap M \subseteq A$.

as i was the identity on $P(\kappa)\cap A$, which was to be proved. ⊣

Uniqueness Theorem: Let M and N both be κ-models, with U a constructing ultrafilter for M, and V a constructing ultrafilter for N. Then $U = V$ and so $M = N$.

⊢ Let λ be a regular cardinal $> \kappa^+$, and F the closed unbounded filter over λ. By the previous theorem, $\mathrm{Ult}_\lambda(M,U) = L[F] = \mathrm{Ult}_\lambda(N,V)$ and $i^U_{0\lambda}(U) = F\cap L[F] = i^V_{0\lambda}(V)$, where $i^U_{0\lambda}: M \to \mathrm{Ult}_\lambda(M,U)$ and $i^V_{0\lambda}: N \to \mathrm{Ult}_\lambda(N,V)$ are the appropriate iterated ultrapower embeddings.

The idea of the proof is to use this wedding of iterated ultrapowers, together with the Definability Lemma. So, let S be a set of ordinals all greater than κ so that S has order type $\geq \kappa^+$, and let θ be a cardinal greater than every element of S. Furthermore, we can assume by 8.9.(iv) that θ and every element of S is fixed by both $i^U_{0\lambda}$ and $i^V_{0\lambda}$.

We will now establish $U \subseteq V$; since the argument is symmetric in U and V, this will complete the proof. So, assume $X \in U$. By the Definability Lemma, there is a formula ϕ so that $X = \{\xi < \kappa \mid L_\theta[U] \models \phi(\xi, a, U)\}$ for some finite $a \subseteq \kappa\cup S$. Now set $Y = \{\xi < \kappa \mid L_\theta[V] \models \phi(\xi, a, V)\}$. As $i^U_{0\lambda}$ and $i^V_{0\lambda}$ both fix every element of a, we have $i^U_{0\lambda}(X) = \{\xi < \lambda \mid L_\theta[F] \models \phi(\xi, a, F\cap L[F])\} = i^V_{0\lambda}(Y)$. In particular, as $X \in U$, it follows that $i^U_{0\lambda}(X) = i^V_{0\lambda}(Y) \in F$, so that $Y \in V$. Finally, as $i^U_{0\lambda}$ and $i^V_{0\lambda}$ both fix every element of κ, $X = i^U_{0\lambda}(X)\cap\kappa = i^V_{0\lambda}(Y)\cap\kappa = Y$, and so $X \in V$. ⊣

Corollary 1: If $V = L[U]$, where U is a normal ultrafilter over κ, then U is the only normal ultrafilter over κ.

Corollary 2: If M is a κ-model, then κ is the only measurable cardinal in M.

⊢ This is a version of Scott's original proof of the incompatibility of $V = L$ and measurability. Argue by contradiction and let $M \models \lambda$ is the least measurable cardinal $\neq \kappa$. Choose any λ-complete ultrafilter V over λ. Then if $j: M \to \mathrm{Ult}_1(M,V)$ is the corresponding embedding into the (transitized) ultrapower, $\mathrm{Ult}_1(M,V)$ is a $j(\kappa)$-model with constructing ultrafilter $j(U)$, so that $\mathrm{Ult}_1(M,V) = L[j(U)]$. However, by 8.9.(iv), as κ is inaccessible in M, we have $j(\kappa) = \kappa$. Thus, by the previous theorem $j(U) = U$ and $\mathrm{Ult}_1(M,V) = L[j(U)] = L[U] = M$. But by elementarity, $\mathrm{Ult}_1(M,V) \models j(\lambda)$ is the least measurable $\neq \kappa$, and this is a contradiction of $\lambda < j(\lambda)$. ⊣

With more attention to detail, Kunen also proved:

(a) If ρ is the least ordinal so that there is a ρ-model M and U is the constructing ultrafilter for M, then any σ-model for $\sigma > \rho$ is of form $\mathrm{Ult}_\alpha(M,U)$.

(b) If M is a κ-model with constructing ultrafilter U, then the following holds in M: whenever V is any κ-complete ultrafilter over κ, V is U^n up to a bijection between the index sets κ and ${}^n\kappa$, for some $n \in \omega$.

(c) If M is a κ-model and V is an arbitrary κ-complete ultrafilter over κ, then $L[V] = M$.

(d) In any κ-model, a cardinal is Jonsson just in case it is Ramsey.

These various results have an attractive clarity, with strong Brucknneresque architectural lines. The extent of structure found in κ-models, their uniqueness of construction, and the hidden global regularity of their generation by iterated ultrapowers show what possibilities lay hidden in the lode of elementary embeddings, just waiting to be tapped.

One of the main problems with large cardinal hypotheses much stronger than measurability (see the next chapter) is that no corresponding natural inner models suggest themselves (we shall discuss this source of perplexion later). At least some sign of encouragement are the models of Mitchell[1974]. Let us define a partial order $\lhd$ between ω_1-complete ultrafilters, by: $U \lhd V$ iff U is a member of (the transitive isomorph of) the ultrapower of V by V. That the order $\lhd$ is non-trivial is consistency-wise a stronger assumption than measurability and follows, for example, from the existence of a strongly compact cardinal, as recently shown by Mitchell. Mitchell fashioned a coherency condition on ultrafilter sequences in the ground model that mirrors long $\lhd$-chains of normal ultrafilters over various measurable cardinals, somewhat reminiscent of the approach in §8 of carrying out the iterated ultrapower construction directly in the ground model. Finally, he showed that constructing from coherent sequences yield inner models which exhibit the major uniformity properties of κ-models such as the GCH, a Δ^1_3-well-ordering of the reals, and corresponding versions of the Definability Lemma and Uniqueness Theorem. Dodd-Jensen(1976) has recently introduced "mice" into the fine structure study of $L[U]$, in connection with establishing the Covering Theorem for the Core Model K (see §29). Mitchell showed that this rodent infestation was also possible in his models.

§10. A Mixed Bag

This section is devoted to three disparate results, all proved first by Kunen in the framework of iterated ultrapowers. Although some of these results can now be proved without iterated ultrapowers, the technique looms large in the background as having fashioned the climate wherein the results were first conceived. Typically a powerful mathematical idea creates a unifying framework, and this imposition of structure then leads to new mathematical intuitions and ramifications in several directions.

The first result involves the size of 2^κ when κ is measurable. As we mentioned in §2 (see also §13), Scott had tied 2^κ to 2^α for $\alpha < \kappa$ by establishing that if $\{\alpha < \kappa \mid 2^\alpha = \alpha^+\}$ is a member of a normal ultrafilter over κ, then $2^\kappa = \kappa^+$. We know the GCH to be consistent with the existence of a measurable cardinal, so the problem remained of how to ever get $2^\kappa > \kappa^+$ to hold, yet preserve the measurability of κ. Any forcing technique which achieves this must be particularly sensitive, since the measurability of κ involves all subsets of κ.

It is historically interesting that it was precisely this problem which motivated Silver to invent his method of forcing (see §25). Once formulated, this method has a persuasive character as being arguably the natural one for iterating forcing to adjust powers of regular cardinals, while preserving a great deal of the structure of the ground model. Thus, it is to the credit of the theory of large cardinals to have motivated this new technique of general applicability to set theory.

In any case, Kunen[1971a] had already realized the strong consistency strength of the violation of the GCH at a measurable cardinal: If there is a measurable cardinal κ so that $2^\kappa > \kappa^+$, then for any ordinal α there is an inner model with α measurable cardinals. The conclusion also follows from any of the following assumptions: (a) every κ-complete filter over κ can be extended to a κ-complete ultrafilter over κ; (b) there is a uniform κ-complete ultrafilter over κ^+; and so also (c) κ is strongly compact. The proof of this result involves a refined use of iterated ultrapowers, and we content ourselves by giving a glimpse at the possibilities by showing: If there is a measurable cardinal κ so that $2^\kappa > \kappa^+$, there is a transitive set model of (ZFC & there is measurable cardinal):

$\vdash$ Let U be a normal ultrafilter over κ, and set $\bar{U} = U \cap L[U]$. For any ordinal α, define $i_{0\alpha}: L[U] \to Ult_\alpha(L[U],\bar{U})$ to be the usual elementary embedding into the αth iterated ultrapower. Set $F = \{X \subseteq i_{0\omega}(\kappa) \mid \exists n \forall m (n<m<\omega \to i_{0m}(\kappa) \in X)\}$. Then by §9, $Ult_\omega(L[U],\bar{U}) = L[F]$ is a $i_{0\omega}(\kappa)$-model with constructing ultrafilter $F \cap L[F]$.

Now set $j: V \to M = V^\kappa/U$. By the Closure Lemma of §2, $\langle i_{0n}(\kappa) \mid n \in \omega\rangle \in M$, and because F is definable from this sequence, we have $F \cap M \in M$. Thus $L[F] \subseteq M$. Now observe that $j(\kappa)$ is inaccessible in M, so it follows that $j(\kappa)$ is inaccessible in $L[F]$, as $L[F]$ models GCH and is an inner model of M.

Finally, by $2^\kappa > \kappa^+$, 8.7.(i), and 8.9.(i), we have:

$$i_{0\omega}(\kappa) < ((2^\kappa)^+)^{L[U]} = \kappa^{++L[U]} \le \kappa^{++} \le 2^\kappa < j(\kappa).$$

Thus, we can ultimately conclude that:

$$L_{j(\kappa)}[F] \models \text{ZFC} \;\&\; F \cap L[F] \text{ is a normal ultrafilter over } i_{0\omega}(\kappa). \qquad \dashv$$

Having seen in the preceding proof the strong techniques available to us, the full result of Kunen[1971a] already seems quite plausible.

The second result of this section is one direction of 7.25., a characterization of $0^\#$. As with 7.22.(a), the existence of $0^\#$ implies the existence of a multitude of elementary embeddings: $L \to L$. In fact, they correspond closely to order-preserving injections $OR \to OR$, as the action of such injections on the canonical indiscernibles in turn lifts via Skolemization to an elementary embedding: $L \to L$. Kunen showed that a converse was possible: If there is an elementary embedding of

L into L which is not the identity, then $0^\#$ exists. If one looks at the least ordinal ρ moved by such an embedding, then it generates as usual an ultrafilter on $P(\rho) \cap L$. This ultrafilter will not be constructible, but adding further conditions to be able to generate well-founded iterated ultrapowers of L ("from the outside") from it, Kunen saw that elementarity considerations indicate that the class of iterates of ρ, $\{i_{0\alpha}(\rho) \mid \alpha \in OR\}$, act as indiscernibles for L. The ideas herein introduced, primarily the emergence of critical points of embeddings as indiscernibles and the need to retain classes of ordinals fixed by embeddings, were then combined with the uniform aspects of L to get proofs of Kunen's theorem which do not use the machinery of iterated ultrapowers. The following proof is due to Paris; another due to Silver is given in Devlin[1973],pp.199-205.

$\vdash$ Let $k: L \to L$ be elementary, with ρ the first ordinal moved. We can define an ultrafilter U over the constructible subsets of ρ by: $X \in U$ iff $X \in P(\rho) \cap L$ & $\rho \in k(X)$. The embedding k is not itself very informative, so we switch to the embedding corresponding to an ultrapower construction using U. First, note that for any formula $\phi(x_1,\ldots,x_n)$ and $f_1,\ldots,f_n \in L \cap {}^\rho L$, we have $\{\alpha < \rho \mid L \models \phi(f_1(\alpha),\ldots,f_n(\alpha))\} \in L$. Hence, we can form the ultrapower of L with respect to U, using only functions in $L \cap {}^\rho L$. The appropriate version of Łoś' Theorem holds, and the ultrapower is easily seen to be well-founded. Its transitive isomorph must be L, being an inner model of the Axiom of Constructibility. Hence, let $j: L \to L$ be the elementary embedding induced by this ultrapower construction. Note that, as usual, we will have $j(\rho) > \rho$, yet $j(\xi) = \xi$ for $\xi < \rho$.

Set $M = \{x \in L \mid j(x) = x\}$. Then $M \prec L$. This is so since M is the Skolem Hull of itself with respect to the definable well-ordering of L, since anything definable from parameters fixed by j is again fixed by j. The reason why the switch from k to j was made was to insure that M is large: If (in V) λ is a cardinal of cofinality $> |\rho|$, then $j(\lambda) = \lambda$. (To show this, note first that if $\xi < \lambda$, a counting argument shows $j(\xi) \le |{}^\rho\xi|^{+L} \le \lambda$. The full result then follows, as $cf(\lambda) > |\rho|$ insures $j(\lambda) = \sup\{j(\xi) \mid \xi < \lambda\}$.) By a stationary class is meant a class of ordinals which meets every closed unbounded class of ordinals. Thus, the foregoing shows in particular that $M \cap OR$ is a stationary class. M must thus be a proper class, so $M \cong L$. The following process is reminiscent of iterating ultrapowers:

For every ordinal α, we will define classes M_α to satisfy: (i) $M_\alpha \prec L$, (ii) $M_\alpha \cap OR = M \cap C \cap OR$ for some closed unbounded class C of ordinals, and so $M_\alpha \cap OR$ is a stationary class, and (iii) $\alpha < \beta$ implies $M_\alpha \supseteq M_\beta$.

Initially, set $M_0 = L$. At limit stages δ, simply set $M_\delta = \bigcap_{\alpha<\delta} M_\alpha$. Then $M_\delta \cap OR$ satisfies condition (ii), and $M_\delta \prec L$, as M_δ is its own Skolem Hull. Finally, at successor stages, having already defined $M_\alpha \prec L$, let $\tau: L \cong M_\alpha$ be the inverse of the unique transitization map for M_α, and set $M_{\alpha+1} = \tau''M$. Then since $M \prec L$, we have $M_{\alpha+1} = \tau"M \prec \tau"L = M_\alpha \prec L$. Note also that if the stationary classes $M \cap OR = \langle \eta_\xi \mid \xi \in OR \rangle$ and $M_\alpha \cap OR = \langle \zeta_\xi \mid \xi \in OR \rangle$ are listed in ascending order, $M_{\alpha+1} \cap OR = \langle \zeta_{\eta_\xi} \mid \xi \in OR \rangle$, which is easily seen to satisfy (ii). Hence, this inductive definition satisfies all the clauses required of it.

We now define many new elementary embeddings. For any ordinal α, let $\tau_\alpha: L \cong M_\alpha$ be the inverse of the unique transitization map for M_α. Then for $\alpha \le \beta$, set $j_{\alpha\beta} = \tau_\beta \cdot \tau_\alpha^{-1}$. Thus, $j_{\alpha\beta}: M_\alpha \to M_\beta$ is an isomorphism, and $\alpha \le \beta \le \gamma$ implies $j_{\beta\gamma} \cdot j_{\alpha\beta} = j_{\alpha\gamma}$. Claim: For any γ, $j_{\alpha\beta} | M_{\alpha+\gamma}: M_{\alpha+\gamma} \to M_{\beta+\gamma}$ is an isomorphism. (By induction on γ. The limit case follows immediately from the induction. For the successor stages, assume true for γ. Thus, if $\overline{j_{\alpha\beta}} = j_{\alpha\beta} | M_{\alpha+\gamma}$ then $\overline{j_{\alpha\beta}}: M_{\alpha+\gamma} \to M_{\beta+\gamma}$ is an isomorphism. It follows that $\overline{j_{\alpha\beta}} \cdot \tau_{\alpha+\gamma} = \tau_{\beta+\gamma}$, as both are inverses of the unique transitization map for $M_{\beta+\gamma}$. However, by definition $M_{\alpha+\gamma+1} = \tau_{\alpha+\gamma}"M$ and $M_{\beta+\gamma+1} = \tau_{\beta+\gamma}"M$, and hence the result follows by tracing through the various embeddings.)

We are now in a position to define our proposed class of indiscernibles. Set $\rho_0 = \rho$, and for any $\alpha > 0$, set $\rho_\alpha = j_{0\alpha}(\rho)$. Let us establish the following several facts:

(a) For any α, ρ_α = least ordinal in $M_\alpha - M_{\alpha+1}$.

(b) If $\alpha < \beta$, then $\rho_\alpha < j_{\alpha\beta}(\rho_\alpha) = \rho_\beta$.

(c) If $\alpha \le \beta$, $\rho_\alpha \cap M_\alpha = \rho_\alpha \cap M_\beta$.

(d) If $\alpha \le \beta$, $j_{\alpha\beta}(\rho_{\alpha+\gamma}) = \rho_{\beta+\gamma}$.

(e) If $\xi_1 < \dots < \xi_n < \eta < \zeta$ and ϕ is a formula in $n+m+1$ free variables, then:

$$L \models \phi(\rho_{\xi_1},\dots,\rho_{\xi_n},\rho_\eta,\rho_{\eta+1},\dots,\rho_{\eta+m}) \leftrightarrow \phi(\rho_{\xi_1},\dots,\rho_{\xi_n},\rho_\zeta,\rho_{\zeta+1},\dots,\rho_{\zeta+m}).$$

To show (a), recall first that $M_0 = L$ and $M_1 = M$, so that $\rho_0 = \rho$ = least ordinal in $M_0 - M_1$, since ρ is the least ordinal moved by j. By the Claim above $j_{0\alpha} | M_1: M_1 \to M_{\alpha+1}$ is an isomorphism, so (a) now follows by a straightforward argument. (b) follows from the fact that whenever $\alpha < \beta$, $\rho_\alpha \notin M_{\alpha+1} \supseteq M_\beta$ by (a), and the fact that $j_{\alpha\beta}$ is order-preserving on ordinals. (c) follows by induction on β, using (a) and (b) along the way. (d) follows from the Claim above, and the characterization of (a).

It is (e) which is the gateway to indiscernibility. To establish it, let us assume for notational simplicity that $m = n = 1$ and $\xi = \xi_1$. It then suffices to show $L \models \phi(\rho_\xi,\rho_\eta,\rho_{\eta+1}) \to \phi(\rho_\xi,\rho_\zeta,\rho_{\zeta+1})$, since the converse direction follows from considering $\neg\phi$. So, suppose to the contrary that $L \models \phi(\rho_\xi,\rho_\eta,\rho_{\eta+1})$ & $\neg\phi(\rho_\xi,\rho_\zeta,\rho_{\zeta+1})$. So $L \models \exists \alpha < \rho_\eta (\phi(\alpha,\rho_\eta,\rho_{\eta+1})$ & $\neg\phi(\alpha,\rho_\zeta,\rho_{\zeta+1}))$. As the parameters appearing in this formula are all in $M_\eta \prec L$, we can pick an $\alpha \in \rho_\eta \cap M_\eta$ so that $L \models (\phi(\alpha,\rho_\eta,\rho_{\eta+1})$ & $\neg\phi(\alpha,\rho_\zeta,\rho_{\zeta+1}))$. However, it readily follows from (c) and (d) above that $j_{\eta\zeta}(\alpha) = \alpha$, $j_{\eta\zeta}(\rho_\eta) = \rho_\zeta$, and $j_{\eta\zeta}(\rho_{\eta+1}) = \rho_{\zeta+1}$. Hence, $M_\eta \models \phi(\alpha,\rho_\eta,\rho_{\eta+1})$ implies that $M_\zeta \models \phi(\alpha,\rho_\zeta,\rho_{\zeta+1})$. This is a contradiction of $M_\zeta \prec L$.

We can finally establish that $\{\rho_\alpha \mid \alpha \in OR\}$ is a proper class of indiscernibles for L: That it is a proper class is immediate from (b). Suppose ϕ is a formula in $k+1$ free variables, and $\alpha_0 < \dots < \alpha_k$. To establish indiscernibility for ϕ amounts to using (e) $k+1$ times:

$$\begin{aligned} L \models \phi(\rho_0,\dots,\rho_k) \quad &\text{iff} \quad L \models \phi(\rho_{\alpha_0},\rho_{\alpha_0+1},\dots,\rho_{\alpha_0+k}) \\ &\text{iff} \quad L \models \phi(\rho_{\alpha_0},\rho_{\alpha_1},\rho_{\alpha_1+1},\dots,\rho_{\alpha_1+k-1}) \\ &\;\;\vdots \\ &\text{iff} \quad L \models \phi(\rho_{\alpha_0},\rho_{\alpha_1},\dots,\rho_{\alpha_k}). \end{aligned}$$

The rest is an anticlimax. To actually get $0^\#$, we can proceed for instance as follows: Note that $\{\rho_\alpha \mid \alpha < \omega_1\}$ is a set of indiscernibles for $L_{\rho_{\omega_1}}$. Hence there is a β so that L_β has a set of indiscernibles of order type ω_1. Let $\bar\beta$ be the least such β, and let H be a set of indiscernibles of order type ω_1 for $L_{\bar\beta}$ so that the ωth element of H is the smallest possible. We can now establish exactly as in the succession of lemmas in §7 that the corresponding EM blueprint has Properties I-IV, and hence must be $0^\#$. ⊣

We remark that the preceding result is the route to $0^\#$ in the proof of Jensen's Covering Theorem (see §29). In Jensen's scheme, one works with Σ_0-embeddings (i.e. embeddings preserving bounded formulas) and, depending on the species of argument, those with domain only some initial segment L_α. Then some details involving cofinality must be attended to, but in the versions when the Σ_0-embedding is extended to all of L, there is no problem, as a $j: L \to L$ is elementary iff it is Σ_0-elementary, by an easy reflection argument. See Gaifman[1974], Part II, for general remarks about Σ_0-embeddings of fragments of ZF.

As once foretold, we can now establish: If there is a Jonsson cardinal, then $0^\#$ exists:

⊢ If κ is Jonsson, let $X \prec L_\kappa$ be such that $|X| = |L_\kappa| = \kappa$ and $X \neq L_\kappa$. Then $X \cong L_\kappa$ and thus the inverse of the unique transitization map for X induces an elementary embedding $L_\kappa \to L_\kappa$ which moves some ordinal $\rho < \kappa$. We are now exactly in the position of being able to define an ultrafilter on $P(\rho) \cap L$ as in the proof of the preceding theorem, as $P(\rho) \cap L \subseteq L_{(\rho^+)^L} \subseteq L_\kappa$. Thus, that proof shows that $0^\#$ exists. ⊣

We close this section with yet another result of Kunen, a further dividend of iterated ultrapowers. The principal result of Kunen[1973] is that in the theory (ZFC & there are ω_1 measurable cardinals) one can show that the Axiom of Choice fails in C^{ω_1}, where C^{ω_1} (due to Chang[1971]) is the least inner model M so that ${}^{\omega}M \subseteq M$. This creates somewhat of an obstacle to an obvious, naive approach to constructing natural inner models with closure properties appropriate, say, for supercompactness (see §14). Toward demonstrating his result, Kunen established a rather striking lemma, using iterated ultrapowers. Let us first reactivate some notation from §2: if U is a ω_1-complete ultrafilter, then j_U is the elementary embedding of V into (the transitization of) the ultrapower of V by U. Kunen showed: For any ordinal η, $\{\kappa \mid \kappa$ is measurable and there is a κ-complete ultrafilter U over κ so that $\eta < j_U(\eta)\}$ is finite. This Lemma was also used very recently by Dodd-Jensen (1976), in connection with the Core Model K. (Specifically, they used it to establish that if $j: K \to K$ is elementary, κ is the least ordinal moved by j, and $j(\kappa)$ is (really) regular, then $j(\kappa)$ is measurable in an inner model.) We hope for further applications in the future. The following is an attractive proof of Kunen's Lemma which does not use the paraphenalia of iterated ultrapowers, due to Fleissner:

⊢ First of all, if U is a κ-complete ultrafilter over κ, designate that a half-open interval of ordinals $[\alpha,\beta)$ is a moving interval for U iff j_U fixes cofinally many ordinals $< \alpha$ as well as β, yet j_U moves every ordinal in $[\alpha,\beta)$. It follows that if $[\alpha,\beta)$ is a moving interval for U, then: (a) $cf(\alpha) = \kappa$, and (b) $\beta = \sup\{j_U^n(\xi) \mid n \in \omega\}$ whenever $\alpha \le \xi < \beta$, where j_U^n is the nth iterate of j_U:

To show (a), first establish in general that if $cf(\gamma) \neq \kappa$, then $j_U(\gamma) = \sup\{j_U(\delta) \mid \delta < \gamma\}$; this involves a κ-completeness argument for $cf(\gamma) < \kappa$ and a cofinality argument for $cf(\gamma) > \kappa$. Since j_U fixes cofinally many ordinals $< \alpha$, we have $\sup\{j_U(\delta) \mid \delta < \alpha\} = \alpha$, and so the fact that $\alpha < j_U(\alpha)$ necessitates $cf(\alpha) = \kappa$. To show (b) is easy enough, noting that $j(\sup\{j_U^n(\xi) \mid n\epsilon\omega\}) = \sup\{j_U^{n+1}(\xi) \mid n\epsilon\omega\}$.

We next observe that if U is a κ-complete ultrafilter over κ and V is a λ-complete ultrafilter over $\lambda > \kappa$, then j_U and j_V commute: For example, for any ordinal γ, $j_U(\gamma)$ = order type of ${}^{\kappa}\gamma/U$. But since λ is the bigger measurable cardinal, κ and U are fixed by j_V, and thus $j_V(j_U(\gamma))$ = order type of $({}^{\kappa}j_V(\gamma))/U = j_U(j_V(\gamma))$.

Now note that if $[\alpha,\beta)$ is a moving interval for U and $[\bar\alpha,\bar\beta)$ is a moving interval for V, then they are either disjoint or strictly nested: This means that if $\alpha < \bar\alpha < \beta$, we must establish that $\bar\beta < \beta$. To this end, first choose a $\gamma = j_V(\gamma)$ so that $\alpha \le \gamma < \bar\alpha$; this is possible by the nature of $\bar\alpha$. By (b) above, there is some $n \in \omega$ so that $\bar\alpha < j_U^n(\gamma) < \beta$. By the previous paragraph, we know that j_U and j_V commute, so $j_V(j_U^n(\gamma)) = j_U^n(j_V(\gamma)) = j_U^n(\gamma)$. Thus $\bar\beta \le j_U^n(\gamma)$ by definition of a moving interval, and we conclude that $\bar\beta < \beta$.

We are finally in a position to complete the proof. So, suppose to the contrary that η is an ordinal so that there is an infinite number of measurable cardinals $\{\kappa_i \mid i \in \omega\}$ with κ_i-complete ultrafilters U_i over κ_i for each $i \in \omega$, and $j_{U_i}(\eta) > \eta$. Clearly, for each $i \in \omega$, η is then in some moving interval $[\alpha_i,\beta_i)$ for U_i. However, from (a) above follows that $i \neq j$ implies $\alpha_i \neq \alpha_j$. Thus, we can assume (by possibly taking a subsequence) that $i < j$ implies $\alpha_i < \alpha_j$.

But then, as the $[\alpha_i,\beta_i)$'s all contain η, by the previous paragraph they are nested intervals, and so $\beta_0 > \beta_1 > \beta_2 > \ldots$ is a descending sequence of ordinals. This is a contradiction. ⊣

§11. Saturated Ideals

We have deferred (from §1!) until now any detailed discussion of the other branch of the bifurcation emanating from the pioneering paper Ulam[1930], the concept of real-valued measurability. It was Solovay[1971] who did the substantial modern work in this area, proving in particular that the existence of a measurable cardinal is equiconsistent with 2^{ω} being a real-valued measurable cardinal. (See §12, §24.) Although his results were proved before Kunen formulated his scheme of iterated ultrapowers, we have delayed until now the discussion of this whole area of research in order to incorporate some aspects of iterated ultrapowers (see §12).

It was Tarski who early on studied a property of real-valued measurable cardinals which conveys many of its strong consequences. We need to establish some definitions, so let κ be a cardinal and I an ideal over κ. An $X \subseteq \kappa$ is of I-positive measure iff $X \notin I$. $I^* = \{X \subseteq \kappa \mid \kappa - X \in I\}$ is the dual filter to I. I is non-trivial iff it is κ-complete (i.e. whenever $\gamma < \kappa$ and $\{X_\alpha \mid \alpha < \gamma\} \subseteq I$, then $\bigcup_\alpha X_\alpha \in I$), and $\{\alpha\} \in I$ for every $\alpha < \kappa$. Throughout this paper, by "ideal" will henceforth be meant "non-trivial ideal". I is λ-saturated iff whenever $\{X_\alpha \mid \alpha < \lambda\}$ consists of I-positive measure sets so that $\alpha < \beta < \gamma$ implies $X_\alpha \cap X_\beta \in I$, then $\gamma < \lambda$. I has saturatedness λ iff I is λ-saturated but not σ-saturated for any $\sigma < \lambda$.

If κ is real-valued measurable, and μ is a measure, then $\{X \subseteq \kappa \mid \mu(X) = 0\}$ is a (non-trivial) ω_1-saturated ideal over κ. (If $\mu(X_\alpha) > 0$ for every $\alpha < \omega_1$, then we can suppose without loss of generality that there is an $n \in \omega$ so that $\mu(X_\alpha) \ge \frac{1}{n}$ for every $\alpha < \omega_1$; but then, clearly there must be $\alpha < \beta < \omega_1$ so that $\mu(X_\alpha \cap X_\beta) > 0$.) As Tarski and later Solovay noticed, most of the large cardinal aspects of real-valued measurability can already be culled from the existence of saturated ideals. One can already see the plausibility of this in light of (two-valued) measurability, as the dual filter to a saturated ideal is rather close to being ultra. In this section, we initially develop the combinatorial theory of saturated ideals. However, one should keep in mind that being real-valued measurable is a rather special property over and beyond merely carrying a ω_1-saturated ideal (see Kunen[1968] and Prikry[1970], and also the end of this section), and there are still interesting open questions distinctly concerning real-valued measurability. At the same time, the study of saturated ideals has broadened the horizon beyond their original raison d'être, and has even led to significant clarifications about such standard set theoretical concepts as stationary sets.

We should make some initial comments about saturated ideals. First, if I is a (non-trivial) ideal over κ, then if $\alpha < \kappa$, $\alpha \in I$ by κ-completeness, and hence an easy argument shows that κ must already be regular. A trichotomy then exists in the theory depending on whether the saturatedness of I is $<\kappa$, κ, or κ^+. Since I must in any case be $(2^\kappa)^+$-saturated, this covers the interesting cases. If the saturatedness of I is $\leq \omega$, then I is a prime ideal, and hence κ is a measurable cardinal. We shall see that a cardinal carrying an ideal which is sufficiently saturated has many properties of measurable cardinals.

One of the main points about the existence of a saturated ideal over κ is that the (strong) inaccessibility of κ need not be a concomitant; κ may even be 2^ω, and then significant statements about sets of reals are being made. The following result makes a basic reduction when strong power set hypotheses are imposed on κ; (a) is a result of Tarski[1939], and (b) was observed by Silver to follow similarly: If I is an ideal over κ such that either (a) I is λ-saturated for a $\lambda < \kappa$ so that $2^{<\lambda} < \kappa$, or (b) I is κ-saturated and κ is weakly compact, then κ is a measurable cardinal.

$\vdash$ Suppose first that I had an atom, i.e. an I-positive measure set A so that if B and C are disjoint sets so that $B \cup C = A$, then either B or C is in I. Then as in §1, $\{X \subseteq \kappa \mid X \cap A \notin I\}$ can be shown to be a κ-complete ultrafilter, and hence κ would be a measurable cardinal. Thus, we can assume by way of contradiction that I has no atoms. Then inductively build a tree T with nodes $\{X_f \mid f \in \bigcup_{\alpha<\kappa} {}^\alpha 2\}$ as follows:

Set $X_{<>} = \kappa$. If $\alpha < \kappa$, and $f \in {}^\alpha 2$, and X_f has already been defined, let $X_{f^\frown<0>} \cup X_{f^\frown<1>} = X_f$ so that $X_{f^\frown<0>}$ and $X_{f^\frown<1>}$ are disjoint, and have I-positive measure whenever X_f does. That I has no atoms makes this possible. Finally, at limits $\delta < \kappa$, if $f \in {}^\delta 2$, set $X_f = \bigcap_{\beta<\delta} X_{f|\beta}$. The tree T thus built has the following property: (*) If $\gamma \leq \kappa$ and $g \in {}^\gamma 2$ then the "offshoots" $\{X_{g|\alpha^\frown<i>} \mid \alpha < \gamma \;\&\; g(\alpha) \neq i < 2\}$ consists of disjoint sets.

Consider now the subtree $\bar{T} = \{X_f \mid X_f \notin I\}$. In case (a) of the theorem, (*) and λ-saturation indicate that $\bar{T}$ has $<\lambda$ levels. Hence, $|\bar{T}| \leq 2^{<\lambda}$ and by looking at the tips in T of branches through $\bar{T}$, it is straightforward to see that κ is the union of at most $2^{<\lambda} < \kappa$ sets in I, contradicting the κ-completeness of I.

In case (b) of the theorem, note first that if $\bar{T}$ had height $<\kappa$ then by the strong inaccessibility of κ and the argument of the previous paragraph, we would again have a contradiction of the κ-completeness of I. Thus, we can suppose that $\bar{T}$ is a κ-tree, and so the weak compactness of κ implies that $\bar{T}$ has a κ-branch. However, by (*) this would violate the κ-saturation of I. $\dashv$

The preceding proof is perhaps the earliest example of the use of trees ("ramification systems"). If κ is inaccessible and not measurable, it thus follows that any ideal over κ must have saturatedness $\geq \kappa$. Kunen-Paris[1971] established the consistency of having an inaccessible (in fact weakly compact), yet not measurable, cardinal κ carrying an ideal of saturatedness κ^+, starting with a

measurable cardinal. Using in part Silver forcing (see §25), Kunen(1974) answered the remaining technical question by establishing the consistency of having an inaccessible nonmeasurable cardinal κ carrying an ideal of saturatedness κ, again starting with a measurable cardinal.

This more or less takes care of the cases where the (strong) inaccessibility of κ is assumed. We now turn to see what can be achieved when no such assumption is made. We shall progressively consider κ^+, κ, and $<\kappa$ saturation to see how stronger hypotheses sharpen the focus, and to illuminate those aspects of measurability which really are consequences of these restricted hypotheses. Generally, anyone working with saturated ideals soon realizes that the κ-saturation of an ideal I over κ is a relatively easy concept to grasp and manipulate, since it just means (by κ-completeness) that κ cannot be partitioned into κ disjoint I-positive measure sets. However, κ^+-saturation has no such clear intuitive feel. What we can do is to look at Boolean algebraic equivalence classes. If I is an ideal over κ, $P(\kappa)/I$ is the usual Boolean algebra consisting of equivalence classes $[X]$ for $X \subseteq \kappa$, where $[X] \leq [Y]$ iff $X - Y \in I$. Note that $[X] = 0$, the zero element of the Boolean algebra, iff $X \in I$. (We shall use the $[X]$ notation without explicit mention when the I involved is obvious from the context.)

The following result of Smith-Tarski[1957] was the first significant comment made on κ^+-saturation: If I is a κ^+-saturated ideal over κ, then $B = P(\kappa)/I$ is a complete Boolean algebra:

$\vdash$ Complete means that whenever $S \subseteq B$, the least upper bound ΣS exists. We divide the proof into three cases, depending on $|S|$:

Case 1: $|S| < \kappa$. Write $S = \{[X_\alpha] \mid \alpha < \gamma\}$ where $\gamma < \kappa$. Then $[\bigcup_{\alpha<\gamma} X_\alpha] = \Sigma S$. This is so since if $[Y] \geq [X_\alpha]$ for every $\alpha < \gamma$, then $\bigcup_{\alpha<\gamma}(X_\alpha - Y) = (\bigcup_{\alpha<\gamma} X_\alpha) - Y \in I$ by κ-completeness, i.e. $[Y] \geq [\bigcup_{\alpha<\gamma} X_\alpha]$.

Case 2: $|S| = \kappa$. Write $S = \{b_\alpha \mid \alpha < \kappa\}$. Using Case 1, we apply a standard contrivance to disjointify the elements: Set $a_\alpha = b_\alpha - \Sigma_{\xi<\alpha} b_\xi$ for every $\alpha < \kappa$. Then a straightforward argument shows that: (a) $a_\alpha \wedge a_{\bar{\alpha}} = 0$ whenever $\alpha \neq \bar{\alpha}$; (b) if $\beta < \kappa$ then $\Sigma_{\alpha<\beta} a_\alpha = \Sigma_{\alpha<\beta} b_\alpha$; and (c) if $\Sigma_{\alpha<\kappa} a_\alpha$ exists, then $\Sigma_{\alpha<\kappa} a_\alpha = \Sigma_{\alpha<\kappa} b_\alpha$.

We now work with the a_α's instead, which we can take without loss of generality to all be $\neq 0$. Extend $\{a_\alpha \mid \alpha < \kappa\}$ to a maximal family A such that $x \neq y$ both in A implies $x \neq 0$ and $x \wedge y = 0$. By κ^+-saturation, we must have $|A| \leq \kappa$. So, we can write $A = \{[A_\delta] \mid \delta < \kappa\}$. Now observe that we can actually take the A_δ's, not only the $[A_\delta]$'s, to be disjoint. (If necessary, replace A_δ by $\bar{A}_\delta = A_\delta - \bigcup_{\xi<\delta} A_\xi = A_\delta - \bigcup_{\xi<\delta}(A_\xi \cap A_\delta)$, noting that the subtracted set must be in I by κ-completeness.) Finally, set $E = \bigcup\{A_\delta \mid [A_\delta] = \text{one of the original } a_\alpha\text{'s}\}$. We finish Case 2 by showing: $[E] = \Sigma_{\alpha<\kappa} a_\alpha = \Sigma_{\alpha<\kappa} b_\alpha$:

Suppose that $[\bar{E}] \geq a_\alpha$ for every $\alpha < \kappa$. We want to show that $[\bar{E}] \geq [E]$, i.e. $E - \bar{E} \in I$. So, set $b = [E - \bar{E}]$. Certainly $b \wedge a_\alpha = 0$ for every $\alpha < \kappa$ by hypothesis on $\bar{E}$. Also, for $[A_\delta] \neq$ an a_α, $A_\delta \cap E = \emptyset$, so $b \wedge [A_\delta] = 0$. Thus, if $b > 0$, we would have a contradiction of the maximality of A. Hence $0 = b = [E - \bar{E}]$, i.e. $E - \bar{E} \in I$.

Case 3: $|S| > \kappa$: Here we can proceed by induction on $|S|$. So, suppose it is true for all $T \subseteq S$ such that $|T| < |S|$, that ΣT exists. Write $S = \{b_\alpha \mid \alpha < \lambda\}$. As before, we can set $a_\alpha = b_\alpha - \sum_{\xi<\alpha} b_\xi$ for every $\alpha < \lambda$, using the inductive hypothesis. Since $\alpha \neq \bar{\alpha}$ implies $a_\alpha \wedge a_{\bar{\alpha}} = 0$, there are at most κ a_α's not zero, by κ^+-saturation. Thus, $\sum_{\alpha<\lambda} a_\alpha$ exists by Case 2, and observe that $\sum_{\alpha<\lambda} a_\alpha = \sum_{\alpha<\lambda} b_\alpha$. ⊣

The preceding result has a partial converse due to Solovay: If $2^\kappa < 2^{\kappa^+}$ and I is an ideal over κ so that $P(\kappa)/I$ is a complete Boolean algebra, then I is κ^+-saturated. (Suppose not, and let $\{b_\alpha \mid \alpha < \kappa^+\} \subseteq P(\kappa)/I$ be different elements so that $\alpha \neq \beta$ implies $b_\alpha \wedge b_\beta = 0$. If $X \subseteq \kappa^+$, set $a_X = \sum_{\alpha \in X} b_\alpha$. Then $X \neq Y$ implies $a_X \neq a_Y$. But then $2^{\kappa^+} \le |P(\kappa)/I| \le 2^\kappa$, a contradiction.)

Indeed, the modern results on saturated ideals are due to Solovay[1971]. The renaissance of measurability in the early sixties invited similar possibilities for saturated ideals, and the new ideas of Scott, Rowbottom, and others were duly lifted by Solovay to the generalized situation.

Solovay's first order of business was to produce normality, in analogy to Scott's original work on measurability. We first need to establish some definitions so suppose I is an ideal over κ and $X \subseteq \kappa$. A function $f \in {}^\kappa\kappa$ is I-infinite on X iff $\{\xi \in X \mid f(\xi) = \alpha\} \in I$ for every $\alpha < \kappa$. A function $f \in {}^\kappa\kappa$ is incompressible on X iff f is I-infinite on X, yet whenever $g \in {}^\kappa\kappa$ and $\{\xi \in X \mid g(\xi) < f(\xi)\} \notin I$, then g is not I-infinite on X. By I being a normal ideal we mean, of course, the dual to the filter notion in §2, i.e. if $\{\xi < \kappa \mid f(\xi) < \xi\} \notin I$ then there is a $\gamma < \kappa$ so that $\{\xi < \kappa \mid f(\xi) = \gamma\} \notin I$.

Theorem: Suppose I is a κ^+-saturated ideal over κ. Then:

(a) Whenever $X \subseteq \kappa$ and $X \notin I$, there is an incompressible function on some $Y \subseteq X$ with $Y \notin I$.

(b) There is an incompressible function on κ.

(c) If f is an incompressible function on κ, then $f_*(I) = \{X \subseteq \kappa \mid f^{-1}(X) \in I\}$ is a normal κ^+-saturated ideal over κ.

⊢ We use Tarski's result on the completeness of the Boolean algebra $P(\kappa)/I$.

To show (a), argue by contradiction and assume: (*) whenever $Y \subseteq X$ with $Y \notin I$ and f is I-infinite on Y, there is an I-infinite g on Y so that $\{\xi \in Y \mid g(\xi) < f(\xi)\} \notin I$.

Let $f_0 \in {}^\kappa\kappa$ be any function I-infinite on X, like the identity function. By κ^+-saturation, let $\{A_\alpha \subseteq X \mid \alpha < \kappa\}$ be a maximal family such that: $A_\alpha \notin I$ but

$\alpha < \beta < \kappa$ implies $A_\alpha \cap A_\beta \in I$, and there is an I-infinite g_α on A_α so that $\{\xi \in A_\alpha \mid g_\alpha(\xi) < f_0(\xi)\} \notin I$ for every $\alpha < \kappa$. Then by assumption (*), we must have $\sum_{\alpha<\kappa} [A_\alpha] = [X]$.

As in Smith-Tarski, we can take the A_α's to be disjoint (else replace A_α by $A_\alpha - \bigcup_{\beta<\alpha} A_\beta$). Now set $f_1 = \bigcup_{\alpha<\kappa} g_\alpha | A_\alpha$ and extend f_1 arbitrarily to all of κ. It is not hard to see that f_1 is I-infinite on X: if to the contrary $\{\xi \in X \mid f_1(\xi) = \gamma\} \notin I$ for some $\gamma < \kappa$, since $[X] = \sum_{\alpha<\kappa} [A_\alpha]$, $\{\xi \in A_\alpha \mid g_\alpha(\xi) = f_1(\xi) = \gamma\} \notin I$ for some α—but this contradicts the I-infiniteness of g_α on A_α. Also, if $X_1 = \{\xi \in X \mid f_1(\xi) < f_0(\xi)\}$, then $[X_1] = \sum_{\alpha<\kappa} [A_\alpha] = [X]$.

We can now similarly define for every $n > 0$ an f_n infinite on X so that if $X_n = \{\xi \in X \mid f_n(\xi) < f_{n-1}(\xi)\}$, then $[X_n] = [X]$. But the rest is easy: we must then have $\bigcap_n X_n \notin I$ since $X - \bigcap_n X_n \in I$, yet any $\xi \in \bigcap_n X_n$ corresponds to a descending sequence $\ldots f_2(\xi) < f_1(\xi) < f_0(\xi)$. Thus, (*) was contradictory, and (a) is proved.

To show (b), by κ^+-saturation let $\{B_\alpha \mid \alpha < \kappa\}$ be a maximal family so that: $B_\alpha \notin I$ but $\alpha < \beta < \kappa$ implies $B_\alpha \cap B_\beta \in I$, and there is an f_α incompressible on A_α. By (a), we must have $\sum_{\alpha<\kappa} [B_\alpha] = [\kappa] = \mathbb{1}$. We can assume that the B_α's are disjoint just as in (a), and set $f = \bigcup_{\alpha<\kappa} f_\alpha | A_\alpha$. Any extension of f to all of κ is then an incompressible function on κ.

(c) is straightforward. ⊣

Thus, we can "normalize" κ^+-saturated ideals over κ, much like κ-complete ultrafilters over κ. Solovay showed that strengthening to κ-saturation opens the door to results about the large size of κ:

Theorem: Suppose I is a normal κ-saturated ideal over κ. Then:

(i) If $X \notin I$ and $\xi \in X$ implies $f(\xi) < \xi$, then there is a $Y \subseteq X$ with $[Y] = [X]$ and a $\gamma < \kappa$, so that $\xi \in Y$ implies $f(\xi) \le \gamma$.

(ii) $\{\alpha < \kappa \mid cf(\alpha) = \alpha\} \in I^*$.

(iii) If $X \in I^*$, then $M(X) \in I^*$, where M is Mahlo's operation: $M(X) = \{\alpha \in X \mid X \cap \alpha$ is stationary in $\alpha\}$.

⊢ For (i), let $A_\alpha = \{\xi \in X \mid f(\xi) = \alpha\}$ and consider $T = \{\alpha < \kappa \mid A_\alpha \notin I\}$. By κ-saturation $|T| < \kappa$, so $\gamma = \sup T < \kappa$. It is not hard to show by normality that $(X - \bigcup_{\alpha \in T} A_\alpha) \in I$, and hence $Y = \bigcup_{\alpha \in T} A_\alpha$ and γ are as desired.

For (ii), first recall that I^* is the filter dual to I, and in particular is normal. Assume to the contrary that $\{\alpha < \kappa \mid cf(\alpha) < \alpha\} \notin I$, so by normality for some $\gamma < \kappa$, $T = \{\alpha < \kappa \mid cf(\alpha) = \gamma\} \notin I$. For each $\alpha \in T$, let $\langle \delta^\alpha_\xi \mid \xi < \gamma \rangle$ be cofinal in α. We can then define regressive functions for $\xi < \gamma$ by $g_\xi(\alpha) = \delta^\alpha_\xi < \alpha$ for $\alpha \in T$. By (i), for each $\xi < \gamma$ there is a $T_\xi \subseteq T$ with $[T_\xi] = [T]$ so that $g_\xi``T_\xi$ is bounded, say by $\rho_\xi < \kappa$. Let $\rho = \sup\{\rho_\xi \mid \xi < \gamma\} < \kappa$. As usual by κ-completeness of I, $[\bigcap_{\xi<\gamma} T_\xi] = [T]$. But if $\alpha \in \bigcap_{\xi<\gamma} T_\xi$, then $\alpha \le \rho$, contradicting the non-triviality of I.

For (iii), assume to the contrary that $X \in I^*$ yet $W = \kappa - M(X) \notin I$. So, for each $\alpha \in W$ there is a C_α closed unbounded in α so that $C_\alpha \cap X = \emptyset$. For every $\xi < \kappa$, define $f_\xi \in {}^\kappa\kappa$ by:

$$f_\xi(\alpha) = \begin{cases} \xi\text{th element of } C_\alpha & \text{if this is possible,} \\ 0 \text{ otherwise.} \end{cases}$$

Then f_ξ is regressive on W, so by (i) let $W_\xi \subseteq W$ with $[W_\xi] = [W]$ so that f_ξ"W_ξ is bounded, say by $\gamma_\xi < \kappa$.

Set $\bar{W} = \{\alpha \in W \mid \xi < \alpha$ implies $f_\xi(\alpha) \le \gamma_\xi\}$. Then we claim: $[\bar{W}] = [W]$. Well, suppose not. Then $\{\alpha \in W \mid \exists h(\alpha) < \alpha$ with $f_{h(\alpha)}(\alpha) > \gamma_{h(\alpha)}\} \notin I$, so by normality $K = \{\alpha \in W \mid f_\xi(\alpha) > \gamma_\xi\} \notin I$ for some $\xi < \kappa$. However, $K \cap W_\xi = \emptyset$, and this contradicts $[W_\xi] = [W]$.

We now set $C = \{\alpha < \kappa \mid \alpha$ is a limit and $\xi < \alpha$ implies $\gamma_\xi < \alpha\}$. Since C is closed unbounded in κ and I^* is normal, $C \in I^*$. Finally, let $\beta \in X \cap C \in I^*$ and $\bar{\beta} \in \bar{W}$ with $\bar{\beta} > \beta$. By (ii), we can assume in addition that $\bar{\beta}$ is regular, and so in particular that $C_{\bar{\beta}}$ has order type $\bar{\beta}$. Claim: $\beta = f_\beta(\bar{\beta}) = \beta$th element of $C_{\bar{\beta}}$: First, $f_\beta(\bar{\beta}) \ge \beta$ is obvious. But also,

$$f_\beta(\bar{\beta}) = \sup_{\delta<\beta} f_\delta(\bar{\beta}) \le \sup_{\delta<\beta} \gamma_\delta \le \beta .$$

(Here, the equality follows since β is a limit ordinal, the first inequality, since $\bar{\beta} \in \bar{W}$, and the last, since $\beta \in C$.) Thus, the Claim is proved, but we now have a contradiction, as $\beta \in C_{\bar{\beta}} \cap X = \emptyset$. ⊣

The combination of (ii) and (iii) above shows that κ is weakly inaccessible, and must moreover be of very high order in the weak Mahlo hierarchy. κ is κ-weakly Mahlo, and using the normality of I, we can keep going a long way further in the hierarchy by diagonalization. (Of course, the same phenomenon obtains for κ-complete ultrafilters over a measurable cardinal κ, and in fact for the Π^1_1-indescribable filter (see the end of §4).) Since we shall see that possibly $\kappa \le 2^\omega$, this is about as much as we can ask.

It is to be remarked that just as Ulam[1930] was already able to establish combinatorially that a measurable cardinal is inaccessible, he showed that (in the present-day terminology) a κ carrying a κ-saturated ideal is weakly inaccessible. The stronger results that we now know ultimately owe their origin to the infusion of model theoretic techniques into set theory. It is a tribute to the fecundity of Ulam[1930] that the device introduced to prove the initial κ-saturated ideal result has become a standard tool in combinatorial set theory, nowadays known as Ulam matrices. This must surely be the first example of an idea introduced in the context of large cardinals readily seen to be of general applicability in set theory.

By weakening the saturation hypothesis, the possibility that there could be a κ^{++}-saturated ideal over κ^+ was conjectured for some time. Then Kunen(1974) established that if a very strong large cardinal hypothesis is consistent, then it is also consistent that ω_1 carry a ω_2-saturated ideal (see §17 for a discussion). The interest in this result lies in the possibility of such phenomena occurring in the low orders of the cumulative hierarchy, as it had already been known (Kunen [1970]) that the existence of such an ideal over ω_1 implied the consistency of the existence of many measurable cardinals.

What about partition properties? Strengthening even further to $<\kappa$ saturation, Solovay proved the following generalization of Rowbottom's theorem on normal ultrafilters: Suppose I is a λ-saturated ideal over κ where λ is a regular cardinal $< \kappa$. Then if $f: [\kappa]^{<\omega} \to \gamma$ where $\gamma < \kappa$, there is an $X \in I^*$ so that $|f"[X]^{<\omega}| < \lambda$.

⊢ Just as in Rowbottom's proof, it suffices to establish the above statement with "$<\omega$" replaced by "n" for every $n \in \omega$. So, proceed by induction:

If $n = 1$, then essentially $f: \kappa \to \gamma$ and if $T = \{\xi \mid f^{-1}(\{\xi\}) \notin I\}$, $|T| < \lambda$ by λ-saturation and clearly $\bigcup_{\xi \in T} f^{-1}(\{\xi\}) \in I^*$.

Suppose already known for n, and assume $f: [\kappa]^{n+1} \to \gamma$ where $\gamma < \kappa$. Define partitions $f_\alpha: [\kappa]^n \to \gamma$ for every $\alpha < \kappa$ by:

$$f_\alpha(s) = \begin{cases} f(\{\alpha\} \cup s) & \text{if } \alpha < \cap s, \\ 0 & \text{otherwise.} \end{cases}$$

By inductive hypothesis, let $H_\alpha \in I^*$ be such that $f_\alpha"[H_\alpha]^n = S_\alpha$ and $|S_\alpha| < \lambda$.

We now whittle down further. First, let $g: \kappa \to \lambda$ be given by $g(\alpha) = |S_\alpha|$. By the case $n = 1$, let $K \in I^*$ so that $|g"K| < \lambda$. As λ is assumed regular, $\sup(g"K) = \bar{\lambda} < \lambda$. For each $\alpha \in K$, let $s_\alpha: \bar{\lambda} \to S_\alpha$ be a surjection. Then define $g_\xi: K \to \gamma$ for every $\xi < \bar{\lambda}$ by $g_\xi(\alpha) = s_\alpha(\xi)$. Again, for every $\xi < \bar{\lambda}$ there are $K_\xi \subseteq K$ with $K_\xi \in I^*$ such that $|g_\xi"K_\xi| < \lambda$. By regularity of λ, if $E = \bigcup_{\xi<\bar{\lambda}} g_\xi"K_\xi$, $|E| < \lambda$.

Finally, take $X = \bigcap_{\xi<\bar{\lambda}} K_\xi \cap \Delta_{\alpha<\kappa} H_\alpha$. $X \in I^*$ by normality, so to finish the proof, it suffices to show $f"[X]^{n+1} \subseteq E$. So, suppose $t \in [X]^{n+1}$, written $t = \{\alpha\} \cup s$ where $\alpha < \cap s$. Then $f(t) = f_\alpha(s) \in S_\alpha$ as $s \in [H_\alpha]^n$, and as $\alpha \in \bigcap_{\xi<\bar{\lambda}} K_\xi$ we have $S_\alpha \subseteq E$. ⊣

Notice that if κ carries a λ-saturated ideal over κ for some $\lambda < \kappa$, by replacing λ by λ^+ to obtain regularity if necessary (as λ^+ is still $< \kappa$), the preceding theorem establishes that κ is at least Jonsson, and thus $0^\#$ exists (by §10). In fact, Solovay showed that in an inner model, κ is measurable. This was the first proof of any such result, though we now know stronger structural theorems along these lines (see §12).

A recent result of Prikry(1974) makes an interesting application of Solovay's partition theorem, of particular applicability to real-valued measurable cardinals: Suppose I is a λ-saturated ideal over 2^ω, where λ is regular and $\nu < \lambda$ implies $2^\nu = 2^\omega$. Then for every $\nu < 2^\omega$, $2^\nu = 2^\omega$.

⊢ We first remark that the non-triviality of I insures that 2^ω is regular. Argue now by contradiction, and let $\bar{\nu} < 2^\omega$ be the least cardinal such that $2^{\bar{\nu}} > 2^\omega$. In particular, $\bar{\nu} \geq \lambda$. Simple cardinal arithmetic insures that $\bar{\nu}$ must be regular.

We now employ a standard sort of device for enumerating initial segments of sets. For each $\alpha < \bar{\nu}$, let $f_\alpha : P(\alpha) \to 2^\omega$ be injections. Then for arbitrary $X \subseteq \bar{\nu}$, let $G_X : \bar{\nu} \to 2^\omega$ be defined by: $G_X(\alpha) = f_\alpha(X \cap \alpha)$. As 2^ω is regular, let $\rho_X < 2^\omega$ with $G_X``\bar{\nu} \subseteq \rho_X$. Surely there is a fixed $\rho < 2^\omega$ so that for some $S \subseteq P(\bar{\nu})$ with $|S| \geq 2^\omega$ we have $X \in S$ implies $\rho_X = \rho$.

Let us now define $F: [S]^2 \to \bar{\nu}$ by $F(\{X,Y\})$ = least α such that $G_X(\alpha) \neq G_Y(\alpha)$. By Solovay's partition theorem, there is an $H \subseteq S$ with $|H| = 2^\omega$ and $|F``[H]^2| < \lambda$. Let $\eta = \sup(F``[H]^2)$. As $\bar{\nu} \geq \lambda$ and $\bar{\nu}$ is regular, we have $\eta < \bar{\nu}$.

We can now derive a contradiction, using the injective aspects of the f_α's. Given any $X \neq Y \in H$, say $F(\{X,Y\}) = \beta$. Then by definition $X \cap \beta \neq Y \cap \beta$. But $\beta < \eta$, and so also $X \cap \eta \neq Y \cap \eta$. Thus, $G_X(\eta) \neq G_Y(\eta) < \rho$. (This is where we needed $\eta < \bar{\nu}$, to insure that η is in the domain of the G_X's.) We have just demonstrated that different X's in H give rise to different ordinals $G_X(\eta) < \rho$. Thus, $|H| = 2^\omega \leq \rho < 2^\omega$, a contradiction. ⊣

An inspection of this proof shows that we could have started with somewhat weaker hypotheses, but they become rather awkward to state. Though combinatorial, perhaps the peculiar interplay between cardinals and their powers needed for this result made it of relatively late vintage. We have as an immediate corollary: If 2^ω is real-valued measurable, then $\nu < 2^\omega$ implies $2^\nu = 2^\omega$. It is well-known that the same conclusion also follows from Martin's Axiom, MA(2^ω); this is interesting, since MA(2^ω) is known to contradict the real-valued measurability of 2^ω (see Kunen[1968] or §5 of Martin-Solovay[1970]).

We have emphasized in the introduction how aspects of the investigation of large cardinal concepts have led to clarification of standard set theoretical concepts. Perhaps the prime example of this is Solovay's solution of an old problem of Fodor on splitting stationary sets. To clarify the interplay of concepts, let us make some definitions for I an ideal over κ. If $A \subseteq \kappa$, $I|A = \{X \subseteq \kappa | X \cap A \in I\}$, the ideal generated by $I \cup \{\kappa - A\}$. I is an M-ideal iff whenever $X \in I^*$, so is $\{\alpha < \kappa | cf(\alpha) > \omega \ \& \ X \cap \alpha$ is stationary in $\alpha\} \in I^*$, i.e. I^* is "closed" under Mahlo's operation. ("$cf(\alpha) > \omega$" is thrown in to avoid trivialities.) By a previous theorem, if I is a normal κ-saturated ideal over κ, then I is an M-ideal. Finally, NS_κ denotes the normal ideal of non-stationary subsets of κ, the ideal dual to the closed unbounded filter over κ. Now for Solovay's result: If κ is a regular uncountable cardinal and $A \subseteq \kappa$ is stationary, then A can be split into κ disjoint stationary sets.

⊢ Assume that this is not true for some stationary A. Then $NS_\kappa|A$ is a (non-trivial) normal ideal over κ, as these properties are inherited from NS_κ. But also, by hypothesis on A, $NS_\kappa|A$ is seen to be κ-saturated. To obtain a contradiction, it suffices to establish:

If $I = NS_\kappa|S$ for some $S \notin I$, then I is not an M-ideal: Assume to the contrary that I is an M-ideal. Since $S \in I^*$, it then follows that $M(S) \in I^*$, i.e. for some C closed unbounded in κ, we have $C \cap S \subseteq M(S)$. Let $\bar{C}$ be the set of limit points of C other than κ, so that $\bar{C}$ is also closed unbounded in κ. Let α be the least element of $S \cap \bar{C}$. Thus, $C \cap \alpha$ must be closed unbounded in α. Also $\alpha \in M(S)$, so an easy limit argument using $cf(\alpha) > \omega$ shows that $\bar{C} \cap \alpha$ is also closed unbounded in α. However, $(\bar{C} \cap \alpha) \cap S \neq \emptyset$ as $S \cap \alpha$ is stationary in α, contradicting the leastness of α. ⊣

It is true that one can give the proof of Solovay's theorem without first developing the theory of saturated ideals, and ending up with a specific splitting of the stationary set. However, there is no denying the historical context in which the theorem was first proved. Solovay's result has become a standard tool in set theory for the study of regular uncountable cardinals. It is a good example of how results on one plane can be proved by mathematical contemplation at a higher plane. Recent uses of it in foundational studies are seen in the work of Kueker [1977] and Barwise-Kaufmann-Makkai[1978] in stationary logic.

So, NS_κ cannot be κ-saturated for regular uncountable κ. An entirely basic combinatorial question soon arose as to whether NS_κ can ever be κ^+-saturated. Those who know $\diamondsuit_\kappa$ will immediately see that it implies that NS_κ is not 2^κ-saturated. Recent work of Baumgartner-Taylor-Wagon[1977] sheds some light on the question. They established: Suppose I is a normal ideal over κ. Then I is κ^+-saturated iff the normal ideals extending I are exactly those of form $I|A$ for some $A \notin I$.

⊢ Suppose first that I is κ^+-saturated, and let $J \supseteq I$ be any normal extension. Let $S \subseteq P(\kappa)$ be maximal with respect to: if $A \neq B$ both in S, then $A \in J - I$ and $A \cap B \in I$. By κ^+-saturation, $|S| \leq \kappa$, so write $S = \{A_\alpha | \alpha < \kappa\}$. Now $\kappa - A_\alpha \in I^*$ for each $\alpha < \kappa$, so $A = \Delta_{\alpha<\kappa}(\kappa - A_\alpha) \in I^*$ by normality. The maximality of S can now be used to show that $I|A = J$. For example, if $X \in J$, then $(X \cap A) \cap A_\alpha \subseteq A \cap A_\alpha \subseteq \alpha+1 \in I$ for every $\alpha < \kappa$, so by maximality of S, $X \cap A \in I$, i.e. $X \in I|A$.

For the converse, suppose that I is not κ^+-saturated, and let $T \subseteq P(\kappa) - I$ be maximal with respect to: $A \neq B$ both in T implies $A \cap B \in I$. We can suppose $|T| \geq \kappa^+$. Define J by: $X \in J$ iff $X \subseteq \kappa$ & $|\{A \in T | X \cap A \notin I\}| \leq \kappa$. Then $J \supseteq I \cup T$.

J must also be normal: Suppose $Y = \{\alpha < \kappa | f(\alpha) < \alpha\} \notin J$. Setting $K = \{A \in T | Y \cap A \notin I\}$, we have $|K| > \kappa$. For any $A \in K$, by normality of I there is a $\gamma_A < \kappa$ so that $A \cap f^{-1}(\{\gamma_A\}) \notin I$. Then for some $\bar{K} \subseteq K$ with $|\bar{K}| > \kappa$ and a fixed $\gamma < \kappa$, we have $A \in \bar{K}$ implies $\gamma = \gamma_A$. This demonstrates that $f^{-1}\{\{\gamma\}\} \notin J$.

Finally, observe that if $\bar{A} \notin I$, we have $J \neq I|\bar{A}$: This is so since by maximality of T, there is an $A \in T$ so that $\bar{A} \cap A \notin I$. But then $A \in J$, yet $A \notin I|\bar{A}$ by definition. $\dashv$

Notice that the proof of Solovay's theorem showed in particular that no M-ideal is of form $NS_\kappa|A$ for any stationary $A \subseteq \kappa$. Baumgartner, Taylor, and Wagon were led to define a cardinal κ to be greatly Mahlo iff κ carries a normal M-ideal. (See also Glöede[1973].) Thus, the above theorem has the corollary: If κ is greatly Mahlo, then NS_κ is not κ^+-saturated. Weakly compact cardinals are greatly Mahlo, since the Π^1_1-indescribable filter is dual to an M-ideal, and the greatly Mahlo cardinals below a weakly compact cardinal in fact comprise a stationary subset. However, greatly Mahloness is inevitably a large cardinal property, and so we still know very little about the possible κ^+-saturation of NS_κ for accessible κ. Although we had mentioned that Kunen(1974) proved the consistency of the existence of a normal ω_2-saturated ideal over ω_1 from a very strong large cardinal assumption, his ideal was not NS_{ω_1}. That strong hypotheses which impose some distinctive character to weakly inaccessible cardinals decides basic combinatorial questions about them for which comparable facts are virtually unknown for successor cardinals is a noteworthy circumstance.

For recent work on the combinatorial theory of ideals, we refer to: Baumgartner-Hajnal-Maté[1975](for results on another question of Fodor: if $\{S_\alpha | \alpha < \kappa\}$ are stationary subsets of a regular uncountable κ, are there stationary $A_\alpha \subseteq S_\alpha$ so that $\alpha \neq \beta$ implies that $A_\alpha \cap A_\beta = \emptyset$?); Taylor(1977)(for a general structural approach to Fodor's question making important connections to the theory of regularity of ultrafilters); Jech-Prikry(1977)(for applications to powers of cardinals); and a forthcoming paper of Baumgartner, Taylor, and Wagon on a theory of (non-trivial) ideals and partitions.

We conclude this section with an early relative consistency result on saturated ideals, due to Prikry[1970]. First, the following lemma of Silver is of independent interest: Lemma: If I is a λ-saturated ideal over κ and λ is a regular cardinal $< \kappa$, then given $\{X_\alpha | \alpha < \lambda\}$ subsets of κ of I-positive measure, there is a $K \subseteq \lambda$ with $|K| = \lambda$ so that $\bigcap_{\alpha \in K} X_\alpha \neq \emptyset$.

$\vdash$ Argue by contradiction. Then for each $\delta < \kappa$, we have $s_\delta = \sup\{\alpha < \lambda | \delta \in X_\alpha\} < \lambda$. If we then set $E_\xi = \{\delta | s_\delta = \xi\}$ for each $\xi < \lambda$, the E_ξ's partition κ into λ parts. Thus if $T = \{\xi < \lambda | E_\xi \notin I\}$, then $|T| < \lambda$ by λ-saturation, and the usual argument establishes $\bigcup_{\xi \in T} E_\xi \in I^*$. However, choose any $\bar{\xi}$ so that $\sup T < \bar{\xi} < \lambda$. Then $(X_{\bar{\xi}} \cap \bigcup_{\xi \in T} E_\xi)$ must have I-positive measure, yet it is empty--a contradiction. $\dashv$

Prikry simply noticed that: If I is a λ-saturated ideal over κ with λ a regular cardinal $< \kappa$, and if P is a λ-c.c. notion of forcing, then in any generic extension through P, I generates a (non-trivial) λ-saturated ideal:

$\vdash$ In any generic extension $V[G]$, I generates the ideal $\bar{I} = \{X \in V[G] \cap P(\kappa) | X \subseteq Y$ for some $Y \in I\}$. We want to establish its non-triviality and λ-saturation.

For non-triviality, we need only check κ-completeness, and the argument is rather standard: Suppose $\gamma < \kappa$ and $p \Vdash \tau: \gamma \to \check{I}$. For each $\alpha < \gamma$ let Q_α be a maximal family of mutually incompatible conditions so that for each $q \in Q_\alpha$: (a) q extends p, and (b) there is an $X^q_\alpha \in I$ such that $q \Vdash \tau(\alpha) = \check{X}^q_\alpha$. Since $|Q_\alpha| < \lambda < \kappa$ by the λ-c.c. of P, we have $Y_\alpha = \bigcup\{X^q_\alpha | q \in Q_\alpha\} \in I$. Thus also $Y = \bigcup_{\alpha<\gamma} Y_\alpha \in I$ and it is now straightforward to establish $p \Vdash \bigcup \tau''\gamma \subseteq \check{Y}$.

For λ-saturation, suppose $p \Vdash (\tau: \lambda \to \dot{P}(\kappa)$ & Range(τ) consists of pairwise disjoint sets). For each $\alpha < \lambda$, set $Z_\alpha = \{\xi | q \Vdash \xi \in \tau(\alpha)$, for some q extending $p\}$. Then it is clear that $p \Vdash \tau(\alpha) \subseteq Z_\alpha$. If it is true that for a $\alpha < \lambda$, $Z_\alpha \in I$, then we are done. Thus, assume by way of contradiction that $Z_\alpha \notin I$ for every $\alpha < \lambda$. By Silver's Lemma, there is a $K \subseteq \lambda$ with $|K| = \lambda$ such that $\eta \in \bigcap_{\alpha \in K} Z_\alpha$ for some η. However, by definition of Z_α, there are q_α extending p for each $\alpha \in K$, so that $q_\alpha \Vdash \eta \in \tau(\alpha)$. Clearly $\{q_\alpha | \alpha \in S\}$ consitute a collection of λ mutually incompatible conditions, a contradiction of the λ-c.c. $\dashv$

The general form of Prikry's result can be used to advantage. For example, if we start with a measurable cardinal κ, and force with the usual ω_1-c.c. notion of forcing for adding κ subsets of ω, we get a model M where there is a ω_1-saturated ideal over (the new) 2^ω. We can also perform a further ω_1-extension to arrange Martin's Axiom, $MA(2^\omega)$, to hold. (For this last extension, the usual way of forcing $MA(2^\omega)$ needs that M satisfies: 2^ω is regular and whenever $\nu < 2^\omega$, $2^\nu \le 2^\omega$. But this is true in M either by the initial forcing construction for getting M, or by a Prikry result presented earlier.) Thus, Con(ZFC & there is a measurable cardinal) implies Con(ZFC & $MA(2^\omega)$ & 2^ω carries a ω_1-saturated ideal). The point about including $MA(2^\omega)$ is that it is known to contradict the real-valued measurability of 2^ω (Kunen[1968]). Hence, to get 2^ω to be real-valued measurable, some care is needed in the forcing construction (see §24).

Recently, Laver has investigated ideals with stronger saturation properties. If I is a (non-trivial) ideal over κ, say that I is (λ,μ,ν)-saturated iff whenever $S \subseteq P(\kappa)$ is a collection of λ subsets of I-positive measure, there is an $\bar{S} \subseteq S$ with $|\bar{S}| = \mu$ so that: whenever $T \subseteq \bar{S}$ and $|T| \le \nu$, then $\bigcap T$ still has I-positive measure. Laver(1976) established for example that if a measurable cardinal exists, there is a forcing extension in which there is a $(2^\omega,2^\omega,\omega)$-saturated ideal over 2^ω. Laver goes on to show that the existence of such an ideal implies (for those who know the terminology) that $2^\omega \to (2^\omega,\alpha)^2_2$ for every $\alpha < \omega_1$. This was formerly known to follow from the real-valued measurability of 2^ω (Kunen) but Laver's construction is generally applicable, and works to get, for instance, $(2^{\omega_1},2^{\omega_1},\omega_1)$-saturated ideals over 2^{ω_1} and $2^{\omega_1} \to (2^{\omega_1},\alpha)^2_2$ for every $\alpha < \omega_2$.

The next section is devoted to producing a strong converse to Prikry's relative consistency result; we shall see that espousing certain ideals necessitates having faith in measurability.

§12. Precipitous Ideals

We bring together in this section several ingredients which will then produce inner models of measurability from ideals with sufficiently strong properties. The first example of such a possibility was the initial result of Solovay proved in the mid-sixties (see Solovay[1971]), to which Prikry's relative consistency result of the previous section was complementary: If $\lambda < \kappa$ and I is a normal λ-saturated ideal over κ, then κ is measurable in $L[I]$. Solovay's proof did not provide much information, and Kunen[1970] considerably improved the result by using iterated ultrapowers in conjunction with another idea of Solovay to establish: If I is a κ^+-saturated normal ideal over κ then in $L[I]$, $I \cap L[I]$ is maximal and hence dual to a normal κ-complete ultrafilter over κ. Finally, Jech and Prikry isolated a property of ideals even weaker than κ^+-saturation which seems just what is necessary to carry out Kunen's argument. This property precipitated during the spate of activity spurred by Silver's result in 1974 on the Singular Cardinals Problem (see §29). The property turns out to have rather neat combinatorial characterizations, and moreover can even be satisfied by the ideal of non-stationary subsets of ω_1. This section deals with this whole development.

First, it is a significant remark that our treatment in §8 of Kunen's technique of iterated ultrapowers was actually a special case of his general scheme. Kunen saw that to take iterated ultrapowers of an inner model M via an ultrapower U, it was not even necessary to assume that $U \in M$, as long as an essential iterability condition was imposed on U, tying it closely to M. This was an important technical generalization, and the insights gained in this richer formalism ultimately led to the results of §10 as well as of this section.

Following Kunen, if M is an inner model of ZF, call U an M-ultrafilter over κ iff U is an ultrafilter on the Boolean algebra $P(\kappa) \cap M$ so that: (a) U is M-κ-complete, i.e. whenever $\gamma < \kappa$ and $\langle X_\alpha \mid \alpha < \gamma \rangle \in M$ so that each $X_\alpha \in U$, then $\bigcap_{\alpha<\gamma} X_\alpha \in U$; and (b) U is M-iterable, i.e. whenever $\langle Y_\alpha \mid \alpha < \kappa \rangle \in M$, then $\{\alpha \mid Y_\alpha \in U\} \in M$. Thus, it is not assumed that $U \subset M$. κ may not be a cardinal (in V), but it will be the case that $M \models \kappa$ is weakly compact. (This can be established by showing $\kappa \to (\kappa)^2_2$ in M, using conditions (a) and (b) and imitating a standard ultrafilter proof of Ramsey's Theorem.) There is a sort of converse: If $P(\kappa) \cap M$ is countable and $M \models \kappa$ is weakly compact, then there is an M-ultrafilter over κ. (Construct an M-ultrafilter of form $U = \bigcup_n F_n$ where each $F_n \in M$ is, in M, a κ-complete filter generated by κ sets, and $m < n$ implies $F_m \subseteq F_n$. Since there are only countably many "conditions" to be met, an adroit use of weak compactness in M will provide an inductive construction of the

F_n's which will assure that U satisfies (a) and (b).) The point is merely that the existence of an M-ultrafilter is consistency-wise much weaker than measurability.

With any ultrafilter on $P(\kappa) \cap M$, we can construct an ultrapower of M by using only those $f: \kappa \to M$ so that $f \in M$. The condition (b) is just what is needed to carry out the iteration of ultrapowers. The point is that for any M-ultrafilter U over κ, the product ultrafilters U^n can be defined: For U^2, we would like typically $X \in U^2$ iff $X \in P(\kappa \times \kappa) \cap M$ & $T_X = \{\alpha \mid \{\beta \mid \langle \alpha,\beta \rangle \in X\} \in U\} \in U$, but for this we need $T_X \in M$, something that is assured by M-iterability. It can now be checked that U^2 also satisfies the M-iterability condition, and so we can define $U^3 \subseteq P(\kappa^3) \cap M$ such that $U^3 = U \times U^2$ in the appropriate sense, and so on to get every U^n. The entire construction in §8 can now be carried out with appropriate modifications. (Indeed, this was the original formulation of Kunen[1970].) Set:

(v)' $Fn_\alpha(M,\kappa) = \{f \in Fn_\alpha(\kappa) \mid f$ has a finite support y so that $f_y \in M\}$.

(vi)' $P_\alpha(M,\kappa) = \{X \in P_\alpha(\kappa) \mid X$ has a finite support y so that $X_y \in M\}$.

(vii)' $U_\alpha = \{X \in P_\alpha(M,\kappa) \mid X$ has a finite support y with $|y| = n$, so that $X_y \in U^n\}$.

These can now be used for the definitions of the iterated ultrapowers $\mathrm{Ult}_\alpha(M,U)$ and embeddings $i_{\alpha\beta}$. When $U \notin M$, these definitions can no longer be carried out within M, but in any case structural facts like 8.4. still hold. The isomorphism result 8.5. must be recast somewhat; since we do not assume $U \in M$, $i_{0\alpha}(U)$ no longer makes any sense. We can however define the appropriate "outside" version: Say $X \in U^{(\alpha)}$ iff $X \in P(i_{0\alpha}(\kappa)) \cap \mathrm{Ult}_\alpha(M,U)$ & if $X = [f]_\alpha$, then $\{s \in {}^\alpha\kappa \mid f(s) \in U\} \in U_\alpha$. $U^{(\alpha)}$ turns out to be a $\mathrm{Ult}_\alpha(M,U)$-ultrafilter over $i_{0\alpha}(\kappa)$, and 8.5. goes through with $i_{0\alpha}(U)$ replaced by $U^{(\alpha)}$. When all the $\mathrm{Ult}_\alpha(M,U)$ are well-founded, 8.8. (except for the inappropriate (v) and (vi)) holds as well as 8.9. (with cardinalities and cofinalities calculated in V only).

What about well-foundedness? Condition (a) in the definition of M-ultrafilter is too weak to guarantee the well-foundedness of ultrapowers, since there could be infinite descending sequences outside of M. It turns out that to impose on an M-ultrafilter the additional condition that all its iterated ultrapowers are to be well-founded is to bolster its consistency strength from that of weak compactness to something at least on the order of the existence of $0^\#$. Indeed, Kunen had an initial characterization of the existence of $0^\#$ as equivalent to the existence of an L-ultrafilter U so that all the $\mathrm{Ult}_\alpha(L,U)$ are well-founded. Later, it was realized that only the first ultrapower $\mathrm{Ult}_1(L,U)$ need be well-founded because of the uniformity of L; the characterization of $0^\#$ in §10 can be construed as a restatement of this fact. It is interesting that, reminiscent of 7.4., Gaifman had essentially already established that if U is an M-ultrafilter, then $\mathrm{Ult}_\alpha(M,U)$ is well-founded for all $\alpha < \omega_1$ iff $\mathrm{Ult}_\alpha(M,U)$ is well-founded for all α. In practice, however, the following sufficient condition is more useful (this attests to a recurring phenomenon in mathematics: one often proves much more than one needs to prove!): If an M-ultrafilter U has the property that arbitrary countable intersections of its elements are non-empty, then all the $Ult_\alpha(M,U)$ are well-founded. (The proof proceeds much as for 8.6.)

The second ingredient for our scheme is the notion of a generic ultrapower, first used significantly in Solovay[1971]. (With the emergence of the two powerful techniques of forcing and ultrapowers, it was perhaps inevitable that they should be amlgamated!) Musing over saturated ideals, with the paradigm of measurability in mind Solovay wanted to construct some sort of ultrapower to establish strong reflection properties. He devised the following scheme: given an ideal I, one can extend (the dual of) I generically to an ultrafilter, which can then be used to take an ultrapower of the ground model. Hopefully, this process would carry the imprint of strong properties initially imposed on I.

Let us attend to the details. (The whole technique is given general exposition in the recent Jech-Prikry(1977), to which we refer.) Fix a (non-trivial) ideal I over a cardinal κ. To I corresponds a natural notion of forcing, which is essentially the Boolean algebra $P(\kappa)/I$ minus the zero element, construed as a partially ordered set: Define $R(I)$ to be the notion of forcing with subsets of κ of I-positive measure as conditions, with the specification: X is a stronger condition than Y iff $[X] \leq [Y]$ (iff $X - Y \in I$). Suppose now that G is an $R(I)$-generic filter over V, considered in this context as the ground model. Then in $V[G]$, G is a (non-principal) V-κ-complete ultrafilter on $P(\kappa) \cap V$ such that $G \cap I = \emptyset$.

$\vdash$ For instance, to establish V-κ-completeness it suffices to show that if $\gamma < \kappa$ and $\kappa = \bigcup_{\alpha<\gamma} X_\alpha$ is a partition of κ so that $\langle X_\alpha \mid \alpha < \gamma \rangle \in V$, then there is an α so that $X_\alpha \in G$. However, because I is κ-complete, $D = \{Y \in P(\kappa) - I \mid Y \subseteq X_\alpha$ for some $\alpha < \gamma\}$ is dense in $R(I)$, and so the V-genericity of G assures the result. The other clauses are evident. $\dashv$

It may be that $G \in V$; for example, I could be a maximal ideal in which case G is just the dual ultrafilter. However, the case that really concerns us is when I does not have an atom, else we know (from §11) that κ would already be measurable in V. Indeed, applying a basic precept of forcing, when I is atomless, $G \notin V$ (otherwise, $(P(\kappa) - I) - G$ would be dense in $R(I)$, a contradiction).

Let us continue. In $V[G]$, we can construct an ultrafilter of V with G, using only those $f: \kappa \to V$ so that $f \in V$. Let us call this (possibly ill-founded) ultrapower the generic ultrapower, and extend our notation $Ult_1(V,G)$ to denote it, even though it is not clear how to iterate ultrapowers as G may not satisfy the V-iterability condition.

There is a fine point that should perhaps be made explicit. We intend that

the construction of $Ult_1(V,G)$ be carried out within $V[G]$, but it is not generally true that the ground model is even definable in a generic extension. However, there is a familiar device available: one simply adds a new name $\check{V}$ to the forcing language, specifying that for any term τ and condition p, $p \Vdash \tau \in \check{V}$ iff for any q stronger than p there is an r stronger than q and an $x \in V$ so that $r \Vdash \tau = \check{x}$. (In Boolean terms, $[\![\tau \in \check{V}]\!] = \Sigma\{[\![\tau = \check{x}]\!] \mid x \in V\}$.) Thus, we forcibly make the ground model V a class in generic extensions, and so in our particular case $Ult_1(V,G)$ can be defined from V and G entirely within $V[G]$.

As always, the crucial question with respect to the possible infusion of large cardinal techniques is whether the ultrapower is well-founded. Solovay's basic realization was that: If I is a κ^+-saturated ideal over κ, then we have that $\kappa \Vdash_{R(I)} Ult_1(\check{V},\underline{G})$ is well-founded. (Here, as always, $\underline{G}$ is the canonical name in the forcing language for the generic filter.)

$\vdash$ Since κ is the weakest condition in $R(I)$, the conclusion is just saying that in any generic extension the generic ultrapower is well-founded. Let us argue by contradiction and assume for some condition X that $X \Vdash$ "$\langle [\tau_n] \mid n \in \omega \rangle$ is an ϵ-descending sequence in $Ult_1(\check{V},\underline{G})$". In particular, for each $n \in \omega$, $X \Vdash \tau_n \in \check{V}$. By the formal definition of $\check{V}$ we can suppose that for each $n \in \omega$ there is a set W_n of conditions all stronger than X, maximal with respect to: (a) for each $Y \in W_n$ there is a function $f_n^Y \in V$ such that $Y \Vdash \tau_n = \check{f}_n^Y$; and (b) $Y \cap Z \in I$ for distinct $Y, Z \in W_n$, i.e. W_n consists of pairwise incompatible conditions.

By κ^+-saturation, we must have $|W_n| \leq \kappa$ for each $n \in \omega$. By a familiar strategem from §11, we can then suppose that the elements of each W_n are pairwise disjoint. Let us define a function $f_n \in V$ with domain κ by: $f_n = f_n^Y$ on $Y \in W_n$ and arbitrary elsewhere. Then surely $X \Vdash \tau_n = \check{f}_n$, for every $n \in \omega$, by a typical forcing argument using the maximality of W_n.

Set $T_n = \{\alpha < \kappa \mid f_{n+1}(\alpha) < f_n(\alpha)\}$ for each $n \in \omega$. Then by assumption $X \Vdash \check{T}_n \in \underline{G}$ for every $n \in \omega$, which by definition of generic object means that X is a stronger condition than T_n, i.e. $X - T_n \in I$. Thus, $X \cap \bigcap_n T_n$ must still have I-positive measure, but any element in this set gives rise to an infinite ϵ-descending sequence, and hence a contradiction. $\dashv$

It was this result that Kunen used to get inner models of measurability. Jech and Prikry observed that simply the conclusion of Solovay's result is enough to push through a variation of Kunen's argument. Thus, they designated: a (non-trivial) ideal I over κ is precipitous iff $\kappa \Vdash_{R(I)} Ult_1(\check{V},\underline{G})$ is well-founded. One feature of Kunen's argument was that he started with a normalized κ^+-saturated ideal I over κ, and indeed shows that in $L[I]$ the dual to $I \cap L[I]$ is a normal ultrafilter over κ. It is not known that if there is a precipitous ideal, then there is a normal one. Jech and Prikry realized that κ can still be proved to be measurable in an inner model, and we shall see how the proof can incorporate Kunen's version.

Precipitous ideals thus precipitated out of an examination of method. The concept is best defined in terms of its original meta-mathematical motivation, but once isolated, it was seen to have some rather neat combinatorial characterizations. For one, whenever I is an ideal over κ and $S \subseteq \kappa$ is of I-positive measure, say that W is an *I-partition of S iff* $W \subseteq P(S) - I$ is maximal with respect to: $Y \cap Z \in I$ for distinct $Y, Z \in W$. For two I-partitions W and $\bar{W}$ of S, write $W \le \bar{W}$ *iff* any member of W is a subset of some member of $\bar{W}$. Then *I is precipitous iff whenever $S \subseteq \kappa$ is of I-positive measure and $W_0 \ge W_1 \ge W_2 \cdots$ are I-partitions of S, there exist $X_n \in W_n$ for every $n \in \omega$ so that $X_0 \supseteq X_1 \supseteq \cdots$ and $\bigcap_n X_n \ne \emptyset$.* Actually, there is a game theoretic characterization which in some sense is more natural than this one—see §27.

We shall now proceed to establish through several steps the main result of this section (see Jech-Magidor-Mitchell-Prikry(1977)): *If there is a precipitous ideal I over κ, then κ is measurable in an inner model.*

$\vdash$ First, let $K = \{\nu \mid \nu$ is a strong limit cardinal with $cf(\nu) > \kappa\}$. The salient property of the class K is that for any inner model M and U an M-ultrafilter over κ all of whose iterated ultrapowers of M are well founded, the corresponding embeddings have the following properties: (i) $i_{0\nu}(\kappa) = \nu$ for every $\nu \in K$, and (ii) $i_{0\gamma}(\nu) = \nu$ for any $\nu \in K$ and $\gamma < \nu$. These facts follow from 8.9.(ii)(iv), remembering our earlier remarks about their generalization to M-ultrafilters. Fix for the duration of the proof an increasing sequence $a = \langle \lambda_n \mid n \in \omega \rangle$ of cardinals $\lambda_n \in K$ so that $|K \cap \lambda_n| = \lambda_n$, and set $\lambda = \sup \lambda_n$. We first show that $L[a]$ is uniform enough that:

Step 1. There is an $L[a]$-ultrafilter U over κ so that all the ultrapowers $Ult_\alpha(L[a], U)$ are well-founded.

Let $id: \kappa \to \kappa$ be as usual the identity function on κ. By precipitousness, there is some S of I-positive measure and an ordinal γ so that $S \Vdash [\check{id}] = \gamma$ in (the well-founded) $Ult_1(\check{V}, \underset{\sim}{G})$. Set

$$U = \{X \in P(\kappa) \cap L[a] \mid X \cap S \notin I\}.$$

We want to establish that U is an ultrafilter on $P(\kappa) \cap L[a]$, and so we *Claim: whenever $X \in P(\kappa) \cap L[a]$, either $X \cap S \in I$ or $S - X \in I$.* This is the main idea of the proof. By precipitousness, we can already procure enough knowledge of a well-founded ultrapower in one condition S so that S becomes an atom for I, at least for sets in a stable inner model.

To establish the Claim, we first show that every $X \in P(\kappa) \cap L[a]$ is definable in $L[a]$ from a finite subset of $\kappa \cup K \cup \{a\}$. This is just like the Definability Lemma of §9: Let A be the elements of $L[a]$ so definable, so that $A \prec L[a]$. If i is the transitizing isomorphism, since $|K \cap \lambda_n| = \lambda_n$ for every n, $i(a) = a$ so that $i: A \cong L[a]$. But $\kappa \subseteq A$, so i is the identity on κ. Hence $i(X) = X$

for $X \in P(\kappa) \cap L[a]$, so that $X \in A$, which is what we wanted to show.

Let us proceed to establish the Claim. Suppose that $X \in P(\kappa) \cap L[a]$ so that $X \cap S \notin I$. We want to show that $S - X \in I$. Let $j_{\underset{\sim}{G}}$ be the name in the forcing language for the natural embedding of V into the generic ultrapower. Since $(X \cap S) \Vdash \check{X} \in \underset{\sim}{G}$, by Łoś' Theorem $(X \cap S) \Vdash \gamma = [\check{id}] \in j_{\underset{\sim}{G}}(\check{X})$. By the previous paragraph, there is formula ϕ and a finite $E \subseteq \kappa \cup K$ so that $X = \{\xi < \kappa \mid L[a] \models \phi(\xi, E, a)\}$. But any condition forces $j_{\underset{\sim}{G}}(\check{E}) = \check{E}$ and $j_{\underset{\sim}{G}}(\check{a}) = \check{a}$ by 8.9.(ii) and the definition of K, so:

$$(X \cap S) \Vdash (L[\check{a}] \models \phi(\gamma, \check{E}, \check{a})).$$

But the statement forced here is a standard one of V, which in fact is absolute between models containing a. Thus, $L[a] \models \phi(\gamma, E, a)$, and therefore any condition forces $\gamma \in j_{\underset{\sim}{G}}(\check{X})$. Hence, $S \Vdash [\check{id}] = \gamma \in j_{\underset{\sim}{G}}(\check{X})$ so that $S \Vdash \check{X} \in \underset{\sim}{G}$. This just means S is a stronger condition than X, i.e. $S - X \in I$.

We have thus established that U is an ultrafilter on $P(\kappa) \cap L[a]$. That U is $L[a]$-κ-complete is immediate, as I is κ-complete. Let us next turn to the $L[a]$-iterability condition. If $F: \kappa \to L[a]$, we want to check that $\{\alpha < \kappa \mid F(\alpha) \in U\} \in L[a]$. But by our work above, $S \Vdash \{\alpha < \kappa \mid \check{F}(\alpha) \in \check{U}\} = \{\alpha < \kappa \mid \gamma \in j_{\underset{\sim}{G}}(\check{F}(\alpha))\}$. However, any condition forces: $\{\alpha < \kappa \mid \gamma \in j_{\underset{\sim}{G}}(\check{F}(\alpha))\} = \{\alpha \mid \gamma \in j_{\underset{\sim}{G}}(\check{F})(\alpha)\} \cap \kappa$ & $j_{\underset{\sim}{G}}(\check{F}) \in L[j_{\underset{\sim}{G}}(\check{a})] = L[\check{a}]$. Thus,

$$S \Vdash \{\alpha < \kappa \mid \check{F}(\alpha) \in \check{U}\} \in L[\check{a}].$$

By absoluteness, this just means that the forced statement holds in V. We have thus established that U is an $L[a]$-ultrafilter.

The preceding argument is due to Kunen, but augmented by Jech and Prikry in the relativization to a condition S. The remaining steps are due entirely to Kunen. Let F be the filter over λ generated from the sequence a, i.e. $X \in F$ *iff* $X \subseteq \lambda$ & $\exists n \forall m > n (\lambda_m \in X)$. Hence, $L[F] \subseteq L[a]$. Properties of the U found above are used to establish:

Step 2. F is a strong filter, i.e. in $L[F]$, $\bar{F} = F \cap L[F]$ is a normal ultrafilter over λ, and hence $L[F]$ is a λ-model.

For any $\alpha \le \beta$, let $i_{\alpha\beta}: Ult_\alpha(L[a], U) \to Ult_\beta(L[a], U)$ be the usual embedding. We first establish some facts about $i_{0\alpha}$ when $\alpha < \lambda$. It is then the case that $i_{0\alpha}(\lambda_n) = \lambda_n$ for every n so that $\alpha < \lambda_n$, as $\lambda_n \in K$. The point is then that the filter F is essentially preserved by $i_{0\alpha}$: whenever X is a member of $P(\lambda) \cap L[a] \cap Ult_\alpha(L[a], U)$, then $X \in F \cap L[a]$ *iff* $X \in i_{0\alpha}(F \cap L[a])$. Thus, the middle equality in the following sequence can be checked level by level in the usual generation of a relatively constructible universe:

$$i_{0\alpha}(L[F]) = i_{0\alpha}(L[F \cap L[a]]) = L[i_{0\alpha}(F \cap L[a])]^{Ult_\alpha(L[a],U)} = L[F \cap L[a]]^{L[a]} = L[F]$$

Secondly, notice that whenever $n < m < \omega$, since $\lambda_n, \lambda_m \in K$, we have: (a) $\lambda_n = i_{0\lambda_n}(\kappa)$ is the critical point of $i_{\lambda_n\lambda_m}$ by 8.8.(i); and (b) $i_{\lambda_n\lambda_m}(\lambda_n) = i_{\lambda_n\lambda_m}(i_{0\lambda_n}(\kappa)) = i_{0\lambda_m}(\kappa) = \lambda_m$.

We can now use these facts to establish that in $L[F]$, $\bar{F}$ is a normal ultrafilter over λ. For example, to establish normality assume to the contrary that $\{\alpha < \lambda \mid f(\alpha) < \alpha\} \in \bar{F}$ yet $\{\alpha < \lambda \mid f(\alpha) = \eta\} \notin \bar{F}$ for any $\eta < \lambda$, and $f \in L[F]$ is the least (in the canonical well-ordering of $L[F]$) with this property. Since f is thus definable in $L[F]$, the important point is that $i_{\alpha\beta}(f) = f$ whenever $\alpha \leq \beta < \lambda$ by the penultimate paragraph. By assumption, there is at least one $n \in \omega$ so that $f(\lambda_n) < \lambda_n$. Then for any $m > n$,

$$\begin{aligned} f(\lambda_n) &= i_{\lambda_n\lambda_m}(f(\lambda_n)) && \text{by (a) above,} \\ &= i_{\lambda_n\lambda_m}(f)(i_{\lambda_n\lambda_m}(\lambda_n)) \\ &= f(\lambda_m) && \text{by (b) above.} \end{aligned}$$

Thus if $\eta = f(\lambda_n)$, surely $\{\alpha < \lambda \mid f(\alpha) = \eta\} \in \bar{F}$, which is a contradiction.

The other requirements for $\bar{F}$ can be similarly established, or the proof can be completed directly as follows: If $\kappa = \bigcup_{\alpha<\delta} X_\alpha$ is a partition of κ so that $\delta < \lambda$ and $\langle X_\alpha \mid \alpha < \delta \rangle \in L[F]$, let $f \in {}^{\lambda}\delta \cap L[F]$ be defined by $f(\xi) = \alpha$ iff $\xi \in X_\alpha$. Then $\{\alpha < \lambda \mid f(\alpha) < \alpha\} \in \bar{F}$, so by the previous paragraph there is an α so that $X_\alpha \in \bar{F}$.

Step 3. There is a κ-model.

This is established by pulling back the λ-model $L[F]$, using some facts about the $i_{0\alpha}$'s. Let H be the (proper) class of ordinals fixed by $i_{0\alpha}$ for every $\alpha < \lambda$. Let $A \prec L[F]$ the Skolem Hull in $L[F]$ of H. Finally, let $t: A \simeq T$ be the transitization. First of all, $\kappa \cup \{\lambda\} \subseteq H$. Yet, if $\kappa \leq \beta < \lambda$, then $\beta \notin H$. (For example, if we set ζ = least α so that $i_{0\alpha}(\kappa) > \beta$, then $\beta < i_{0\zeta}(\beta)$.) Hence, for such β, we have $\beta \notin A$, as any set definable from ordinals fixed by an $i_{0\alpha}$ must itself be fixed by $i_{0\alpha}$. It follows that $t(\lambda) = \kappa$, and so by elementarity T is a κ-model and we are done. ⊣

The main result is established; there is some κ-model. Kunen had noticed the very special nature of a normal ideal, which in the present context can be cast as: If I is a normal, precipitous ideal over κ, then in $L[I]$, $I \cap L[I]$ is a maximal ideal and hence dual to a normal ultrafilter over κ.

⊢ We outline a demonstration of this, which amounts to tracing normality through the steps of the previous proof. First, in the proof of Step 1, when I is normal we can take $S = \kappa$ and $\gamma = \kappa$, so that $U = (P(\kappa) - I) \cap L[a]$. This U would then be a normal $L[a]$-ultrafilter (in the appropriate sense: if $f \in {}^{\kappa}\kappa \cap L[a]$ and $\{\alpha < \kappa \mid f(\alpha) < \alpha\} \in U$, then for some $\eta < \kappa$, $\{\alpha < \kappa \mid f(\alpha) = \eta\} \in U$). Then Step 3 is augmented with the following argument. U is also a normal T-ultrafilter since t fixes every $\alpha < \kappa$ and so $T \cap P(\kappa) = L[a] \cap P(\kappa)$. Let $j: T \to \mathrm{Ult}_\lambda(T, U)$ be the usual embedding. Taking $U^{(\alpha)}$ as defined earlier in this section in connection with M-ultrafilters and assuming $X \in U$, $j(X) \in U^{(\lambda)}$ By normality of U the Lemma just before the Strong Filters Theorem of §9 holds with $i_{0\alpha}(U)$ replaced by $U^{(\alpha)}$, so thus we can assert $j(X) \in G$, where G is the cardinal filter over λ. Hence, $j(X)$ is in the constructing ultrafilter for the λ-model, and so by elementarity X is in the constructing ultrafilter for the κ-model, which is T. Since X was arbitrary, U is the constructing ultrafilter for the κ-model, which is what we set out to prove. ⊣

It is not true that all precipitous ideals I over κ yield maximal $I \cap L[I]$ in $L[I]$. Wagon(1975) gave a characterization of those that do, assuming that the κ-model exists.

As discussed in §11, it is still unknown whether the ideal of non-stationary subsets of ω_1 can be ω_2-saturated. Jech-Magidor-Mitchell-Prikry(1977) gives a proof of: Con(ZFC & there is a measurable cardinal) is equivalent to Con(ZFC & NS_{ω_1} is precipitous). Mitchell had noticed that the dual of a normal ultrafilter over κ generates a normal precipitous ideal in any generic extension by the Lévy collapse (see §18) of κ to ω_1. Magidor was then able to make a further forcing extension to make a normal precipitous ideal the NS_{ω_1} of the extension. This involved shooting through new closed unbounded sets, iterating a standard forcing technique for generically adding closed unbounded subsets of ω_1 (see Baumgartner-Harrington-Kleinberg[1976]), and essentially used the fact that a Lévy collapse had previously been carried out. It turns out that ω_1 is not a typicality here, but a speciality; adding closed unbounded subsets of $\nu > \omega_1$ involves difficulties, so it is not known for example whether NS_{ω_2} can be precipitous.

We have ridden on a train of thought that started with real-valued measurability and developed through saturated ideals, and have now come to an actual equiconsistency of measurability with the existence of a certain ideal over ω_1. The ideal essence of measurability has precipitated, and it was found to be intimately bound up with well-foundedness of ultrapowers.

§13. Some Remarks on Ultrafilters

We comment in this section on how large cardinal concepts have impinged on the theory of ultrafilters. (For a good secondary reference hereabouts, see the book Comfort-Negrepontis[1974]. We remind the reader that all ultrafilters are to be non-principal.) A basic relation used in the study of ultrafilters is the Rudin-Keisler (RK) partial ordering on ultrafilters defined as follows: Suppose U is an ultrafilter over I and V is an ultrafilter over J. Then $V \leq U$ iff there is a function $f: I \to J$ so that $f_*(U) = V$, where $f_*(U) = \{X \subseteq J \mid f^{-1}(X) \in U\}$. Thus, Scott's normalization process in §2 was simply a canonical way of relating to any κ-complete ultrafilter U over κ a normal ultrafilter $N \leq U$. As usual, there are the derived relationships: $U \cong V$ iff $U \le V$ and $V \le U$; and $V < U$ iff $V \le U$ yet not $U \le V$. M.E. Rudin proved Rudin's Lemma: $U \cong V$ iff $V \le U$ and whenever $f: I \to J$ so that $f_*(U) = V$, there is an $X \in U$ so that $f|X$ is injective. This gives a rather clear picture of the RK ordering: maps between index sets induce the ordering, and an equivalence class under the ordering consists of those ultrafilters which are generated by one-to-one relabelings of the index sets.

There is an extensive, attractive theory on (non-principal) ultrafilters over ω ; they comprise a collection of objects which can be identified with $\beta N - N$ (where βX is the Stone-Čech compactification of a topological space X and N is traditionally ω with the discrete topology), and after all, βN is the simplest non-trivial example of a Stone-Čech compactification that one can study. Since these ultrafilters are just the "ω-complete" ultrafilters over ω , one would expect the study of κ-complete ultrafilters over κ in general to have definite analogies and yield similar dividends; see Kanamori[1977] for such a study with special attention to distinctive features when $\kappa > \omega$, like the use of the closed unbounded filter. Both the cases $\kappa = \omega$ and $\kappa > \omega$ turn out to depend heavily on additional set theoretic assumptions beyond ZFC.

A development in the general study of ultrafilters in which the above cases naturally arise is the consideration of minimal points in RK. An ultrafilter U is RK-minimal iff there is no $V < U$. This notion is certainly well defined for RK equivalence classes and so in looking at RK-minimal ultrafilters U we can without loss of generality assume that U is over a cardinal, say κ , and moreover that U is uniform. (This last assertion is justified since we could always have considered instead $\bar{U} = \{X \cap Y | X \in U\}$ where $Y \in U$ is of minimal cardinality; $\bar{U} \cong U$ by Rudin's Lemma.)

The first thing to notice is that if U is a RK-minimal, uniform ultrafilter over κ , then U is κ-complete, and hence κ is ω or a measurable cardinal. (Otherwise, we can get a $\lambda < \kappa$ and a partition $\bigcup_{\alpha<\lambda} X_\alpha = \kappa$ so that each $X_\alpha \notin U$. Then if $f: \kappa \to \lambda$ is defined by $f(\xi) = \alpha$ iff $\xi \in X_\alpha$, we have $f_*(U) < U$.) Thus, RK-minimal ultrafilters are already quite special. Kunen[1970a] probably had Rowbottom's partition theorem for normal ultrafilters in mind when he established the following attractive characterizations:

Theorem: The following are equivalent for a uniform ultrafilter U over κ :

(i) U is RK-minimal.

(ii) U is selective, i.e. whenever $f \in {}^\kappa\kappa$ so that $f^{-1}(\{\xi\}) \notin U$ for every $\xi < \kappa$, there is an $X \in U$ so that $f|X$ is injective.

(iii) U is quasi-normal, i.e. whenever $\{X_\alpha | \alpha < \kappa\} \subseteq U$, there is a $Y \in U$ so that $\eta < \zeta$ both in Y implies $\zeta \in X_\eta$.

(iv) U is Ramsey, i.e. whenever $f: [\kappa]^2 \to 2$, there is an $X \in U$ homogeneous for f .

$\vdash$ (i) $\leftrightarrow$ (ii) is almost immediate from the definitions, using Rudin's Lemma above.

(iii) $\to$ (iv). Given $f: [\kappa]^2 \to 2$, for each $\alpha < \kappa$ there is an $i_\alpha < 2$ so that $X_\alpha = \{\beta < \kappa | f(\{\alpha,\beta\}) = i_\alpha\} \in U$. Surely there is a $Z \in U$ and an $i < 2$ so that $\{\alpha < \kappa | i_\alpha = i\} \in U$. Let $Y \in U$ be as in quasi-normality for the family $\{X_\alpha | \alpha < \kappa\}$. Then $Y \cap Z \in U$ is homogeneous for f .

(iv) $\to$ (i). Suppose $f_*(U) = V \le U$, where $f: \kappa \to OR$. Then define $g: [\kappa]^2 \to 2$ simply by $g(\{\alpha,\beta\}) = 0$ iff $f(\alpha) = f(\beta)$. Let $X \in U$ be homogeneous for f . If $g''[X]^2 = \{0\}$, then $f''X = \{\gamma\}$ for some $\gamma < \kappa$, and so $\{\gamma\} \in V$ contradicting the non-principality of V . Thus, $g''[X] = \{1\}$. Clearly $f|X$ is one-to-one, so by Rudin's Lemma $V \cong U$.

(ii) $\to$ (iii). This is really the only non-trivial part, and we have saved it until last. So, suppose $\{X_\alpha | \alpha < \kappa\} \subseteq U$. Using (i) $\leftrightarrow$ (ii) and the comment just before the theorem, we can assume that U is κ-complete, and hence that $\alpha < \beta < \kappa$ implies $X_\alpha \supseteq X_\beta$. Also, if $\bigcap_\alpha X_\alpha \in U$, this set can be taken as our desired Y , so without loss of generality assume $\bar{S} = \kappa - \bigcap_\alpha X_\alpha \in U$.

We now use selectivity to thin down to the desired set Y . Define $f: \bar{S} \to \kappa$ by $f(\xi)$ = least α such that $\xi \notin X_\alpha$. f cannot have any constant value α on any set in U as $X_\alpha \in U$, so by selectivity there is an $S \subseteq \bar{S}$ with $S \in U$ so that $f|S$ is injective. Now let $g \in {}^\kappa\kappa$ be any strictly increasing function such that: $g(\alpha) > \sup\{\xi | \xi \in S \ \& \ f(\xi) \le \alpha\}$. This is possible since $f|S$ is injective, and κ is regular (as U is κ-complete). Finally, define $h \in {}^\kappa\kappa$ by iterated applications of g , i.e. $h(0) = 0$, $h(\alpha+1) = g(h(\alpha))$ and at limits γ , $h(\gamma) = \sup_{\alpha<\gamma} h(\alpha)$.

We can certainly partition κ into ordinal intervals with $p \in {}^\kappa\kappa$ by: $p(\xi) = \alpha$ iff $h(\alpha) \le \xi < h(\alpha+1)$. By selectivity, let $\bar{T} \in U$ so that $p|\bar{T}$ is injective. Let $T \in U$ consist of alternate elements of $\bar{T}$. (This is possible since if $\{\delta_\xi | \xi < \kappa\}$ is the ascending enumeration of $\bar{T}$ we can take T to be either $\{\delta_\xi | \xi$ is an even ordinal $< \kappa\}$ or $\{\delta_\xi | \xi$ is an odd ordinal $< \kappa\}$, whichever is in U .)

Set $Y = S \cap T \in U$. We claim Y is as desired. Indeed, let $\eta < \zeta$ both be in Y . By our construction of T , $\eta < g(\alpha) < gg(\alpha) \le \zeta$ for some $\alpha < \kappa$. As g is increasing, $g(\eta) \le gg(\alpha)$. So by definition of g , we have $f(\zeta) > \eta$, i.e. $\zeta \in X_\eta$ as required. $\dashv$

When $\kappa > \omega$, we could also have added $U \cong N$ for some normal ultrafilter N over κ , by Scott's original normalization idea, and (iii) above. Normal ultrafilters over $\kappa > \omega$ are in fact the unique representatives of their RK equivalence classes: If $N_1 \cong N_2$ are both normal, then $N_1 = N_2$. The characterizations (iii) and (iv) are particularly interesting, especially for $\kappa = \omega$. These are indeed aspects of normal κ-complete ultrafilters over $\kappa > \omega$ which are immutable under permutations of the index set. However, when $\kappa = \omega$, we cannot have actual normality, essentially because there are no limit processes taking place below ω. What does remain is the near approximation (iii) to diagonal intersection and, of course, the nice Rowbottom aspect (iv). It is certainly not unreasonable to say that these properties for ultrafilters over ω emerged through the study of measurability. With CH or even $MA(2^{\omega})$ an inductive construction of a Ramsey ultrafilter over ω is possible. Kunen[1976] first showed that it is consistent with ZFC that there are no Ramsey ultrafilters over ω. In 1977, strong versions of this result were established by Miller, and particularly by Shelah, who constructed a model of ZFC without even p-points in its $\beta N - N$, solving a well-known topological problem.

In any case, we have seen that for $\kappa > \omega$ whenever U is a κ-complete ultrafilter over κ there is an RK-minimal one $\leq U$, namely a normal one. Another pleasing aspect of large-cardinal-type ultrafilters is the following result of Solovay: The RK ordering on ω_1-complete ultrafilters is well-founded.

$\vdash$ For each $n \in \omega$ let U_n be a ω_1-complete ultrafilter over index set I_n, and for $n < m$ let $f_{nm}: I_n \to I_m$ such that $f_{nm*}(U_n) = U_m$ and $f_{nm} \circ f_{0n} = f_{0m}$. Thus $\ldots U_2 \leq U_1 \leq U_0$. We must show that $U_{n+1} \simeq U_n$ for some n, and for this it certainly suffices by Rudin's Lemma to establish that for some n, $f_{n,n+1}|X$ is injective for some $X \in U_n$.

Consider an equivalence relation $\simeq$ defined on I_0 by $x \simeq y$ iff there is an n so that $f_{0n}(x) = f_{0n}(y)$. Fix one element in each equivalence class as a representative, and for $x \in I_0$ set $f(x) =$ the least n such that $f_{0n}(x) = f_{0n}(r)$ and $r \simeq x$ is the representative of the class of x. By ω_1-completeness there is an $\bar{n}$ such that $X = \{x \in I_0 | f(x) = \bar{n}\} \in U_0$. Then it is simple to see that $f_{\bar{n},\bar{n}+1}$ is injective on $f_{0\bar{n}}"X \in U_{\bar{n}}$. $\dashv$

One begins to see how imposing strong hypotheses on ultrafilters introduces nice new structural features into a known framework.

We now briefly look at hypotheses on ultrafilters weaker than ω_1-completeness. During the period of initial preoccupation in model theory with ultrafilters in the sixties, Keisler formulated several related concepts: An ultrafilter U is (κ,λ)-regular iff there are λ sets in U any κ of which have empty intersection. An ultrafilter U is λ-descendingly incomplete iff there are $\{X_\alpha | \alpha < \lambda\} \subseteq U$ so that $\alpha < \beta < \lambda$ implies $X_\alpha \supseteq X_\beta$, and $\bigcap_{\alpha<\lambda} X_\alpha = \emptyset$. An ultrafilter U over an index set I is λ-decomposable iff there is a partition $I = \bigcup_{\alpha<\lambda} Y_\alpha$ into disjoint sets Y_α so that whenever $S \subsetneq \lambda$ and $|S| < \lambda$, $\bigcup_{\alpha \in S} Y_\alpha \notin U$.

Thus, an ultrafilter U over I is λ-decomposable iff there is an $f: I \to \lambda$ so that $f_*(U)$ is a uniform ultrafilter over λ. Straightforward arguments establish that: (a) λ-descendingly incomplete is equivalent to $cf(\lambda)$-descendingly incomplete; (b) ω-decomposable is the same as ω-incomplete; (c) λ-decomposable implies λ-descendingly incomplete, and (d) the converse holds if λ is regular. λ-decomposability can be regarded as a more general notion than λ-descendingly incompleteness, since the former can be distinctive for singular λ. Finally, observe that if U is (κ,λ)-regular and ν is a regular cardinal so that $\kappa \leq \nu \leq \lambda$, then U is ν-decomposable.

$\vdash$ Let $\{X_\alpha | \alpha < \lambda\} \subseteq U$ such that any κ of them has empty intersection. Set $Y_\gamma = \bigcup\{X_\alpha | \gamma \leq \alpha < \nu\}$ for every $\gamma < \nu$. Then $Y_\gamma \in U$ and $\gamma < \delta < \nu$ implies $Y_\gamma \supseteq Y_\delta$. Finally,

$\bigcap_{\gamma<\nu} Y_\gamma = \bigcap_{\gamma<\nu} \bigcup\{X_\alpha | \gamma \leq \alpha < \nu\} = \bigcup\{\bigcap_{\alpha<\nu} X_{f(\alpha)} | f \in {}^{\nu}\nu$ and $f(\alpha) \geq \alpha$ for every $\alpha < \nu\}$,

and this last union is the union of empty sets, by the regularity of ν and $\kappa \leq \nu$. Thus, $\bigcap_{\gamma<\nu} Y_\gamma = \emptyset$, and this finishes the proof by comment (d) above. $\dashv$

Regularity is a measure of width for ultrafilters, studied in general after an initial use was made in giving an ultraproduct proof of the Compactness Theorem. Sufficient regularity assumptions on ultrafilters yield ultraproducts of large cardinality and model theoretic universality (see Chang-Keisler[1973],4.3.). Note that for every λ there is a (ω,λ)-regular ultrafilter over λ. (Instead of λ itself, for convenience we take $P_\omega\lambda = \{x \subseteq \lambda | |x| < \omega\}$ as the index set: Set $A_\alpha = \{x \in P_\omega\lambda | \alpha \in x\}$ for every $\alpha < \lambda$, and define $S = \{A_\alpha | \alpha < \lambda\}$. S has the finite intersection property, and any extension of S to an ultrafilter over $P_\omega\lambda$ will have S witnessing its (ω,λ)-regularity.)

Regularity and decomposability can be regarded as hypotheses about the existence of nice scalings for ultrafilters. It was somewhat unexpected that their denials turn out to be properties of large cardinal character. The first inkling of this was in Chudnovsky-Chudnovsky[1971], which showed for example that if there is an ultrafilter over a regular uncountable κ which is ν-indecomposable for arbitrary large $\nu < \kappa$, then κ is ω-weakly Mahlo.

Prikry[1973] then carried out a more systematic study, having discovered a way of incorporating some aspects of the theory of normal ultrafilters over a measurable cardinal. Let U be an ultrafilter over a cardinal κ. $f \in {}^{\kappa}\kappa$ is unbounded (mod U) iff $\{\xi < \kappa | f(\xi) \geq \alpha\} \in U$ for every $\alpha < \kappa$. $f \in {}^{\kappa}\kappa$ is a least function for U iff f is unbounded (mod U) but there is no g unbounded (mod U) so that $\{\xi < \kappa | g(\xi) < f(\xi)\} \in U$. Finally, U is weakly normal iff the identity fuction $\kappa \to \kappa$ is a least function for U, i.e. if $\{\alpha < \kappa | g(\alpha) < \alpha\} \in U$, we have $\{\alpha < \kappa | g(\alpha) \leq \gamma\} \in U$ for some $\gamma < \kappa$. Without κ-completeness, this is as much of normality as we can hope to achieve. Clearly, if f is a least function for U then $f_*(U)$ is weakly normal. Of course, if U is a ω_1-complete ultrafilter, then there is a least function for U. Somewhat unexpectedly, Prikry established that with some weaker indecomposabiltiy conditions on U and some power set hypotheses, U can also be shown to have a least function. Thus, a weak normalization of U was possible, and this opened the door to some interesting structural theorems reminiscent of the formalism attendant to normal ultrafilters over a measurable cardinal.

Stronger least function results were soon forthcoming. Kanamori[1976] established: Suppose U is a uniform ultrafilter over a regular cardinal κ. (a) If $\kappa = \lambda^+$, then U is not (λ,λ^+)-regular iff there is a least function f for U so that $\{\alpha < \kappa \mid cf(f(\alpha)) = \lambda\} \in U$. (b) If U is not (ω,ν)-regular for some $\nu < \kappa$, then there is a least function for U. Ketonen's ideas figured prominently here, and in particular (b) was proved by him independently. (b) subsumes Prikry's result alluded to earlier. Least functions were thus firmly established as a possible feature of ultrafilters which may not be ω_1-complete. It is questionable whether these results about irregularity could have ever been formulated without the experience of measurability.

A direct road to large cardinals was paved when some ingenious ideas of Silver [1974] were imaginatively combined by Ketonen[1976] with a restricted ultrapower of the Vopěnka-Hrbáček type (see the end of §3) to yield: If there is a weakly normal ultrafilter over a regular uncountable cardinal κ which is not (ν,κ)-regular for any $\nu < \kappa$, then $0^\#$ exists. Jensen (in Jensen-Koppelberg(1977)) saw how to amend the argument with definability considerations to establish: If there is a weakly normal ultrafilter over a regular uncountable cardinal κ and $2^{<\kappa} = \kappa$, then $0^\#$ exists.

It has not yet been shown that in L every ultrafilter over a regular cardinal λ is (ω,λ)-regular, though the various results above come rather close. So far, Prikry[1971] established that in L every uniform ultrafilter over ω_1 is (ω,ω_1)-regular, and Jensen amplified the argument to show that in L, for every $n \in \omega$ every uniform ultrafilter over ω_n is (ω,ω_n)-regular. Then Benda (in Benda-Ketonen[1974]) discovered a clearer argument for achieving these results. Nothing is known in L about regularity or decomposability for uniform ultrafilters over ω_ω, or indeed any singular cardinal.

Benda's argument has actually become widely applicable in the study of regularity; it is a kind of argument using what in combinatorial set theory is called almost disjoint transversals. We illustrate it, and the sort of possibilities with weak normality that we have been talking about all along, in the following: If U is a weakly normal ultrafilter over a regular cardinal $\kappa > \omega$ which is not (ν,κ)-regular for any $\nu < \kappa$, then: if $T = \{\alpha < \kappa \mid 2^\alpha = \alpha^+\} \in U$, then $2^\kappa = \kappa^+$.

$\vdash$ It should be kept in mind that a weakly normal ultrafilter is in particular uniform, since the identity function is unbounded modulo the ultrafilter. Throughout the proof, we shall use the shorthand $f < g \pmod U$ for $\{\alpha < \kappa \mid f(\alpha) < g(\alpha)\} \in U$.

Argue by contradiction, and suppose $\{X_\xi \mid \xi < \kappa^{++}\}$ are distinct subsets of κ. For each $\alpha \in T$, let $\psi_\alpha : 2^\alpha \to \alpha^+$ be an injection. Define $f_\xi : T \to \kappa$ for each $\xi < \kappa^{++}$ by: $f_\xi(\alpha) = \psi_\alpha(X_\xi \cap \alpha)$. Then $\{f_\xi \mid \xi < \kappa^{++}\}$ constitute κ^{++} functions which are eventually different on T, i.e. whenever $\xi \neq \bar\xi$ there is a $\gamma_{\xi\bar\xi} < \kappa$ so that $\alpha \in T - \gamma_{\xi\bar\xi}$ implies $f_\xi(\alpha) \neq f_{\bar\xi}(\alpha)$.

Now observe that there must be a $\xi_0 < \kappa^{++}$ so that if $g = f_{\xi_0}$, then $f_\xi < g \pmod U$ for at least κ^+ many ξ's. By a suitable relabeling, we can assume $f_\xi < g \pmod U$ for each $\xi < \kappa^+$. Since $g(\alpha) < \alpha^+$ for each $\alpha \in T$, there must be injections $\sigma_\alpha : g(\alpha) \to \alpha$ for each $\alpha \in T$. Now define $g_\xi : T \to \kappa$ for $\xi < \kappa^+$ by

$$g_\xi(\alpha) = \begin{cases} \sigma_\alpha(f_\xi(\alpha)) & \text{if } f_\xi(\alpha) < g(\alpha), \\ 0 & \text{otherwise.} \end{cases}$$

We can now apply weak normality. Since each g_ξ is regressive, there is a $\rho_\xi < \kappa$ so that $\{\alpha < \kappa \mid g_\xi(\alpha) < \rho_\xi\} \in U$. By a cardinality argument, we might as well assume that $\rho_\xi = \rho < \kappa$ for every $\xi < \kappa^+$. Also, in what follows, we can assume furthermore that ρ is a cardinal. (For this, let $t: \rho \to |\rho|$ be injective, and define:

$$\bar g_\xi(\alpha) = \begin{cases} t(g_\xi(\alpha)) & \text{if } g_\xi(\alpha) < \rho, \\ g_\xi(\alpha) & \text{otherwise.} \end{cases}$$

Then the $\bar g_\xi$'s are bounded by $|\rho| \pmod U$, and it will be seen that they can be used in place of the g_ξ's in what follows.)

We again cut back: There is some $\xi_1 < \kappa^+$ so that if $h = g_{\xi_1}$ then $g_\xi < h \pmod U$ for at least κ many ξ's. Again, by a suitable relabeling we can assume $g_\xi < h \pmod U$ for each $\xi < \kappa$. Finally, define Y_ξ for $\xi < \kappa$ by:

$$Y_\xi = \{\alpha < \kappa \mid \alpha > \sup\{\gamma_{\xi\bar\xi} \mid \bar\xi < \xi\} \ \& \ f_\xi(\alpha) < g(\alpha) \ \& \ g_\xi(\alpha) < h(\alpha) < \rho\}.$$

Each $Y_\xi \in U$, so by the denial of (ρ,κ)-regularity, let $S \subseteq \kappa$ so that $|S| \geq \rho$ and $\bar\alpha \in \bigcap_{\xi \in S} Y_\xi$ for some $\bar\alpha$. Then by tracing through the injections and using eventual difference, we find that $\{g_\xi(\bar\alpha) \mid \xi \in S\}$ comprise ρ different ordinals all $< h(\bar\alpha) < \rho$. This is a contradiction. $\dashv$

This theorem is a typical example of reflection phenomena that one begins to expect for large cardinals. As mentioned in §2, the prototype of this sort of result is due to Scott, with "U is a normal ultrafilter over κ" replacing the weaker hypothesis on U above. It was this type of least function argument that led Silver to his solution of most cases of the Singular Cardinals Problem (see §29). See also Jech-Prikry(1977) for many related results about saturated ideals.

Of relative consistency results, Prikry[1970] had established (via Prikry forcing on a measurable cardinal) the consistency of the existence of a uniform ultrafilter over a singular cardinal κ of cofinality ω, which is ν-indecomposable for $\omega < \nu < \kappa$. Then more recently Magidor(1977), using a variant of an argument of Kunen, established (see also §17): Con(ZFC & there is a huge cardinal) implies Con(ZFC & there is a uniform, weakly normal ultrafilter over ω_2 which is not (ω,ω_2)-regular). Whether there can be a uniform ultrafilter over ω_1 which is not (ω,ω_1)-regular is still a prominent open question. Magidor also established (see also §14): Con(ZFC & there is a cardinal κ which is κ^+-supercompact) implies Con(ZFC & there is a uniform ultrafilter over $\omega_{\omega+1}$ which is ν-indecomposable for $\omega < \nu < \omega_\omega$), and also the analogous conclusion with $\omega_{\omega+1}$ replaced by ω_ω .

Just as saturation of ideals can be considered a weakening of measurability where κ-completeness is retained but ultrafiltration is no longer assumed, irregularity and indecomposability of ultrafilters can be regarded as weakenings of measurability where ultrafiltration is retained but ω_1-completeness is no longer assumed. That real-valued measurability led to the former concept and model theoretic ideas led to the latter are interesting developments in the flow of history. κ-completeness rather than ultrafiltration is the stronger hypothesis; it can be shown that: (i) If I is a normal κ-saturated ideal over κ , then any extension of the dual filter to an ultrafilter U is a weakly normal ultrafilter which is not (ν,κ)-regular for any $\nu < \kappa$. (ii) If in addition I is ν-saturated for some regular $\nu < \kappa$, then U will be ν-indecomposable. Closer connections between saturation and irregularity are established in Taylor(1977). It is remarkable how so many aspects of measurability, even in such weakened strains as irregularity, have spread across the canopy of set theory.

IV. LARGE LARGE CARDINALS

§14. Supercompactness

We now embark on a study of axioms of infinity stronger than measurability. The atmosphere becomes at once rarefied, as we leave behind the considerable elaboration of measurability in the previous chapter, and study global principles based on gross features of elementary embeddings. Realizing that most of the strong reflection properties of measurable cardinals can be culled from the Closure Lemma (§2), in the second half of the 1960's Reinhardt and Solovay formulated several principles by imposing strong closure conditions on the range inner models of elementary embeddings. There was already a principle stronger than measurability known, that of strong compactness, but somehow its elementary embedding formulations were not very malleable. So, a crucial closure assumption was added by Solovay, in the spirit of "what if we also know this?". Then all the desired fruit, suddenly ripened, were easily plucked, and appropriately enough the new concept was dubbed supercompactness. Initially, it was thought that the new closure assumption will eventually be shown to follow from strong compactness, but it is now known that supercompactness and strong compactness are not the same concept. The emergence of supercompactness is a good contemporary example of the discovery of new principles through the formal investigation of set theory.

Much of this section as well as this whole chapter follows the exposition of Solovay-Reinhardt-Kanamori(1978), which we henceforth refer to as SRK. Let us (re)activate some definitions. The notation $j: V \to M$ signifies that j is an elementary embedding, not the identity, of the universe into an inner model M. The critical point of such a j is the least ordinal α such that $j(\alpha) > \alpha$. When U is an ω_1-complete ultrafilter over I , we denote by $j_U: V \to M_U \simeq V^I/U$ the corresponding elementary embedding of the universe into the transitive isomorph M_U of the ultrapower by U . Finally, $[f]_U$ denotes that element of M_U corresponding to the ultrapower equivalence class of f . (The subscripts U will often be dropped when clear from the context.)

In the study of elementary embeddings, it will be the case that several properties can already be rendered by embeddings into ultrapowers. Moreover, these have concrete features which aid their investigation; for example, as in Paris' proof in §10 they fix a stationary class of ordinals. Further characteristics can be discerned:

Generalized Closure Lemma: Suppose U is an ω_1-complete ultrafilter over a set I, $j = j_U$, $M = M_U$, etc.:

(i) If $j"x \subset M$ and $y \subseteq M$ is such that $|y| \leq |x|$, then $y \in M$.
(ii) $j"(|I|^+) \notin M$.
(iii) $U \notin M$.

$\vdash$ (i) is proved just as the Closure Lemma of §2 is, by producing a g so that in M, $[g]$ is a function with domain $j"x$ and range y.

For (ii), let $[f] \in M$ be arbitrary. If $A = \{i \in I \mid |f(i)| \leq |I|\} \in U$, we can certainly find an $\alpha \in |I|^+ - \bigcup\{f(i) \mid i \in A\}$. Thus, $j(\alpha) \notin [f]$. If $B = \{i \in I \mid |f(i)| > |I|\} \in U$, we can certainly define an injective function h on B so that $h(i) \in f(i)$ for every $i \in B$. Thus, $[h] \in [f]$, yet h is not constant on any set in U. Hence, in either case we have established that $[f] \neq j"(|I|^+)$.

For (iii), assume to the contrary that $U \in M$. It follows that $P(I) \subseteq M$ and ${}^I I \subseteq P(I \times I) \subseteq M$. Note that for any ordinal α, $j(\alpha)$ = the order type of $\{[f] \mid f \in {}^I\alpha\}$. Thus, $j"|I| \in M$, since $j"|I|$ is just the collection of such order types for α ranging over ordinals below $|I|$, and can be properly defined in M as $U \in M$ and ${}^I I = ({}^I I)^M$. By (i), it follows that M is "closed under $|I|$-sequences", and in particular ${}^I|I|^+ = ({}^I|I|^+)^M$. Using this, we can now show similarly that $j"(|I|^+) \in M$, contradicting (ii). $\dashv$

(i) is an important structural fact about ultrapowers; if $|x| = \lambda$ there, we simply write ${}^\lambda M \subseteq M$ to indicate that M is closed under the taking of arbitrary λ-sequences. (ii) and (iii) (which in a special case has already been seen in 8.7. (ii)) impose limitations on the breadth of ultrapowers. Perhaps they can be interpreted as saying: whatever ultrafilters seem to be capable of, they are still only <u>sets</u>.

(ii) says for a κ-complete ultrafilter U over a measurable cardinal κ that M_U is not closed under κ^+-sequences. The following concept, formulated by Solovay seems the proper generalization of measurability: If $\kappa \leq \lambda$, κ is <u>λ-supercompact</u> <u>iff</u> there is an elementary embedding $j: V \to M$ so that:

(a) j has critical point κ and $j(\kappa) > \lambda$, and

(b) ${}^\lambda M \subseteq M$.

κ is <u>supercompact</u> <u>iff</u> κ is λ-supercompact for all $\lambda \geq \kappa$.

Here, (b) is the crucial closure assumption, implying in particular that M contains all sets hereditarily of cardinality $\leq \lambda$. Comparison with the analogous characterization of λ-compactness (see §15) will indicate the strength of (b). In (a), the addition of $j(\kappa) > \lambda$ is merely to be more definite; in any case, whenever $j: V \to M$ with critical point κ so that ${}^\lambda M \subseteq M$, then some nth iterate j^n has these same properties, <u>and</u> $j^n(\kappa) > \lambda$ (Kunen[1971]; follows from §17). Note that κ is κ-supercompact <u>iff</u> κ is measurable. It will shortly be shown that if κ is 2^κ-supercompact, then κ is already the κth measurable cardinal.

Let us now turn to the task of getting a characterization of λ-supercompactness which is more concrete. If $j: V \to M$ is as in the definition of the λ-supercompactness of κ, the lemma above suggests considering the following ultrafilter:

$$X \in U \quad \text{iff} \quad j"\lambda \in j(X)$$

We now embark on a short course of discovery: First, what is an index set for U? Let us remember the notation $P_\nu I = \{x \subseteq I \mid |x| < \nu\}$. Note that since $|j"\lambda| = \lambda < j(\kappa)$ in M, $j"\lambda \in (P_{j(\kappa)}j(\lambda))^M$. Hence, $P_\kappa\lambda \in U$, and we can consider this set to be the underlying index set for U. We can also check that U has the following properties:

(i) U is a κ-complete ultrafilter.

(ii) For any $a \in \lambda$, $\{x \mid a \in x\} \in U$.

(iii) If f is a function defined on a set in U so that $\{x \mid f(x) \in x\} \in U$, then there is an $a \in \lambda$ so that $\{x \mid f(x) = a\} \in U$.

By (i) and (ii) if $y \in P_\kappa\lambda$, $\{x \mid y \subseteq x\} \in U$. For a proof of (iii), note that if $j(f)(j"\lambda) \in j"\lambda$ then $j(f)(j"\lambda) = j(a)$ for some $a \in \lambda$. Let us now designate:

<u>Definition</u>: If $\kappa \leq \lambda$, an ultrafilter U over $P_\kappa\lambda$ is <u>normal</u> <u>iff</u> it satisfies (i),(ii), and (iii) above. More generally, an ultrafilter U over $P_\kappa I$, where I is any set, is <u>normal</u> <u>iff</u> it satisfies (i),(ii), and (iii) with λ replaced by I. Finally, without reference to κ, an ultrafilter U over $P(I)$ (i.e. $U \subseteq PP(I)$) is <u>normal</u> <u>iff</u> it satisfies (ii) and (iii) with λ replaced by I.

Hence, an ω_1-complete ultrafilter U over $P(I)$ is <u>normal</u> <u>iff</u> $[\mathrm{id}]_U = j_U"I$ where $\mathrm{id}: P(I) \to P(I)$ is the identity map. To anticipate a possible source of confusion, we remind the reader that the index sets of these various ultrafilters consist of <u>subsets</u> of a given set I, not its elements.

Just as in §2, we now take an ultrapower of the universe: Let U be a normal ultrafilter over $P_\kappa\lambda$ and consider the canonical $j: V \to M = V^{P_\kappa\lambda}/U$. We have:

(a) ${}^\lambda M \subseteq M$: This follows from $[\mathrm{id}] = j"\lambda$ and (i) of the Generalized Closure Lemma.

(b) κ is the critical point of j and $j(\kappa) > \lambda$: We have $\{x \mid |x| < \kappa\} \in U$ so $|[\mathrm{id}]| = |j"\lambda| < j(\kappa)$. But $|j"\lambda| = \lambda$ in M, since M is closed under λ-sequences.

We have established: <u>If $\kappa \leq \lambda$, then κ is λ-supercompact iff there is a normal ultrafilter over $P_\kappa\lambda$</u>. Thus, λ-supercompactness is equivalent to the existence of a certain set. This reduction by Solovay makes possible many structure theorems, and introduces a new and interesting set, $P_\kappa\lambda$. The following observations are also due to Solovay: <u>If U is normal over $P_\kappa\lambda$, then: (a) M_U is actually closed under $\lambda^{<\kappa}$-sequences, and (b) if $F: P(\lambda) \to \lambda$ is defined by $F(x) = \sup x$, then there is an $X \in U$ such that $F|X$ is injective</u>. See SRK for proofs.

We shall soon see in several typical arguments how supercompactness is a strong reflection property. In abstract form we can formulate this as follows: for any transitive class X, say that Σ_n (respectively, Π_n) <u>relativizes down to X</u> <u>iff</u>

whenever $P(\cdot)$ is Σ_n (Respectively, Π_n), if $a \in X$ and $P(a)$ holds, then $\langle X,\in\rangle \models P(a)$. Note that Σ_n relativizes down to X iff Π_{n+1} does. According to Lévy[1965], if $|V_\theta| = \theta$, then Σ_1 (and hence Π_2) relativizes down to V . We have the following result: If κ is supercompact, Σ_2 (and hence Π_3) relativize down to V_κ :

$\vdash$ Suppose $P(x)$ is $\exists y Q(x,y)$ where Q is Π_1 . Let $a \in V_\kappa$ so that $P(a)$ holds, and fix b such that $Q(a,b)$ holds. By supercompactness, let $j\colon V \to M$ with critical point κ so that $b \in M \cap V_{j(\kappa)}$. This is possible by previous remarks, by taking a j corresponding to λ-supercompactness for a λ sufficiently large. Note that $j(a) = a$. Thus, $(V_{j(\kappa)} \models P(a))^M$ since Π_1 formulas are preserved under restriction. It follows by elementarity that $V_\kappa \models P(a)$, as desired. $\dashv$

§16 contains a corresponding result for extendibility. Certainly, a more exact, level-by-level analysis of the preceding result could have been given in terms of λ-supercompactness. However, it is typical of the global postulation of supercompactness that we do not want to tie our hands beforehand, though any particular application will of course only use λ-supercompactness for some λ sufficiently large. A related result of Magidor[1971] establishes a characterization of supercompactness as a second-order Löwenheim-Skolem property. Finally, we give an illustrative corollary, interesting in its own right: If $\kappa < \lambda$, κ is α-supercompact for every $\alpha < \lambda$, and λ is supercompact, then κ is supercompact.

$\vdash$ By the characterization of supercompactness via normal ultrafilters, the hypotheses certainly show that $V_\lambda \models \kappa$ is supercompact. Now "x is supercompact" is a Π_2 property of x , again by use of normal ultrafilters. Hence, by the previous result κ (really) is supercompact. $\dashv$

It is possible to establish this result by a level-by-level construction of normal ultrafilters; see 5.8. of SRK. Note that this corollary shows in particular that the Σ_3 sentence "there is a supercompact cardinal" does not hold in V_κ if κ is the least supercompact cardinal. Hence, Σ_2 was optimal in the penultimate result.

The new notion of normality turns out to be the natural generalization of the familiar one. In particular, there is a one-to-one correspondence between normal ultrafilters over $P_\kappa\kappa$ and normal ultrafilters over κ : If V is normal over κ, $U = \{X \subseteq P_\kappa\kappa \mid X \cap \kappa \in V\}$ is normal over $P_\kappa\kappa$. Conversely, if U over $P_\kappa\kappa$ is normal, then $\kappa \in U$: If not, then $\{x \in P_\kappa\kappa \mid x \text{ is not an ordinal}\} \in U$. For such x , let $f(x) \in x$ be so that $f(x)$ is the least above some ordinal not in x . By normality $[f] = j(\gamma)$ for some $\gamma < \kappa$, but this contradicts $\{x \mid \gamma \subseteq x\} \in U$.

Determining the number of normal ultrafilters possible over a measurable cardinal has turned out to be an interesting problem, a focus for the introduction of new methods. We saw in §9 how Kunen established that if U is normal ultrafilter over κ , then $U \cap L[U]$ is the only normal ultrafilter in $L[U]$. Kunen-Paris[1971]

showed (see §24) that if κ is measurable in the ground model, there is a forcing extension in which κ carries the maximal number of normal ultrafilters, i.e. 2^{2^κ}. Then Mitchell[1974] showed that if κ is 2^κ-supercompact and τ is $\leq \kappa$ or one of the terms κ^+ or κ^{++}, then there is an inner model in which κ is measurable and carries exactly τ normal ultrafilters. (Mitchell's models were described at the end of §9; he has recently succeeded in getting a version of the result from just strong compactness.) It is still not known whether one can get Mitchell's conclusion starting from just the measurability of κ . In Mitchell's model with exactly two normal ultrafilters over κ , one contains the set $\{\alpha < \kappa \mid \alpha \text{ is measurable}\}$ and the other does not; in this regard, consider the next two results:

Theorem: If κ is 2^κ-supercompact, there is a normal ultrafilter U over κ so that $\{\alpha < \kappa \mid \alpha \text{ is measurable}\} \in U$. Hence, κ is the κth measurable cardinal.

$\vdash$ Let $j\colon V \to M$ with critical point κ so that M is closed under 2^κ-sequences. If U is defined by:

$$X \in U \text{ iff } X \subseteq \kappa \ \& \ \kappa \in j(X),$$

then U is a normal ultrafilter over κ , as can easily be verified (recall §2). But since M is closed under 2^κ-sequences, it is not hard to see that every ultrafilter over κ is a member of M . Hence, κ is measurable in M , i.e. $\{\alpha < \kappa \mid \alpha \text{ is measurable}\} \in U$ by the definition of U . $\dashv$

Theorem: Any measurable cardinal κ carries a normal ultrafilter U so that $\{\alpha < \kappa \mid \alpha \text{ is not measurable}\} \in U$.

$\vdash$ By induction. Let V be a normal ultrafilter over κ . Set $T = \{\alpha < \kappa \mid \alpha \text{ is measurable}\}$. If $T \notin V$, we are done, so assume $T \in V$. For each $\alpha \in T$, by the inductive hypothesis let U_α be normal over α so that $\{\beta < \alpha \mid \beta \text{ is not measurable}\} \in U_\alpha$. Define U over κ by:

$$X \in U \text{ iff } \{\alpha < \kappa \mid X \cap \alpha \in U_\alpha\} \in V .$$

This "fusion" of ultrafilters is then a normal ultrafilter over κ as can easily be checked, and moreover $\{\alpha < \kappa \mid \alpha \text{ is not measurable}\} \in V$. $\dashv$

The preceding results show that if κ is 2^κ-supercompact, there are at least two normal ultrafilters over κ . In fact, there are 2^{2^κ} normal ultrafilters over κ , and this is a special case of a general result on the number of normal ultrafilters over $P_\kappa\lambda$. See SRK for proofs; the reflection phenomena in these proofs are so strong that, rather paradoxically, the following question is still open: If κ is 2^κ-supercompact, is it provable that there is more than one normal ultrafilter over κ containing the set $\{\alpha < \kappa \mid \alpha \text{ is not measurable}\}$?

One can already well imagine the strength of supercompactness as a structural principle about the existence of a multitude of embeddings. Given this view, it is notable that supercompactness can be given characterizations which have a combinatorial flavor (see Magidor[1974] and DiPrisco-Zwicker(1977)). We should also mention here that Menas[1975] has investigated partition properties for normal ultrafilters over $P_\kappa\lambda$ in analogy to Rowbottom's result for normal ultrafilters over a measurable cardinal; the interesting thing here is that not all normal ultrafilters over $P_\kappa\lambda$ may possess the partition property.

Many of these results hinge on a combinatorial study of $P_\kappa\lambda$ initiated in Jech[1973]. Jech in particular formulated the following "two cardinal" versions of well-known concepts: Suppose κ is regular and $\lambda \geq \kappa$. An $X \subseteq P_\kappa\lambda$ is closed iff whenever $\gamma < \kappa$ and $\{x_\xi \mid \xi < \gamma\} \subseteq X$ so that $\xi < \bar{\xi}$ implies $x_\xi \subseteq x_{\bar{\xi}}$, we have $\bigcup_{\xi<\gamma} x_\xi \in X$. (Jech's original definition was slightly more complicated, but equivalent to the one given.) An $X \subseteq P_\kappa\lambda$ is unbounded iff whenever $y \in P_\kappa\lambda$, there is an $x \in X$ so that $y \subseteq x$. Finally, an $S \subseteq P_\kappa\lambda$ is stationary iff $S \cap C \neq \emptyset$ for every closed unbounded $C \subseteq P_\kappa\lambda$.

For any $y \in P_\kappa\lambda$, the set $\{x \in P_\kappa\lambda \mid y \subseteq x\}$ is closed unbounded. Jech established that the intersection of $< \kappa$ closed unbounded subsets of $P_\kappa\lambda$ is still closed unbounded. Thus, the closed unbounded subsets of $P_\kappa\lambda$ generate a κ-complete filter. Jech also proved that this filter is normal in the following sense: whenever $\{x \in P_\kappa\lambda \mid f(x) \in x\}$ is stationary, for some $\gamma < \lambda$, $\{x \in P_\kappa\lambda \mid f(x) = \gamma\}$ is stationary. This is certainly an interesting development. A closed unbounded subset of $P_\kappa\lambda$ can be collectively considered a close approximation of λ through $< \kappa$ cardinality sets, with some attendant apparatus. (See Kueker[1977] for an implementation in model theory.) We hasten to remind the reader that the idea of $P_\kappa\lambda$ arose directly from the fount of large cardinal theory.

On the one hand, $P_\kappa\lambda$ has an engaging appeal as a generalizing concept. On the other hand, without a natural well-ordering, it is much more difficult to work with, and many easily stated combinatorial problems remain unsolved. One crucial difficulty from the standpoint of large cardinals is that if U is a normal ultrafilter over $P_\kappa\lambda$, then this property does not relativize to $U \cap L[U] \in L[U]$. This of course is because $P_\kappa\lambda \cap L[U]$ may be comparatively small (compare: for any ordinal α, $\alpha \cap L[U] = \alpha$ trivially). Thus, unlike for measurability no natural inner models for supercompactness seem to immediately suggest themselves.

In forthcoming sections we shall see the efficacy of supercompactness, especially in the establishment of relative consistency results, some of which are known to necessitate a hypothesis stronger than the existence of many measurable cardinals. The weaker concept of strong compactness, though providing interesting direct consequences (see §15, §21) does not presently enjoy any similar confidence in its consistency strength. We shall clarify the relationship among measurability, strong compactness, and supercompactness in a succeeding section (§26), showing in particular that strong compactness is a varying concept which does not fit neatly into the hierarchical picture. Finally, refer to §16 for further results on supercompactness, including a characterization.

§15. Strong Compactness and the GCH

As discussed in §3, strong compactness was historically motivated by efforts to generalize the usual Compactness Theorem to infinitary languages $L_{\kappa\kappa}$. We present here the modern characterization of the concept in terms of elementary embeddings and ultrafilters, the approach which in part inspired the formulation of supercompactness. In addition, an interesting result of Solovay on the GCH above a strongly compact cardinal will be discussed.

Definitions: If $\kappa \leq \lambda$ and U is an ultrafilter over $P_\kappa\lambda$, then U is fine iff U is κ-complete and for each $\alpha < \lambda$, $\{x \mid \alpha \in x\} \in U$. (Thus, we leave out the crucial clause (iii) of the definition of normality in §14.) If $\kappa \leq \lambda$, κ is λ-compact iff there is a fine ultrafilter over $P_\kappa\lambda$.

Trivially, if κ is λ-supercompact, then κ is λ-compact; and κ is measurable iff κ is κ-compact. Let us proceed forthwith to some characterizations.

Theorem: If $\kappa \leq \lambda$, the following are equivalent:

(i) κ is λ-compact.

(ii) There is a $j: V \to M$ with critical point κ so that: $X \subseteq M$ and $|X| \leq \lambda$ implies that there is a $Y \in M$ so that $X \subseteq Y$, and $M \models |Y| < j(\kappa)$.

(iii) If F is any κ-complete filter over an index set I so that F is generated by $\leq \lambda$ sets, then F can be extended to a κ-complete ultrafilter over I.

$\vdash$ (i) $\to$ (ii). Let U be fine over $P_\kappa\lambda$, and consider $j: V \to M = V^{P_\kappa\lambda}/U$. If $X = \{[f_\alpha] \mid \alpha < \lambda\} \subseteq M$, set $G(x) = \{f_\alpha(x) \mid \alpha \in x\}$ and $Y = [G]$. Then $X \subseteq Y$, and $M \models |Y| < j(\kappa)$, by Łoś' Theorem.

(ii) $\to$ (iii). Suppose F is as hypothesized, and generated by elements of $T \subseteq P(I)$, where $|T| \leq \lambda$. By (ii) let $Y \supseteq j"T$ so that $Y \in M$ and $M \models |Y| < j(\kappa)$. In M, $j(F)$ is a $j(\kappa)$-complete filter and $j(F) \cap Y$ is a subset of cardinality $< j(\kappa)$. Hence, there is a $c \in M$ so that $c \in \bigcap(j(F) \cap Y)$. We can now use this c to generate an ultrafilter in the usual fashion: Set $X \in U$ iff $X \subseteq I$ & $c \in j(X)$. It is easy to show that U is a κ-complete ultrafilter over I, which extends F.

(iii) $\to$ (i). Extend the κ-complete filter over $P_\kappa\lambda$ generated by the sets $\{x \mid \alpha \in x\}$ for $\alpha < \lambda$ to a κ-complete ultrafilter. $\dashv$

Notice that by (iii) above and a characterization in §3, we have: κ is strongly compact iff κ is λ-compact for all $\lambda \geq \kappa$. The weakness of λ-compactness which Solovay sought to correct is illustrated in (ii); with λ-supercompactness we can always take $X = Y$. Ketonen[1972] gives another characterization: If $\kappa \leq \lambda$ are both regular, κ is λ-compact iff for every regular μ such that $\kappa \leq \mu \leq \lambda$ there is a uniform κ-complete ultrafilter over μ. This demonstrates a conjecture of Kunen, and seems to say that fine ultrafilters over $P_\kappa\lambda$ are not that hard to come by. Normality is another story; the following result of Menas[1974] indicates that strong compactness and supercompactness are not the same concept.

Lemma:

(i) If κ is measurable and a limit of strongly compact cardinals, then κ is strongly compact.

(ii) If κ is the least cardinal as in (i), then κ is not 2^{κ}-supercompact.

$\vdash$ For (i), let U be a κ-complete ultrafilter over κ so that $A = \{\alpha < \kappa \mid \alpha$ is strongly compact$\} \in U$. (This is easy to achieve; if V is any κ-complete ultrafilter over κ and $f: \kappa \to A$ is an injection, then $f_*(V)$ will do.) If $\lambda \geq \kappa$, for $\alpha \in A$ let U_α be fine over $P_\alpha \lambda$. Define V_λ by:

$$X \in V_\lambda \ \text{ iff } \ X \subseteq P_\kappa \lambda \ \ \& \ \ \{\alpha \mid X \cap P_\alpha \lambda \in U_\alpha\} \in U .$$

Then it can be checked that V_λ is fine over $P_\kappa \lambda$.

For (ii), argue by contradiction, and suppose κ were 2^{κ}-supercompact. Let $j: V \to M$ with critical point κ so that M is closed under 2^{κ}-sequences. By definition of κ and elementarity, we have in M that $j(\kappa)$ is the least measurable cardinal which is a limit of strongly compact cardinals. But M is closed under 2^{κ}-sequences so that κ is measurable in M, and also, if $\alpha < \kappa$ is strongly compact, $j(\alpha) = \alpha$ is strongly compact in the sense of M. This contradicts the leastness of $j(\kappa) > \kappa$. $\dashv$

It is a consequence of the existence of an extendible cardinal that there are many cardinals as in (i) above (see §16). The preceding lemma was the conceptual beginning point of Menas' [1974] consistency result on the least strongly compact cardinal not being supercompact, since subsumed by work of Magidor (see §25).

Of consistency results involving strong compactness, not much is known beyond Kunen's result, cited in §10, though recent work of Mitchell seems to reveal some unexpected strength. Some interesting direct results, however, are known; the Vopěnka-Hrbáček result has already been outlined in §3, and a relatively recent result of Solovay [1974] states in particular that if κ is strongly compact, then $2^\lambda = \lambda^+$ for every singular strong limit cardinal $\lambda > \kappa$. It is interesting to observe that the conclusion, for all singular strong limit λ, also follows from the hypothesis "$0^\#$ does not exist" by Jensen's Covering Theorem (see §29). Jensen's result says something about the L-like quality of V in the absence of large cardinal perturbations. Solovay's result, on the other hand, says that a large cardinal hypothesis imposes a superstructure on V, which then provides new controls.

We now embark on a proof of Solovay's result. After seeing an argument of Ketonen [1972] for establishing the regularity of certain ultrafilters, Solovay realized how the argument could be applied to establish: If $\kappa \leq \lambda$ are regular and κ is λ-compact, then $\lambda^{<\kappa} = \lambda$.

$\vdash$ Let U be fine over $P_\kappa \lambda$, and $j: V \to M \approx V^{P_\kappa \lambda}/U$. Set $[g] = \sup\{j(\alpha) \mid \alpha < \lambda\}$. We make the following important Claim: $\{x \mid g(x) < \lambda \ \& \ \mathrm{cf}(g(x)) < \kappa\} \in U$.

To establish this, it certainly suffices to show that $\{x \mid g(x) = \sup(x \cap g(x))\} \in U$. If this were false, by definition of g there would be some $\alpha < \lambda$ such that $\{x \mid \sup(x \cap g(x)) < \alpha\} \in U$. But clearly, $\{x \mid \alpha \in x \ \& \ \alpha < g(x)\} \in U$, which is a contradiction. Thus, the Claim is proved.

We now produce certain sets $\{X_\alpha \mid \alpha < \lambda\} \subseteq U$ (this was Ketonen's idea in a similar context for getting a particular family to witness (κ,λ)-regularity). For every $\alpha < \lambda$ let $a_\alpha \subseteq \alpha$ be cofinal in α of type $\mathrm{cf}(\alpha)$. Thus, $\{x \mid |a_{g(x)}| < \kappa\} \in U$ by the Claim. Inductively define for every $\alpha < \lambda$ increasing disjoint ordinal intervals $[\eta_\alpha, \rho_\alpha) \subseteq \lambda$, with $\eta_\alpha < \rho_\alpha < \lambda$, and functions h_α as follows:

If defined already for $\beta < \alpha$, let $\eta_\alpha = \sup\{\rho_{\beta+1} \mid \beta < \alpha\}$. Let h_α be defined by: $h_\alpha(x)$ = least element of $a_{g(x)}$ greater than or equal to η_α whenever this is possible. h_α is defined at least on a set in U as $j(\eta_\alpha) < [g]$. Now $[h_\alpha] < [g]$, so by definition of $[g]$, let ρ_α = least ordinal $> \eta_\alpha$ such that $[h_\alpha] < j(\rho_\alpha)$. This inductive definition will proceed through all $\alpha < \lambda$ as λ is regular. Clearly, for every $\alpha < \lambda$, $X_\alpha = \{x \mid \eta_\alpha \leq h_\alpha(x) < \rho_\alpha\} \in U$.

Solovay's idea now is as follows: we will find a set $S \subseteq P_\kappa \lambda$ so that: (a) $|S| \leq \lambda$, and (b) whenever $y \in P_\kappa \lambda$ there is an $s \in S$ so that $y \subseteq s$. This will then finish the proof, since $P_\kappa \lambda \subseteq \bigcup\{P(s) \mid s \in S\}$, and so $\lambda^{<\kappa} \leq \lambda \cdot \kappa \leq \lambda$.

For each $x \in P_\kappa \lambda$, set $t_x = \{\alpha \mid x \in X_\alpha\} = \{\alpha \mid \eta_\alpha \leq h_\alpha(x) < \rho_\alpha\}$. We show that $S = \{t_x \mid x \in P_\kappa \lambda \ \& \ |a_{g(x)}| < \kappa\}$ works: Firstly, for any $x \in P_\kappa \lambda$, we have $t_x \in P_\kappa \lambda$. This is because $h_\alpha(x) \in a_{g(x)}$, different α's in t_x pepper their $h_\alpha(x)$ into disjoint ordinal intervals, and $|a_{g(x)}| < \kappa$. Thus, $S \subseteq P_\kappa \lambda$. Secondly, if $g(x) = g(\bar{x})$ then surely $t_x = t_{\bar{x}}$, and since by the Claim we can suppose $\mathrm{Range}(g) \subseteq \lambda$, we have $|S| \leq \lambda$. Finally, let $y \in P_\kappa \lambda$ be arbitrary. Then $\bigcap_{\alpha \in y} X_\alpha \in U$ by κ-completeness, so choose some $\bar{x} \in (\bigcap_{\alpha \in y} X_\alpha \cap \{x \mid |a_{g(x)}| < \kappa\}) \in U$. Hence $y \subseteq t_{\bar{x}}$, and S has all the properties desired. $\dashv$

We are now in a position to prove Solovay's GCH result: If κ is λ^+-compact and λ is a singular strong limit cardinal $> \kappa$, then $2^\lambda = \lambda^+$.

$\vdash$ Note first that $2^\lambda = \lambda^{\mathrm{cf}(\lambda)}$ for singular strong limit λ by cardinal arithmetic. There are now two cases:

Case I: $\mathrm{cf}(\lambda) < \kappa$. Then $2^\lambda = \lambda^{\mathrm{cf}(\lambda)} \leq (\lambda^+)^{\mathrm{cf}(\lambda)} = \lambda^+$, the last equality following from the previous result.

Case II: $\mathrm{cf}(\lambda) \geq \kappa$. We here call upon the following important result of Silver (discussed in §29): if ν is singular with $\mathrm{cf}(\nu) > \omega$ so that $\{\alpha < \nu \mid 2^\alpha = \alpha^+\}$ is stationary in ν, then $2^\nu = \nu^+$. In the case at hand, $S = \{\alpha < \lambda \mid \alpha$ is a singular strong limit cardinal of cofinality $< \kappa\}$ is stationary in λ. By Case I, $\alpha \in S$ implies $2^\alpha = \alpha^+$. Thus, by Silver's result, $2^\lambda = \lambda^+$. $\dashv$

Silver's result, proved after Solovay's, provides a simplification in Case II of the original proof. However, the further results in Solovay [1974] on powers of cardinals cannot ostensibly be simplified in this way.

The scaling on power sets that strong compactness imposes also affects combinatorial principles; see §21.

§16. Extendibility

At the same time that Solovay was formulating supercompactness, Reinhardt, a few doors down the hall at Berkeley, was first considering extendibility. That such closely intertwined concepts should be independently arrived at in such close proximity is quite a coincidence. Supercompactness has the flavor of generalization from measurability, but extendibility reflects more ethereal ambitions. Reinhardt [1974] motivates extendibility via considerations involving strong principles of reflection and resemblance formalized in an extended theory which allows transfinite levels of higher type objects over V. Essentially, OR is hypothesized to be extendible in this setting. With the natural reflection down into the realm of sets, we have the concept of an extendible cardinal.

If $\eta > 0$, a cardinal κ is called <u>η-extendible</u> <u>iff</u> there is a ζ and a $j: V_{\kappa+\eta} \to V_\zeta$ with critical point κ, where $\kappa+\eta < j(\kappa) < \zeta$. <u>$\kappa$ is extendible</u> <u>iff</u> κ is η-extendible for every $\eta > 0$.

Thus, we consider embeddings which are sets, but whose range structures have the ultimate closure property—they are initial segments of the universe. Since $\eta \geq \kappa\cdot\kappa$ implies $\kappa+\eta = \eta$, the exact form of the above definition is distinctive only for small η. The condition $\kappa+\eta < j(\kappa)$ is for definiteness, in analogy with λ-supercompactness; (full) extendibility as a concept is not altered if we do away with this condition (see 5.2. of SRK). When $\eta < \kappa$, it is not hard to see that $\zeta = j(\kappa)+\eta$. Note that in such cases η-extendibility is just a postulate of resemblance: with $j: V_{\kappa+\eta} \to V_{j(\kappa)+\eta}$, V_κ and $V_{j(\kappa)}$ are indistinguishable as far as $(\eta+1)$-order properties are concerned. Finally, observe that if κ is η-extendible and $0 < \delta < \eta$, then κ is δ-extendible: Since the term V_α is definable from α, if ζ and j are as in the definition of η-extendibility, we have that $j|V_{\kappa+\delta}: V_{\kappa+\delta} \to V_{j(\kappa+\delta)}$ is elementary.

1-extendibility is already quite strong: <u>If κ is 1-extendible, then κ is measurable and there is a normal ultrafilter U over κ so that $\{\alpha < \kappa | \alpha$ is measurable$\} \in U$.</u>

$\vdash$ Let $j: V_{\kappa+1} \to V_{j(\kappa)+1}$ with critical point κ. Notice that $P(\kappa) \subseteq V_{\kappa+1}$. Thus U, defined by $X \in U$ <u>iff</u> $X \subseteq \kappa$ & $\kappa \in j(X)$, is normal over κ, as usual. Certainly $U \in V_{j(\kappa)+1}$, so $V_{j(\kappa)+1} \models \kappa$ is measurable, i.e. $\{\alpha < \kappa | \alpha$ is measurable$\} \in U$. $\dashv$

If κ is supercompact, it is consistent that there is no inaccessible cardinal $> \kappa$, since if there were one, we can cut off the universe at the least one and still have a model of set theory in which κ is supercompact. However, suppose κ is even 1-extendible, with $j: V_{\kappa+1} \to V_{j(\kappa)+1}$. Then by elementarity $j(\kappa)$ is inaccessible in $V_{j(\kappa)+1}$, hence in V. Similarly, if κ is 2-extendible with $j: V_{\kappa+2} \to V_{j(\kappa)+2}$, by elementarity $j(\kappa)$ is measurable in $V_{j(\kappa)+2}$, and hence in V. Thus, the extendibility of a cardinal κ implies the existence of large cardinals $> \kappa$. These considerations begin to show how strongly the existence of an extendible cardinal affects the higher levels of the cumulative hierarchy, and why η-extendibility cannot be formulated, as λ-supercompactness can, merely in terms of the existence of certain ultrafilters. (Recall (ii) and (iii) of the Generalized Closure Lemma, §14.)

We have seen several times how the various large cardinals can be characterized by aspects of recurring and unifying themes. It was a pleasing discovery of Magidor [1971] that extendibility can be cast as a direct analogue of strong compactness for higher order languages. First, the inevitable definitions: By L^n_κ we denote the nth order logic using $L_{\kappa\omega}$, i.e. $< \kappa$ conjunctions are allowed. A collection of sentences of L^n_κ is <u>satisfiable</u> <u>iff</u> it has a model, under the natural interpretation of infinitary conjunction, in the sense of full nth order logic (e.g. if X is the (first-order) domain, then second-order variables are to range over <u>all</u> of $P(X)$), and is <u>κ-satisfiable</u> <u>iff</u> every subcollection of cardinality $< \kappa$ is satisfiable. A cardinal κ is <u>L^n_κ-compact</u> <u>iff</u> whenever Σ is a κ-satisfiable collection of L^n_κ sentences (with any number of non-logical symbols), then Σ is satisfiable.

Observe that κ is strongly compact <u>iff</u> κ is L^1_κ-compact. (A technicality: the characterization of strong compactness in §3 shows that $L_{\kappa\omega}$ can replace $L_{\kappa\kappa}$.) We now establish: <u>κ is extendible iff κ is L^n_κ-compact for every $n \in \omega$ iff κ is L^2_κ-compact.</u>

$\vdash$ First, suppose that κ is extendible, and assume Σ is an arbitrary, κ-satisfiable collection of L^n_κ sentences. We can suppose that Σ is coded as a set, via a Gödelization where the logical symbols of L^n_κ are coded by some elements of V_κ. Let $\lambda > \kappa$ be sufficiently large so that $|\Sigma| \leq \lambda$ and $V_\lambda \models \Sigma$ is κ-satisfiable. By λ-extendibility, let $j: V_\lambda \to V_\zeta$ with critical point κ and $\lambda < j(\kappa)$. We have by elementarity that $V_\zeta \models j(\Sigma)$ is $j(\kappa)$-satisfiable.

Now note that $j"\Sigma \subseteq j(\Sigma)$ and $|j"\Sigma| \leq \lambda < j(\kappa)$. Thus, in V_ζ, $j"\Sigma$ has a model A, whence A really is a model of $j"\Sigma$ (in V). However, A is then a model of Σ, since the formulas of $j"\Sigma$ are like those of Σ with at most the non-logical symbols renamed. Thus, we have established that κ is L^n_κ-compact.

To finish the proof, it suffices to establish that if κ is L^2_κ-compact, then κ is extendible. First of all, it is well known that there is a Π^1_1 sentence σ so that for any transitive set X, $X \models \sigma$ <u>iff</u> $X = V_\alpha$ for some α. (Roughly, σ would say that X is closed under the definable function $F(\alpha) = V_\alpha$, and that a "class" which is a subclass of an element is itself an element.) Secondly, there is a Π^1_1 sentence τ saying that the membership relation is well-founded.

Now let $\lambda > \kappa$. We seek a ζ and a $j: V_\lambda \to V_\zeta$ with critical point κ so that $\lambda < j(\kappa)$. Let Σ be the union of: (a) $\{\sigma,\tau\}$, (b) the L^2_κ theory of $\langle V_\lambda, \in, x\rangle_{x\in V_\lambda}$, using constants c_x for $x \in V_\lambda$, and (c) the sentences "d_α is an ordinal and $d_\alpha < d_\beta < c_\kappa$" for every $\alpha < \beta \leq \lambda$, where $\{d_\alpha | \alpha \leq \lambda\}$ is a set of new constants. By L^2_κ-compactness, κ is regular, being at least strongly compact. Thus, by a judicious assignment of ordinals $< \kappa$ to constants d_α, it is clear that any subcollection of Σ of cardinality $< \kappa$ is satisfiable in $\langle V_\lambda, \varepsilon, x\rangle_{x \varepsilon V_\lambda}$ itself.

Σ is thus κ-satisfiable, and so by hypothesis it is satisfiable. By (a) we can suppose that Σ is satisfied by a structure of form $\langle V_\zeta, \varepsilon, \bar{x}, \gamma_\alpha\rangle_{x\varepsilon V_\lambda;\ \alpha \leq \lambda}$, where γ_α interprets d_α. Clearly, the map $j(x) = \bar{x}$ is an elementary embedding $j\colon V_\lambda \to V_\zeta$. Σ has $L_{\kappa\omega}$ sentences specifying the members of each $x \varepsilon V_\kappa$, so by induction on rank, $x \varepsilon V_\kappa$ implies $j(x) = x$. Finally, $j(\kappa) > \lambda$ since $\{\gamma_\alpha \mid \alpha \leq \lambda\}$ constitutes a subset of $j(\kappa)$ of order type $\lambda+1$. The proof is thus complete. $\dashv$

This result is an example of a recurring phenomenon: going from first to second order logic involves a definite strengthening, but there is no further difference among the nth order versions for $n > 1$. In the next several results we shall see how closely interlaced are the local versions of supercompactness and extendibility.

Lemma 1: If κ is $|V_{\kappa+\eta}|$-supercompact and $\eta < \kappa$, there is a normal ultrafilter U over κ so that $\{\alpha < \kappa \mid \alpha$ is η-extendible$\} \varepsilon U$.

$\vdash$ Let $j\colon V \to M$ be as in $|V_{\kappa+\eta}|$-supercompactness. Then $V_{\kappa+\eta} \varepsilon M$, since $V_{\kappa+\eta}$ is hereditarily of cardinal $\leq |V_{\kappa+\eta}|$. Similarly, if we set $e = j|V_{\kappa+\eta}$, $e \varepsilon M$. Now in M, $e\colon V_{\kappa+\eta} \to j(V_{\kappa+\eta})$ is an elementary embedding with critical point κ and $\kappa+\eta < e(\kappa)$, since this is all true in V. Thus, κ is η-extendible in M. Define U by: $X \varepsilon U$ iff $X \subseteq \kappa$ & $\kappa \varepsilon j(X)$. Then U is the usual normal ultrafilter over κ corresponding to j, and $\{\alpha < \kappa \mid \alpha$ is η-extendible$\} \varepsilon U$, as $j(\eta) = \eta$. $\dashv$

Lemma 2: If κ is η-extendible and $\delta+1 < \eta$, then κ is $|V_{\kappa+\eta}|$-supercompact. Hence, if κ is extendible, then κ is supercompact.

$\vdash$ Suppose $j\colon V_{\kappa+\eta} \to V_\zeta$ is as in η-extendibility. Since $j(\kappa)$ is (really) inaccessible and $\kappa+\delta < j(\kappa)$, we have $|V_{\kappa+\delta}| < j(\kappa)$. Hence, since $\delta+1 < \eta$ so that $P(P_\kappa V_{\kappa+\delta}) \subseteq V_{\kappa+\eta}$, we can define a normal ultrafilter over $P_\kappa V_{\kappa+\delta}$ as usual:

$$X \varepsilon U \text{ iff } j''V_{\kappa+\delta} \varepsilon j(X). \quad \dashv$$

Note that this result establishes in particular that: If there is an extendible cardinal, then Con(ZFC & there is a supercompact cardinal). (Let κ be extendible, and $\lambda > \kappa$ any inaccessible cardinal; then $V_\lambda \models \kappa$ is supercompact.) The methods of the preceding two lemmas yield a characterization of supercompactness, noticed in Magidor[1971]: κ is supercompact iff for every $\eta > \kappa$ there is an $\alpha < \kappa$ and a $j\colon V_\alpha \to V_\eta$ with critical point γ so that $j(\gamma) = \kappa$.

$\vdash$ For the forward direction, fix $\eta > \kappa$ and let $j\colon V \to M$ be as in the $|V_\eta|$-supercompactness of κ. Then just as in Lemma 1, $j|V_\eta\colon V_\eta \to (V_{j(\eta)})^M$ is an elementary embedding which is in M. Thus, $M \models$ "there is an $\alpha < j(\kappa)$ and an elementary embedding $e\colon V_\alpha \to V_{j(\eta)}$ with a critical point γ such that $e(\gamma) = j(\kappa)$". The result now follows from the elementarity of j.

For the converse, fix $\eta > \kappa$ and let $j\colon V_{\alpha+\omega} \to V_{\eta+\omega}$ for some $\alpha < \kappa$, with critical point γ so that $j(\gamma) = \kappa$. As in Lemma 2, since $P(P_\gamma \alpha) \subseteq V_{\alpha+\omega}$, j determines a normal ultrafilter U over $P_\gamma \alpha$. But $U \varepsilon V_{\alpha+\omega}$ and so $j(U)$ is a normal ultrafilter over $P_{j(\gamma)} j(\alpha) = P_\kappa \eta$. $\dashv$

The next result shows that the combination of even 1-extendibility with supercompactness transcends supercompactness: If κ is supercompact and 1-extendible, then there is a normal ultrafilter U over κ such that $\{\alpha < \kappa \mid \alpha$ is supercompact$\} \varepsilon U$. Hence, the least supercompact cardinal is not 1-extendible.

$\vdash$ Let $j\colon V_{\kappa+1} \to V_{j(\kappa)+1}$ be as in 1-extendibility, and let U be the usual normal ultrafilter over κ defined from j. Since $j(\kappa)$ is inaccessible, $V_{j(\kappa)+1} \models \kappa$ is δ-supercompact for every $\delta < j(\kappa)$. Hence, $A = \{\alpha < \kappa \mid \alpha$ is δ-supercompact for every $\delta < \kappa\} \varepsilon U$. However, by a result in §14 (the corollary to the "relativizing down" result), since κ is supercompact, $\alpha \varepsilon A$ implies α is supercompact. $\dashv$

On the other hand, the next result shows that supercompactness can control extendibility to some extent: If $\kappa < \lambda$, κ is extendible and λ is supercompact, then $V_\lambda \models \kappa$ is extendible.

$\vdash$ The formula "x is extendible" is Π_3, and so relativizes down to V_λ as λ is supercompact, by a result in §14. $\dashv$

We thus see that the extendibility of a cardinal κ can already be comprehended in V_λ, where λ is a supercompact cardinal $> \kappa$. In particular, it is consistent to assume that there is no supercompact cardinal above an extendible cardinal. Leapfrogging from large cardinals to larger cardinals, this remark may serve to allay initial suspicions about extendibility which might arise from the fact that it has as a consequence the existence of proper classes of various large cardinals.

Finally, the next result is the analogue of a result in §14, from which the terminology is carried over: If κ is extendible, then Σ_3 (and hence Π_4) relativize down to V_κ:

$\vdash$ Actually, we only use the fact that there are arbitrarily large inaccessibles $\lambda > \kappa$ with $V_\kappa \prec V_\lambda$. Suppose $P(x)$ is $\exists y Q(x,y)$ where Q is Π_2. Let $a \varepsilon V_\kappa$ so that $P(a)$ holds, and fix b such that $Q(a,b)$ holds. Let $\lambda > \kappa$ be inaccessible so that $b \varepsilon V_\lambda$ and $V_\kappa \prec V_\lambda$. Now Π_2 relativizes down to V_λ, by a remark in §14 just after the notion "relativizing down" is introduced. Thus, $V_\lambda \models Q(a,b)$. Hence, $V_\lambda \models P(a)$, and by elementarity $V_\kappa \models P(a)$. $\dashv$

This result is optimal, since by the result before, the Σ_4 sentence "there is an extendible cardinal" is false in V_λ if λ is the least extendible cardinal.

All in all, supercompactness and extendibility have many similar features, especially in the local versions. The characterization of supercompactness given in this section graphically illustrates how it is a sort of "pull back" version of extendibility. In the next section we steer suddenly upward, surveying the heights toward the ultimate inconsistency.

§17. Higher Principles and an Inconsistency Result of Kunen

We now embark on a course that culminates in a result of Kunen, which will delimit our further efforts. The way is charted with principles which have an even grosser character, and appeal less to basic motivating ideas. Once the possible spectrum was delineated by Kunen's result, at least in one possible direction guided by our motivations, landmarks along the way were established with principles perhaps somewhat ad hoc but nonetheless of some interest.

We shall first make an approach to Kunen's result from below, although the concepts to be initially discussed were clarified only in light of Kunen's result. This is historically inaccurate and will be less well motivated, but is done to avoid a dissolution into anticlimax. (Hopefully, this aside will be adequate to answer the question: Quo Vadis?)

The following principle first arose in the context of model theory: Vopěnka's Principle: Given a proper class of (set) structures of the same similarity type, there exists one that can be elementarily embedded into another.

This is a principle about the comparative richness of proper classes: at least one member should be injectible into another, while preserving some structure. Vopěnka's Principle can also be viewed via Skolemization in terms of universal algebra. It may not be immediately clear that the principle is a very strong axiom of infinity at all. Indeed, it was a significant latter-day realization that the principle was amenable to investigation via large cardinal techniques, and that it actually has a natural place in the emerging hierarchy.

One way in which Vopěnka's Principle will differ from our previous axioms is that it does not merely assert the existence of a large cardinal with higher order properties, but provides a framework in which many such cardinals can be shown to exist. To carry out a study within set theory, we shall consider inaccessible κ so that $V_\kappa \models$ Vopěnka's Principle. The following treatment is from Kanamori[1978] and §6 of SRK.

For definiteness, let us define a sequence of structures $\langle M_\alpha \mid \alpha < \kappa \rangle$ to be natural iff each $M_\alpha = \langle V_{f(\alpha)}, \in, \{\alpha\}, R_\alpha \rangle$ where $R_\alpha \subseteq V_{f(\alpha)}$, and $\alpha < \beta < \kappa$ implies $\alpha < f(\alpha) \leq f(\beta) < \kappa$. Clearly, we can construe as natural those sequences where R_α is replaced by a finite number of relations. The specification of $\{\alpha\}$ in M_α insures that whenever $\alpha < \beta$ and $j: M_\alpha \to M_\beta$ is elementary, j moves some ordinal, since $j(\alpha) = \beta$. Finally, any sequence $\langle M_\alpha \mid \alpha < \kappa \rangle$ of structures of the same similarity type so that each $M_\alpha \in V_\kappa$, can be augmented to be a natural sequence, whenever κ is inaccessible.

We now define, for κ inaccessible, a notion of a large subset of κ: $X \subseteq \kappa$ is Vopěnka iff whenever $\langle M_\alpha \mid \alpha < \kappa \rangle$ is a natural sequence, there is an elementary embedding of one into another with critical point $\in X$. The following is a typical definition of a filter relating to a large cardinal (recall for example the remarks at the end of §4): $F_\kappa^V = \{X \subseteq \kappa \mid \kappa - X \text{ is not Vopěnka}\}$. We have: F_κ^V is a (proper) filter iff $V_\kappa \models$ Vopěnka's Principle iff κ is Vopěnka. We now establish: If κ is Vopěnka, F_κ^V is a normal κ-complete filter over κ (and hence contains every closed unbounded subset of κ).

⊢ It suffices to show that if $\{X_\gamma \mid \gamma < \kappa\} \subseteq F_\kappa^V$, then the diagonal intersection $Y = \Delta_\gamma X_\gamma \in F_\kappa^V$. (This is part would also establish κ-completeness.)

Assume to the contrary that $Y \notin F_\kappa^V$. Thus, $\kappa - Y$ is Vopěnka. Fix a function $F: (\kappa - Y) \to \kappa$ so that $F(\delta) < \delta$ and $\delta \notin X_{F(\delta)}$. For each $\gamma < \kappa$, since $X_\gamma \in F_\kappa^V$, choose a natural sequence $\langle M_\alpha^\gamma \mid \alpha < \kappa \rangle$ so that: whenever there is an elementary embedding of one into another with critical point ρ, then $\rho \in X_\gamma$.

Now define a natural sequence $\langle N_\alpha \mid \alpha < \kappa \rangle$ by:

$$N_\alpha = \langle V_{g(\alpha)+\omega}, \in, \{\alpha\}, \langle M_\alpha^\gamma \mid \gamma < \alpha \rangle, F|(\alpha - Y) \rangle,$$

where $V_{g(\alpha)}$ is the union of the domains of M_α^γ for $\gamma < \alpha$. Since by assumption $\kappa - Y$ is Vopěnka, let $\eta \in (\kappa - Y)$ so that η is the critical point of a $j: N_\alpha \to N_\beta$. Using F, we have that if $\bar\gamma = F(\eta) < \eta$, then $F(j(\eta)) = j(F(\eta)) = F(\eta) = \bar\gamma$, as η is the critical point of j. However, we then have that $j|M_\alpha^{\bar\gamma}: M_\alpha^{\bar\gamma} \to M_\beta^{\bar\gamma}$ is elementary with critical point η, contradicting the definition of F. ⊣

The above result is the case $n = 1$ of a larger scheme in Kanamori[1978], which owes a direct debt to Baumgartner[1975]. We can now establish: If κ is Vopěnka, then $\{\alpha < \kappa \mid V_\kappa \models \alpha \text{ is extendible}\} \in F_\kappa^V$.

⊢ Define $F \in {}^\kappa\kappa$ by:

$$F(\delta) = \begin{cases} \delta & \text{if } V_\kappa \models \delta \text{ is extendible,} \\ \delta+\eta & \text{where } \eta \text{ is least so that } V_\kappa \models \delta \text{ is not } \eta\text{-extendible, otherwise.} \end{cases}$$

If $C = \{\rho < \kappa \mid F|\rho: \rho \to \rho\}$, then C is closed unbounded. $C \in F_\kappa^V$ by the previous result, so let $\langle M_\alpha \mid \alpha < \kappa \rangle$ be a natural sequence so that whenever there is an elementary embedding of one into another with critical point ρ, then $\rho \in C$.

Consider the natural sequence $\langle N_\alpha \mid \alpha < \kappa \rangle$ defined by:

$$N_\alpha = \langle V_{\gamma_\alpha}, \in, \{\alpha\}, M_\alpha, C \cap \gamma_\alpha \rangle,$$

where γ_α is the least limit point of C greater than every ordinal in the domain of M_α. It suffices to show that if $j: N_\alpha \to N_\beta$ with critical point ζ, then ζ is extendible. Well, assume not, so that $F(\zeta) = \xi > \zeta$. Since $\zeta < \gamma_\alpha$ and $\gamma_\alpha \in C$, we have $\xi < \gamma_\alpha$ by definition of C. So, $j|V_\xi: V_\xi \to V_{j(\xi)}$ is elementary with critical point ζ. Also, note that $\zeta \in C$, as M_α is encoded in N_α. Thus,

$j(\zeta) \in C$ as $C \cap \gamma_\alpha$ is encoded in N_α. Hence, $\zeta < j(\zeta)$ implies $\xi = F(\zeta) < j(\zeta)$. But if $\zeta = \xi + \eta$, all these facts show that $V_\kappa \models \zeta$ is η-extendible, which contradicts the definition of F. ⊣

Although κ being Vopěnka is a Π^1_1 property of V_κ and hence does not even imply the weak compactness of κ, we have thus established that Vopěnka's Principle implies the existence of extendible cardinals in an appropriately strong sense. For further results, including some characterizations, see §6 of SRK.

We now turn to strong principles which should at least have a familiar ring from the experience of supercompactness. Let us first establish some notation which we intend to be in effect through the rest of this section: If j is some elementary embedding with critical point κ, then we shall set $\kappa_0 = \kappa$ and for each integer n, $\kappa_{n+1} = j(\kappa_n)$ if κ_n is still in the domain of j, and $\kappa_\omega = \sup\{\kappa_n | n \in \omega\}$, again, if definable at all. Notice that, when defined, κ_ω is the least ordinal $> \kappa$ fixed by j, as $j(\kappa_\omega) = \sup\{j(\kappa_n) | n \in \omega\} = \kappa_\omega$. (Of course, the κ_n's depend on j, but the j being discussed should be clear from the text.)

We now formulate: If n is an integer, κ is <u>n-huge</u> <u>iff</u> there is a $j: V \to M$ with critical point κ so that ${}^{\kappa_n}M \subseteq M$. κ is <u>huge</u> (Kunen) <u>iff</u> κ is 1-huge.

Observe that κ is 0-huge <u>iff</u> κ is measurable, and κ is n-huge implies κ_n is (really) inaccessible. The n-huge cardinals certainly have an analogous flavor to λ-supercompact cardinals, but there is an important difference: While λ-supercompactness is hypothesized with an <u>a priori</u> λ in mind as a proposed degree of closure for M (and results in $\lambda < j(\kappa)$), n-hugeness has closure properties only <u>a posteriori</u>: M here is to be closed under κ_n-sequences, how ever large the κ_n turn out to be. This is a strengthening of an essential kind. Indeed, it is not clear how to motivate n-hugeness as a postulate of reflection at all. However, we have not left behind everything familiar: n-hugeness can be given a characterization via the existence of certain ultrafilters: <u>κ is n-huge iff there is a κ-complete normal ultrafilter U over some $P(\lambda)$, and cardinals</u> $\kappa = \lambda_0 < \lambda_1 < \ldots < \lambda_n = \lambda$ <u>so that for each $i < n$, $\{x \subseteq \lambda | \overline{x \cap \lambda_{i+1}} = \lambda_i\} \in U$</u>. (We use here that if y is a set of ordinals, then $\bar{y}$ denotes its order type.)

⊢ First, if $j: V \to M$ as in the n-hugeness of κ, define U over $P(\kappa_n)$ by:

$$X \in U \ \underline{\text{iff}}\ X \subseteq P(\kappa_n) \ \& \ j''\kappa_n \in j(X).$$

Then it is straightforward to show that U is normal and κ-complete. Also, note that $\overline{j''\kappa_n \cap j(\kappa_i)} = \overline{j''\kappa_i} = \kappa_i$ for $0 \leq i \leq n$, so that we can set $\kappa_i = \lambda_i$.

Conversely, take $j: V \to M \approx V^{P\lambda}/U$. Then $[id] = j''\lambda$ and M is closed under λ-sequences. Also, by hypothesis we have for $0 \leq i < n$,

$$j(\lambda_i) = [\langle \overline{x \cap \lambda_{i+1}} | x \subseteq \lambda \rangle]$$
$$= \overline{j''\lambda \cap j(\lambda_{i+1})} = \lambda_{i+1}.$$ ⊣

We can also prove the usual sort of hierarchical result: <u>If κ is n+1-huge, then there is a normal ultrafilter U over κ so that $\{\alpha < \kappa | \alpha$ is n-huge$\} \in U$</u>:

⊢ Suppose that $j: V \to M$ as in the n+1-hugeness of κ. Since M is closed under κ_{n+1}-sequences, M certainly contains the ultrafilter described in the previous result for the n-hugeness of κ, arising from j. Hence, $M \models \kappa$ is n-huge, and so we can take U to be the usual normal ultrafilter over κ corresponding to j. ⊣

Finally, the following result will establish the transcendence to the concepts that came before: <u>If κ is huge, there is a normal ultrafilter U over κ such that: whenever $\langle M_\alpha | \alpha < \kappa \rangle$ is a natural sequence, there is an $Y \in U$ so that $\alpha < \beta$ both in Y implies that there is an elementary embedding: $M_\alpha \to M_\beta$ with critical point α</u>.

⊢ Let $j: V \to M$ witness the hugeness of κ, and let U be the usual normal ultrafilter over κ corresponding to j. We show that this U works:

Let $\langle M_\alpha | \alpha < \kappa \rangle$ be any natural sequence. <u>Claim</u>: If $\alpha < \kappa$ and we set $X_\alpha = \{\xi < \kappa |$ there is an elementary embedding $M_\alpha \to M_\xi$ with critical point $\alpha\}$, then $T = \{\alpha < \kappa | X_\alpha \in U\} \in U$. Suppose for the moment that the Claim has been verified. Then by the normality of U, we have $Y = \{\alpha \in T | (\beta < \alpha \ \& \ \beta \in T)$ imply $\alpha \in X_\beta\} \in U$. Clearly this Y satisfies the conclusion, and we would be done.

Thus, it remains to establish the Claim. First, some notation:

$$j(\langle M_\alpha | \alpha < \kappa \rangle) = \langle M^*_\alpha | \alpha < j(\kappa) \rangle$$
$$j(\langle M^*_\alpha | \alpha < j(\kappa) \rangle) = \langle M^{**}_\alpha | \alpha < j^2(\kappa) \rangle$$
$$j(\langle X_\alpha | \alpha < \kappa \rangle) = \langle X^*_\alpha | \alpha < j(\kappa) \rangle$$

Notice that $\alpha < \kappa$ implies $M^*_\alpha = M_\alpha$. Now for any $\alpha < \kappa$, $X_\alpha \in U$ <u>iff</u> $\kappa \in j(X_\alpha)$ <u>iff</u> M_α is elementarily embeddable into M^*_κ with critical point α. So, if $\alpha < j(\kappa)$, by elementarity, $X^*_\alpha \in j(U)$ <u>iff</u> in M, M^*_α is elementarily embeddable into $M^{**}_{j(\kappa)}$ with critical point α. Hence, $T \in U$ <u>iff</u> $\kappa \in j(T)$ <u>iff</u> $X^*_\kappa \in j(U)$ <u>iff</u> in M, M^*_κ is elementarily embeddable into $M^{**}_{j(\kappa)}$ with critical point κ.

Now $j|M^*_\kappa: M^*_\kappa \to M^{**}_{j(\kappa)}$ is elementary. Also, $j|M^*_\kappa$ is just a set of ordered pairs of M of cardinality $|M^*_\kappa| < j(\kappa)$. Thus, since M is closed under $j(\kappa)$-sequences, we have $j|M^*_\kappa \in M$, and the Claim is proved by the previous paragraph. ⊣

Notice that in the above, surely $M \models (V_\kappa \models$ Vopěnka's Principle). Hence, $\{\alpha < \kappa | V_\alpha \models$ Vopěnka's Principle$\} \in M$. Though we have thus established the comparative strength of hugeness to Vopěnka's Principle, the proof typically shows much more. In fact, in §8 of SRK are isolated many principles which spread across from one concept to the other. We mention only one: a cardinal κ is <u>almost huge</u> <u>iff</u> there is a $j: V \to M$ with critical point κ so that ${}^{\lambda}M \subseteq M$ for every $\lambda < j(\kappa)$. This is $A_3(\kappa)$ of SRK, and it is there established that the principle is equivalent to the existence of a sequence of normal ultrafilters which satisfies a coherency property.

Hugeness has begun to make itself felt in various ways in set theory. As we mentioned in §11, Kunen(1974) established that Con(ZFC & there is a huge cardinal) implies Con(ZFC & there is an ω_2-saturated ideal over ω_1). Laver then amplified Kunen's conclusion to: Con(ZFC & there is a $(\omega_2,\omega_2,\omega)$-saturated ideal over ω_1)(see the end of §11 for the terminology). The existence of such an ideal has as a consequence the following polarized partition relation (for those familiar with the terminology):

$$\binom{\omega_2}{\omega_1} \rightarrow \binom{\omega_1}{\omega_1}_{\omega}$$

This shows that it is consistent that the answer to Problem 27 of Erdős-Hajnal[1971] is no (relative to the existence of a huge cardinal!). Magidor also used a variant of the Kunen argument to establish the relative consistency results about weakly normal, irregular ultrafilters cited at the end of §13. The following significant result on the Singular Cardinals Problem is also due to Magidor: Con(ZFC & there is a huge cardinal with a supercompact cardinal below it) implies Con(ZFC & $2^{\omega_\omega} = \omega_{\omega+2}$ yet $2^{\omega_n} = \omega_{n+1}$ for every $n \in \omega$) (see also §29).

At present, it is unclear how strong consistency-wise these various propositions about the lower orders of the cumulative hierarchy are. Indeed, hugeness may not be necessary, but the very fact that some upper bound on consistency strength has been established is quite significant, both as a plausibility argument for these propositions and as empirical evidence about hugeness. After some vicissitudes, Kunen's result is known to proceed already from almost hugeness. In this regard, it is interesting that the existence of a ω_2-saturated ideal over ω_1 implies that has a property which might appropriately be term "generic almost hugeness". This at least suggests the possibility of the converse to Kunen's result, i.e. the demonstration of the equiconsistency of the existence of an almost huge cardinal and the existence of a ω_2-saturated ideal over ω_1. Of course, this would make an elegant and unexpected direct connection between two ostensibly disparate large cardinal concepts.

We now discuss Kunen's result on the inconsistency of a possible extension of our guiding ideas. Supercompactness and extendibility were formulated with the realization that strong reflection properties can be achieved by imposing stringent closure properties on range structures of elementary embeddings. In the first flush of experience with these ideas, Reinhardt speculated on the possibility of an ultimate extension: Could there be an elementary embedding $j: V \to V$? Notice that by the Generalized Closure Lemma of §14, such a j cannot be rendered by an ultrapower. Partial results indicated that j would have extremely strong properties, and then Kunen[1971] was able to prove that there can be no such j, at least in the theory ZFC. To do this, he used a simple case of a combinatorial result of

Erdős-Hajnal[1966]. However, we present a proof of the general case, as it shows in particular that if we allow infinitary operations, there are Jonsson algebras (see §6) of every infinite cardinality. The proof relies on the Axiom of Choice, and so the result will constrast with certain Choice-less situations which we shall encounter in which "infinite exponent" partition relations hold (see §28).

For any set x, a function f is called ω-Jónsson over x iff $f: {}^{\omega}x \to x$ and whenever $y \subseteq x$ and $|y| = |x|$, then $f"^{\omega}y = x$. The Erdős-Hajnal result is: For every infinite cardinal λ, there is an ω-Jónsson function over λ. The following elegant presentation of the proof is due to Galvin-Prikry(1976).

$\vdash$ For the special case $\lambda = \omega$, there is a simple inductive argument available: Let $\{\langle X_\alpha, n_\alpha\rangle \mid \alpha < 2^{\omega}\}$ enumerate $[\omega]^{\omega} \times \omega$. (Remember that $[\omega]^{\omega}$ is the collection of infinite subsets of ω.) By induction on $\alpha < 2^{\omega}$ pick $s_\alpha \in {}^{\omega}X_\alpha$ so that $s_\alpha \neq s_\beta$ for every $\beta < \alpha$ and set $f(s_\alpha) = n_\alpha$. Then any extension of f to all of ${}^{\omega}\omega$ is ω-Jónsson over ω.

Suppose now that $\lambda > \omega$. Let S be a maximal collection of subsets of λ so that members of S have order type ω and are mutually almost disjoint (i.e. $x \neq y$ both in S implies $x \cap y$ is finite). By the special case above, we can assume that for each $x \in S$ there is a function f_x ω-Jónsson over x. Define now a function $g: {}^{\omega}\lambda \to \lambda$ by:

$$g(s) = \begin{cases} f_x(s) & \text{if the range of } s \text{ is infinite and } s \in {}^{\omega}x \text{ for some } x \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Then g is well-defined by the maximality of S.

It suffices to find an $A \subseteq \lambda$ so that $|A| = \lambda$, yet whenever $B \subseteq A$ and $|B| = \lambda$, then $g"^{\omega}B \supseteq A$. For such an A, an ω-Jónsson function over A can easily be derived from g. So, assume that no such A exists. Then there are sets $\lambda \supseteq A_0 \supseteq A_1 \supseteq A_2 \ldots$ each of cardinality λ, and $a_n \in (A_n - A_{n+1})$ so that $a_n \notin g" A_{n+1}$. If $y = \{a_n \mid n \in \omega\}$, by maximality of S there is an $x \in S$ so that $x \cap y$ is infinite. Let $x \cap y = \{a_{n_0}, a_{n_1}, \ldots\}$, where $n_0 < n_1 < \ldots$. Now $t = \{a_{n_1}, a_{n_2}, \ldots\} \subseteq A_{n_1}$. But by definition of f_x, there is an $s \in {}^{\omega}t$ so that $g(s) = a_{n_0}$. Hence $a_{n_0} \in g" A_{n_1}$, which is a contradiction. $\dashv$

We now establish Kunen's result. He actually showed: If $j: V \to M$ with critical point κ, then $P(\kappa_\omega) \not\subseteq M$.

$\vdash$ Consider the set $X = j"\kappa_\omega$. It suffices to establish that $X \notin M$. So, assume to the contrary that $X \in M$. Let f be ω-Jónsson over κ_ω. Then in M, $j(f)$ is ω-Jónsson over $j(\kappa_\omega) = \kappa_\omega$. So in M we have: since $|X| = \kappa_\omega$ there is an $x \in {}^{\omega}X$ so that $j(f)(x) = \kappa$. However, if $x(n) = j(\alpha_n)$ for $n \in \omega$, then $x = j(y)$, where $y \in {}^{\omega}\kappa_\omega$ with $y(n) = \alpha_n$. Hence, $\kappa = j(f)(j(y)) = j(f(y))$, contradicting the fact that the critical point of an elementary embedding is never in its range. $\dashv$

As Kunen himself remarks, since κ_ω is a strong limit cardinal of cofinality ω, the argument for the special case $\lambda = \omega$ in the proof of the Erdős-Hajnal result already suffices to produce an ω-Jónsson function over κ_ω.

Well, there we have it. That there is no $j: V \to V$ is cast into fairly specific form, and it is now readily seen how the n-huge cardinals fit into the scheme of things: they assert stronger and stronger closure properties, until their natural ω-ary extension turns out to be inconsistent. It is perhaps appropriate that once the "trick" was found, an inconsistency result should have a fairly simple form. Yet, Kunen's proof has a fortuitous feel, not being very closely tied to structural aspects of the elementary embedding. For example, the following assertions are not known to be inconsistent with ZFC:

I1. There is a $j: V_{\kappa_\omega+1} \to V_{\kappa_\omega+1}$.

I2. There is a $j: V \to M$ with $V_{\kappa_\omega} \subseteq M$.

I3. There is a $j: V_{\kappa_\omega} \to V_{\kappa_\omega}$.

Notice that in I1 we have specified that the range of j be included in $V_{\kappa_\omega+1}$, but this will be true since $j(\kappa_\omega) = \kappa_\omega$. Similarly for I3. To motivate these propositions, note that Kunen's argument establishes that there is no $j: V_{\kappa_\omega+2} \to V_{\kappa_\omega+2}$. This follows from the observation that what was needed for the argument to work was to have an ω-Jónsson function over κ_ω in the domain of the elementary embedding. But such a function is of form ${}^\omega\kappa_\omega \to \kappa_\omega$, and so occurs first in $V_{\kappa_\omega+2}$. In truth, I1 and I3 are the only possible forms that an axiom of the type "there is a non-trivial elementary embedding of some V_α into itself" can now take.

That I1 implies I2 follows from a self-extending technique of Gaifman[1974]; see its IV.8. That I2 implies I3 follows from the fact that if j is as in I2, since $j(\kappa_\omega) = \kappa_\omega$, $j(V_{\kappa_\omega}) = V_{\kappa_\omega}^M = V_{\kappa_\omega}$, so that $j|V_{\kappa_\omega}: V_{\kappa_\omega} \to V_{\kappa_\omega}$. Finally, that I3 implies there is a cardinal which is n-huge for every $n \in \omega$ is clear, since the ultrafilters which characterize n-hugeness are all retreivable from the embedding of I3.

In any case, I1, I2, and I3 ostensibly seem to differ in an inessential way from the proposition proved inconsistent by Kunen. Perhaps there is a more intrinsic argument about elementary embeddings which would disallow these propositions and possibly more. Or is the ω-Jónsson function an essential ingredient which incorporates the Axiom of Choice in its role of negating infinite exponent partition relations? Indeed, can one prove that there is no $j: V \to V$ without the Axiom of Choice? The situation at this veritable Gotterdämmerung for large cardinals needs to be clarified, but pending an answer to this last question, we can perhaps best view Kunen's result as a limitation imposed by the Axiom of Choice on the extent of reflection possible in the universe.

This section is concluded with an illustrative result: Suppose I3 holds, i.e. there is a $j: V_\lambda \to V_\lambda$ with a critical point κ so that $\lambda = \sup\{j^n(\kappa) \mid n \in \omega\}$. Then there is a $k: V_\lambda \to V_\lambda$ with critical point $j(\kappa)$ and $\lambda = \sup\{k^n(j(\kappa)) \mid n \in \omega\}$. Thus, we are dealing here with a large cardinal property for κ which implies the very same large cardinal property for $j(\kappa) > \kappa$. This illustrates the great strength of I3, especially in the context of our experience with extendibility.

$\vdash$ Given the j and κ, the idea behind the proof is simply to look at what would be $j(j)$, which exhibits all the desired properties. Alas, the set j is not in the domain of j, so we employ a piecing together argument.

Let us revert to the notation $\kappa_n = j^n(\kappa)$ for $n \in \omega$. Set $j_n = j|V_{\kappa_n}$ for each $n \in \omega$, so that $j_n: V_{\kappa_n} \to V_{\kappa_{n+1}}$ is elementary. The j_n's are certainly members of V_λ, so by an application of j we have in V_λ (and hence in V) that $j(j_n): V_{\kappa_{n+1}} \to V_{\kappa_{n+2}}$ is elementary with critical point κ_1. Set $k_n = j(j_n)$. Then observe that $k_0 \subseteq k_1 \subseteq k_2 \ldots$. Consider $k = \bigcup_n k_n$. It now suffices to establish the Claim: $k: V_\lambda \to V_\lambda$ is elementary with critical point κ_1 so that $k(\kappa_n) = \kappa_{n+1}$ for every $n > 0$.

Only elementarity need be checked. For this purpose, notice that by elementarity of j, we have $V_\kappa \prec V_{\kappa_1}$. It follows by repeated application of j that $V_{\kappa_n} \prec V_{\kappa_{n+1}}$ for every $n \in \omega$. Hence, $V_{\kappa_n} \prec V_\lambda$ for every $n \in \omega$ by union of elementary chains. Now argue as follows to finish the proof: If $V_\lambda \models \phi[a_1,\ldots,a_m]$ let n be sufficiently large so that $a_1,\ldots,a_m \in V_{\kappa_{n+1}}$. Since $V_{\kappa_{n+1}} \prec V_\lambda$, $V_{\kappa_{n+1}} \models \phi[a_1,\ldots,a_m]$. By an application of $k|V_{\kappa_{n+1}} = k_n$, we have $V_{\kappa_{n+2}} \models \phi[k(a_1),\ldots,k(a_m)]$. Thus, $V_\lambda \models \phi[k(a_1),\ldots,k(a_m)]$, as $V_{\kappa_{n+2}} \prec V_\lambda$. $\dashv$

V. CARDINAL COLLAPSING AND COMBINATORIAL PRINCIPLES

§18. The Lévy Collapse, Lebesgue Measurability, and Scattered Sets

Having dealt in the preceding chapters with the formulation and elaboration of large cardinal concepts, in the rest of this paper we shift our focus of attention to their relative consistency results, tracing the history through the post-Cohen era. The present chapter in particular deals primarily with consistency results involving interesting combinatorial hypotheses on the lower orders of the cumulative hierarchy.

We have already seen, for example in §6 and §12, how the method of inner models provides a way of getting the consistency of the existence of large cardinals. Some interesting existential assumption is made about an accessible cardinal κ say, and somewhat unexpectedly, κ assumes enormous proportions in some inner model, as the assumption translates in the slender inner domain into a clear, large cardinal assertion. The method of forcing provides a converse process. A large cardinal hypothesis imposed on the ground model can often be transmuted into a non-trivial combinatorial property in a forcing extension, typically by a forcing which collapses the large cardinal to some accessible cardinal like ω_2. It is a noteworthy phenomenon that actual equiconsistency results have been established in this fashion, between natural but ostensibly disparate hypotheses. The crystalline clarity of an inner model reveals a large cardinal, and forcing adds the needed camouflage to view it in a combinatorial guise.

This is perhaps the main source of empirical evidence for the efficacy and inevitability of the theory of large cardinals. The spectrum of large cardinals provides a linear scale, which acts as a standard to measure the consistency strength of many hypotheses superposed on ZF and intended to further elucidate the nature of sets. (We remind the reader of the obvious analogy to the ordinals which measure the proof-theoretic strength of various subsystems of analysis.) Not only can consistency strengths be compared in this way, but upper bounds can be established on the spectrum, and this provides definite plausibility arguments. In the absence of convincing philosophical reasons for accepting various new set theoretic axioms or combinatorial principles, we can at best provide mathematical elucidation of such formal interrelationships as these.

Soon after Cohen established forcing as a general meta-mathematical technique, Lévy devised a basic method of generically collapsing a limit cardinal which, significantly enough, necessitates an inaccessible cardinal. This method has strong uniform properties which makes it widely applicable, and in particular was used by Solovay already in 1964 to establish a result which even today rivals any other as the most mathematically significant result obtained by forcing since Cohen's initial

work: the demonstration of the consistency, relative to the existence of an inaccessible cardinal, of the proposition that every set of reals is Lebesgue measurable. In the context of large cardinals, we have already cited a fairly typical use of Lévy's method at the end of §12 in connection with precipitous ideals.

Let us turn forthwith to the definition of the notion of forcing which in a special case gives the Lévy collapse. So assume λ is a regular cardinal and X is any set. By $\mathrm{Col}(\lambda,X)$ we denote the notion of forcing where the conditions are partial functions f with domain $\subseteq \lambda\times X$ so that: $f(\langle\alpha,x\rangle) \in x$ for every $\langle\alpha,x\rangle \in \mathrm{Domain}(f)$, and $|f| < \lambda$. A condition f is to be <u>stronger than</u> a condition g <u>iff</u> $f \supseteq g$.

First note the preliminary <u>Fact 1: $\mathrm{Col}(\lambda,X)$ is $<\lambda$-closed</u>, i.e. whenever $\gamma < \lambda$ and $\langle f_\xi \mid \xi < \gamma\rangle$ is a sequence of conditions such that $\xi < \zeta$ implies f_ζ is stronger than f_ξ, then there is a condition stronger than all the f_ξ's—just take the set-theoretic union, using the regularity of λ. Thus, by standard forcing lore, <u>λ is a cardinal in any generic extension via $\mathrm{Col}(\lambda,X)$.</u>

We now proceed to the principal case $X = \kappa$ an inaccessible cardinal $> \lambda$; then $\mathrm{Col}(\lambda,\kappa)$ is a notion of forcing for a <u>Lévy collapse</u>. The intent of the Lévy collapse is clear: if G is any $\mathrm{Col}(\lambda,\kappa)$-generic filter over V construed as the ground model, then $\bigcup G$ is a function in $V[G]$. Moreover, whenever $\lambda \leq \beta < \kappa$, f_β defined by $f_\beta(\alpha) = (\bigcup G)(\langle\alpha,\beta\rangle)$ is a surjection of λ onto β, so that $|\beta|^{V[G]} = \lambda$. It follows that $\kappa \leq \lambda^{+V[G]}$. We would like κ to be a cardinal in $V[G]$, and this is where the inaccessibility of κ in V is essentially invoked: <u>Fact 2: $\mathrm{Col}(\lambda,\kappa)$ has the κ-c.c. (and hence κ, being a cardinal in $V[G]$, $\kappa = \lambda^{+V[G]}$).</u>

$\vdash$ The argument is of a familiar sort, the prototype of which is due to Cohen himself, but with an essential use of inaccessibility. Suppose that $Q \subseteq \mathrm{Col}(\lambda;\kappa)$ consists of pairwise incompatible conditions; we must establish that $|Q| < \kappa$. To show this, we inductively construct a sequence $\langle Q_\alpha \mid \alpha < \lambda\rangle$ so that each $Q_\alpha \subseteq Q$ with $|Q_\alpha| < \kappa$, and $\alpha < \beta < \lambda$ implies $Q_\alpha \subseteq Q_\beta$. The construction will be aided by setting $s_\alpha = \bigcup\{\mathrm{Domain}(p) \mid p \in Q_\alpha\}$ along the way, so that in particular $|s_\alpha| < \kappa$.

To begin, set $Q_0 = \{q\}$ where $q \in Q$ is arbitrary. At limits $\gamma < \lambda$, just set $Q_\gamma = \bigcup\{Q_\alpha \mid \alpha < \gamma\}$; by the regularity of κ and inductive assumptions, $|Q_\gamma| < \kappa$. Finally, at successors $< \lambda$, having already defined Q_α with the corresponding s_α, first consider an equivalence relation on Q by: $p \underset{\alpha}{\sim} q$ <u>iff</u> $p,q \in Q$ and $p|s_\alpha = q|s_\alpha$. Then let $Q_{\alpha+1}$ consist of members of Q_α together with exactly one member from each $\underset{\alpha}{\sim}$-equivalence class. By the inaccessibility of κ, a counting argument shows that $|Q_{\alpha+1}| \leq |Q_\alpha| + 2^{|s_\alpha|} < \kappa$. The inductive definition is complete.

The claim is now that $Q = \bigcup\{Q_\alpha \mid \alpha < \lambda\}$, and by the regularity of κ this would surely establish $|Q| < \kappa$. To demonstrate this claim, suppose $p \in Q$ is arbitrary. If we set $s = \bigcup\{s_\alpha \mid \alpha < \lambda\}$, then since $|p| < \lambda$ and λ is regular, it follows that $p|s = p|s_\alpha$ for some $\alpha < \lambda$. So, by the construction of $Q_{\alpha+1}$, there is a $\bar{p} \in Q_{\alpha+1}$ so that $\bar{p}|s_\alpha = p|s_\alpha$. To complete the argument we need only observe that $\bar{p} = p$. If there were not true, then p and $\bar{p}$ would have to be incompatible. However, $\mathrm{Domain}(\bar{p}) \subseteq s_{\alpha+1}$ and $p|s_{\alpha+1} \subseteq p|s = p|s_\alpha$, and so p and $\bar{p}$ must disagree on s_α. This contradicts $\bar{p}|s_\alpha = p|s_\alpha$, and we are done. $\dashv$

Hence, forcing with $\mathrm{Col}(\lambda,\kappa)$ preserves cardinals $\leq \lambda$ and $\geq \kappa$, and closes the gap precisely by making κ the successor cardinal of λ in the extension. One of the endearing qualities of the Lévy collapse is that it is amenable to a simple product analysis. For any δ so that $\lambda < \delta < \kappa$, we have $\mathrm{Col}(\lambda,\kappa) \cong \mathrm{Col}(\lambda,\delta) \times \mathrm{Col}(\lambda,\kappa-\delta)$ under an obvious isomorphism (where a product of partially ordered sets is understood to carry the natural partial order defined componentwise). A basic precept of forcing can then be invoked to show Fact 3: If G is $\mathrm{Col}(\lambda,\kappa)$-generic over V, then $G \cap \mathrm{Col}(\lambda,\delta)$ is $\mathrm{Col}(\lambda,\delta)$-generic over V, $G \cap \mathrm{Col}(\lambda,\kappa-\delta)$ is $\mathrm{Col}(\lambda,\kappa-\delta)$-generic over $V[G\cap \mathrm{Col}(\lambda,\delta)]$, and $V[G] = V[G\cap\mathrm{Col}(\lambda,\delta)][G\cap\mathrm{Col}(\lambda,\kappa-\delta)]$. Intuitively, this is saying that forcing with $\mathrm{Col}(\lambda,\kappa)$ can be divided into two steps, one which first adds the generic surjective maps: $\lambda \to \beta$ for every $\beta < \delta$, and then one which adds the rest. This component analysis is useful in conjunction with the following, proved by a chain argument like in the preceding proof: Fact 4: If G is $\mathrm{Col}(\lambda,\kappa)$-generic over V, and in $V[G]$, $f: \lambda \to a$ where $a \in V$, then $f \in V[G\cap\mathrm{Col}(\lambda,\delta)]$ for some $\delta < \kappa$.

Solovay went further to establish an important lemma. Taking $\lambda = \omega$, notice that since $\mathrm{Col}(\omega,\kappa)$ consists of finite conditions, it is absolute for transitve models of set theory containing κ. Solovay proved the Factoring Lemma: If G is $\mathrm{Col}(\omega,\kappa)$-generic over V, and in $V[G]$, $f: \omega \to a$ where $a \in V$, then $V[G] = V[f][H]$ for some H which is $\mathrm{Col}(\omega,\kappa)$-generic over $V[f]$. He also noticed the following "homogeneity" result: If $a_1,\ldots,a_n \in V$ and ϕ is a formula in the forcing language for $\mathrm{Col}(\omega,\kappa)$ with no constants and n free variables, then the empty condition forces $\phi[\check{a}_1,\ldots,\check{a}_n]$ or $\neg\phi[\check{a}_1,\ldots,\check{a}_n]$.

(There is an attractive characterization of $\mathrm{Col}(\omega,\kappa)$ in Boolean algebraic terms which incorporates these features. Jensen showed that the usual complete Boolean algebra corresponding to $\mathrm{Col}(\omega,\kappa)$ is a unique κ-saturated structure of cardinality κ in a precise, model-theoretic sense. See Mathias[1977], Theorem 3.17.)

Combining these facts about the Lévy collapse with further genericity and definability arguments, Solovay was able to achieve his impressive result. Say that a set x is real,ordinal definable iff there is an ordinal α, a set $a \subseteq \omega$, and a formula $\phi(\cdot,\cdot)$ so that: $y \in x$ iff $V_\alpha \models \phi(y,a)$. This definition within set theory adequately captures the intuitive notion, by the Reflection Principle. Let HROD be the class of hereditarily real, ordinal definable sets, i.e. those x such that every element in the transitive closure of $\{x\}$ is real, ordinal definable. HROD can be shown to be an inner model of ZF, much as the better

known HOD is, in Myhill-Scott[1971]. Now for the considerable harvest of Solovay's efforts:

If there is an innaccessible cardinal κ in the ground model V and G is $\mathrm{Col}(\omega,\kappa)$-generic over V, then $(\mathrm{HROD})^{V[G]}$ is a model of ZF satisfying:

(L) Every set of reals is Lebesgue measurable.

(B) Every set of reals has the Baire property.

(P) Every uncountable set of reals contains a perfect subset.

(DC) The Principle of Dependent Choices.

Some definitions for the uninitiated: A set of reals X has the Baire property iff there is an open set O so that the symmetric difference of X and O is of first category. A set of reals is perfect iff it is closed and contains no isolated points. For good measure, we might as well state that a random real is a real generic for the following notion of forcing: conditions are Lebesgue measurable sets of reals of positive measure, and X is a stronger condition than Y iff $X - Y$ has Lebesgue measure zero. Solovay introduced the important concept of random real to establish (L) in the model, and companion arguments using Cohen-generic reals instead establish (B). (P) is a significant statement about the structure of sets of reals, which we shall discuss at some length below.

The Principle of Dependent Choices states that whenever R is a binary relation so that for every x there is a y so that $\langle x,y\rangle \in R$, there is a function with domain ω so that for every $n \in \omega$, $\langle f(n),f(n+1)\rangle \in R$. It is just the restricted choice principle needed to demonstrate the existence and main properties of Lebesgue measure. It is well-known that various other choice principles must fail in Solovay's model; for instance, there can be no (non-principal) ultrafilter over ω, as such an object, construed as a set of reals, cannot be Lebesgue measurable. A poignant failure of AC is the result of Mathias[1977] that: $\omega \to (\omega)^\omega_2$ holds in Solovay's model. Infinite exponent partition relations of considerably stronger consistency strength will play a key role in §28.

It is generally thought that the consistency strength of inaccessibility is not needed to get the consistency of (L) + (DC). But barring any fast-breaking developments, concerted efforts have not thus far overcome this obstacle. However, even if this were surmounted, Solovay's ideas will no doubt remain basic cornerstones.

One can measure partial successes along the projective hierarchy. It is a classical result of descriptive set theory that Σ^1_1 sets are Lebesgue measurable. Gödel's demonstration of a Δ^1_2-well-ordering of the reals in L indicates that this is as much as the outright provability strength of ZF can muster, since any well-ordering of the reals in type ω_1 cannot be Lebesgue measurable in the plane, by Fubini's Theorem. Part of Solovay's windfall was the following elegant characterization of the Lebesgue measurability of Σ^1_2 sets: For any real a, every Σ^1_2(in a) set of reals is Lebesgue measurable iff almost every real (in the sense of

Lebesgue) is a random generic real over L[a]. This condition follows, for example, from the assertion that $\omega_1^{L[a]}$ is countable. A forcing argument (via $MA(2^{\omega})$ & $\omega_1 < 2^{\omega}$ if desired; see Martin-Solovay[1970],p.169) yields: Con(ZF) implies Con (ZFC & every Σ^1_2 set of reals is Lebesgue measurable). At present, it is not known whether Σ^1_2 here can be replaced by Δ^1_3 .

The bearing that all this has on this paper is simply that the existence of a measurable cardinal (or just the existence of $a^{\#}$ for every $a \subseteq \omega$) certainly implies $\omega_1^{L[a]}$ is countable for every real a by §7 . Thus, if there is a measurable cardinal, then every Σ^1_2 set of reals is Lebesgue measurable. This result is an elegant and unexpected foray of measurability into descriptive set theory. Even then, Silver's demonstration of a Δ^1_3-well-ordering of the reals in $L[U]$, where U is a normal ultrafilter over a measurable cardinal, establishes this as a limit on the provability strength of (ZF & there is a measurable cardinal).

We conclude this section with a discussion of aspects of the proposition (P). It is an interesting thread through history that (P) is the ultimate statement of the sort that Cantor himself sought in his initial efforts to affirm the Continuum Hypothesis, after observing the simple fact that any perfect set of reals must have cardinality 2^{ω} . Unfortunately, Cantor could only proceed as far as the Cantor-Bendixon Theorem, which shows in particular that every closed uncountable set of reals contains a perfect subset. It is well-known nowadays that (P) violates the Axiom of Choice. In a result remarkable because of its early date, Specker[1957] §2.32 established that (P) implies that ω_1 is inaccessible in L . Hence, Solovay's result combines with this to show that (in contradistinction to the prevalent opinion about (L) + (DC)): Con(ZFC & there is an inaccessible cardinal) iff Con(ZFC & (P)). This is the earliest significant equiconsistency result concerning large cardinals. With the plethora of relative consistency results now available to us, there is perhaps a tendency to get a bit jaded by it all, but through fresh eyes this is undoubtedly a striking statement about the structure of sets of reals, indicative of how far set theory has progressed since Cantor.

Let us call a set of reals scattered iff it has no perfect subset. Thus, (P) says that every scattered set of reals is countable. Of partial versions of (P) along the projective hierarchy, it is a classical result that every scattered Σ^1_1 set of reals is countable. Again, L delimited the provability strength of ZF : Gödel[1938] established that in L there is an uncountable, scattered Π^1_1 set of reals. (More recently, Guaspari(1973), Sacks[1976] and others have observed that there is a largest scattered (lightface) Π^1_1 set of reals, i.e. a set C_1 which is scattered and Π^1_1 , yet whenever X is also scattered and Π^1_1 , then $X \subseteq C_1$. This set has the characterizations $C_1 = \{a \mid a \in L_{(\omega_1^a)}\} = \{a \mid$ for every real b, $\omega_1^a \leq \omega_1^b$ iff a is hyperarithmetic in b $\}$, where ω_1^a is the least ordinal not recursive in a . The first characterization makes it clear that $C_1 \subseteq L$, but the second

is more amenable to recursion theoretic arguments. See Kechris[1975] for details, as well as generalizations under Projective Determinacy.)

Since Gödel, scattered Π^1_1 sets have been fully investigated: We might as well look at Σ^1_2 sets, since an argument using Π^1_1-uniformization shows that every scattered Π^1_1 set is countable iff every scattered Σ^1_2 set is countable. In a germinal paper for contemporary descriptive set theory, Solovay[1969] established the following elegant characterization: For any real a , every scattered Σ^1_2(in a) set is countable iff $\omega_1^{L[a]}$ is countable. For the forward direction, if to the contrary $\omega_1^{L[a]} = \omega_1$, then the Δ^1_2(in a)-well-ordering of the reals in L[a], being especially good, can be used to concoct a scattered Π^1_1(in a) set which is uncountable. The converse direction was given a clear formulation by Mansfield [1970]: If a Σ^1_2(in a) set contains a real $\notin$ L[a], then it is not scattered. Mansfield's argument really brings out the essence; in modern terminology, it uses the fact that Σ^1_2(in a) is ω_1-Souslin over L[a].

Again, measurability impinges on the theory. Since the existence of a measurable cardinal (or even $a^{\#}$ for every $a \subseteq \omega$) implies that $\omega_1^{L[a]}$ is countable for every real a , we have: If there is a measurable cardinal, then every scattered Σ^1_2 set is countable. And again, if U is a normal ultrafilter over a measurable cardinal, then $L[U]$ sets a delimitation: If $\omega_1^{L[U]} = \omega_1$, then the Δ^1_3-well-ordering of the reals in $L[U]$, being especially good, can be used to concoct a scattered Π^1_2 set which is uncountable.

As with Lebesgue measurability, questions about scatteredness can be decided exactly one level further on the projective hierarchy by adding the strength of measurability to ZF . Further results in descriptive set theory attest to this phenomenon; see for example Martin-Solovay[1969]. That such a precise state of affairs can exist about definable sets of reals and large cardinals was quite a revelation about the structure of the set theoretical universe.

§19. Kurepa's Hypothesis and Chang's Conjecture

A classical question posed by Kurepa[1935][1936] provided a topic for discussion which eventually led to an early independence result, essentially involving large cardinals, concerning a transparently combinatorial assertion about infinite sets. Kurepa asked whether there can be an $F \subseteq P(\omega_1)$ with $|F| \geq \omega_2$ so that $\{X \cap \alpha \mid X \in F\}$ is countable for every $\alpha < \omega_1$. Such an F is nowadays called a Kurepa family, and Kurepa's Hypothesis (KH) is the assertion that there is a Kurepa family (although apparently Kurepa himself conjectured that KH was false!).

There is a convenient formulation of KH in terms of trees. Let us define a Kurepa tree to be, in the terminology of §5, an ω_1-tree with at least ω_2 ω_1-branches. Then KH holds iff there is a Kurepa tree.

$\vdash$ Suppose that F is a Kurepa family. For each $X \subseteq \omega_1$ let $f_X : \omega_1 \to 2$ denote its representing function. Then the tree with elements $\{f_X | \alpha \mid X \in F \ \& \ \alpha < \omega_1\}$

ordered by inclusion is a Kurepa tree. Conversely, if $\langle T, <_T \rangle$ is a Kurepa tree, we can inductively reconstitute it so that $T \subseteq \omega_1$ and $\alpha <_T \beta$ implies $\alpha < \beta$. It can then be checked that the ω_2 ω_1-branches of T comprise a Kurepa family. ⊣

Interestingly enough, the first consistency result bearing on KH employed the Lévy collapse. As early as 1963, Lévy and Rowbottom (independently) observed that if there is an inaccessible cardinal κ in the ground model V, and κ is Lévy collapsed to ω_1 (i.e. force with $\mathrm{Col}(\omega,\kappa)$), then $P(\kappa)^V$ is a Kurepa family in the extension. Actually, the inaccessible cardinal turned out to be a red herring here, as Stewart[1966] was able to concoct forcing conditions for adjoining a Kurepa tree to models of just ZFC. Where inaccessibility became essential was in establishing the independence of KH. In 1967, Silver (see Silver[1971a]) showed that if there is an inaccessible cardinal κ and κ is Lévy collapsed to ω_2 (i.e. force with $\mathrm{Col}(\omega_1,\kappa)$), then KH fails in the extension.

⊢ The proof uses the product analysis of the Lévy collapse explicated in Fact 3 of §18. Let G be $\mathrm{Col}(\omega_1,\kappa)$-generic over V, so that in particular $\omega_1 = \omega_1^{V[G]}$ and $\kappa = \omega_2^{V[G]}$ by Facts 1 and 2 of §18. We must extablish that in $V[G]$ there is no Kurepa tree.

So, suppose $\langle T, <_T \rangle$ is an ω_1-tree in $V[G]$; as $\omega_1 = \omega_1^{V[G]}$ there can be no ambiguity here or hereafter in the proof. We can assume as in the previous proof that $T \subseteq \omega_1$ and $<_T \subseteq \omega_1 \times \omega_1$, and hence by Fact 4 of §18, that $\langle T, <_T \rangle \in V[G \cap \mathrm{Col}(\omega_1,\delta)]$ for some $\delta < \kappa$. Now $|\mathrm{Col}(\omega_1,\delta)| < \kappa$, so forcing with $\mathrm{Col}(\omega_1,\delta)$ preserves the (strong) inaccessibility of κ, by standard arguments. Thus, within $V[G \cap \mathrm{Col}(\omega_1,\delta)]$, $\langle T, <_T \rangle$ has $< \kappa$ ω_1-branches.

By Fact 3 of §18, $V[G] = V[G \cap \mathrm{Col}(\omega_1,\delta)][G \cap \mathrm{Col}(\omega_1,\kappa-\delta)]$ where $G \cap \mathrm{Col}(\omega_1,\kappa-\delta)$ is $\mathrm{Col}(\omega_1,\kappa-\delta)$-generic over $V[G \cap \mathrm{Col}(\omega_1,\delta)]$. Notice that $\mathrm{Col}(\omega_1,\kappa-\delta)$ is a ω-closed notion of forcing. So, the following lemma will complete the proof (if we apply it with $V[G \cap \mathrm{Col}(\omega_1,\delta)]$ in the role of the ground model V):

Lemma: If in a ground model V, T is an ω_1-tree and Q is a ω-closed notion of forcing, then every ω_1-branch through T in any generic extension via Q is already in V.

To show this, suppose $q \in Q$ and $q \Vdash \dot{b}$ is a $\check{\omega}_1$-branch through $\check{T}$. Assume that there is no p stronger than q and $c \in V$ so that $q \Vdash \check{c} = \dot{b}$, else a standard density argument will insure the result. We derive a contradiction from this by building a perfect subtree of T in V. For every $s \in \bigcup_{n\in\omega} {}^n 2$ define conditions $q_s \in Q$ and elements $t_s \in T$ as follows:

Set $q_{\langle\rangle} = q$. If q_s is already produced for some $s \in \bigcup_{n\in\omega} {}^n 2$, let $t_{s^\frown\langle 0\rangle}$ and $t_{s^\frown\langle 1\rangle}$ be at the same level of T and conditions $q_{s^\frown\langle 0\rangle}$ and $q_{s^\frown\langle 1\rangle}$ both stronger than q_s so that:

$$q_{s^\frown\langle 0\rangle} \Vdash \check{t}_{s^\frown\langle 0\rangle} \in \dot{b} \text{, and } q_{s^\frown\langle 1\rangle} \Vdash \check{t}_{s^\frown\langle 1\rangle} \in \dot{b} .$$

(If this were not possible, then we could define in V a $c \subseteq T$ by: $t \in c$ iff $r \Vdash \check{t} \in \dot{b}$ for some r stronger than q_s. Then c would be an ω_1-branch through T in V, and clearly $q_s \Vdash \check{c} = \dot{b}$, contradicting our assumption about q.)

Since $\{t_s \mid s \in \bigcup_{n\in\omega} {}^n 2\}$ is countable, there must be a $\gamma < \omega_1$ so that all the t_s are at levels $< \gamma$. By the ω-closure of Q, for every $f \in {}^\omega 2$ there is a condition q_f stronger than $q_{f|n}$ for every $n \in \omega$. Since all these conditions are stronger than q, there must be a p_f stronger than q_f and a t_f at the γth level of T so that: $p_f \Vdash \check{t}_f \in \dot{b}$. Clearly, $f \neq g$ implies $t_f \neq t_g$, so that the γth level of T has 2^ω elements, contradicting the fact that T is an ω_1-tree.

This completes the proof of the Lemma, and hence of the theorem. ⊣

Constructibility considerations completed the picture. Solovay's construction of a Kurepa tree in L, together with Jensen's demonstration of the failure of Souslin's Hypothesis in L, were the earliest examples (after Godel's!) of the use of the constructible universe to establish the relative consistency of combinatorial propositions. Solovay's construction (like Jensen's) was so uniform that in fact he was able to show: If $V = L[A]$ for some $A \subseteq \omega_1$, then KH holds. (For a proof see Jech[1971], which is a good reference for this section. Jensen's study of generalized versions of KH holding in L is discussed in §20.) The point of Solovay's general version is that we can derive from it that: If KH fails, then ω_2 is inaccessible in L.

⊢ Suppose by way of contradiction that ω_2 is not inaccessible in L. In any case ω_2 is still regular in L, so as L models GCH, there must then be a $\rho < \omega_2$ so that $(\rho^+)^L = \omega_2$. Surely we can find an $A \subseteq \omega_1$ coding enough maps so that $\omega_1^{L[A]} = \omega_1 = |\rho|^{L[A]}$. It follows that $\omega_2 = (\rho^+)^L \leq (\rho^+)^{L[A]} = \omega_2^{L[A]} \leq \omega_2$, so equality pervades. By Solovay's result, there is a Kurepa tree T in the sense of $L[A]$. However, $\omega_1^{L[A]} = \omega_1$ and $\omega_2^{L[A]} = \omega_2$, so that T (really) is a Kurepa tree, contradicting the hypothesis. ⊣

We can now conclude with the attractive: Con(ZFC & there is an inaccessible cardinal) iff Con(ZFC & KH fails). This is an exact measurement of the strength of the failure of KH. Even before this result, Prikry had established a direct connection between a large cardinal property and the failure of a weak form of KH. By the weak Kurepa's Hypothesis (wKH) let us mean the assertion that there are functions $\{f_\alpha \mid \alpha < \omega_2\} \subseteq {}^{\omega_1}\omega$ which are eventually different, i.e. whenever $\alpha < \beta < \omega_2$ there is a $\gamma_{\alpha\beta} < \omega_1$ so that $f_\alpha(\delta) \neq f_\beta(\delta)$ for $\gamma_{\alpha\beta} \leq \delta < \omega_1$. KH easily implies wKH (yet recently Baumgartner established that Con(ZFC & there is an inaccessible cardinal) iff Con(ZFC & wKH holds yet KH fails)). Prikry's early result was that if there is a (non-trivial) ω_2-saturated ideal over ω_1, then wKH fails. However, the hypothesis here is unduly strong, with consistency strength at least that of the existence of many measurable cardinals (as cited in §11). Silver then showed that the failure of wKH is a consequence of a Rowbottom-type property, and investigated its consistency strength. Because of several thematic connections, we take time to deal with this property in some detail.

Recalling the ideas and notation of §6, among the non-trivial $\twoheadrightarrow$ relations studied by Chang and others in the context of model theory, the one at lowest level not known to be refutable in ZFC has come to be called the Chang's Conjecture: $\langle\omega_2,\omega_1\rangle \twoheadrightarrow \langle\omega_1,\omega\rangle$. By a version of Rowbottom's model theoretic characterization (§6), Chang's Conjecture holds iff whenever $f\colon [\omega_2]^{<\omega} \to \omega_1$, there is an $X \subseteq \omega_2$ so that $|X| = \omega_1$ and $|f``[X]^{<\omega}| \leq \omega$. The chief interest in Chang's Conjecture lies in the possibility of such a strong partition property obtaining at a low level of the cumulative hierarchy. Silver (and probably others) observed that: If Chang's Conjecture holds, then wKH fails, and $0^{\#}$ exists.

$\vdash$ A weak version of the partition formulation of Chang's Conjecture already serves to disallow wKH. If $\{f_\alpha \mid \alpha < \omega_2\} \subseteq {}^{\omega_1}\omega$ were a collection of ω_2 eventually different functions, then define $G\colon [\omega_2]^2 \to \omega_1$ by $G(\{\alpha,\beta\})$ = least γ so that $f_\alpha(\delta) \neq f_\beta(\delta)$ for $\gamma \leq \delta < \omega_1$. Assume that there is an $X \subseteq \omega_2$ with $|X| = \omega_1$ so that $|G``[X]^2| \leq \omega$. Let $\eta = \sup(G``[X]^2) < \omega_1$. Then $\{f_\alpha(\eta) \mid \alpha \in X\}$ constitute ω_1 different ordinals $< \omega$, a contradiction.

To get $0^{\#}$ from Chang's Conjecture, let $\langle A,R,E\rangle \prec \langle L_{\omega_2},\omega_1,\varepsilon\rangle$, where $|A| = \omega_1$ and $|R| = \omega$. Let $j\colon \langle L_\gamma,\delta,\varepsilon\rangle \simeq \langle A,R,E\rangle$ be the inverse of the unique transitization map, where $\delta < \omega_1 \leq \gamma$. So $j\colon \langle L_\gamma,\delta,\varepsilon\rangle \to \langle L_{\omega_2},\omega_1,\varepsilon\rangle$ is elementary and shifts some ordinal $< \omega_1$, as $j(\delta) = \omega_1$. Hence, if ρ is the critical point of j, then $P(\rho)\cap L \subseteq L_{(\rho^+)^L} \subseteq L_{\omega_1} \subseteq L_\gamma$, so we are exactly in the position of being able to define an ultrafilter on $P(\rho)\cap L$ as in the proof in §10 for the existence of $0^{\#}$. $\dashv$

With this last result in mind, the following theorem of Silver establishes a reasonable upper bound on the consistency strength of Chang's Conjecture: Con(ZFC & there is a Ramsey cardinal) implies Con(ZFC & Chang's Conjecture). For the experts, we give a sketch of Silver's attractive proof, which is a cocktail of many ideas:

$\vdash$ Suppose that λ is Ramsey. Then by results cited in the second paragraph of §6, λ is inaccessible and $\lambda \to (\lambda)^{<\omega}_\gamma$ for every $\gamma < \lambda$.

We need the following Fact about "mild" Cohen extensions: If P is a notion of forcing such that $|P| < \lambda$, then in any generic extension via P, λ is still Ramsey. To show this, if $p \in P$ and $p \Vdash f\colon [\lambda]^{<\omega} \to 2$, define $g\colon [\lambda]^{<\omega} \to P(P\times 2)$ by $g(s) = \{\langle q,i\rangle \mid q$ is stronger than p and $q \Vdash f(\check{s}) = i\}$. As $|P(P\times 2)| < \lambda$, by a remark in the previous paragraph let $X \subseteq \lambda$ be homogeneous for g so that $|X| = \lambda$. Then $p \Vdash (\check{X}$ is homogeneous for f & $|\check{X}| = \lambda)$, the last conjunct being immediate from the λ-c.c. of P.

By appealing to this Fact, we can assume from now on that we have Martin's Axiom $MA(2^\omega)$ and $\omega_1 < 2^\omega < \lambda$, since this involves only a small cardinality extension.

Consider the following modification of $Col(\omega_1,\lambda)$, the usual Lévy collapse of λ to ω_2: Let Q be the collection of partial functions with domain $\subseteq \omega_1 \times \lambda$ so

that $f(\langle\alpha,\beta\rangle) \in \beta$ for every $\langle\alpha,\beta\rangle \in$ Domain(f), meeting the following two conditions: (i) $|\{\beta \mid \exists\alpha(\langle\alpha,\beta\rangle \in \text{Domain}(f))\}| \leq \omega_1$, and (ii) $\sup\{\alpha \mid \exists\beta(\langle\alpha,\beta\rangle \in \text{Domain}(f))\} < \omega_1$. Thus, instead of the usual condition deciding a $\omega\times\omega$ part of the generic map, we decide an oblong $\omega\times\omega_1$ part. Ordering Q by inclusion, combinatorial arguments as in §18 establish that Q is a ω-closed, λ-c.c. notion of forcing which collapses λ to ω_2. We now show that any generic extension via Q satisfies Chang's Conjecture.

To this end, suppose that $q \in Q$ and $q \Vdash f\colon |\dot{\omega}_2|^{<\omega} \to \dot{\omega}_1$. (Here, of course, we can set the forcing terms $\dot{\omega}_2 = \lambda$ and $\dot{\omega}_1 = \omega_1$.) We must produce a condition $\bar{q}$ extending q and a term τ so that $\bar{q} \Vdash (|f``[\tau]^{<\omega}| \leq \omega$ & $|\tau| = \dot{\omega}_1)$.

Define a relation $R \subseteq ([\lambda]^{<\omega} \times \omega_1 \times Q)$ by: $\langle s,\delta,p\rangle \in R$ iff p extends q and $p \Vdash f(\check{s}) = \delta$. Consider the structure $\mathfrak{A} = \langle V_\lambda,\varepsilon,\omega_1,Q,\{q\},R,\ldots\rangle$, where ... denotes a complete set of Skolem functions (in a countable language). By the Ramseyness of λ, let $H \subseteq \lambda$ be λ indiscernibles for this structure, and let $\mathfrak{B} = \langle B,\ldots\rangle \prec \mathfrak{A}$ be the Skolem Hull of the first ω_1 members of H in $\mathfrak{A}$ via the listed Skolem functions. Notice that $B\cap\omega_1$ is countable, by a standard indiscernibility argument: First of all, for any term t in n free variables and $\vec{x} < \vec{y}$ both in ${}^nH$, whenever $t(\vec{x}) < \omega_1$ we must have $t(\vec{x}) = t(\vec{y})$. (Otherwise, by indiscernibility we either get an infinite descending sequence of ordinals or else λ distinct ordinals $< \omega_1$.) Thus, every element of $B\cap\omega_1$ can be rendered as $t(\vec{x})$ for some term t and $\vec{x}$ a finite sequence taken from the first ω elements of H, and so $B\cap\omega_1$ is countable as we are dealing with a countable language.

A similar argument establishes that $Q\cap B$ has the ω_1-c.c.: If $Q\cap B$ had ω_1 mutually incompatible conditions, there must be at least one term t, say in n free variables, and $\vec{x} < \vec{y} \in {}^nH$ so that $t(\vec{x})$ and $t(\vec{y})$ denote two incompatible members of $Q\cap B$. But then by elementarity we can use n-tuples from H to produce λ mutually incompatible members of Q, a contradiction.

By elementarity using the relation R, we have that for any $s \in B\cap[\lambda]^{<\omega}$, the set $D_s = \{p \mid p$ extends q & $\exists\beta(\beta \in B\cap\omega_1$ & $p \Vdash f(\check{s}) = \beta)\}$ is dense below q in $Q\cap B$. Thus, since $|B\cap[\lambda]^{<\omega}| \leq |B| = \omega_1$, by $MA(2^\omega)$ and $\omega_1 < 2^\omega$ let $G \subseteq Q\cap B$ be generic for all these D_s.

By the special way that Q was defined, we have $\bigcup G \in Q$, as $B\cap\omega_1$ is only countable. Finally, $\bigcup G$ extends q, and it is clear from elementarity using the relation R that if we set $B\cap\lambda = K$, then $\bigcup G \Vdash f``[\check{K}]^{<\omega}$ is countable. As K is uncountable and Q is ω-closed, K is forced to remain uncountable, so we are done. $\dashv$

There is an unusual speciality in this proof, namely in the applicability of Martin's Axiom only to ω_1-c.c. partially ordered sets. Can one show, for example, that Con(ZFC & there is a Ramsey cardinal) implies Con(ZFC & $\langle\omega_3,\omega_2\rangle \twoheadrightarrow \langle\omega_2,\omega_1\rangle$) ?

See Magidor[1977] for a recent application of Chang's Conjecture.

§20. Ineffability and Subtlety

In the previous section, we saw how the independence of Kurepa's Hypothesis turned out to be a large cardinal concept; in the present section, we recount how the further investigation of the hypothesis in L uncovered a new and distinctive large cardinal idea. (Properly speaking, we could have discussed these matters in an early chapter; however, the topic fits thematically here, and its development comprises a veritable cocktail of applications of practically all of the concepts discussed in Chapters I and II.)

After Solovay built a Kurepa tree in L, Jensen was able to isolate the essential features of L that made the construction possible by studying its generalization. For a regular uncountable cardinal κ, Kurepa's Hypothesis for κ (KH_κ) is the assertion that there is an $F \subseteq P(\kappa)$ with $|F| \geq \kappa^+$ so that $|\{X \cap \alpha \mid X \in F\}| \leq |\alpha|$ for every infinite $\alpha < \kappa$. Jensen (see Jensen-Kunen[1971] or Devlin[1973a], Chapter 10) showed that this generalized version of Kurepa's Hypothesis holds practically always in L, and characterized those κ at which it fails. Formulating the concept of ineffability for regular uncountable cardinals, he showed that: If κ is ineffable, then KH_κ fails; and the converse holds if $V = L$.

Ineffability is defined as follows: if κ is a regular uncountable cardinal, then κ is ineffable iff for any sequence $\langle S_\alpha \mid \alpha < \kappa\rangle$ with $S_\alpha \subseteq \alpha$ for every $\alpha < \kappa$, there is an $S \subseteq \kappa$ so that $\{\alpha < \kappa \mid S_\alpha = S \cap \alpha\}$ is stationary in κ. The emergence of ineffability is an interesting episode in set theory, both for its relatively recent occurrence and for its notably original flavor. Kunen quickly brought the new concept into line with more familiar ones by providing the following characterization: The following are equivalent for regular uncountable κ:

(a) κ is ineffable.

(b) Whenever $f: [\kappa]^2 \to \kappa$ is regressive (i.e. $f(\{\alpha,\beta\}) < \min(\alpha,\beta)$), there is a $X \subseteq \kappa$ stationary in κ so that $|f''[X]^2| = 1$.

(c) Whenever $f: [\kappa]^2 \to 2$, there is an $X \subseteq \kappa$ stationary in κ so that $|f''[X]^2| = 1$.

$\vdash$ (a) $\to$ (b). Suppose $f: [\kappa]^2 \to \kappa$ is regressive. Define $f_\alpha \in {}^\alpha\alpha$ for each $\alpha < \kappa$ by: $f_\alpha(\beta) = f(\{\beta,\alpha\})$ for $\beta < \alpha$. We need to efficiently code pairs of ordinals with ordinals, so let $G: OR \times OR \twoheadrightarrow OR$ be Gödel's pairing function, so in particular $C = \{\alpha < \kappa \mid G|\alpha\times\alpha: \alpha\times\alpha \twoheadrightarrow \alpha\}$ is closed unbounded in κ. Then by the ineffability of κ, there is a function $g \in {}^\kappa\kappa$ so that $Y = \{\alpha \in C \mid f_\alpha = g|\alpha\}$ is stationary in κ. Now $g|Y$ is regressive on Y, so there is a $\gamma < \kappa$ so that $X = \{\alpha \in Y \mid g(\alpha) = \gamma\}$ is stationary in κ. But this insures that X is homogeneous for the original f, since if $\beta < \alpha$ are both in X, then $f(\{\beta,\alpha\}) = f_\alpha(\beta) = g(\beta) = \gamma$.

(b) $\to$ (c) is trivial.

(c) $\to$ (a). Suppose that $\langle S_\alpha \mid \alpha < \kappa\rangle$ is such that $S_\alpha \subseteq \alpha$ for every $\alpha < \kappa$. Define $f: [\kappa]^2 \to 2$ by: for $\alpha < \beta < \kappa$, $f(\{\alpha,\beta\}) = 0$ iff ($S_\alpha = S_\beta \cap \alpha$ or else

the least element $< \alpha$ in the symmetric difference of S_α and $S_\beta \cap \alpha$ is in S_β). By (c), let $X \subseteq \kappa$ be stationary in κ and homogeneous for f. We need some cosmetics, so notice that for each $\gamma < \kappa$, there is a $\rho_\gamma \geq \gamma$ so that for any $\alpha \in (X - \rho_\gamma)$, $S_\alpha \cap \gamma = S_{\rho_\gamma} \cap \gamma$. (This follows by induction on γ, using the homogeneity of X for f, noting that limit stages are automatic.) Set $C = \{\alpha < \kappa \mid \alpha$ is a limit ordinal & $\gamma < \alpha$ implies $\rho_\gamma < \alpha\}$, so that C is closed unbounded in κ. Now if $\alpha < \beta$ are both in $C \cap X$, then for any $\gamma < \alpha$ we have $S_\alpha \cap \gamma = S_{\rho_\gamma} \cap \gamma = S_\beta \cap \gamma$, and hence $S_\alpha = S_\beta \cap \alpha$. Thus, if we set $S = \bigcup\{S_\alpha \mid \alpha \in C \cap X\}$, then $\{\alpha \mid S_\alpha = S \cap \alpha\} \subseteq C \cap X$, which is stationary. This establishes the ineffability of κ. $\dashv$

It follows immediately that a measurable cardinal is ineffable, and that an ineffable cardinal is weakly compact. We now sharpen these implications considerably by applying the technology we have developed in our first two chapters.

First, it was an interesting observation that, in a general setting, ineffability is an internal attribute of critical points of elementary embeddings: Suppose $\langle M,\in\rangle$ is a transitive model of ZF (except possibly Replacement), and there is an elementary embedding $j: \langle M,\in\rangle \to \langle M,\in\rangle$ with critical point δ. Then $\langle M,\in\rangle \models \delta$ is ineffable.

$\vdash$ Standard arguments establish that $M \models \delta$ is regular and uncountable. Suppose now that $f \in M$ is a function with domain δ, so that $f(\alpha) \subseteq \alpha$ for every $\alpha < \delta$. Then $j(f) \in M$ is a function with domain $j(\delta) > \delta$ and $E = j(f)(\delta) \subseteq \delta$ with $E \in M$. Setting $S = \{\alpha < \delta \mid f(\alpha) = E \cap \alpha\}$, M models enough so that we can assert $S \in M$. Notice that $\delta \in j(S)$, since $j(f)(\delta) = E = j(E) \cap \delta$ because δ is the critical point of j. Also, if $C \in M$ is any closed unbounded subset of δ, since $j(C) \cap \delta = C$, δ is a limit point of $j(C)$ and so $\delta \in j(C)$. Thus, $j(S) \cap j(C) \neq \emptyset$, and so by elementarity $S \cap C \neq \emptyset$. This shows that S is a stationary subset of δ in the sense of M. Hence, we have established that $M \models \delta$ is ineffable. $\dashv$

Similar arguments show that M thinks δ to be highly indescribable. Recalling a familiar situation from §7, we can conclude that any of the canonical indiscernibles given by $0^\#$ is ineffable in L. Silver, Reinhardt, and Jensen observed a sharper result: Below any cardinal κ such that $\kappa \to (\omega)_2^{<\omega}$, there is an ineffable cardinal.

$\vdash$ We can assume that κ is strongly inaccessible by taking $\kappa = \kappa(\omega)$, recalling some comments at the beginning of §6 on Erdös cardinals. Consider the structure $\langle V_\kappa,\in,\ldots\rangle$, where the $\ldots$ denote a complete set of Skolem functions (in a countable language). As in §7, we can suppose from our hypothesis that there are ω indiscernibles $\gamma_0 < \gamma_1 < \gamma_2 \ldots < \kappa$ for this structure, so let $\mathfrak{A} \prec \langle V_\kappa,\in,\ldots\rangle$ be the Skolem Hull determined by these indiscernibles. As usual, the map that sends γ_i to γ_{i+1} induces an elementary embedding $k: \mathfrak{A} \to \mathfrak{A}$. Let $t: \mathfrak{A} \cong \mathfrak{B}$ be the transitization of $\mathfrak{A}$, and $j: \mathfrak{B} \to \mathfrak{B}$ the elementary embedding corresponding to k.

j has a critical point δ and fits into the scheme of the previous result, so that $\mathfrak{B} \models \delta$ is ineffable. Hence, $\mathfrak{A} \models \eta$ is ineffable, where $\eta = t^{-1}(\delta)$. Thus, $\langle V_\kappa, \varepsilon, \ldots \rangle \models \eta$ is ineffable, which means that η really is ineffable, as κ was strongly inaccessible. ⊣

With a bit more work, one can actually show that $\{\eta < \kappa(\omega) \mid \eta$ is ineffable$\}$ is stationary in $\kappa(\omega)$. Anyhow, the point is that the existence of $\kappa(\omega)$ is already compatible with $V = L$ (see §7). Also, one can see that if λ is ineffable, then (λ is ineffable)^L. (Suppose $\langle S_\alpha \mid \alpha < \kappa \rangle \in L$ with $S_\alpha \subseteq \alpha$ for every $\alpha < \kappa$. By ineffability, in V, there is an $S \subseteq \kappa$ so that $A = \{\alpha < \kappa \mid S_\alpha = S \cap \alpha\}$ is stationary in κ. However, κ is weakly compact, so since arbitrarily large initial segments of S are εL, $S \varepsilon L$ (by an indescribability argument; see §4). Thus also $A \varepsilon L$, and A is stationary in L, as A really is stationary.) Thus, ineffability is a relatively mild concept, although we saw that it has an interesting natural occurrence in the context of elementary embeddings.

We can bring in the paraphernalia of indescribability (§4) to measure the strength of ineffability. The ineffability of κ is a Π^1_3 property of $\langle V_\kappa, \varepsilon \rangle$, so this sets an upper limit; Kunen proved the interesting fact that if κ is ineffable, then κ is Π^1_2-indescribable.

⊢ To show this, we shall find it convenient to use ineffability in the following form: If $A_\alpha \subseteq V_\alpha$ for every $\alpha < \kappa$, then there is an $A \subseteq V_\kappa$ so that $\{\alpha < \kappa \mid A_\alpha = A \cap V_\alpha\}$ is stationary in κ. (Since an ineffable cardinal is inaccessible, we can call upon a bijection $g\colon \kappa \leftrightarrow V_\kappa$ for coding, noting that $\{\alpha < \kappa \mid g|\alpha\colon \alpha \leftrightarrow V_\alpha\}$ is closed unbounded in κ and hence has stationary intersection with any stationary subset of κ.)

Suppose now that $R \subseteq V_\kappa$ and $\langle V_\kappa, \varepsilon, R \rangle \models \phi(R)$, where ϕ is Π^1_2. We assume by way of contradiction that for every $\alpha < \kappa$, $\langle V_\alpha, \varepsilon, R \cap V_\alpha \rangle \models \neg\phi(R \cap V_\alpha)$. Writing $\phi(\cdot)$ as $\forall X \exists Y \psi(X, Y, \cdot)$ where ψ is first-order, it follows that for every $\alpha < \kappa$ there is an $A_\alpha \subseteq V_\alpha$ so that $\langle V_\alpha, \varepsilon, R \cap V_\alpha \rangle \models \forall Y \neg\psi(A_\alpha, Y, R \cap V_\alpha)$. By previous remarks, we can now apply ineffability to produce $A \subseteq V_\kappa$ so that $D = \{\alpha < \kappa \mid \alpha$ is a limit ordinal & $A_\alpha = A \cap V_\alpha\}$ is stationary in κ.

It is now claimed that $\langle V_\kappa, \varepsilon, R \rangle \models \forall Y \neg\psi(A, Y, R)$, which would yield a contradiction and finish the proof. Well, suppose not. Then for some $\bar{Y} \subseteq V_\kappa$, $\langle V_\kappa, \varepsilon, R \rangle \models \psi(A, \bar{Y}, R)$. As ψ is first-order, a standard argument show that $C = \{\alpha < \kappa \mid \langle V_\alpha, \varepsilon, R \cap V_\alpha \rangle \models \psi(A \cap V_\alpha, \bar{Y} \cap V_\alpha, R \cap V_\alpha)\}$ is closed unbounded in κ. But D was stationary, so any $\alpha \varepsilon C \cap D$ yields a contradiction. ⊣

It is now immediate from §4 that if κ is ineffable, then $\{\alpha < \kappa \mid \alpha$ is weakly compact$\}$ is stationary in κ. We mention stronger results at the end of this section. Let us now attend to some remarks concerning technical questions that might arise from these proceedings. The new idea of ineffability offers various variations. One generalization even gives a characterization of supercompactness (see

Magidor[1974]), while another fits into a general scheme of "flipping" properties for characterizing large cardinals (see Abramson-Harrington-Kleinberg-Zwicker[1977]). A natural weakening of ineffability is: a regular uncountable cardinal κ is almost ineffable iff for any sequence $\langle S_\alpha \mid \alpha < \kappa \rangle$ with $S_\alpha \subseteq \alpha$ for every $\alpha < \kappa$, there is an $S \subseteq \kappa$ so that $|\{\alpha < \kappa \mid S_\alpha = S \cap \alpha\}| = \kappa$. As we are foregoing stationariness here, one might conjecture from Kunen's characterization of ineffability that almost ineffability is equivalent to weak compactness. This is not so, as the work of Baumgartner[1975] indicates. In analogy to ineffability, almost ineffability of κ is equivalent to: whenever $f\colon [\kappa]^2 \to \kappa$ is regressive (i.e. $f(\{\alpha, \beta\}) < \min(\alpha, \beta)$), there is an $X \subseteq \kappa$ with $|X| = \kappa$ so that $|f''[X]^2| = 1$. Hence, an almost ineffable cardinal is weakly compact, but the former notion turns out to be much stronger. It is also interesting that, as essentially observed by Jensen (see Boos[1975]), a Ramsey cardinal is almost ineffable, although Ramseyness being a Π^1_2 property cannot imply ineffability itself.

Another question is the following: We saw in §5 that if $\kappa \to (\kappa)^2_2$, then $\kappa \to (\kappa)^n_2$ for every $n \in \omega$. Can we similarly raise the exponent to any $n \in \omega$ for ineffability? The answer turns out to be no, with the requirement of stationariness of homogeneous sets constituting an essential block. Let us define for any $n \in \omega$: a regular uncountable cardinal κ is n-ineffable iff whenever $f\colon [\kappa]^{n+1} \to 2$ there is an $X \subseteq \kappa$ stationary in κ so that $|f''[X]^{n+1}| = 1$. Thus, ineffability is equivalent to 1-ineffability. Baumgartner[1975] studied the n-ineffable cardinals in some detail using natural normal filters, and showed in particular that for any $n > 1$, if κ is n-ineffable, then $\{\alpha < \kappa \mid \alpha$ is $(n-1)$-ineffable$\}$ is stationary in κ.

Let us finally turn to another natural weakening of ineffability: a regular uncountable cardinal κ is subtle iff for any sequence $\langle S_\alpha \mid \alpha < \kappa \rangle$ with $S_\alpha \subseteq \alpha$ for every $\alpha < \kappa$, if $C \subseteq \kappa$ is closed unbounded, there are $\beta < \gamma$ both in C so that $S_\beta = S_\gamma \cap \beta$. It may have been the case that Jensen's inspiration for ineffability came from his formulation of the principle $\diamondsuit_\kappa$: There is a sequence $\langle S_\alpha \mid \alpha < \kappa \rangle$ with $S_\alpha \subseteq \alpha$ for every $\alpha < \kappa$, so that for any $S \subseteq \kappa$, $\{\alpha < \kappa \mid S_\alpha = S \cap \alpha\}$ is stationary in κ. Jensen had established that if $V = L$, then $\diamondsuit_\kappa$ holds for every regular uncountable κ. Notice that $\diamondsuit_\kappa$ is a sort of $\exists\forall$ version of the $\forall\exists$ statement of ineffability. Jensen showed that if κ is ineffable then $\diamondsuit_\kappa$ holds, by mimicking his argument for L, and noted in fact that if κ is subtle, then $\diamondsuit_\kappa$ holds.

Subtlety was thus born out of a technical observation, but has since achieved quite a distinctive place among the ranks of large cardinals. Although the subtlety of κ is just a Π^1_1 property of $\langle V_\kappa, \varepsilon \rangle$ and hence does not even imply the weak compactness of κ, it is known that a subtle cardinal κ is (strongly) inaccessible, and that $\{\alpha < \kappa \mid \alpha$ is Π^m_n-indescribable for every $m, n\}$ is stationary in κ. Also, the investigations of Baumgartner[1975] reveal that, in a natural sense, ineffability is made up of subtlety and Π^1_2-indescribability. One can thus regard subtlety as the pure, distilled form of the ineffability idea. Recent work of Stavi also supports this contention.

Whether parent or child, ineffability and subtlety have emerged as significant hypotheses about the possible coherence of sequences of sets. The awareness of subtle phenomena is an interesting latter-day theme.

§21. The Tree Property, E^μ_λ, and $\Box_\lambda$

Expanding on a theme from the previous section, we now turn to discuss how the further elucidation of abstract structural features of L have uncovered new large cardinal relationships. It is to Jensen's penetrating analysis of L that we largely own the emergence of and contemporary metamathematical interest in several unifying combinatorial principles. Jensen had begun his investigation of the fine structure underlying Gödel's notion of constructibility over a decade ago, but no one could then have guessed at the succession of profound and beautiful results that he has since achieved. (One is reminded of Wordsworth's Newton: "...a mind forever/ Voyaging through strange seas of thought alone.") By the early 1970's, Jensen had formulated several deep combinatorial principles which hold in L; it is a remarkable coincidence that the denials of many of these turn out to be equiconsistent to the existence of large cardinals.

We have already encountered (in §5) the first combinatorial property to be discussed. This is the tree property, which in combination with inaccessibility yielded a characterization of weak compactness. The possibility had been voiced that the tree property for a cardinal κ may not in itself imply the inaccessibility of κ. The Axiom of Constructibility in any case precludes this contingency: Jensen's[1972] investigations revealed that in L, weak compactness is equivalent to possessing the tree property, and in fact, to the assertion of a generalized version of Souslin's Hypothesis. Jensen here had found the appropriate generalization to his initial demonstration of the failure of Souslin's Hypothesis in L.

It was the work of Mitchell[1972] as supplemented by Silver that firmly established the tree property as a definite possibility for small cardinals. Since Aronszajn had show that there is always an ω_1-Aronszajn tree (recall §5), a counterexample to the tree property for ω_1, the least candidate was ω_2. Both Aronszajn's and Specker's generalized constructions had actually built κ-Aronszajn trees which were subtrees of $< \bigcup_{\alpha<\kappa} {}^\alpha\kappa, \subseteq >$. Such trees are called special κ-Aronszajn trees. Mitchell developed a method for collapsing cardinals which, like the Lévy collapse, was amenable to a product analysis, and with it he was able to establish in particular that: If there is a Mahlo cardinal κ, then there is a forcing extension V[G] in which $\kappa = \omega_2^{V[G]}$, and there are no special κ-Aronszajn trees. This held true to form, for he also showed that: If κ is a successor cardinal and there are no special κ-Aronszajn trees, then κ is Mahlo in L.

Thus, Con(ZFC & there is a Mahlo cardinal) iff Con(ZFC & there are no special ω_2-Aronszajn trees).

Silver was then able to show that Mitchell's forcing construction establishes: If there is a weakly compact cardinal κ, then there is a forcing extension V[G] in which $\kappa = \omega_2^{V[G]}$ and there are no κ-Aronszajn trees. He also produced the companion result: If κ is the successor of a regular cardinal and there are no κ-Aronszajn trees, then κ is weakly compact in L. Thus, Con(ZFC & there is a weakly compact cardinal iff Con(ZFC & ω_2 has the tree property). (Recently, Baumgartner announced an alternate scheme to Mitchell's: If there is a weakly compact cardinal κ, and Sacks forcing is iterated κ times, then $\kappa = \omega_2^{V[G]}$ and there are no κ-Aronszajn trees.)

These are certainly satisfying results. The tree property has been scaled down from an inaccessible to fit ω_2, and a natural weakening corresponds exactly to Mahloness. Versions of the tree property have thus revealed the intrinsic graded relationship between the ostensibly dissimilar concepts of Mahloness and weak compactness.

We next discuss another combinatorial question which has several interesting points of contact with various large cardinals. The question in general form is when stationariness is reflected: If A is a stationary subset of a regular cardinal κ, is there an $\alpha < \kappa$ so that $A \cap \alpha$ is a stationary subset of α?

If κ were weakly compact, then from its Π^1_1-indescribability follows directly that such an α exists, as being stationary is a Π^1_1-property of V_κ. Jensen[1972] first raised interest in such considerations by showing that in L, this property characterizes weak compactness: If V = L, then a regular cardinal κ is weakly compact iff for every stationary subset A of κ, there is an $\alpha < \kappa$ so that $A \cap \alpha$ is stationary in α.

What about this property for accessible cardinals? Without modification, the property trivially fails for κ^+ whenever κ is regular, as $A = \{\delta < \kappa^+ \mid cf(\delta) = \kappa\}$ is an easy counterexample. This turns out to be essentially the only restriction here; we can admit successor cardinals into our study by introducing the following modified assertions for regular cardinals $\lambda < \kappa$:

E^λ_κ: There is an $A \subseteq \{\delta < \kappa \mid cf(\delta) = \lambda\}$ stationary in κ such that for every limit ordinal $\alpha < \kappa$, $A \cap \alpha$ is not stationary in α.

Jensen in fact could prove: If V = L, then a regular cardinal κ is weakly compact iff E^λ_κ fails for some regular $\lambda < \kappa$.

It is perhaps expected that the failure of E^λ_κ, being a reflection property, should be a consequence of further large cardinal hypotheses. What is interesting is that an assumption on ultrafilters from §13 seems just what is needed to establish the failure of E^λ_κ. The following theorem evolved from Prikry[1973]Theorem 4, which is subsumed by it: If $\lambda < \kappa$ are regular and there is a uniform, λ-indecomposable ultrafilter over κ, then E^λ_κ fails.

$\vdash$ Let us fix for each $\delta < \kappa$ such that $cf(\delta) = \lambda$ an increasing sequence $\langle \gamma^\delta_\xi \mid \xi < \lambda \rangle$ of order-type λ, closed and cofinal in δ. For any $X \subseteq \{\delta < \kappa \mid cf(\delta) = \lambda\}$, say that a function $f \colon X \to \lambda$ is a disjointer for X iff for $\delta \neq \eta$ both in X, whenever $f(\delta) < \xi < \lambda$ and $f(\eta) < \zeta < \lambda$, we have $\gamma^\delta_\xi \neq \gamma^\eta_\zeta$.

To establish the failure of E^λ_κ, we argue as follows: Suppose $A \subseteq \{\delta < \kappa \mid cf(\delta) = \lambda\}$ has the property that $A \cap \alpha$ is not stationary in α for every $\alpha < \kappa$. We proceed to demonstrate that A is not stationary in κ. For this purpose, we shall first show by induction that for every $\alpha < \kappa$, there is a disjointer f_α for $A \cap \alpha$:

So, suppose that there are already disjointers f_β for $A \cap \beta$, for every $\beta < \alpha$. Since $A \cap \alpha$ is not stationary in α, let $C_\alpha \subseteq \alpha$ be closed unbounded in α so that $C_\alpha \cap A = \emptyset$. We can then define a function f_α on $A \cap \alpha$ according to the following well-defined recipe: Given a $\delta \in A \cap \alpha$, first find the unique, consecutive elements ζ, β of C_α so that $\zeta < \delta < \beta$. (We might as well assume $0 \in C_\alpha$.) Then let $f_\alpha(\delta)$ be the maximum of: $f_\beta(\delta)$, or the least $\xi < \lambda$ so that $\gamma^\delta_\xi > \zeta$. This definition (once understood!) renders f_α a disjointer for $A \cap \alpha$.

We now suppose U is a uniform λ-indecomposable ultrafilter over κ. As λ is assumed regular, this means that whenever $g \colon \kappa \to \lambda$, there is a $\rho < \lambda$ so that $\{\alpha < \kappa \mid g(\alpha) \leq \rho\} \in U$. We use U to take a sort of ultraproduct of the f_α's, to get a disjointer for A itself:

For each $\delta \in A$, by considering the function $\alpha \mapsto f_\alpha(\delta)$, there must be an $f(\delta) < \lambda$ so that $X_\delta = \{\alpha < \kappa \mid \delta < \alpha \ \& \ f_\alpha(\delta) \leq f(\delta)\} \in U$. We claim that f is a disjointer for A. Indeed, if $\delta \neq \eta$ both in A, suppose $f(\delta) < \xi < \lambda$ and $f(\eta) < \zeta < \lambda$. Let $\alpha \in X_\delta \cap X_\eta \in U$, so that $f_\alpha(\delta) \leq f(\delta)$ and $f_\alpha(\eta) \leq f(\eta)$. But f_α is a disjointer, so $\gamma^\delta_\xi \neq \gamma^\eta_\zeta$.

The rest is easy. We can use the disjointer f for A to show that A cannot be stationary in κ. If A were stationary, since the function g given by $g(\delta) = \gamma^\delta_{f(\delta)+1}$ is regressive on A, there is a stationary $B \subseteq A$ on which g is constant. This contradicts the fact that f is a disjointer for A. $\dashv$

This result, of course, has the following wake: If λ is regular and there is a uniform ω_1-complete ultrafilter over λ, then E^ω_λ fails. Hence, if $\kappa \leq \lambda$ are regular and κ is λ-compact (see §15), then E^ω_λ fails.

Before turning to relative consistency results involving E^λ_κ, we attend to another, and better known, combinatorial principle. This is the weighty principle $\square_\kappa$ (variously read box or square), first isolated in the laboratory of Jensen's fine structure study of L. $\square_\kappa$ states: There is a sequence $\langle C_\xi \mid \xi < \kappa^+ \rangle$ so that for any $\xi < \kappa^+$, we have:

(i) $C_\xi \subseteq \xi$ and if ξ is a limit ordinal, C_ξ is closed unbounded in ξ.

(ii) If $cf(\xi) < \kappa$, then $|C_\xi| < \kappa$.

(iii) If γ is a limit point of C_ξ, then $C_\gamma = C_\xi \cap \gamma$.

(It follows from (ii) and (iii) that if $cf(\xi) = \kappa$, then C_ξ has order-type κ.) Jensen[1972] showed that if V = L, then $\square_\kappa$ holds for every $\kappa > \omega$. As with other combinatorial principles discovered by Jensen in L, $\square_\kappa$ encapsulates something about the uniform generation of L which can sometimes be used in place of the Axiom of Constructibility. We now establish: Suppose $\kappa > \omega$ and $\square_\kappa$ holds. Then $E^\lambda_{\kappa^+}$ holds for every regular $\lambda < \kappa^+$.

$\vdash$ We shall in fact show that if T is a stationary subset of κ^+, then there is a stationary $S \subseteq T$ so that for any limit ordinal $\alpha < \kappa^+$, $S \cap \alpha$ is not stationary in α. For this purpose, we might as well assume that T consists of limit ordinals. Let $\langle C_\xi \mid \xi < \kappa^+ \rangle$ be a sequence witnessing $\square_\kappa$. For $\eta \leq \kappa$, set $S_\eta = \{\xi \in T \mid C_\xi$ has order-type $\eta\}$. Then, as the S_η's partition T into κ parts, there is some $\bar\eta \leq \kappa$ so that $S = S_{\bar\eta}$ is still stationary in κ^+. We now claim that for any limit ordinal $\alpha < \kappa^+$, $S \cap \alpha$ is not stationary in α. There are three cases:

(a) $cf(\alpha) = \omega$. Then as S consists of limit ordinals, S is disjoint from any sequence of type ω of successor ordinals, cofinal in α.

(b) $cf(\alpha) > \omega$ and C_α has order-type $\leq \bar\eta$. Let $\bar C_\alpha$ consist of the limit points of C_α. Then as $cf(\alpha) > \omega$, $\bar C_\alpha$ is closed unbounded in α, and $(S \cap \alpha) \cap \bar C_\alpha = \emptyset$ by property (iii) of $\square_\kappa$.

(c) $cf(\alpha) > \omega$ and C_α has order-type $> \bar\eta$. Let $\gamma \in C_\alpha$ so that $C_\alpha \cap \gamma$ has order-type $\bar\eta$. Then if $\bar C_\alpha$ is defined as in the previous case, we have by property (iii) of $\square_\kappa$ that $(S \cap \alpha) \cap (\bar C_\alpha - (\gamma + 1)) = \emptyset$.

Thus, the claim is proved, and so is the theorem. $\dashv$

The following corollary is immediate: If κ is λ^+-compact, then $\square_\lambda$ fails. Thus, the scalings by ultrafilters imposed by strong compactness on the final levels of the cumulative hierarchy disallows $\square_\lambda$ (and E^ω_λ) for λ sufficiently large. In essence, what is at play, of course, is the indecomposability of ultrafilters.

Turning to consistency results, Jensen[1972]p.286 himself had noted that if κ^+ is not Mahlo in L, then $\square_\kappa$ holds. Roughly, the attributes of a $\square_\kappa$ sequence, whether constructed in V or within an inner model, are quite absolute, and what seems to be needed to make the length of the sequence (the real) κ^+ is precisely that κ^+ is not Mahlo in L. Solovay complemented this result for ω_1; he showed that if a Mahlo cardinal is Lévy collapsed to ω_2, then $\square_{\omega_1}$ fails in the extension. Thus, Con(ZFC & there is a Mahlo cardinal) iff Con(ZFC & $\square_{\omega_1}$ fails).

Recent work of Dodd-Jensen(1976) casts some light on the consistency strength of the failure of $\square_\kappa$ when κ is singular. They show that if there is no inner model with a measurable cardinal, then the Covering Theorem holds for the Core Model K, which in particular implies that $(\kappa^+)^K = \kappa^+$ for singular κ (see §29). It can also be verified that in K, $\square_\kappa$ holds for every $\kappa > \omega$. Now a $\square_\kappa$ sequence is quite absolute, so that one existing in K really is one if $(\kappa^+)^K = \kappa^+$. Thus, we have: If κ is singular and $\square_\kappa$ fails (e.g. $E^\lambda_{\kappa^+}$ fails for some regular $\lambda < \kappa$), then there is an inner model with a measurable cardinal. Recent work of Mitchell indicates that we can in fact get inner models with many measurable cardinals from the hypothesis. So, the consistency strength of the failure of $\square_\kappa$ for singular κ has been reasonably measured on our scale: it lies somewhere between the strengths of strong compactness and the existence of many measurable cardinals.

(We can also recall from §13 that Magidor established: Con(ZFC & there is a κ which is κ^+-supercompact) implies Con(ZFC & there is a uniform ultrafilter over $\omega_{\omega+1}$ which is ν-indecomposable for $\omega < \nu < \omega_\omega$). Since $\square_{\omega_\omega}$ must fail if such an ultrafilter exists by results of this section, this establishes a reasonable bound on the consistency strength of the existence of such an ultrafilter: it lies somewhere between the existence of a κ^+-supercompact cardinal κ and the existence of many measurable cardinals.)

It is now appropriate to deal with the consistency strength of the failure of principles E^λ_κ. Reminiscent of Mitchell's result on the tree property, Baumgartner [1976]§7 showed that a cardinal collapse preserves a non-trivial attribute of weak compactness: If there is a weakly compact cardinal κ in the ground model, then if κ is Lévy collapsed to ω_2 (i.e. force with $Col(\omega_1,\kappa)$), then $E^\omega_{\omega_2}$ fails in the extension.

$\vdash$ In brief, the elements of Baumgartner's proof are as follows: Let G be $Col(\omega_1,\kappa)$-generic over the ground model V, so that $\kappa = \omega_2^{V[G]}$. Since $Col(\omega_1,\kappa)$ is ω-closed, $\{\delta < \kappa \mid cf(\delta) = \omega\}$ will be extensionally the same set in what follows. We must show that the following holds in $V[G]$: if $A \subseteq \{\delta < \kappa \mid cf(\delta) = \omega\}$ is stationary in κ, then there is an $\alpha < \kappa$ so that $A \cap \alpha$ is stationary in α.

Baumgartner first recalls the following well-known fact: If in $V[G]$, $C \subseteq \kappa$ is closed unbounded in κ, then there is a $D \in V$ closed unbounded in κ so that $D \subseteq C$. This calls upon the regularity of κ and the κ-c.c. of the Lévy collapse.

Using this, he is able to apply the Π^1_1-indescribability of κ in V on enough of the forcing apparatus to show that: If in $V[G]$, $A \subseteq \{\delta < \kappa \mid cf(\delta) = \omega\}$ is stationary in κ, then there is an inaccessible $\alpha < \kappa$ so that in $V[G \cap Col(\omega_1,\alpha)]$ $A \cap \alpha$ is stationary in α.

By the usual product analysis, $V[G]$ is a generic extension of $V[G \cap Col(\omega_1,\alpha)]$ via (the ω-closed) $Col(\omega_1,\kappa-\alpha)$. Hence, the proof is concluded with the following observation (with $V[G \cap Col(\omega_1,\alpha)]$ in the role of the ground model, and $\alpha = \lambda$):

Suppose λ is a regular uncountable cardinal, $S \subseteq \{\delta < \lambda \mid cf(\delta) = \omega\}$ is a stationary subset of λ, and Q is an ω-closed notion of forcing. Then in any generic extension via Q, S is still stationary in λ. $\dashv$

The complementary result here, of course, would be a proof of: if $E^\omega_{\omega_2}$ fails, then ω_2 is weakly compact in L. Unfortunately, this has not been forthcoming thus far. All we presently know is that by previous remarks, if $E^\omega_{\omega_2}$ fails, then

since $\square_{\omega_1}$ fails, ω_2 is Mahlo in L. However, it seems likely that the full desired result holds, since with a slight strengthening of the hypothesis (which can be checked to hold in Baumgartner's model) one gets: Suppose that κ is a regular uncountable cardinal, and there is stationary subset S of κ such that: whenever $A_1, A_2 \subseteq S$ are stationary subsets of κ, there is an $\alpha < \kappa$ so that $A_1 \cap \alpha$ and $A_2 \cap \alpha$ are both stationary in α. Then κ is weakly compact in L.

We conclude by citing a result of Shelah related to Baumgartner's: If there is a supercompact cardinal κ and κ is Lévy collapsed to ω_2, then the extension satisfies the following statement: whenever $\lambda > \omega_1$ is regular and $A \subseteq \{\delta < \lambda \mid cf(\delta) = \omega\}$ is stationary in λ, then there is an $\alpha < \lambda$ so that $A \cap \alpha$ is stationary in α (that is, E^ω_λ fails for every regular $\lambda > \omega_1$).

§22. The Closed Unbounded Filter over ω_1

Among all regular uncountable cardinals, we have already encountered some special situations (at the ends of §12 and §19) obtaining at ω_1. They seem to rely on the fact (which does not generalize!) that ω_1 is the least uncountable cardinal. The closed unbounded filter over ω_1 is quite distinctive for this reason, and it is worth devoting a short section to its possible extent.

Throughout this section, let F denote the closed unbounded filter over ω_1. So F is the minimal ω_1-complete filter over ω_1 which is normal. How close is it to being an ultrafilter? With the Axiom of Choice, F cannot be an ultrafilter since otherwise ω_1 would be measurable. Without the Axiom of Choice, ω_1 can be measurable, if the existence of a measurable cardinal is consistent (Jech[1968]). However, it is still open whether the assertion that F itself is an ultrafilter is consistent relative to the existence of some large cardinal. Martin and Mitchell showed that if F were an ultrafilter, then for every α there is an inner model of ZFC with α measurable cardinals. We shall see in §28 that the Axiom of Determinacy directly implies that F is an ultrafilter, an example of the axiom's tremendous consistency strength.

As we are continuing to assume the Axiom of Choice as a standing hypothesis, it naturally becomes of interest to see to what extent F is an ultrafilter at least over classes of simply defined subsets of ω_1. In other words, how simple can a partition of ω_1 into two disjoint stationary subsets be? Let us introduce a "projective" hierarchy of subsets of ω_1:

Fix throughout a recursive bijection $\omega \leftrightarrow \omega\times\omega$. Thus, we can regard a subset of ω as a relation on ω, and when this relation is a well-ordering, ask to what countable ordinal it corresponds. We now define: $X \subseteq \omega_1$ is a Σ^1_n (respectively, Π^1_n) subset of ω_1 iff there is a Σ^1_n (respectively, Π^1_n) subset A of $P(\omega)$ (in the usual projective hierarchy) so that: whenever $a \subseteq \omega$ codes a well-ordering of order-type $\alpha < \omega_1$, then $\alpha \in X$ iff $a \in A$.

Somewhat reminiscent of results in §18 about Lebesgue measurable and scattered sets, Silver showed that at least in the first levels of this hierarchy we cannot find a partition of ω_1 into two disjoint stationary subsets:

(i) Every $\underset{\sim}{\Sigma}^1_1$ subset of ω_1 either contains or is disjoint from a closed unbounded subset of ω_1.

(ii) If $a^{\#}$ exists for every $a \subseteq \omega$, every $\underset{\sim}{\Sigma}^1_2$ subset of ω_1 either contains or is disjoint from a closed unbounded subset of ω_1.

What is the consistency strength of the assertion that F is an ultrafilter over the class of all projective subsets of ω_1? Harrington(1975) established that ineffability was enough: Con(ZFC & there is an ineffable cardinal) implies Con(ZFC & for every $n \in \omega$, every $\underset{\sim}{\Sigma}^1_n$ subset of ω_1 either contains or is disjoint from a closed unbounded subset of ω_1).

$\vdash$ The idea briefly is to Lévy collapse an ineffable cardinal κ to ω_1. The usual product analysis shows that every $\underset{\sim}{\Sigma}^1_n$ subset of ω_1 in the extension essentially appears at an early stage of the forcing. Using the ineffability of κ in the ground model, one can then get a stationary $S \subseteq \kappa$ such that: every $\underset{\sim}{\Sigma}^1_n$ subset of ω_1 in the extension contains or is disjoint from a final segment of S. (Observe that stationariness of subsets of κ is preserved through a Lévy collapse of κ, as in a claim in the outline of Baumgartner's result in §21.) Finally, one can shoot a closed unbounded subset through S by a further forcing extension which adds no new subsets of ω (see Baumgartner-Harrington-Kleinberg[1976]) and hence no new $\underset{\sim}{\Sigma}^1_n$ subsets of ω_1. $\dashv$

As for getting a lower bound on consistency strength, one can show: Suppose for every $n \in \omega$, every $\underset{\sim}{\Sigma}^1_n$ subset of ω_1 either contains or is disjoint from a closed unbounded subset of ω_1. Then for every $m,n \in \omega$, we have that $\{\alpha < \omega_1 \mid \alpha \text{ is } \Pi^m_n\text{-indescribable in } L\}$ is uncountable. This brackets the consistency strength rather closely, but further elucidation, possibly leading to an equiconsistency result, is still desirable.

Recall now that a set x is ordinal definable iff there is an α and a formula $\phi(\cdot)$ so that $y \in x$ iff $V_\alpha \models \phi(y)$. (See Myhill-Scott[1971]; ordinal parameters can be allowed in ϕ without affecting the definition.) If we want to get all ordinal definable subsets of ω_1 to be "decided" by F, we know exactly what is needed: Con(ZFC & there is a measurable cardinal) iff Con(ZFC & every ordinal definable subset of ω_1 contains or is disjoint from a closed unbounded subset of ω_1).

$\vdash$ First, if F possesses this strong property of deciding all ordinal definable subsets of ω_1, consider $L[F]$. As F is ordinal definable, every member of $L[F]$ is ordinal definable (in the sense of V). Hence, in $L[F]$, $F \cap L[F]$ is an ultrafilter over (the real) ω_1, and so ω_1 is measurable in $L[F]$.

We outline the converse direction. Suppose κ is measurable and U is a normal ultrafilter over κ. We Lévy collapse κ to ω_1; let G be $Col(\omega,\kappa)$-generic over the ground model V. By arguments first devised by Solovay (for his

model for Lebesgue measurability—see §18), if in $V[G]$, x is an ordinal definable subset of κ $(= \omega_1^{V[G]})$, then either x or $\kappa - x$ has a subset in U.

We can now conclude this argument somewhat like Harrington's above: Since U is normal in V, every member of U is stationary in κ in V, and hence in $V[G]$ (again, by a claim in the outline of Baumgartner's result, §21). Because of this, one can devise a notion of forcing in $V[G]$ which will add a closed unbounded subset C of $\kappa = \omega_1^{V[G]}$ such that: for every $Y \in U$, there is an $\alpha < \kappa$ so that $C \subsetneq Y - \alpha$. This further forcing extension will not alter the class of ordinal definable sets, and hence the result follows. (Throughout this argument, "ordinal definable" could in fact have been replaced by "real,ordinal definable" as defined in §18.) $\dashv$

VI. FORCING DEVELOPED FOR LARGE CARDINALS

§23. Prikry Forcing

In this chapter, we continue to elaborate on the general theme of relative consistency results, turning to forcing techniques developed to handle more intrinsic aspects of the theory of large cardinals. As the various large cardinals emerged, new questions about consistency relationships among them also arose, and it is natural that strong set theoretic assumptions should engender new ideas and possibilities.

The first such development, due to Prikry[1970], exploited strong properties satisfied by a measurable cardinal. At first, its main import seems to be to answer somewhat technical questions about the capabilities of the forcing method. But growing familiarity with Prikry's technique is accompanied by a corresponding greater appreciation for its further elucidation of the concept of measurability, as well as for its applicability in relative consistency results involving the large large cardinals.

Let us fix a measurable cardinal κ and a normal ultrafilter U over κ. Prikry's idea is to add generically a new countable sequence of ordinals cofinal in κ, such that the set of possible extensions of any finite approximation to the sequence is large in the precise sense that it is to be a member of U. This motivates the following formulation of Prikry's notion of forcing: Let $P_U = \{\langle s,X\rangle \mid s$ is a finite subset of κ & $X \in U$ with $\bigcap X > \bigcup s\}$, with the proviso that $\langle s,X\rangle$ is a stronger condition than $\langle t,Y\rangle$ iff t is an initial segment of s (i.e. $s \cap \alpha = t$, for some α), $X \subseteq Y$, and $s - t \subseteq Y$. Thus, $\langle s,X\rangle \in P_U$ should be regarded as a finite approximation s to a generic sequence, together with a bookkeeping set X of possible ordinals from which extensions of s are to be comprised.

It is not hard to see that if $\langle s,X\rangle$ and $\langle t,Y\rangle$ are compatible, then either s is an initial segment of t or vice versa. Hence, if G is a P_U-generic filter over the ground model V, then $s_G = \bigcup\{s \mid \langle s,X\rangle \in G\}$ is a countable set, which by a simple density argument must be cofinal in κ. Hence, $cf^{V[G]}(\kappa) = \omega$, and since P_U obviously satisfies the κ^+-c.c., all cardinals $> \kappa$ are preserved in the extension from V to $V[G]$.

So far, any uniform filter over κ could have replaced our U, and there is no guarantee that κ is not collapsed by this process. Large cardinals ideas enter into the picture through the route of Rowbottom's partition theorem for normal ultrafilters (§6), in the following key lemma: Prikry's Lemma: If $\langle s,X\rangle \in P_U$ and ϕ is any formula of the forcing language, then there is a $\langle s,Y\rangle$ stronger than $\langle s,X\rangle$ so that $\langle s,Y\rangle \Vdash \phi$, or $\langle s,Y\rangle \Vdash \neg\phi$.

$\vdash$ What we must do is to find a $Y \subseteq X$ with $Y \in U$ so that $\langle s,Y\rangle$ "decides" ϕ. For this purpose, define a partition $f\colon [X]^{<\omega} \to 3$ by:

$$f(t) = \begin{cases} 0 & \text{if } \langle s\cup t,T\rangle \Vdash \phi \text{, for some } \langle s\cup t,T\rangle \in P_U, \\ 1 & \text{if } \langle s\cup t,T\rangle \Vdash \neg\phi \text{, for some } \langle s\cup t,T\rangle \in P_U, \\ 2 & \text{otherwise.} \end{cases}$$

By Rowbottom's theorem, there is a $Y \subseteq X$ with $Y \in U$ so that for every $n \in \omega$, $|f``[Y]^n| = 1$.

We can now establish that this Y is as desired: If to the contrary $\langle s,Y\rangle$ does not decide ϕ, then there are $\langle s\cup t,Z\rangle$ and $\langle s\cup \bar{t},\bar{Z}\rangle$ both stronger than $\langle s,Y\rangle$ so that: $\langle s\cup t,Z\rangle \Vdash \phi$, and $\langle s\cup \bar{t},\bar{Z}\rangle \Vdash \neg\phi$. By extending one condition if necessary, we can suppose that $|t| = |\bar{t}| = m$. Thus $t,\bar{t} \in [Y]^m$, yet clearly $f(t) = 0$ and $f(\bar{t}) = 1$. This contradicts the homogeneity of Y. $\dashv$

Of course, for any $\langle s,X\rangle \in P_U$ and forcing statement ϕ, there is a $\langle t,Y\rangle \in P_U$ stronger than $\langle s,X\rangle$ which decides ϕ; the point of Prikry's Lemma is that we can in fact take $s = t$. The crux of several notions of forcing involving large cardinals (like those touched upon in §26 and §29) lies just in some analogous fact. Continuing to assume that G is a P_U-generic filter, we can now establish: If $|y| < \kappa$ in V, and $x \in V[G]$ with $x \subseteq y$, then $x \in V$.

$\vdash$ Let $\underset{\sim}{x}$ be a term in the forcing language for x, and suppose $\langle s,X\rangle \Vdash \underset{\sim}{x} \subseteq \check{y}$. We must find a $z \in V$ and a condition $\langle t,Y\rangle$ stronger than $\langle s,X\rangle$ so that $\langle t,Y\rangle \Vdash \underset{\sim}{x} = \check{z}$. But this is now easy: Working in V, for each $a \in y$ by Prikry's Lemma let $Y_a \subseteq X$ with $Y_a \in U$ and either $\langle s,Y_a\rangle \Vdash \check{a} \in \underset{\sim}{x}$, or $\langle s,Y_a\rangle \Vdash \check{a} \notin \underset{\sim}{x}$. As $|y| < \kappa$, $Y = \bigcap_{a\in y} Y_a \in U$. Then if z is defined (in V) by: $a \in z$ iff $\langle s,Y\rangle \Vdash \check{a} \in \underset{\sim}{x}$, then $\langle s,Y\rangle \Vdash \underset{\sim}{x} = \check{z}$. $\dashv$

It follows immediately from this last fact that $(V_\kappa)^V = (V_\kappa)^{V[G]}$, so all cardinals $< \kappa$ are preserved, and finally, κ too remains a cardinal, being a limit cardinal. In summary, Prikry forcing for a measurable cardinal κ preserves all cardinals, yet changes the cofinality of κ to ω. Changing a cofinality while preserving all cardinals was a new technical possibility about the forcing method. We did not exactly need a normal ultrafilter for significant aspects of Prikry forcing to go through, but in any case a filter with rather strong partition properties seems necessary—see Devlin[1974] for some characterizations.

In previous sections (§6, §9, §12), we have seen how in special cases the cardinal filter generated by some countable increasing sequence of cardinals is capable of yielding a normal ultrafilter, and hence a measurable cardinal, in some inner model. Prikry forcing provides the structural rationale: If G is P_U-generic over V, then $s_G = \bigcup\{s \mid \langle s,X\rangle \in G\}$ generates U in $V[G]$, i.e. $X \in U$ just in case $s_G - X$ is finite. Mathias[1973] in fact established that this last property characterizes Prikry genericity: Suppose that M is an inner model of ZFC so that in M, κ is measurable and U is a normal ultrafilter over κ. Assume that $s \subseteq \kappa$ has order-type ω. Then there is a G P_U-generic over M so that $s_G = s$ iff s generates U, i.e. $X \in U$ just in case $s - X$ is finite. Actually, Mathias first conceived his ideas in an ultrafilter-free context about ω, and this led to his result (see his [1977]) that $\omega \to (\omega)^{\omega}_2$ holds in Solovay's Lebesgue measurability model (§18). Ramsey ultrafilters over ω (recall §13), having the requisite partition properties, were later found to play a useful role in these proceedings.

How about changing cofinalities to cardinals other than ω? Magidor(1975) devises a generalization of Prikry forcing to show: Suppose $\lambda < \kappa$ are regular, and κ is a measurable cardinal such that there is an ascending sequence of order-type λ of normal ultrafilters over κ in Mitchell's order $\lhd$ (see the end of §9). Then there is a notion of forcing which preserves all cardinals yet changes the cofinality of κ to λ. The method here does not have the exactitude of Prikry's in that new bounded subsets of κ will be added, in particular the infinite initial segments of the new generic sequence cofinal in κ. Informal arguments exist to show why something like this must happen.

It is an interesting sidelight of forthcoming work in §29 that the existence of a (set) notion of forcing, which preserves all cardinals but changes a cofinality, is equi-consistent with the existence of a measurable cardinal. For those reading ahead, the point is that a (set) notion of forcing preserves the Core Model K, and so the covering property would be violated if some cardinal is newly singularized. Similar arguments using Mitchell's generalization of K show that Magidor's hypotheses in the preceding paragraph are problably necessary.

§24. Kunen-Paris Forcing

We discuss here a framework for forcing involving measurable cardinals which has aspects that filter through to the general methods of the next section. It provided an early analysis of the interplay of elementary embeddings and forcing defined via products, and is especially useful for proving technical relative consistency results about ultrafilters and ideals. As an application, we provide a proof using the method of a result of Solovay[1971] once promised: the relative consistency of real-valued measurability with respect to (two-valued) measurability.

The scheme introduced in Kunen-Paris[1971] is, slightly simplified, as follows: Let us assume that in the ground model V, U is a normal ultrafilter over $\kappa > \omega$, and $j: V \to M \cong V^{\kappa}/U$. Suppose that P is a notion of forcing in V. Ideally, we would like to extend U to a κ-complete ultrafilter in a generic extension via P. Of course, this will not always be possible, but with some restrictions on P, we will be able to extend U in a weak sense. So, the following assumptions are made:

(i) We can identify $j(P) = P \times Q$ for some Q in such a way that:

(ii) For every $p \in P$, $j(p) = \langle p, \mathbb{1} \rangle$ (where $\mathbb{1}$ is the weakest condition of Q).

Those looking ahead to §25 will notice that (ii) is a special case of a general compatibility requirement there.

Regarding $j(P)$ as a notion of forcing for V, we define a term $\underline{U^*}$ in the forcing language appropriate for $j(P)$ by:

$$r \Vdash \tau \in \underline{U^*} \text{ iff } r \Vdash \tau = \bar{\tau} \subseteq \kappa \text{, for some term } \bar{\tau} \text{ in the forcing language appropriate for } P \text{, and } M \models (r \Vdash \kappa \in j(\bar{\tau})).$$

Let us try to make sense of this definition. Firstly, we can construe the forcing language for P as a sublanguage of the forcing language for $j(P) = P \times Q$, so the intrusion of $\bar{\tau}$ is not incongruous. Secondly, we are trying to define $\underline{U^*}$ as a term for forcing with $j(P)$ over the full V, so the first two forcing statements about $r \in j(P)$ should be regarded in this way. However, the last statement $r \Vdash \kappa \in j(\bar{\tau})$ should be read, of course, in terms of the forcing relation with $j(P)$ within M. Finally, remember that for any $X \subseteq \kappa$ in the ground model V, $X \in U$ iff $\kappa \in j(X)$, as U is normal. Thus, we are trying to extend U to P-terms $\bar{\tau}$ which denote subsets of κ, so that $j(\bar{\tau})$ is a $j(P)$-term denoting a subset of $j(\kappa)$, and seeing whether $\kappa \in j(\bar{\tau})$ is forced. What goes on in §25 is essentially a generalization of this idea cast in terms of extending elementary embeddings; in the present section we have in mind more combinatorial results.

Suppose now that G is $j(P)$-generic over V. Since $j(P) = P \times Q$, we can consider $G = G_1 \times G_2$, where G_1 is P-generic over V and G_2 is Q-generic over $V[G_1]$. Our labors were directed toward the following observation, which says that the desired extension of U for a forcing extension via P exists tantalizingly in the further extension obtained by forcing with Q. (Recall a similar situation in §12.) Suppose U^* is the realization in $V[G]$ of the term $\underline{U^*}$. Then in $V[G]$, U^* is a $V[G_1]$-κ-complete ultrafilter on $P(\kappa) \cap V[G_1]$ such that $U^* \supseteq U$.

$\vdash$ We shall only check $V[G_1]$-κ-completeness. So, assume $\gamma < \kappa$, τ is a P-term, and $r \in j(P)$ with: $r \Vdash \tau: \gamma \to \dot{P}(\kappa)$, and $M \models \forall\alpha<\gamma(r \Vdash \kappa \in j(\tau(\alpha)))$. Let $\bar{\tau}$ be a P-term so that any condition $\in P$ forces $\bar{\tau} = \bigcap_{\alpha<\gamma} \tau(\alpha)$. Then in M, any condition $\in j(P)$ forces $j(\bar{\tau}) = \bigcap_{\alpha<\gamma} (j(\tau))(\alpha) = \bigcap_{\alpha<\gamma} j(\tau(\alpha))$, as $j(\gamma) = \gamma$. Thus, $M \models (r \Vdash \forall\alpha<\gamma(\kappa \in j(\tau)(\alpha))$ directly implies $M \models (r \Vdash \kappa \in j(\bar{\tau}))$. $\dashv$

With a bit more difficulty, one can also show that U^* is $V[G_1]$-normal. The point of all this is that in many cases, forcing with the Q part of $j(P) = P \times Q$ is tame in one of two senses: (i) it adds no new subsets of κ, so in $V[G]$, U is an ultrafilter over κ, and hence κ is measurable, or (ii) it has a strong chain condition on it so that a pull-back of U^* to a non-trivial filter in $V[G_1]$ is possible. When P is a uniformly defined, iterative notion of forcing (see concrete examples in §25), we can often write $j(P) = P \times Q$ and arrange for the Q portion to be tame in this way. Among several results using constructions of this sort, Kunen-Paris[1971] established: Con(ZFC & there is a measurable cardinal κ) implies Con(ZFC & there is a measurable cardinal κ with $2^{2^{\kappa}}$ normal ultrafilters over κ). Here, many copies of the tail Q are used to generate different normal ultrafilters; see §14 for remarks about the possible number of normal ultrafilters.

We shall apply the technique when P is a product measure algebra, and get the aforementioned consistency of the real-valued measurability of 2^{ω}, first achieved by Solovay using different means. For this purpose, let us quickly review the necessary facts about measure algebras. Suppose that B is a <u>σ-algebra</u>, i.e. a Boolean algebra all of whose countable suprema exist. A <u>measure on B</u> is a function μ: $B \to [0,1]$ so that: (a) $\mu(\mathbb{1}) = 1$, and (b) whenever $\{b_n \mid n \in \omega\} \subseteq B$ with $b_n \wedge b_m = 0$ for $n \neq m$, then $\mu(\bigvee_n b_n) = \Sigma_n \mu(b_n)$. If in addition μ is <u>positive</u> (i.e. $\mu(b) = 0$ <u>iff</u> $b = 0$), then we say that $\langle B,\mu\rangle$ is a <u>measure algebra</u>. A measure algebra is always a complete Boolean algebra.

Suppose now that I is a set, and $\langle B_i,\mu_i\rangle$ for $i \in I$ are measure algebras. Call $C \in \prod_{i\in I} B_i$ a <u>cylinder</u> <u>iff</u> $C(i)$ is the unit element of B_i except for a finite number of coordinates i. Let $B \subseteq \prod_{i\in I} B_i$ be the σ-algebra generated by the cylinders. It is known that there is a unique measure μ on B so that $\mu(C) = \prod_{i\in I}\mu_i(C(i))$ for any cylinder C. μ may not be positive, but there is a standard strategem: Let $I = \{b \in B \mid \mu(b) = 0\}$. Then I is an ideal, and $\bar{B} = B/I$ as usual is a σ-algebra consisting of equivalence classes $[b]$ for $b \in B$ (where $[b] = [c]$ <u>iff</u> the symmetric difference $(b - c) \vee (c - b) \in I$). We can define a positive measure $\bar{\mu}$ on $\bar{B}$ by: $\bar{\mu}([b]) = \mu(b)$. Thus, $\langle \bar{B},\bar{\mu}\rangle$ is a measure algebra, called the <u>product measure algebra</u> of the $\langle B_i,\mu_i\rangle$'s.

Let $\mathbb{2}$ be the basic measure algebra $\langle P(2),\mu\rangle$ where μ is the measure: $\mu(\emptyset) = 0$, $\mu(\{0\}) = \mu(\{1\}) = \frac{1}{2}$, and $\mu(\{0,1\}) = 1$. For any set I, let $\mathbb{2}^I$ denote the product measure algebra of I copies of $\mathbb{2}$. We can then force with $\mathbb{2}^I$ with the natural proviso: b is a <u>stronger</u> condition than c <u>iff</u> $0 < b \leq c$ in $\mathbb{2}^I$. This forcing obviously has the ω_1-c.c. .

With all these preliminaries, we can now proceed to provide a proof of Solovay's result: <u>Suppose κ is a measurable cardinal in the ground model V. If G is generic over V for forcing with the measure algebra $\mathbb{2}^{\kappa}$, then in V[G], $\kappa = (2^{\omega})^{V[G]}$ is real-valued measurable.</u>

⊢ The initial observation that $\kappa = (2^{\omega})^{V[G]}$ is standard: First, $(2^{\omega})^{V[G]} \leq (|\mathbb{2}^{\kappa}|^{\omega})^V$ by the usual counting argument, using the ω_1-c.c.; but the measure algebra $\mathbb{2}^{\kappa}$ has cardinality κ, so $(2^{\omega})^{V[G]} \leq \kappa$. What remains is to exhibit κ distinct subsets of ω in V[G]. For this purpose, let h: $\kappa\times\omega \leftrightarrow \kappa$ be a bijection in V. For $\delta < \kappa$ and $k < 2$, let C_k^{δ} be the cylinder $\in {}^{\kappa}\mathbb{2}$ with every coordinate the unit element $2 = \{0,1\}$ except the δth, and $C_k(\delta) = \{k\}$. Then define for every $\alpha < \kappa$ terms s_{α} for subsets of ω by: $p \Vdash n \in s_{\alpha}$ <u>iff</u> p is stronger than $[C_1^{h(\alpha,n)}]$. For any condition p and $\alpha < \beta < \kappa$, there is an $n \in \omega$ sufficiently large so that $p \wedge [C_k^{h(\alpha,n)}] \wedge [C_{1-k}^{h(\beta,n)}] > 0$ for some $k < 2$. This is so since $p > 0$ means that if $p = [b]$, then b could not be $\leq$ infinitely many different cylinders, by definition of a product measure. We have thus shown that for any condition p and $\alpha < \beta < \kappa$, there is a stronger condition q so that $q \Vdash s_{\alpha} \neq s_{\beta}$. Hence, $\kappa \leq (2^{\omega})^{V[G]}$.

We now turn to the demonstration of the real-valued measurability of κ in V[G]. We first <u>argue in V</u>: Let U be a normal ultrafilter over κ, and j: $V \to M \approx V^{\kappa}/U$. Then $j(\mathbb{2}^{\kappa}) = (\mathbb{2}^{j(\kappa)})^M = \mathbb{2}^{j(\kappa)}$, the last equality being an easy consequence of the ω_1-c.c. of measure algebras, and the fact that ${}^{\omega}M \subseteq M$. We can certainly write $\mathbb{2}^{j(\kappa)} = \mathbb{2}^{\kappa} \times \mathbb{2}^{j(\kappa)-\kappa}$. Also, this identifies $j(p) \in j(\mathbb{2}^{\kappa}) = \mathbb{2}^{j(\kappa)}$ with $\langle p,\mathbb{1}\rangle$, since if $p = [b]$, b has at most a countable "support" $s \subseteq \kappa$ of coordinates on which it is not the unit element, and $j(b)$ has the same support $j(s) = s$.

All this, of course, makes the Kunen-Paris method applicable. Hence, there is a term U^* so that, continuing to assume that G is $\mathbb{2}^{\kappa}$-generic over V, if H is any $\mathbb{2}^{j(\kappa)-\kappa}$-generic filter over V[G], then $K = G \times H$ is $\mathbb{2}^{j(\kappa)}$-generic over V, and the realization $U^* \in V[K]$ of U^* is a V[G]-κ-complete ultrafilter on $P(\kappa)\cap V[G]$. The following lemma now concludes the proof (with our V[G] in the role of its ground model V):

<u>Lemma</u>: Suppose that $\langle B,\mu\rangle$ is a measure algebra in the ground model V so that there is a term τ whose realization in any forcing extension via $\langle B,\mu\rangle$ is a V-κ-complete ultrafilter on $P(\kappa) \cap V$. In V, define $\nu(X)$ for any $X \subseteq \kappa$ by:

$$\nu(X) = \mu(\bigvee\{b \in B \mid b \Vdash \check{X} \in \tau\})$$

Then ν is a κ-additive measure on $P(\kappa)$, and so κ is real-valued measurable in V.

To establish this lemma, we argue in V. First, $\nu(\kappa) = 1$ and $\nu(\{\alpha\}) = 0$ for any $\alpha < \kappa$. Suppose now that $\gamma < \kappa$, and X_{α} for $\alpha < \gamma$ are disjoint subsets of κ. If $b \in B$ and $b \Vdash (\bigcup_{\alpha<\gamma} X_{\alpha})\check{} \in \tau$, then for some $\alpha < \gamma$ and some $\bar{b}$ stronger than b, we have $\bar{b} \Vdash \check{X}_{\alpha} \in \tau$, as τ is forced to be V-κ-complete. Thus,

$$\nu(\bigcup_{\alpha<\gamma} X_{\alpha}) = \mu(\bigvee\{b \in B \mid b \Vdash (\bigcup_{\alpha<\gamma} X_{\alpha})\check{} \in \tau\}) = \mu(\bigvee_{\alpha<\gamma}(\bigvee\{b \in B \mid b \Vdash \check{X}_{\alpha} \in \tau\})).$$

Let $q_{\alpha} = \bigvee\{b \in B \mid b \Vdash \check{X}_{\alpha} \in \tau\}$. The q_{α}'s are pairwise incompatible as τ is forced to be ultra, and hence if $T = \{\alpha \mid q_{\alpha} > 0\}$, $|T| \leq \omega$ by the ω_1-c.c. on measure algebras. Thus, we can continue our equality chain:

$$\nu(\bigcup_{\alpha<\gamma} X_{\alpha}) = \mu(\bigvee_{\alpha<\gamma} q_{\alpha}) = \mu(\bigvee_{\alpha\in T} q_{\alpha}) = \sum_{\alpha\in T}\mu(q_{\alpha}) = \sum_{\alpha<\gamma}\mu(q_{\alpha}) = \sum_{\alpha<\gamma}\nu(X_{\alpha}).$$

We have established that ν is κ-additive, and so the proof of the Lemma, and thereby the theorem, is complete. ⊣

§25. Silver's Forcing Method

This section is devoted to an exposition of an important technique for making forcing constructions which preserve large cardinals, while creating some desired situation in the universe. It is a prominent technique for independence results in set theory involving strong axioms of infinity. The basic framework is often given the descriptive name "Backwards Easton Forcing" (we shall comment on why later), and was probably first seen in the work of Jensen. Silver then molded the ideas into a general method, and, in particular, solved the problem of getting a model in which there is a measurable cardinal κ such that $2^\kappa > \kappa^+$ (recall §10). For a detailed presentation of Silver's method in the Boolean algebraic setting, we refer to Menas [1976].

We shall confine our attention to those large cardinals characterizable by the existence of an appropriate elementary embedding. In most cases, the domain of this elementary embedding can be assumed to be of the form $\langle V_\beta, \varepsilon, R\rangle$, where $R \subseteq V_\beta$, and the large cardinal is usually the critical point of the embedding, i.e. the least ordinal moved by it. Thus, the Keisler property for weak compactness (recall §3) shows that weak compactness fits our specifications, with the embedding being in this case the identity map. The measurability of a cardinal κ can be characterized by the existence of some $j\colon \langle V_{\kappa+1}, \varepsilon\rangle \to \langle M, \varepsilon\rangle$ with critical point κ, where M is transitive. Similarly the supercompactness of κ can be characterized as follows: for every $\beta > \kappa$ there is a $j_\beta\colon \langle V_\beta, \varepsilon\rangle \to \langle M_\beta, \varepsilon\rangle$ with critical point κ, where: (i) M_β is transitive, and (ii) M_β is closed under the taking of arbitrary β sequences of its members, provided the members of the sequence have, in M_β, rank bounded below some fixed $\alpha \in M_\beta$. We can go on to make similar comments on extendible cardinals, and so forth.

If we want to extend the universe by some forcing extension which will preserve the size of κ, where κ is the critical point of an elementary embedding $j\colon \langle V_\beta, \varepsilon\rangle \to \langle M, \varepsilon\rangle$, a natural approach to try is to do the forcing carefully enough so that in the extension, j can be extended to an elementary embedding of the new $\langle V_\beta, \varepsilon\rangle$ into some new transitive structure M' extending M, which has all the relevant properties that M had for guaranteeing that κ is a large cardinal.

Mild Extensions. As an easy example in which this can be done, consider the case in which the set of forcing conditions is "mild", i.e. has cardinality less than κ. In this case the notion of forcing P can be assumed to be a member of V_κ, and the salient fact about this is that j fixes P as well as every member of it. Suppose that G is a P-generic filter over the ground model V. Then $(V_\beta)^{V[G]}$ may have new elements, but we can assume that every such element has a "name" in V_β. (Here, $V_\beta = (V_\beta)^V$, the ground model's conception of the set of sets of rank $< \beta$.) To guarantee this, the standard way of naming elements of the generic extension by elements of the ground model has to be modified in an inessential way (especially when β is a successor ordinal), but this can be done using the fact that $P \in V_\kappa \subseteq V_\beta$. Let us denote by $\mathrm{Rel}(G,a)$ that element of $V[G]$ which is the realization of the forcing term $a \in V$. By preceding remarks, for every $x \in (V_\beta)^{V[G]}$, there is an $a \in V_\beta$ so that $x = \mathrm{Rel}(G,a)$.

We are now faced with the task of defining the extended elementary embedding j' for the domain $(V_\beta)^{V[G]}$. The idea is to define $j'(x)$ by referring to a name for x in V_β, applying j to it, and then taking the realization of the

resulting name as the value of $j'(x)$. That is, if $x = \mathrm{Rel}(G,a)$ where $a \in V_\beta$, then set $j'(x) = \mathrm{Rel}(G,j(a))$. An obvious difficulty with this approach is that an element in the generic extension can have many different names, and the j' thus defined may well depend on the particular name picked for x. However, this difficulty does not really arise in this case:

If $\mathrm{Rel}(G,a) = \mathrm{Rel}(G,b)$ for some names $a,b \in V_\beta$, there must be a condition $p \in G$ so that $p \Vdash a = b$. This is a property of p which can be expressed in V_β as a first-order property of p, P, a, and b. Hence, by the fact that j is an elementary embedding, the same property holds in M for $j(p) = p$, $j(P) = P$, $j(a)$, and $j(b)$. The definition of forcing is rather absolute, and M being elementarily equivalent to V_β is behaving well enough so that we can deduce that $p \Vdash j(a) = j(b)$. Hence, $\mathrm{Rel}(G,j(a)) = \mathrm{Rel}(G,j(b))$, and j' is well defined.

We have not yet specified into what structure $(V_\beta)^{V[G]}$ is going to be embedded by j', and a natural candidate for this is $M' = \{\mathrm{Rel}(G,a) \mid a \in M\}$. M' can quite easily be checked to be transitive, with $M \subseteq M'$ (provided we had been reasonable about the "naming" procedure). Also, j' is an elementary embedding: Suppose that $(V_\beta)^{V[G]} \models \phi[\mathrm{Rel}(G,a_1),\ldots,\mathrm{Rel}(G,a_n)]$. Then some $p \in G$ must force a corresponding forcing assertion. Again, this can be expressed as a first-order property in V_β of p, P, $a_1,\ldots,a_n$. The same property holds in M of p, P, $j(a_1),\ldots,j(a_n)$, which implies that p forces the forcing assertion corresponding to: $M' \models \phi[j'(\mathrm{Rel}(G,a)),\ldots,j'(\mathrm{Rel}(G,a_n))]$, since $j'(\mathrm{Rel}(G,a_i)) = \mathrm{Rel}(G,j(a_i))$ for $1 \leq i \leq n$. The reader should be aware of the important role that the fact that $j(p) = p$ for $p \in P$ and $j(P) = P$ played in these arguments.

In the last few paragraphs we essentially outlined the proof that if κ is weakly compact, measurable, supercompact, extendible, or huge, respectively, it retains this property in a generic extension obtained by a set of forcing conditions having cardinality less than κ. (This in general form is the basic remark of Lévy-Solovay [1967].) This implies that many important properties of small sets like the Continuum Hypothesis or Souslin's Hypothesis are not affected by the existence of large cardinals, because the proof of the independence of these various statements call upon notions of forcing of small cardinality. This is a disappointment to the hopes (first expressed by Gödel) of deciding these questions by assuming the existence of large cardinals, but on the other hand the results which do show some effect of large cardinals on small sets, like the existence of $0^\#$, seem all the more surprising.

The problem of preserving a large cardinal κ under forcing extensions becomes much more difficult if we have to use a set of forcing conditions having cardinality at least κ. This is necessary if we want to affect some properties of the large cardinal κ itself, like getting $2^\kappa > \kappa^+$. Let us continue to assume that κ is characterizable as the critical point of some $j\colon \langle V_\beta, \varepsilon\rangle \to \langle M, \varepsilon\rangle$. The problems we are faced with if we try to follow the approach we outlined above for the case of a mild forcing notion are mainly that $j(P)$ is not equal to P, and for $p \in P$, $j(p)$ is not necessarily equal to p. Now, if a is a name in the forcing language for P, $j(a)$ is a name in the forcing language for $j(P)$, rather than for P. Hence, in order to realize $j(a)$, we have to use a generic filter for $j(P)$ rather than for P.

In many important cases this is not a major problem, in view of the fact that what is needed is a $j(P)$-generic filter over M, and not over the full V. It certainly helps here if M is meager enough compared with V so that forcing with P over V immediately guarantees that we also have a $j(P)$-generic filter over M. In important situations this does not happen, and we have to force further to procure a $j(P)$-generic filter over M. These cases can still be handled if the new forcing does not introduce any further sets of rank $< \beta$, so that our carefully nurtured embedding is preserved. (We had similar concerns in §24.)

Let us now assume that G is a P-generic filter over V, and that in $V[G]$, we can get a $j(P)$-generic filter G' over M. Extrapolating from the case of mild extensions, the intention is to extend our given embedding j to j' by: if $x \in (V_\beta)^{V[G]}$ with $x = \mathrm{Rel}(G,a)$ for an $a \in V_\beta$, then $j'(x) = \mathrm{Rel}(G',j(a))$, in the sense of $M[G']$. Our troubles are not over yet because if we want j' to be a well-defined elementary embedding, we need to have that whenever $p \in G$, $j(p) \in G'$. This is a major difficulty, as the following example indicates:

Example. Suppose we want to introduce a new subset of a measurable cardinal κ, without collapsing any cardinals or introducing subsets of smaller cardinals. The obvious choice is of course to introduce a Cohen generic subset of κ: Let the notion of forcing $\bar{P}$ be the set of functions from a bounded subset of κ into 2, ordered by extension. If G is $\bar{P}$-generic, then the subset of κ introduced by G is $\{\alpha < \kappa \mid p(\alpha) = 1 \text{ for some } p \in G\}$. Suppose also that we want to preserve the measurability of κ. That is, we had in the ground model V an elementary embedding $j: \langle V_{\kappa+1},\epsilon\rangle \to \langle M,\epsilon\rangle$, and we want to extend j to $(V_{\kappa+1})^{V[G]}$. Note that we can consider $\bar{P} \in V_{\kappa+1}$. If M was obtained by the usual ultrapower construction, and if $2^\kappa = \kappa^+$, then it can easily be verified that even in V we can find a $j(\bar{P})$-generic filter over M: Because $|M| \le (\kappa^+)^\kappa = \kappa^+$, we have at most κ^+ dense subsets of $j(\bar{P})$ in M. M considers $j(\bar{P})$ to be $<j(\kappa)$-closed (i.e. whenever $\gamma < j(\kappa)$ and $\langle p_\alpha \mid \alpha < \gamma\rangle$ is a sequence of elements of $j(\bar{P})$ increasing in strength, there is a $q \in j(\bar{P})$ stronger than every p_α). Note also that every subset of $j(\bar{P})$ of cardinality $\le \kappa$ is in M. Using all this, we can construct by induction a filter on $j(\bar{P})$ which will meet each of the κ^+ dense subsets in M. (During this construction we use the fact that the sequence constructed so far has cardinality $\le \kappa$, so is in M, and hence has an upper bound in M). Thus, we get a $j(\bar{P})$-generic filter over M. The problem is getting such a filter G' which will satisfy the extra requirement that for some $\bar{P}$-generic filter G over V,

(*) if $p \in G$, then $j(p) \in G'$.

For this particular forcing, this simply cannot be done:

Assume otherwise. Note that since $\bar{P} \subseteq V_\kappa$, j is the identity on $\bar{P}$. Hence, if A and A' are subsets of κ and $j(\kappa)$ introduced respectively by G and G', the condition (*) would imply that $A' \cap \kappa = A$. However, A is not in V whereas every initial segment of A' is in $M \subseteq V$, which is a contradiction.

In fact, it is not surprising that we ran into trouble in the above example, because at least in some cases we cannot introduce a Cohen generic subset of κ while preserving the measurability of κ. Such a case is when V is $L[U]$, where U is a normal ultrafilter over κ. If we could find a $\bar{P}$-generic filter G over this V which introduces a subset A of κ and preserves the measurability of κ, then we would have in $V[G]$ an elementary embedding $\bar{j}$ of $(V_{\kappa+1})^{V[G]}$ into some transitive $\bar{M}$. In $(V_{\kappa+1})^{V[G]}$ we can express the fact that every initial segment of A is in $L[U]$. (We need not refer to U to do this.) The same is true for $\bar{j}(A)$ in $\bar{M}$, and so $A = \bar{j}(A) \cap \kappa$ is in $L[j(U)]$. However, it is a well-known result of Kunen (see 8.8(iii) and fact (a) toward the end of §9) that A is then in $L[U]$. Therefore, A is in V which is a contradiction. (If we had started with a different ground model it may be possible that forcing with $\bar{P}$ preserves the measurability of κ; see Kunen(1974).)

So, the main problem in the general setting is to get a $j(P)$-generic filter G' over M satisfying (*) above, for some P-generic filter G over V. Note that if p,q are both in G, then they are compatible, and therefore $j(p)$ and $j(q)$ are compatible as elements of $j(P)$. The problem is that the set $\{j(p) \mid p \in G\}$ may give too much information so that it cannot be extended to a generic filter.

Silver's Method. Let us now discuss the way Silver handles this problem. Assume that P was defined in such a way that $j(P)$ entails P, i.e. forcing with $j(P)$ over V automatically introduces a P-generic filter over V. In Boolean algebraic terminology, this means that the Boolean algebra corresponding to P is a complete subalgebra of the Boolean algebra corresponding to $j(P)$: If one forces over V with P to get $V[G]$, then $V[G]$ can be further extended to a forcing extension of V using $j(P)$. We shall denote the notion of forcing in $V[G]$ which achieves this by $j(P)/P$, and use this notation in similar contexts. Let us make two further assumptions about P. The first is that for all $p \in P$, $j(p)$ is compatible with p. (Note that p can be considered an element of $j(P)$, in view of the assumption that $j(P)$ entails P.) This implies that $j(p)$ can be considered an element of $j(P)/P$. (For instance, if $j(p) = p$, then $j(p)$ will be identified with the weakest condition of $j(P)/P$.) Further, assume that $V[G]$ considers $j(P)/P$ to be a $|P|$-closed notion of forcing.

Under all these assumptions, it is possible to find for every P-generic filter G a $j(P)$-generic filter G' satisfying (*): In $V[G]$ consider the set $\{j(p) \mid p \in G\}$ as a set of elements of $j(P)/P$ which lies in $V[G]$, such that any two of them are compatible. Since its cardinality is $|P|$, it has an upper bound q . Let H be any $(j(P)/P)$-generic filter over $V[G]$ which contains q . Then the pair $\langle G,H\rangle$ can be combined to give a $j(P)$-generic filter G' over V whose restriction to P is G . The fact that $q \in H$ guarantees that if $p \in G$, then $j(p) \in G'$, which was our objective.

Now recall that we only need a $j(P)$-generic filter over M , and not over the full V . This is especially significant, since we may not need to extend $V[G]$ at all. If we are lucky, we may be able to produce, within $V[G]$, a $(j(P)/P)$-generic filter H' over $M[G]$ which contains the q above, so that $\langle G,H'\rangle$ can be identified with a $j(P)$-generic filter over M .

The requirements that we imposed on P seem rather stringent, and so we should see how we can satisfy them in a rather general setting. Suppose that we want to force some situation to occur at a large cardinal κ (at the very least inaccessible!), for instance, to render $2^{\kappa} > \kappa^{+}$. Since κ will have certain reflection properties, it is quite natural to assume that we shall have to do the same construction for $\alpha < \kappa$. So, we perform an iterated forcing construction, where at the αth stage we already have P_{α} which takes care of ordinals $< \alpha$, and we are given a term τ_{α} which denotes a forcing notion in the universe obtained by forcing with P_{α} , which is supposed to take care of α . Let us assume that τ_{α} is forced to be $<|P_{\alpha}|$-closed. $P_{\alpha+1}$, the next stage of the construction, will then be the iterated forcing notion corresponding to forcing with P_{α} followed by forcing with the realization of τ_{α} . (The first occurrence of this sort of iteration is in Solovay-Tennenbaum[1971].) Let us denote the combined forcing by: $P_{\alpha+1} = P_{\alpha} * \tau_{\alpha}$. (We shall use this notation in similar contexts.) At limit stages the construction is usually more technical, but let us assume that it can be done in such a way that for inaccessible $\alpha \leq \kappa$, P_{α} is the usual direct limit of $\langle P_{\beta} | \beta < \alpha \rangle$. We shall also assume that for all $\alpha < \kappa$, τ_{α} is forced to have cardinality $< \kappa$, so this will insure that for all $\alpha < \kappa$, P_{α} will have cardinality $< \kappa$. The technicalities which are involved in the definition at the limit stages of the iteration come about mainly because we want to guarantee that for $\delta < \gamma$, P_{γ}/P_{δ} is forced to be $<|P_{\delta}|$-closed. In the next paragraph, we elaborate on a related point which is probably for the more sophisticated reader.

Anyone aware of some of the possible applications of this construction may ask whether this complicated iteration scheme is really necessary. If we want to create a certain situation at every regular cardinal $\alpha \leq \kappa$, why not let Q_{α} be the appropriate forcing notion for α and force with an appropriate product of the Q_{α}'s? This is exactly what is done by Easton[1964] when he wants to blow up 2^{α} to α^{++}, say, for every regular $\alpha \leq \kappa$: Q_{α} is the usual notion of forcing in V which introduces α^{++} new subsets of α , and the final notion of forcing R is simply a kind of direct product of the Q_{α}'s. Note that Q_{α} is $<\alpha$-closed. However, observe the following: If $R_{\alpha+1}$ is the product of the Q_{β}'s for $\beta \leq \alpha$, $R/R_{\alpha+1}$ is

unfortunately not closed enough in spite of the fact that for $\alpha < \gamma \leq \kappa$, Q_{γ} $|R_{\alpha+1}|$-closed as a member of V : Once we have forced with $R_{\alpha+1}$, there will new sequences of elements of Q_{γ} . The virtue of the iteration scheme in the p vious paragraph is that for our ultimate notion of forcing P , $P/P_{\alpha+1}$ will b $|P_{\alpha+1}|$-closed, because for every $\gamma > \alpha$, τ_{γ} was assumed to be a notion of fo ing in the universe obtained after forcing with P_{γ} , and is $|P_{\alpha+1}|$-closed in this larger universe. This step-by-step iteration instead of the simultaneous istering to all $\alpha \leq \kappa$ is the reason for the often used term "Backwards Easton"

Let us continue the discussion initiated in the penultimate paragraph. Th notion of forcing that we shall ultimately use is $P = P_{\kappa+1}$, i.e. the notion o forcing that should take care of all $\alpha \leq \kappa$. Let us assume that the definitio the sequence $\langle \tau_{\alpha} | \alpha \leq \kappa \rangle$ is so absolute that if we repeat the same definitio M , we shall get a sequence (of length $j(\kappa)+1$) which extends the old sequence, that: $j(\langle P_{\alpha} | \alpha < \kappa \rangle)(\kappa+1) = P_{\kappa+1} = P$. We are now in a situation where $j(P)$ entails P .

Observe that whenever $p \in P$, $j(p)$ is compatible with p . (To see thi notice that j can be assumed to be the identity on P_{κ} , as $|P_{\kappa}| = \kappa$. So si $P = P_{\kappa+1} = P_{\kappa} * \tau_{\kappa}$, every $p \in P$ can be considered to be a pair $\langle q,s\rangle$, where $q \in P_{\kappa}$ and s is a term forced by q to be in τ_{κ} . Thus, $j(p) = \langle j(q),j(s)\rangle = \langle q,j(s)\rangle$. $j(s)$ is a term forced to belong to $\tau_{j(\kappa)}$, and hence $j(p)$ is clearly compatible with p , since s and $j(s)$ refer to different notions of forcing.) Using this remark, it is routine for the forcing practitioner to show the desired property that $j(P)/P$ will be $|P|$-closed, provided:

(a) M contains every subset of $j(P)$ of cardinality $\leq |P|$, and

(b) $P/P_{\alpha+1}$ is actually $|P_{\alpha+1}|$-closed for every $\alpha < \kappa$,

so that we can invoke the elementarity of j . Condition (b) can be easily met letting τ_{α} be the empty forcing notion most of the time. For example, if the typical forcing notion τ_{α} that we have in mind will be forced to have a cardin ity which is easily accessible from α , then we could agree to only introduce t τ_{α} for inaccessible $\alpha < \kappa$, letting τ_{α} for other α be the empty forcing not

As an example of the above construction, let us survey the proof of Silver' theorem: Con(ZFC & κ is a κ^{++}-supercompact cardinal) implies Con(ZFC & κ is a κ^{++}-supercompact cardinal and $2^{\kappa} > \kappa^{+}$). (Since such a κ is at least measura we have a model in which the GCH is violated at a measurable cardinal.)

$\vdash$ κ is κ^{++}-supercompact iff there is an elementary embedding of $V_{\kappa+3}$ into a transitive structure M such that every sequence of elements of M of cardinal $\leq \kappa^{++}$ can be coded as an element of M . Without loss of generality we can assu that $2^{\kappa} = \kappa^{+}$ (else we are finished) and that $2^{\kappa^{++}} = \kappa^{+++}$ (which can be arrange by a simple forcing extension which does not destroy the κ^{++}-supercompactness of κ).

We now follow the scheme presented above. For $\alpha \leq \kappa$, let τ_{α} be a term denoting the standard forcing notion for adding α^{++} new subsets to α <u>if α is inaccessible</u>, and the empty forcing notion otherwise. We define the sequence $\langle P_\alpha \mid \alpha \leq \kappa \rangle$, and set $P = P_{\kappa+1} = P_\kappa * \tau_\kappa$. Note that the forcing notion denoted by τ_α is in the universe obtained by iterating $\langle \tau_\beta \mid \beta < \alpha \rangle$ first. Note also that the sequence $\langle P_\alpha \mid \alpha \leq \kappa \rangle$ (or rather some trivial coding of it) is definable in $V_{\kappa+3}$. Since M is closed under κ^{++} sequences (note this!), the definition is absolute enough for M so that if we use the same definition in M, we get a sequence which extends $\langle P_\alpha \mid \alpha \leq \kappa \rangle$. Also, M considers $j(P)/P$ to be $<\nu$-closed, where in M, ν is the first inaccessible above κ. This follows, of course, from our choice of the terms τ_α. Since M is closed under sequences of length $\leq \kappa^{++}$, $j(P)/P$ is really κ^{++}-closed. This will yield that if G is a P-generic filter over V, in $V[G]$ we can inductively construct a $(j(P)/P)$-generic filter over $M[G]$, if we can show for example that there are at most κ^{+++} dense subsets of $j(P)$ in M. But this is true, because it can be verified that $|P| = \kappa^{++}$ and P satisfies the κ^+-c.c., hence in V there are at most $(\kappa^{++})^\kappa = \kappa^{++}$ dense subsets of P. Therefore in M, there are at most $j(\kappa^{++})$ dense subsets of $j(P)$. However, if M was obtained by the usual ultrapower construction, $j(\kappa^{++})$ has (real) cardinality $(\kappa^{++})^{\kappa^{++}} = 2^{\kappa^{++}} = \kappa^{+++}$. (The fact that $j(P)/P$ is $|P|$-closed is of course important in getting an upper bound for $\{j(p) \mid p \in G\}$.) Thus, we can find a $j(P)$-generic filter G' over M such that $p \in G$ implies $j(p) \in G'$, and the general construction allows us to extend j to j': $(V_{\kappa+3})^{V[G]} \to M'$, where $M' = \{Rel(G,a) \mid a \in M\}$. ⊣

We essentially used the fact that M is closed under κ^{++} sequences, and this is exactly the point where we had to assume that we started with a large cardinal stronger than measurable. Silver's theorem with just "measurable" in place of "κ^{++}-supercompact" cannot be proved, since as we saw in §10, Kunen established that the consistency strength of the existence of a measurable cardinal κ with $2^\kappa > \kappa^+$ is at least that of the existence of many measurable cardinals.

By amplifying the type of argument we have encountered in this section, Laver (1974) was able to give a very strong version of these techniques for supercompact cardinals as follows: <u>Let κ be a supercompact cardinal. Then there exists a forcing notion P, having cardinality κ and satisfying the κ-c.c., such that in the universe obtained by forcing with P, κ is still supercompact, and forcing over this universe with any $<\kappa$-closed forcing notion will preserve the supercompactness of κ.</u>

⊢ Laver's theorem is proved by using the following combinatorial principle: Let κ be supercompact. Then there exists a sequence $\langle x_\alpha \mid \alpha < \kappa \rangle$ of elements of V such that for every set A and for every β there exists an elementary embedding $j: V \to M$ with critical point κ, where M is closed under β sequences, and $j(\langle x_\alpha \mid \alpha < \kappa \rangle)(\kappa) = A$.

With such a sequence $\langle x_\alpha \mid \alpha < \kappa \rangle$, Laver constructs an iterated forcing sequence $\langle P_\alpha \mid \alpha < \kappa \rangle$, where the corresponding τ_α is x_α if x_α is a term in the forcing language appropriate for P_α which is forced to be a $<\alpha$-closed forcing notion and $|P_\alpha| = \alpha$, and the empty forcing notion otherwise. P is simply P_κ, the direct limit of $\langle P_\alpha \mid \alpha < \kappa \rangle$. The argument that shows that, after forcing with P, any $<\kappa$-closed forcing notion preserves the supercompactness of κ follows the scheme of this section: Given a term τ which denotes (with respect to P) a $<\kappa$-closed forcing notion, we can find an elementary embedding j such that $j(\langle x_\alpha \mid \alpha < \kappa \rangle)(\kappa) = \tau$ and with such a j, $P_{\kappa+1} = P_\kappa * \tau$ satisfies the requirements of our scheme. ⊣

§26. The Least Strongly Compact Cardinal

A pervading feature of the theory of large cardinals is that the different postulations form into a nice linear hierarchy: given two large cardinal concepts, one is typically of transcendingly stronger kind, both consistency-wise and in the direct implicational sense. It is in fact usually the case that if two large cardinal concepts can be directly defined in terms of the existence of a cardinal with certain properties, then the least cardinal satisfying one definition is much smaller than the least cardinal satisfying the other definition, in a strong sense given <u>in terms</u> of the context established by the stronger concept.

Strong compactness presents an anomalous case in its relationship to its neighbors, measurability and supercompactness. There is an equivocation here which is all the more interesting since although supercompactness is now thought the proper concept transcending measurability, strong compactness was formulated first with quite natural motivations. We saw in §14 that a supercompact cardinal κ is the κth measurable cardinal in a strong sense, and the simple proof of this fact has the right feel as a reflection argument. In this section, we briefly outline consistency results involving the various possibilities for the least strongly compact cardinal. This is decidedly a varying concept, as we shall see.

Concerning strong compactness and measurability, the result of Vopěnka and Hrbáček (see §3) established that these concepts provably are not the same. If U is a normal ultrafilter over κ, then κ remains measurable in $L[U]$, while κ cannot be strongly compact in $L[U]$ by their result. Similar arguments apply to several measurable cardinals. Concerning consistency strength, Kunen showed that (as cited in §10) if there is a strongly compact cardinal, then for every α there is an inner model with α measurable cardinals. Hence, strong compactness and measurability cannot be equiconsistent. However, the question still remained whether the classes of strongly compact cardinals and measurable cardinals are provably different (provided strongly compact cardinals exist!). In Magidor [1976] it is established that the answer is in the negative, namely: <u>If there is a strongly compact cardinal, then there is a forcing extension in which it remains strongly</u>

compact, but is now the least measurable cardinal. Let us describe the idea of the proof:

⊢ Suppose κ is a strongly compact cardinal. Very likely, there are many measurable cardinals below κ. We would like to deprive them of their measurability, and one way of achieving this is to make each of them singular. Prikry forcing (§23) gives us the tool for turning one measurable cardinal into a singular cardinal with cofinality ω. What we have to do is to iterate Prikiry forcing in such a way that κ remains strongly compact. Prikry's conditions for λ, say, have the form $<s,X>$ where s is a finite approximation to an ω sequence that will eventually be cofinal in λ, and X is a subset of λ which is the set of possible candidates for extending the finite sequence s. The point was that the X's were always to be taken from some normal ultrafilter over λ. When we iterate Prikry forcing for every measurable cardinal $\alpha < \kappa$, a condition will have the form $\{<s_\alpha,X_\alpha>|\ \alpha < \kappa,\ \alpha \text{ is measurable}\}$, where $<s_\alpha,X_\alpha>$ is a Prikry condition for α, with the change that X_α is not a particular subset of α but rather a term denoting such a set, which belongs to the universe obtained by first iterating the Prikry forcing $< \alpha$. One can check that α remains measurable at this train stop along the way to κ, and so X_α is required to belong to a certain normal ultrafilter over α, which belongs to this universe. One further requirement is that s_α is to be the empty sequence except for finitely many α.

With all these constraints, one can then show that a generalized form of Prikry's Lemma (§23) goes through, with all the desired consequences. Clearly every measurable cardinal below κ becomes singular, and one can verify that no new measurable cardinals are created in the process. The main point is that κ will remain strongly compact, by arguments reminiscent of §24. ⊣

In this model, κ is definitely not supercompact, since it is the least measurable cardinal. Therefore, strong compactness and supercompactness provably are not the same concept. That this may be the case was indicated by an earlier result of Menas, who noted that below an extendible cardinal there are many strongly compact cardinals which are not supercompact. This can be culled from Menas' Lemma in §15: a measurable cardinal which is a limit of strongly compact cardinals must itself be strongly compact. Note that since an extendible cardinal is a limit of supercompact cardinals, by a reflection argument there are many measurables below it which are limits of strongly compacts. Any measurable which is a limit of strongly compacts, but not a limit of such measurables, is strongly compact but not supercompact (by the argument for (ii) of that Lemma). Armed with this, Menas [1974] had been able to show: If there is a measurable cardinal which is a limit of strongly compact cardinals, then there is a forcing extension in which there is a strongly compact cardinal, and it is not supercompact. Of course, this result is subsumed by Magidor's above.

The question still remains whether the first supercompact cardinal is provably larger than the first strongly compact cardinal, and again Magidor[1976] answers this in the negative: If there is a supercompact cardinal, then there is a forcing extension in which it remains supercompact, but it has no strongly compact cardinals below it.

⊢ The idea is to perform the construction in two steps. In the first step we use Silver's forcing method (§25) to get an unbounded subset of κ of measurable cardinals which violate the Generalized Continuum Hypothesis. (A supercompact cardinal has unboundedly many cardinals below it with enough degree of supercompactness so that Silver's construction can be applied to them.)

The second step is to change the cofinality of each of these measurable cardinals to ω using the iterated Prikry forcing described above. Each of these newly singularized cardinals still violates the GCH, and one can verify that they remain strong limit cardinals. This in itself is enough to guarantee that there can be no strongly compact cardinals below κ, by Solovay's GCH result (§15): Any singular strong limit cardinal above a strongly compact cardinal satisfies the GCH.

We still have to preserve the supercompactness of κ, and for this we have to be more cautious about the set of measurable cardinals which are to be made singular. But one can show that by picking this set to be sparse enough in κ, iterating Prikry forcing on it preserves the supercompactness of κ. Hence, we get a model in which the first supercompact cardinal is also the first strongly compact cardinal. ⊣

If we turn our attention to the second strongly compact cardinal, etc., then Stern(1973) was able to show that under the existence of an extendible cardinal, one can get a model in which the first strongly compact cardinal is the first supercompact cardinal but the second is not. Similarly one can have that the first two strongly compact cardinals are not supercompact. Using other methods, one can show that it is consistent to have α supercompact cardinals which are exactly all of the strongly compact cardinals. However, it is not known whether the first two strongly compact cardinals can be the first two measurable cardinals.

The prominent open question left unanswered by these various results is whether strong compactness and supercompactness are equiconsistent. The accepted guess is that the answer is no.

VII. MISCELLANY

§27. Infinitary Games

Over the last couple of decades, it has become increasingly popular to formulate concepts in terms of games. It is helpful to regard, say, alternating quantifiers or nested sequences in a topology in this way. The theory of games as such is often preoccupied with rather different situations, and we shall be dealing with infinitary versions of what in the jargon is the rather simplistic "zero-sum two person game with perfect information". The classic von Neumann-Morgenstern[1944] had already provided the basic theorem for finite versions of such games: there is a winning strategy for one player. Infinitary versions were first studied by Gale-Stewart[1953], and by Mycielski-Zieba[1955] and Mycielski-Swierczkowski-Zieba[1956], and a rich and bizarre world was soon to open for (interested)set theorists, a world, as subsequent work of Solovay indicated, populated by many large cardinals.

The following is the basic infinitary game: Let A be a set, and $B \subseteq {}^{\omega}A$. Then $G(A,B)$ is the game with two players I and II who alternately choose elements of A. I initially picks a_0, then II picks a_1, then I picks a_2, and so forth, with each a_n picked in full knowledge of the initial play $\langle a_0,\ldots,a_{n-1}\rangle$. After ω steps, a sequence $\langle a_0,a_1,a_2,\ldots\rangle \in {}^{\omega}A$ is generated, and player I wins the game if this sequence is in B, and player II wins otherwise.

A <u>winning strategy</u> for either player in $G(A,B)$ is a function $S: \bigcup_{n\in\omega} {}^{n}A \to A$ so that if at each of his turns the player plays $S(\langle a_0,\ldots,a_{n-1}\rangle)$ after the initial play $\langle a_0,\ldots,a_{n-1}\rangle$, then he wins every play of the game. Clearly, at most one player can have a winning strategy. We say that <u>$G(A,B)$ is determined</u> <u>iff</u> one player has a winning strategy.

It turns out that (in ZF) there is always a $B \subseteq {}^{\omega}\omega_1$ so that $G(\omega_1,B)$ is not determined (Mycielski[1964]). Also, as pointed out by Gale-Stewart[1953], if $|A| > 1$, from a well-ordering of ${}^{\omega}A$ one can construct a $B \subseteq {}^{\omega}A$ so that $G(A,B)$ is <u>not</u> determined, by diagonalizing over all possible strategies. In a sense, this B can be regarded as a typically "paradoxical" set made available by the Axiom of Choice, and so Mycielski-Steinhaus[1962] introduced for further study the <u>Axiom of Determinacy</u>(AD):

$$G(\omega,B) \text{ is determined for every } B \subseteq {}^{\omega}\omega .$$

The choice $A = \omega$ is made here because it is really the simplest non-trivial case; AD is known by coding to be equivalent to the proposition that $G(2,B)$ is determined for every $B \subseteq {}^{\omega}2$.

Since the Axiom of Determinacy contradicts the Axiom of Choice, two approaches have been taken in the investigation of the determinacy of games:

(a) Abandoning AC, one can study the consequences of the full ZF + AD.

(b) Retaining AC, one can study weak versions of AD which are possibly consistent with ZFC.

The continuing progress being made along the first alternative will be discussed in the next section. <u>In the present section, we shall continue to assume the Axiom of Choice</u>, and discuss the recent experience as regards the second alternative.

The classical result in the program of determining the extent in ZFC of the class of determinate games is the following result of Gale-Stewart[1953]: <u>If B is an open subset of ${}^{\omega}\omega$, then $G(\omega,B)$ is determined.</u>

$\vdash$ The salient point here is that a member of B is in B by virtue of an initial segment. Hence, if player I does <u>not</u> have a winning strategy, then player II has a winning strategy which can be informally described as follows: At any turn, player II picks the least integer so that after it is played, player I does not have a winning strategy in the truncated game which starts with his next move. If player II cannot do this at some turn, or if he plays according to this recipe but still loses, then in either case this misfortune occurs after a finite number of plays, and we can derive a contradiction by looking at the move before. $\dashv$

This simple result is actually a pivotal one for the theory, as in many cases the determinacy of a game can be reduced in some sense to the determinacy of a derived "open" game. It is natural to look for such a reduction, since a strategy is a function on finite sequences, and an open game epitomizes the determinate situation when a finite approximation to an infinite play is really enough.

As we shall discuss in the next section, in the mid-sixties Solovay inaugurated a new era for AD when he established that it implied the relative consistency of (ZFC & there is a measurable cardinal). This was a totally unexpected result about the structure of the reals, and propelled set theorists to chart out how large cardinals come into play in games with reals.

Descriptive set theory was the natural area to look for clarifications. In order to discuss degrees of determinacy along the Projective Hierarchy, let us denote by $\mathbf{\Pi}^1_n$-Determinacy the assertion that $G(\omega,B)$ is determined for any $B \subseteq {}^{\omega}\omega$ which is $\mathbf{\Pi}^1_n$. (Variations on this notation, like the "lightface" version, have the obvious meaning; for once we need not consider the Σ^1_n versions, as we can look at complements and see that what would be $\mathbf{\Sigma}^1_n$-Determinacy is the same as $\mathbf{\Pi}^1_n$-Determinacy.) By <u>Projective Determinacy (PD)</u> is meant the joint assertion of $\mathbf{\Pi}^1_n$-Determinacy for every $n \in \omega$.

Classical results indicated that if $V = L$, then Π^1_1-Determinacy fails. It transpires that an uncountable, scattered Π^1_1 set in the sense of §18 is a counterexample. But reminiscent of some comments toward the end of that section about the effect of measurables on the Projective Hierarchy, Martin[1970] showed that if there is a measurable cardinal, then $\mathbf{\Pi}^1_1$-Determinacy holds. More precisely, he was able to invoke the Silver theory of indiscernibles to establish the following statement:

If $a \subseteq \omega$ and $a^\#$ exists, then $G(\omega,B)$ is determined for any $B \subseteq {}^\omega\omega$ which is Π^1_1(in a):

$\vdash$ Suppose $B \subseteq {}^\omega\omega$ is Π^1_1(in a), so that it has (normal) form: $f \in B$ **iff** $\forall g \in {}^\omega\omega \exists n \in \omega R(f|n, g|n, a|n)$, where R is recursive. (Here, we have styled $a \in {}^\omega\omega$ by an identification of $P(\omega)$ and ${}^\omega\omega$, to bring the predicate into consistent form; in any case, a does not play any dynamic role in the proceedings.) For any $f \in {}^\omega\omega$, let us denote $T_f = \{ s \in \bigcup_{n\in\omega} {}^n\omega \mid \forall n \leq \text{length}(s)(\neg R(f|n, s|n, a|n))\}$. Then the usual tree analysis of Π^1_1 sets indicates that **$f \in B$ iff T_f is well-ordered by $<_{KB}$** , where $<_{KB}$ is the Kleene-Brouwer ordering of finite sequences: $s <_{KB} t$ **iff** t is a proper initial segment of s , or else s is lexigraphically less than t .

One of the precepts of set theory is that well-ordering can be regarded as an existential statement about order-preserving injectivity into the ordinals. Keeping this in mind, we shall define an auxiliary game $\bar{G}$ to $G(\omega,B)$. First, fix an enumeration $\langle s_i \mid i \in \omega \rangle$ of $\bigcup_{n\in\omega} {}^n\omega$ so that $\text{length}(s_i) \leq i$ for every i . $\bar{G}$ is now described as follows: Player I initially picks an $a_0 \in \omega$ and also an $\alpha_0 \in \omega_1$; then player II picks an $a_1 \in \omega$; then player I picks an $a_2 \in \omega$ and an $\alpha_1 \in \omega_1$; then player II picks an $a_3 \in \omega$; then player I picks an $a_4 \in \omega$ and an $\alpha_2 \in \omega_1$; and so forth. Thus, $\bar{G}$ is like $G(\omega,B)$ except that player I must also choose ordinals at his turns. Player I is said to win a play of this game just in case the following holds: if $f = \langle a_0, a_1, a_2, \ldots \rangle$ is the sequence of integers produced by this play, then the map $s_i \mapsto \alpha_i$ extends an order-preserving injection: $\langle T_f, <_{KB} \rangle \to \langle \omega_1, < \rangle$. In particular, if player I wins a play of $\bar{G}$, then he wins the corresponding play of $G(\omega,B)$, where the ordinals are left out, since the resulting $f \in B$.

Notice that each player knows at the ith play of the game whether $s_i \in T_f$, since we had arranged that $\text{length}(s_i) \leq i$. Now if player I lapses in $\bar{G}$ in his order-preservation efforts, he can never rectify his mistake again; hence, whenever player II wins $\bar{G}$, he has done so on the basis of an initial segment. Hence, $\bar{G}$ is an "open" game, i.e. one for which the Gale-Stewart analysis can be invoked, and so **$\bar{G}$ is determined**. Using this fact, we can now proceed to show that the original game $G(\omega,B)$ must be determined:

Case 1 player I has a winning strategy for $\bar{G}$. By an earlier comment, if he plays according to this strategy in $G(\omega,B)$, deviously modified by keeping his ordinal moves secret from his opponent, then he will always win. Thus, player I has a winning strategy in $G(\omega,B)$.

Case 2 player II has a winning strategy for $\bar{G}$. We now call upon the existence of $a^\#$ to produce a winning strategy for player II in $G(\omega,B)$. Since the predicate R was recursive, we could surely have carried out the construction of the game $\bar{G}$ entirely within L[a] (but using the **real** ω_1 in the definition of $\bar{G}$). Thus, we could argue as before to show that there is an $S \in L[a]$ which in the sense of

L[a] is a winning strategy for one player in game $\bar{G}$.

Notice that if (S is a winning strategy for I)$^{L[a]}$, then S is a winning strategy for I (in V). This is because of the open nature of $\bar{G}$: If there were a play of $\bar{G}$ in V in which player I plays according to S but still loses, then he must have failed to preserve order at a finite stage, and is lost from then on. But then this failure would take place just the same in L[a], and so S was not winning for player I inside L[a] either.

Hence, as we are supposed to be in Case 2, we can conclude that: (S is a winning strategy for II)$^{L[a]}$. Again by the open nature of $\bar{G}$, this means that **S is a winning strategy for II (in V)**, as the following argument shows: As $\bar{G}$ is open, an ordering $R_S \subseteq \langle \bigcup_{n\in\omega} {}^n\omega_1, \supseteq \rangle$ can be defined in L[a] so that: S is a winning strategy for II **iff** R_S is a well-founded relation. But this last statement holds relativized to L[a], and also: (R_S is a well-founded relation)$^{L[a]}$ **iff** R_S is a well-founded relation. (This argument is exactly like for 7.9 of §7.)

By the existence of $a^\#$, let $H \subseteq \omega_1$ with $|H| = \omega_1$ be a set of indiscernibles for L[a]. We now describe a winning strategy for player II in $G(\omega,B)$: Given an initial play $\langle a_0, \ldots, a_{2n} \rangle$ toward some eventual $f \in {}^\omega\omega$, player II proceeds as follows: He first chooses ordinals α_i for $i \leq n$ such that: if $s_i \notin T_f$, $\alpha_i = 0$; otherwise α_i is some ordinal picked $\in H$ so that $s_i \mapsto \alpha_i$ is order-preserving for $<_{KB}$. (Notice that whether $s_i \in T_f$ or not can be decided without knowing the full f , as $\text{length}(s_i) \leq i < 2n$.) Then as if he were playing in $\bar{G}$, player II responds according to the winning strategy S to the initial play

$$\langle \langle a_0, \alpha_0 \rangle, a_1, \langle a_2, \alpha_1 \rangle, a_3, \ldots, \langle a_{2n}, \alpha_n \rangle \rangle .$$

This is a well-defined strategy for player II in $G(\omega,B)$ which does not depend on the particular α_i's chosen, since H is a set of indiscernibles for L[a] and $\bar{G}, S \in L[a]$.

It is now claimed that this is a winning strategy for player II in $G(\omega,B)$: Assume to the contrary that f is the result of a play of $G(\omega,B)$ when player II uses his strategy, yet $f \in B$, i.e. T_f is well-ordered by $<_{KB}$. Since $|T_f| = \omega < |H|$, let $h: T_f \to H$ be an order-preserving injection, and extend h arbitrarily to a function $\bar{h}: \bigcup_{n\in\omega} {}^n\omega \to \omega_1$. Then surely if player II plays according to S in $\bar{G}$, and player I plays $\langle f(2i), \bar{h}(s_i) \rangle$ at his (i+1)st turn, then player II still loses by indiscernibility of H for L[a]. This is a contradiction of the fact that S is a winning strategy for II **in V**. $\dashv$

This rather attractive proof of Martin first brought contemporary large cardinal ideas into the arena of infinite games. Recently, Harrington(1975) skillfully synthesized ideas of Friedman, Sami, Steel, and others to show that, somewhat unexpectedly, the direct converse holds: **If $a \subseteq \omega$ and $G(\omega,B)$ is determined for every B which is Π^1_1(in a), then $a^\#$ exists.** Thus, our outline of 7.25 of §7 is now complete, and we have before us the fascinating revelation that the existence of a non-trivial elementary embedding: $L \to L$ is equivalent to Π^1_1-Determinacy!

As soon as Martin established his result in the late sixties, set theorists became enamored with the idea of using indiscernibles to effect a reduction to open games. The prominent open question at the time was whether Borel(Δ^1_1-)Determinacy can be established outright in ZFC. Indiscernibles were laboriously pieced together by Martin and others to push through various arguments, the best result along these lines being that of Paris[1972] who first established Π^0_4-Determinacy. Actually, this turned out to be a false start toward Borel Determinacy, as Martin[1975] later established in late 1974 that Borel Determinacy holds in ZFC using systematic reductions down along the Borel hierarchy. Sometimes, one can go overboard, even with large cardinal ideas!

Borel Determinacy is a prominent example of an assertion which was amenable to predictive analysis via the strength of axiomatic systems. Friedman[1971] had shown early on that Borel Determinacy is not provable in Zermelo set theory (ZF - Replacement); in fact no particular countable iteration of the power set operation suffices. Thus, set theorists were already aware of the axiomatic strength that must essentially be employed in an eventual proof of Borel Determinacy. Martin's proof used uncountable iterations of the power set operation. Measuring the strength of propositions undecided by ZFC by using the scale of large cardinals is a natural extension of such axiomatic analyses.

The consistency strength of full Projective Determinacy is still completely unknown. The work in the early seventies of Kechris, Kunen, Martin, Moschovakis, and Solovay (most unpublished--but see Kechris[1973][1974][1975], and Moschovakis [1970][1973] and his forthcoming book), have charted the extensive consequences of PD for the descriptive theory of projective sets. These researches have certainly helped isolate the essence of arguments in descriptive set theory from a foundational point of view. However, although the Martin-Harrington results have fully clarified Π^1_1-Determinacy, even the strength of Π^1_2-Determinacy has not been satisfactorily ascertained. It is known to be a remarkably strong proposition; extending work of Solovay, Green(1977) established that Π^1_2-Determinacy implies the existence of an inner model of: ZFC & there is a measurable cardinal κ carrying a normal ultrafilter U with $\{\alpha < \kappa \mid \alpha \text{ is measurable}\} \in U$. Martin then remarked that Green's methods actually show the existence of inner models with measurable cardinals carrying normal ultrafilters of high type in Mitchell's order (see end of §9).

(Very recently, Martin has announced that Π^1_2-Determinacy is a consequence of a very strong axiom of infinity, I2 of §17. At the time of this writing (early 1978) it is too early to get the full story. However, this would certainly be a major breakthrough: an upper bound on the strength of Π^1_2-Determinacy has been established. We eagerly await further details, with the hope that an elaboration

of the strength of Projective Determinacy is finally forthcoming.)

To give an idea of the implications of Π^1_2-Determinacy, we sketch Solovay's ideas in a proof of: If Π^1_2-Determinacy holds, then there is a transitive set model of ZFC with a measurable cardinal.

$\vdash$ Already from the existence of $a^\#$ for every $a \subseteq \omega$, we know that there is for every $a \subseteq \omega$ a minimal transitive model $m(a)$ satisfying "$V = L[a]$". Clearly, $m(a)$ must have form $L_\gamma[a]$ for some $\gamma < \omega_1$ depending on a.

For every sentence σ in the language of set theory, define: $A_\sigma = \{a \subseteq \omega \mid m(a) \models \sigma\}$. A_σ can be shown to be a Δ^1_2 set of reals. Moreover, if a is in A_σ, any real having the same Turing degree as a is in A_σ, since it has the same minimal model. Hence, we can consider A_σ and $P(\omega) - A_\sigma$ to be sets of Turing degrees.

We must now appeal to some facts which are consequences of forthcoming results in §28: Call a set X of Turing degrees large iff $X \supseteq \{d \mid \bar{d} \leq_T d\}$ for some degree $\bar{d}$. Then Π^1_2-Determinacy implies that for every sentence σ, either A_σ or $P(\omega) - A_\sigma$ is large, by the relativization to Π^1_2 of a coming argument in §28 about the filter of large sets, due to Martin.

Let $\Gamma = \{\sigma \mid A_\sigma \text{ is large}\}$. Note that by the above remark Γ is a complete theory. Since for any countable collection $\{a_i \mid i \in \omega\} \subseteq P(\omega)$ there is a $b \subseteq \omega$ so that $a_i \leq_T b$ for every i, we can invoke the largeness of the A_σ's in Γ to get an $x \subseteq \omega$ so that: whenever $x \leq_T a$, then $m(a)$ is a model of Γ.

Let F be the filter over $(\omega_\omega)^{m(x)}$ defined by: $X \in F$ iff $X \in P((\omega_\omega)^{m(x)})$ & $\exists i \forall j > i ((\omega_j)^{m(x)} \in X)$. Thus, $F \cap m(x)$ is the usual cardinal filter in $m(x)$ over $(\omega_\omega)^{m(x)}$. It is now claimed that $(F \cap L[F])^{m(x)}$ is a $(\omega_\omega)^{m(x)}$-complete ultrafilter inside $(L[F])^{m(x)}$.

We just indicate how to show ultrafiltration, the proof of $(\omega_\omega)^{m(x)}$-completeness being similar. The reader should notice the thematic connection to arguments we have encountered in §6 and §9.

Note that $(L[F])^{m(x)}$ has a canonical well-ordering definable in $m(x)$. If our filter were not ultra, let $Y \in P((\omega_\omega)^{m(x)}) \cap (L[F])^{m(x)}$ be the least in this ordering such that neither Y nor $(\omega_\omega)^{m(x)} - Y$ is in F. Y is therefore definable in $m(x)$. We now proceed to establish that if $(\omega_1)^{m(x)} \in Y$, then $(\omega_n)^{m(x)} \in Y$ for every $n > 0$; this argument works just as well for $(\omega_\omega)^{m(x)} - Y$, so we will then have before us a contradiction.

So, assume that $(\omega_1)^{m(x)} \in Y$. This can be stated as the fact that for some sentence σ, $m(x) \models \sigma$. Hence for every $a \geq_T x$, $m(a) \models \sigma$. Now fix an $n > 0$. Let z be a real coding the usual generic collapse over $m(x)$ of $(\omega_n)^{m(x)}$ to $(\omega_1)^{m(x)}$, and let a be a real coding z and x. So, we can consider that the first $\omega+1$ infinite cardinals of $m(a)$ are: $\omega, (\omega_n)^{m(x)}, (\omega_{n+1})^{m(x)}, \ldots (\omega_\omega)^{m(x)}$. Therefore, the cardinal filter over $(\omega_\omega)^{m(x)}$ defined inside $m(a)$ is just

$F\cap m(a)$. Thus, $(L[F])^{m(a)} = (L[F])^{m(x)}$, and $(F\cap L[F])^{m(a)} = (F\cap L[F])^{m(x)}$ Since the canonical well-orderings must also be the same, Y is defined by the same formula in both $m(a)$ and $m(x)$. Now $m(a) \models \sigma$, where σ is as at the beginning of this paragraph, since $a \geq_T x$. But $(\omega_1)^{m(a)} = (\omega_n)^{m(x)}$, so we can finally conclude that $(\omega_n)^{m(x)} \in Y$. Since the $n > 0$ here was arbitrary, we have established what we set out to prove: $\{(\omega_n)^{m(x)} \mid n > 0\} \subseteq Y$. ⊣

This proof exhibits a remarkable connection between a hypothesis on simply defined sets of reals and a highly non-constructive existential postulation in set theory, which seems to hinge ultimately on Skolem's "paradox": if set theory is consistent, it has countable models, and these can be simply coded into reals.

We pass now to a few examples of other infinite games. A variation on the game G(A,B) is the game G*(A,B) where the players have different roles. They still construct an ω-sequence of members of A, but player I at his turns can determine finitely many members of the sequence whereas player II at his turns can determine just one such member. Player I wins as before iff the resulting sequence is in B. Morton Davis[1964] gave a complete solution for the games G*(2,B) as follows: (a) player II has a winning strategy in G*(2,B) iff B is countable. (b) player I has a winning strategy in G*(2,B) iff B as a subset of the topological space ${}^{\omega}2$ contains a perfect subset. (Here and hereafter, it should be clear what is meant by a winning strategy, although we do not bother to define it formally.)

The corresponding axiom of determinacy is AD*: one player has a winning strategy in G*(2,B), for every $B \subseteq {}^{\omega}2$. By Davis' result AD* is equivalent to the proposition that every subset of ${}^{\omega}2$ is either countable or contains a perfect subset—that is, to (P) of §18. Hence, by results cited there, AD* is still inconsistent with the Axiom of Choice, and Con(ZF & AD*) iff Con(ZFC & there is an inaccessible cardinal). This is historically the first link-up between large cardinals and infinite games. AD certainly implies AD* by a simple coding, but we shall see in §28 that AD is an incomparably stronger proposition; in fact, it is rather striking that such an apparently slight alteration in the rules of the game should result in such a drastic diminution in strength.

We finish this section with a discussion of a class of infinite games mainly introduced by Ulam(see his [1964]). We are given an infinite set A. The two players determine in their play a decreasing sequence of subsets of A, and the result of the game is a function of the size of the intersection of the resulting sequence. A few examples of this kind of game are:

(A) Player I splits the set into two parts. Player II chooses one of the parts, and player I splits the part chosen by II. Player II then chooses one of the parts, and so on. Player I wins the game iff the intersection of the chosen parts is non-empty.

(B) Like game (A), except that player II splits, and player I picks. Player I wins iff the intersection contains at least two points.

(C) Both players pick and split. Player I wins iff the resulting intersection is non-empty.

(D) Each player picks in his turn a subset of the set obtained so far. He is not allowed to pick a small subset, where "small" is a part of the definition of the game. Player II gets to start this game, for technical reasons. Player I wins iff the resulting intersection is non-empty.

It is clear that winning these games depends just on the cardinality of A. Game (A) has nothing to do with large cardinals: Player I has a winning strategy in the game iff the cardinality of A is at least 2^{ω}; and player II has a winning strategy iff A is countable. Games (B) and (C) are closely related in the sense that: (i) player I cannot have a winning strategy in either game if the cardinality of A is at most 2^{ω}; and (ii) if the cardinality of A is less than the first measurable cardinal then if player I has a winning strategy in game (B) then he has a winning strategy in game (C). The consistency strength of player I having a winning strategy in either game (B) or (C) follows from the next two theorems:

(Laver) Con(ZFC & there is a measurable cardinal) implies Con(ZFC & Player I has a winning strategy in game (B) (hence in game (C)) played on a set having cardinality ω_2. Note that if game (B) is played on a set A having cardinality at least the first measurable, then player I trivially has a winning strategy by always picking the set which lies in some fixed ω_1-complete ultrafilter over A. Laver's model is obtained by Lévy collapsing a measurable cardinal to ω_2. For the other direction we have: (Silver, Solovay) If for some set A, player I has a winning strategy in either game (B) or game (C), then there is an inner model of ZFC with a measurable cardinal.

⊢ The argument is rather interesting. Let us take game (B) for definiteness. We first generically collapse $|P(A)|$ to ω. In the resulting model we play the game where player I plays according to his strategy. The idea is that player II plays an enumeration of $P(A)^V$, where $P(A)^V$ is the power set of A in the sense of of the ground model V. Let $\langle B_n \mid n \in \omega \rangle$ be such an enumeration. If A_n is the set picked by player I at the (2n-1)th move of the game, for the 2nth move player II splits A_n into $A_n \cap B_n$, and $A_n - B_n$. The set $U = \{B_n \mid$ at the (2n+1)th move player I picked $A_n \cap B_n\}$ is then a ultrafilter on the Boolean algebra $P(A)^V$, and so one can form the ultrapower V^A/U, using only functions: $A \to V$ which are members of A. The salient fact about U is that V^A/U is well-founded. (It is here that the play according to the winning strategy for player I is used.) Hence, it is isomorphic to a transitive class M into which V can be elementarily embedded by an embedding which is not the identity. We can now invoke arguments as in §12 to conclude that there is an inner model with a measurable cardinal. ⊣

The situation for game (D) depends on the meaning of "small". (Consult Galvin-Jech-Magidor(1976) as a reference for this part.) For instance, if small means having cardinality less than $|A|$, then it is known that player II always has a winning strategy. If $A = \kappa$ a cardinal, a generalized notion of small is belonging to some (non-trivial) ideal I over κ. For such <u>ideal games</u>, the following are of interest to player I : (a) he cannot have a winning strategy if $\kappa \leq 2^{\omega}$; (b) he cannot have a winning strategy if I is normal with $\{\alpha < \kappa \mid cf(\alpha) = \omega\} \notin I$; and (c) if U is a normal ultrafilter over a measurable cardinal λ , and λ is Lévy collapsed to ω_2 , then the dual ideal to U generates an ideal I in the extension for which player I has a winning strategy.

We are not neglecting player II in the ideal game. The following nice characterization was heralded in §12, and is really the best way to look at precipitous ideals without meta-mathematical phraseology: <u>An ideal I over a cardinal κ is precipitous iff player II has no winning strategy in the corresponding ideal game.</u>

$\vdash$ Recalling the notation of §12, if I were not precipitous, there must be a condition $X \in P(\kappa) - I$ so that $X \Vdash_{R(I)}$ "$<[\tau_n] \mid n \in \omega>$ is an ε-decreasing sequence in $Ult_1(\check{V},\underline{G})$". The following is a winning strategy for player II: He starts the game by playing X for the first move S_0. In general, if player I has just played S_{2n-1}, player II then finds a $Y \subsetneq S_{2n-1}$ with $Y \in P(\kappa) - I$ and two functions f_n and f_{n+1} so that $Y \Vdash_{R(I)} \tau_n = \check{f}_n$ & $\tau_{n+1} = \check{f}_{n+1}$, then plays $Y \cap \{\alpha < \kappa \mid f_{n+1}(\alpha) \in f_n(\alpha)\}$ for his next move S_{2n}. This strategy guarantees $\bigcap_{n \in \omega} S_n = \emptyset$.

Suppose now that I is precipitous, and assume to the contrary that there is a winning strategy S for player II. If $V[G]$ is a generic extension via $R(I)$, then a density argument with S shows that there is a play of the game $<S_n \mid n \in \omega>$ where player II follows his strategy S <u>and</u> each $S_n \in G$. By precipitousness, in $V[G]$ let $j: V \to M = V^{\kappa}/G$, where of course V^{κ}/G is constructed using only functions $\in V$. We can consider that the ideal game is coded as some set $G_I \in V$. Thus, $<j(S_n) \mid n \in \omega>$ is a play of the game $j(G_I)$ where player II plays according to strategy $j(S)$. Moreover, if $\gamma = [id]_G$ where id is as usual the identity function on κ , then $\gamma \in \bigcap_{n \in \omega} j(S_n)$. By a well-foundedness argument exactly as in 7.9 of §7, it follows that there must be such a play of the game $j(G_I)$ which is a member of M . Hence, by elementarity there is a play $<T_n \mid n \in \omega>$ in V of the game G_I in which player II plays according to S , yet $\alpha \in \bigcap_{n \in \omega} T_n$ for some $\alpha < \kappa$. Thus, S was not a winning strategy for player II. $\dashv$

For generalizations of the ideal game in the Boolean algebraic setting, see Jech(1977).

<u>§28. The Axiom of Determinacy</u>

Those passing through these portals heed the warning: Abandon the Axiom of Choice.

The fascination that the Axiom of Determinacy holds for set theorists lies primarily in its undaunted ability to decide combinatorial propositions, and thereby provide a surrealist landscape with a pathological but finely detailed structure. Left behind is the insecurity of formalist agnosticism, and the malaise of living in an indefinite world cluttered with independence results. Of course, the experience with determinacy so far may be just a prelude to a major inconsistency result, and some may even be attracted to the field by the likelihood of participating in some such event. After all, there is no convincing reason why the axiom should be adopted as a basis for mathematics, especially in view of its consequences. (The situation is somewhat reminiscent of Quine's New Foundations.) But in the absence of any overriding philosophical impetus for the adoption of genuinely new set theoretical principles, or even any idea of what these principles ought to look like, the Axiom of Determinacy provides a retreat into a structured world, one in which opportunities present themselves for new kinds of arguments as well as new syntheses of known ones.

Certainly, the study of AD has refined our set theoretical knowledge. For one thing, the continual necessity of having to avoid the Axiom of Choice (which undoubtedly has induced paranoic tendencies in some!) has sharpened our intuition about the nature of this axiom. For another, our awareness of game theoretic formulations has considerably increased, as §27 indicates. Finally, the axiom provides plausibility arguments for a whole series of new and distinctive propositions, like infinite exponent relations, possible only in Choice-less situations. These propositions may now serve as a focus for further investigations, possibly emerging someday as interesting consistent consequences of ZFC plus some strong axiom of infinity.

In this section we limit ourselves to briefly summarizing consequences of large cardinal character from AD, knowing beforehand that we cannot do justice to all the new styles of argumentation that have been devised. The early results concerning AD dealt with direct consequences about the structure of the real numbers, like (L), (B), and (P) of §18(see Mycielski[1964] and Mycielski-Swierczkowski[1964]). This in itself recommended AD as a substantial, antithetical alternative to AC.

(AD itself implies the somewhat useful choice principle that choice functions exist for countable collections of reals. However, recent work of Solovay indicates that it is unlikely that AD implies DC, the Principle of Dependent Choices (see §18). Specifically, he showed that Con(ZF & AD_R) implies Con(ZF & AD & $\neg$DC), where AD_R is the very strong axiom of determinacy for games played with reals instead of integers chosen at each step. <u>Throughout this section, we assume DC</u>, which is needed in various constructions.)

It was Solovay who in 1966 injected the study of AD with the fresh meta-mathematical arguments of the time to show that: <u>AD implies ω_1 is measurable; in fact,</u>

the closed unbounded filter over ω_1 is an ultrafilter. Thus, the closed unbounded filter over ω_1 first occurred in a large cardinal context, with the ultimate statement of the sort considered in §22. Solovay's result provided a new insight on how the bifurcation postulation of AD about games with integers can somehow carry over to a complete dichotomy about subsets of ω_1, with respect to the closed unbounded subsets as a basis. Of course, the derivation of the existence of a measurable cardinal from AD is real evidence for the strength of AD, since one can then construct from the corresponding ultrafilter an inner model of (ZFC & there is a measurable cardinal) and thereupon cite the standard facts about the strength of measurability in a Choice-ful situation.

The measurability of ω_1 can most quickly be understood as a consequence of a later 1968 result of Martin, who with the eye of a recursion theorist, spotted the following scheme: Let D be the set of Turing degrees of reals. Call an $X \subseteq D$ large iff $X \subseteq \{d \mid \bar{d} \leq_T d\}$ for some degree $\bar{d}$. Let M denote the collection of large subsets of D. Then Martin established that: AD implies that M is a ω_1-complete ultrafilter over D.

$\vdash$ That M is a ω_1-complete filter is immediate, since whenever $\langle d_i \mid i \in \omega \rangle$ is a sequence of degrees, there is a degree $d \geq d_i$ for every $i \in \omega$. It remains to show that M is ultra:

Given $X \subseteq D$, set $B = \{f \in {}^{\omega}\omega \mid \text{degree}(f) \in X\}$, and consider the game $G(\omega,B)$. By AD, one player has a winning strategy S for this game. Now a strategy is just a function: $\bigcup_{n\in\omega} {}^{n}\omega \to \omega$, so we can consider the degree $\bar{d}$ of S. It is claimed that if S is a winning strategy for player I, then $\{d \mid \bar{d} \leq_T d\} \subseteq X$. (The companion argument will show that if S is winning for player II, then $\{d \mid \bar{d} \leq_T d\} \cap X = \emptyset$, and so the proof will be complete.)

So suppose that $d \geq_T \bar{d}$ and $g \in {}^{\omega}\omega$ has degree d. If $G(\omega,B)$ is played where player I follows his strategy S and player II picks successively $a_1 = g(0)$, $a_3 = g(1)$, $a_5 = g(2), \ldots$, then the resulting sequence f is in B as S was winning for player I. Surely f has degree $\max(\bar{d},d) = d$, and so $d \in X$. $\dashv$

This result, specialized to Π^1_2 sets, is what was elicited in the latter-day result about Π^1_2-Determinacy in §27. It is now straightforward to show that AD implies that ω_1 is measurable:

$\vdash$ Let $G: {}^{\omega}\omega \to \omega_1$ be surjective; such a function exists in ZF via codes for countable well-orderings. Define $F: D \to \omega_1$ by: $F(d) = \sup\{G(f) \mid \text{degree}(f) \leq_T d\}$. Then as in §13, $F_*(M) = \{X \subseteq \omega_1 \mid F^{-1}(X) \in M\}$ is an ω_1-complete ultrafilter over ω_1. (To establish that $F_*(M)$ is non-principal, notice that if $f \in {}^{\omega}\omega$ with $G(f) = \alpha$, then $d \geq \text{degree}(f)$ implies $F(d) \geq \alpha$, so that $\{\beta < \omega_1 \mid \alpha \leq \beta\} \in F_*(M)$). $\dashv$

Actually, a careful choice of G will result in $F_*(M)$ being the closed unbounded filter, thereby procuring Solovay's full result. Martin's ultrafilter predictably has much more universality, as we now indicate. For ADeists, Θ names

a well-known ordinal: Θ is the least ordinal η so that there is no surjection: ${}^{\omega}\omega \to \eta$. Intuitively, Θ is the outer limit on ordinals which can be affected by properties of reals. ZF implies that Θ is a cardinal $\geq \omega_2$, and AD implies that Θ is very large: In his early efforts to popularize Determinacy, Friedman invent a game that showed that under AD, if for any ordinal α there is a surjection $G: {}^{\omega}\omega \to \alpha$, there is a surjection $H: {}^{\omega}\omega \to P(\alpha)$. This result allows closure argumen which show that AD implies $\Theta = \omega_\Theta$, Θ is the Θth ordinal α so that $\alpha = \omega_\alpha$, is greater than the least weakly Mahlo cardinal, and so forth. (On the other hand although DC can be invoked to establish $cf(\Theta) > \omega$, work of Solovay indicates that without DC, AD alone probably cannot establish $cf(\Theta) > \omega$.) We can re-run the arg ment from M about the measurability of ω_1 to establish, in self-refined form, that: If $\lambda < \Theta$ and F is a ω_1-complete filter over λ, then F can be extend to a ω_1-complete ultrafilter over λ.

$\vdash$ By Friedman's result, let $G: {}^{\omega}\omega \to P(\lambda)$ be surjective, and define $H: D \to \lambda$: $H(d)$ = least member of $\bigcap\{G(f) \mid G(f) \in F \;\&\; \text{degree}(f) \leq_T d\}$. This last intersec tion is a countable one, so since F is assumed ω_1-complete, H is well-defined. before, consider $U = H_*(M)$. U is a ω_1-complete ultrafilter over λ, and it is hard to see that $U \supseteq F$.

Observe that if we had already started with an ultrafilter V for F, the proof shows in particular that $V = H_*(M)$ for some H, i.e. $V \leq_{RK} M$ in the ter nology of §13. This is indeed a strong universality property for M. In particu ω_1 has a property under AD which from §3 and §15 is like saying that it is strong compact below Θ. Without AC however, the large cardinal equivalences of this fi ter extension property do not ensue.

Almost immediately after Martin introduced his M, Solovay was able to use i to prove: AD implies that ω_2 is measurable. The power of AD was seemingly boun less in establishing the measurability of cardinals, until Martin in 1970 announced the bizarre result that AD implies ω_n for $2 < n < \omega$ are all singular with cofi ality ω_2. The Baroque feel of all this was further amplified soon after when it was shown that AD implies $\omega_{\omega+1}$ and $\omega_{\omega+2}$ are both measurable.

This last result is part of a larger format. Moschovakis[1970] introduced th following ordinals: δ^1_n is the least ordinal η so that there is no surjection: ${}^{\omega}\omega \to \eta$ which is Δ^1_n. (Here, a surjection $G: {}^{\omega}\omega \to \eta$ is Δ^1_n iff the relation $\{\langle f,g\rangle \mid G(f) < G(g)\}$ is a Δ^1_n set.) The δ^1_n's are the analogues of Θ which nat rally arise in the investigation of levels of the Projective Hierarchy. A great d of effort has been invested in their study, both under PD and under the full AD. cite the prominent facts, known by 1971, if AD is assumed: each δ^1_n is a measura cardinal and they occur in pairs $\delta_{2n+2} = (\delta_{2n+1})^+$ (Martin(1971) and Kunen); δ_{2n+1} $(\lambda_n)^+$ for some cardinal λ_n of cofinality ω (Kechris[1974]); $\delta^1_1 = \omega_1$ (Kleene—classical); $\delta^1_2 = \omega_2$, $\delta^1_3 = \omega_{\omega+1}$, $\delta^1_4 = \omega_{\omega+2}$ (Martin(1971)). It is not known what δ^1_5 is under AD, but the prominent conjecture (Kunen) is that $\delta^1_5 = \omega_{\omega^3+1}$ (!). Th fascination of AD lies in such results which exhibit the pathological behavior of uniformly defined ordinals, and the intricate web of combinatorial detail they reinforce. Little is left of what is familiar in this Looking Glass world of set theory.

AD set the stage for another Choice-less motif, infinite exponent partition relations. In the late sixties Kleinberg (see his [1970]) showed that such relations were capable of producing measurable cardinals. Without the Axiom of Choice, it seems that strong dichotomies are possible in rather concrete situations. If $\lambda < \kappa$ are regular cardinals, a set $C \subseteq \kappa$ is λ-closed unbounded iff C is unbounded in κ, and whenever $\alpha < \kappa$ with $cf(\alpha) = \lambda$ so that $C \cap \alpha$ is unbounded in α, then $\alpha \in C$. The λ-closed unbounded subsets of κ generate a (non-principal) filter $\mathcal{C}^{\lambda}_{\kappa}$ which with the Axiom of Choice cannot be ultra. Kleinberg established that: If $\lambda < \kappa$ are regular cardinals so that $\kappa \to (\kappa)^{\lambda+\lambda}_2$, then $\mathcal{C}^{\lambda}_{\kappa}$ is a normal ultrafilter over κ, and hence κ is a measurable cardinal.

$\vdash$ This is proved through several lemmata which isolate the essentials:

Claim 1. If $\kappa \to (\kappa)^{\lambda}_2$, then $\mathcal{C}^{\lambda}_{\kappa}$ is an ultrafilter. To show this, let $X \subseteq \kappa$ be arbitrary. Then define $f\colon [\kappa]^{\lambda} \to 2$ by: $f(s) = 0$ iff $\bigcup s \in X$. By hypothesis let $H \subseteq \kappa$ with $|H| = \kappa$ be homogeneous for f, and then set $C = \{\alpha < \kappa \mid cf(\alpha) = \lambda \ \& \ H \cap \alpha$ is unbounded in $\alpha\}$. It is not hard to see that C is λ-closed unbounded. If $f''[H]^{\lambda} = \{0\}$, then $C \subseteq X$; and if $f''[H]^{\lambda} = \{1\}$, then $C \cap X = \emptyset$. This establishes the claim.

Claim 2. If $\kappa \to (\kappa)^{\lambda}_{\gamma}$ for every $\gamma < \kappa$, then $\mathcal{C}^{\lambda}_{\kappa}$ is a normal κ-complete filter.

We first establish κ-completeness. (Notice that although it can be shown that the intersection of less than κ λ-closed unbounded sets is λ-closed unbounded, this does not in itself suffice, since without AC we cannot simultaneously choose for many members of $\mathcal{C}^{\lambda}_{\kappa}$ corresponding λ-closed unbounded subsets of them.) Let $\gamma < \kappa$ and $F\colon \kappa \to \gamma$. We must find an $\beta < \gamma$ so that $F^{-1}(\{\beta\}) \in \mathcal{C}^{\lambda}_{\kappa}$. Let us define $G\colon [\kappa]^{\lambda} \to \gamma$ by: $G(s) = \alpha$ iff $F(\bigcup s) = \alpha$. Let $H \subseteq \kappa$ with $|H| = \kappa$ be homogeneous for G, and consider as before $C = \{\alpha < \kappa \mid cf(\alpha) = \lambda \ \& \ H \cap \alpha$ is unbounded in $\alpha\}$. C is λ-closed unbounded, and if $G''[H]^{\lambda} = \{\beta\}$, then $C \subseteq F^{-1}(\{\beta\})$, i.e. $F^{-1}(\{\beta\}) \in \mathcal{C}^{\lambda}_{\kappa}$.

To show normality, suppose $f(\alpha) < \alpha$ for every $0 < \alpha < \kappa$. By the previous paragraph, it suffices to get a $C \subseteq \kappa$ which is λ-closed unbounded so that $|f|C| < \kappa$. To do this, define $g\colon [\kappa]^{\lambda} \to 2$ by: $g(s) = 0$ iff $f(\bigcup s) < \bigcap s$. Let $H \subseteq \kappa$ with $|H| = \kappa$ be homogeneous for g. It is not hard to show that it must be the case that $g''[H]^{\lambda} = \{0\}$. Once again, consider $C = \{\alpha < \kappa \mid cf(\alpha) = \lambda \ \& \ H \cap \alpha$ is unbounded in $\alpha\}$. Then if $\delta = \bigcap H$, the least member of H, it is straightforward to show that $f''C \subseteq \delta < \kappa$.

Claim 3. If $\kappa \to (\kappa)^{\lambda+\lambda}_2$, then $\kappa \to (\kappa)^{\lambda}_{\gamma}$ for every $\gamma < \kappa$. To show this, given $F\colon [\kappa]^{\lambda} \to \gamma$ for some $\gamma < \kappa$, define an auxiliary $G\colon [\kappa]^{\lambda+\lambda} \to 2$ by $G(s \cup t) = 0$ iff $\bigcup s \leq \bigcap t \ \& \ F(s) = F(t)$. Let $H \subseteq \kappa$ with $|H| = \kappa$ be homogeneous for G. Then as $\gamma < \kappa$, a straightforward argument shows that it must be the case that

$G''[H]^{\lambda+\lambda} = \{0\}$, and so the claim follows.

The preceding steps in combination establish the full result.

Soon after Kleinberg proved his result, Martin and later Kunen established a context for it by demonstrating that AD implies that strong infinite partition rel[illegible] tions of the relevant sort were satisfied by many cardinals. This in particular [illegible] a new way to establish the measurability of cardinals under AD. By 1973, the str[illegible] est possible partition result was established; Martin showed that: AD implies th[illegible] $\omega_1 \to (\omega_1)^{\omega_1}_2$. Thus, the great dichotomy that AD postulates for games, after havi[illegible] been transferred to ultrafiltration properties, has now been transferred to infini[illegible] exponent partition relations.

Various partition relations for other cardinals were quickly seen to follow [illegible] rectly from Martin's result for ω_1. In 1975, Kleinberg gave an elegant abstract formulation that summed up the conbinatorial gist of what was going on. (See his [1977], which is a good reference for most of this section.) He established the [illegible] lowing in ZF + DC alone:

Assume that κ is an uncountable cardinal satisfying $\kappa \to (\kappa)^{\kappa}_2$. Then ther[illegible] are cardinals $\kappa = \kappa_1 < \kappa_2 < \kappa_3 \ldots$ so that:

(i) κ_1 and κ_2 are measurable; in fact $\kappa_2 \to (\kappa_2)^{\alpha}_2$ for every $\alpha < \omega_{[illegible]}$ yet $\kappa_2 \not\to (\kappa_2)^{\kappa_2}_2$.

(ii) κ_n for $n > 2$ are singular Jónsson cardinals with cofinality κ_2

(iii) $\kappa_{\omega} = \bigcup_{n \in \omega} \kappa_n$ is a Rowbottom cardinal.

Moreover, if there is a normal ultrafilter U over κ so that κ^{κ}/U has order-type κ^+, then the κ_n's can be taken to satisfy:

(iv) $\kappa_{n+1} = \kappa_n^+$ for each n, and

(v) $\kappa_2 \to (\kappa_2)^{\alpha}_2$ for every $\alpha < \kappa_2$.

Now Solovay had shown that if F is the closed unbounded filter over ω_1, [illegible] AD implies $\omega_1{}^{\omega_1}/F$ has order-type ω_2. Thus, with Martin's result, Kleinberg's scheme immediately yields: AD implies that ω_2 is measurable and in fact $\omega_2 \to ($[illegible] for every $\alpha < \omega_2$ (previously known); ω_n for $n > 2$ are singular Jónsson cardin[illegible] with cofinality ω_2; and ω_{ω} is a Rowbottom cardinal. The influence of $\omega_1 \to (\omega$[illegible] seems tremendous, but M.Spector(1978) has shown that it does not extend beyond $\omega_{[illegible]}$ It is known for instance that AD implies $\omega_{\omega+1} \to (\omega_{\omega+1})^{\alpha}_2$ for every $\alpha < \omega_{\omega+1}$ (Ku[illegible] and Spector's result assures that this is not a provable consequence of $\omega_1 \to (\omega_1$[illegible]

Recent work of Kechris(1977) has made Kleinberg's result applicable under AD [illegible] cardinals $> \omega_1$. Kechris established that: AD implies that there is a κ so th[illegible] $\kappa \to (\kappa)^{\kappa}_2$, and $\{\lambda < \kappa \mid \lambda$ is a limit cardinal $\& \ \lambda \to (\lambda)^{\lambda}_2\}$ is stationary in κ. It is certainly difficult to assess such pronouncements!

Of the large large cardinals, we have already mentioned strong compactness w[illegible] respect to ω_1 and Θ. Martin recently established that AD implies that there [illegible] a normal ultrafilter over $P_{\omega_1}(\omega_2)$. (See DiPrisco-Henle(1977) for a proof, as w[illegible]

as a related result, that AD implies that there is a fine ultrafilter over $P_{\omega_2}(\omega_3)$.)
Without the availability of an inner model construction for supercompactness (recall §14), we cannot evaluate the strength of this statement by referring to the standard ZFC context.

AD has certainly peppered the landscape with quite a number of new propositions which seem to be of large cardinal type. Without AC rather small cardinals seem capable of possessing properties which with AC would be attributable only to highly inaccessible cardinals. One way of getting a better understanding of AD and its plausibility would be to see if these new propositions are relatively consistent to ZFC plus the existence of some large cardinal. In other words, how far can known consequences of AD be replicated by using large cardinal hypotheses? Since we want models in which AC fails, it would seem that the constructions will have to be rather involved, calling upon both forcing and inner models. We refer to Bull(1976) and M. Spector(1978) for recent developments along this program.

Our discussion of AD is at an end; in future sections we again assume the Axiom of Choice.

§29. The Singular Cardinals Problem and Covering Properties

This section is devoted to recent developments which relate the theory of large cardinals to the elaboration of a fundamental problem of set theory. It is remarkable that large cardinals should play any role in such a basic context at all, and all the more remarkable that their intervention, as we shall see, is a necessary one.

The generalized form of Cantor's original continuum problem is: given the cardinality of a set, to determine the cardinality of its power set. Hindsight tells us that Cantor's own, frustrating efforts directed toward the solution of this problem were bound to fail. Indeed, one had to await the introduction of Cohen's forcing method to see that the axioms of set theory do not give a definite answer to this problem. Expected or not, given this state of affairs, there naturally arises the problem of at least determining what constraints can be imposed on the functional $\kappa \mapsto 2^\kappa$.

For regular cardinals, the problem was duly and satisfactorily solved by Easton [1970], who proved that the only restrictions probable in ZFC about this functional for regular cardinals are:

(i) If $\kappa \leq \lambda$, then $2^\kappa \leq 2^\lambda$.

(ii) $\kappa < cf(2^\kappa)$.

His result is: Suppose that in the ground model, the GCH holds, and there is a class function F from regular cardinals into cardinals such that: (a) F is non-decreasing, and (b) $\kappa < cf(F(\kappa))$; then there is a forcing extension preserving all cardinalities and cofinalities in which $2^\kappa = F(\kappa)$ for every regular κ. Thus, the theory ZFC has little to say about powers of regular cardinals. (Easton's result can also be achieved with Silver's forcing method (§25), so that most large cardinals will be preserved, if F has a uniform definition amenable to reflection arguments;

see for example Menas[1976]).

The Singular Cardinals Problem is to determine the situation with respect to powers of singular cardinals. There are some complications here, since further constraints relative to powers of smaller cardinals are known theorems of ZFC:

(a) (Bukovsky[1965]; Hechler) If κ is singular and there is a $\gamma < \kappa$ so that $\gamma \leq \beta < \kappa$ implies $2^\beta = 2^\gamma$, then $2^\kappa = 2^\gamma$.

(b) (Silver[1974a]) If κ is singular with uncountable cofinality such that $2^\kappa > \kappa^+$, then $\{\alpha < \kappa \mid 2^\alpha > \alpha^+\}$ contains a closed unbounded subset of κ.

Silver's result is well worth some discussion. It immediately inspired a whole series of results, including significant technical improvements by Galvin-Hajnal [1975] and, of course, Jensen's Covering Theorem (see below). Silver's result was an unexpected theorem of ZFC about the surprising control wielded on powers of many singular cardinals by powers of smaller cardinals, but once the collective intuition of set theorists were sharpened (or rather corrected) by it, ideas both standard and latent were fashioned into new results about the subtle underpinnings that exist in the set theoretical universe.

It may seem paradoxical that there now exist combinatorial proofs of Silver's result (due to Baumgartner, Prikry, and others) which are relatively straightforward and can be regarded as in the genre of semi-classical results about cardinal arithmeti Yet, Silver had to light the way, and it is significant from the viewpoint of this paper that his own proof used elements usually associated with large cardinals: the taking of a sort of ultrapower, and the existence of a least function there. Indeed Silver's result can be seen as a culmination of efforts which started with a result of Scott on the GCH at a measurable cardinal, proceeded through various weakenings (see the last theorem proved in §13), and finally evolved into a standard theorem of ZFC through the use of scaling at a singular cardinal of uncountable cofinality. As with Solovay's theorem on splitting stationary sets (§11), ideas first molded in the workshop of large cardinal theory emerged to establish an important result about the structure of sets throughout the ramified hierarchy.

Returning now to the Singular Cardinals Problem, we can first ask: what is 2^κ for singular κ in Easton's models? It turns out that 2^κ is the minimal cardinal λ satisfying the necessary restrictions: $\lambda \geq \sup\{2^\alpha \mid \alpha < \kappa\}$, and $cf(\lambda) > \kappa$. This devolves into two cases: (i) For some $\gamma < \kappa$, $\gamma \leq \beta < \kappa$ implies $2^\beta = 2^\gamma$. Then $2^\kappa = 2^\gamma$ by the Bukovsky-Hechler result. (ii) Otherwise. Then $\delta = \sup\{2^\alpha \mid \alpha < \kappa\}$ has the same cofinality as κ, so $2^\kappa = \delta^+$. Thus, powers of singular cardinals here are completely determined by the powers of regular cardinals. We shall see that what happens in Easton's models pertains to some contexts involving large cardinals. To cast some recent results in general terms, let us make the following definition:

Suppose that $M \subseteq N$ are two inner models of ZFC. Then N has the covering property with respect to M iff M is a definable class in N, and whenever

$N \models X$ is an uncountable set of ordinals, then there is a $Y \in M$ so that $X \subseteq Y$ and $N \models |X| = |Y|$.

The restriction to uncountable sets turns out to be an essential one in specific cases, by non-trivial arguments involving forcing (see Bukovsky[1976]). That N has the covering property with respect to M is a deep structural statement about the close distance between M and N. In fact, some of the cardinal structure of M is preserved in the passage to N, as the next two lemmas indicate:

Lemma 1. Assume that N has the covering property with respect to $M \subseteq N$. Then if κ is a singular cardinal in N, then κ is a singular cardinal in M, and $(\kappa^+)^N = (\kappa^+)^M$.

$\vdash$ Let $X \in N$ be a set of order-type $< \kappa$, cofinal in κ. By the covering property, get $Y \in M$ so that $X \subseteq Y$ and in N, $|Y| = |X| + (\omega_1)^N < \kappa$. Without loss of generality we can assume that $Y \subseteq \kappa$. Since in N, κ is a cardinal and Y has cardinality less than κ, Y must have order-type $< \kappa$. Hence, κ is singular in M, as Y is clearly cofinal in κ.

Assume now that $\gamma = (\kappa^+)^M < (\kappa^+)^N$. Then in N, γ is a singular ordinal and its cofinality is less than κ, since κ is singular in N. Let $X \in N$ be a set of order-type $< \kappa$, cofinal in γ. We can now proceed as in the preceding paragraph to show that γ is singular in M, which is a contradiction. $\dashv$

The connection between the covering property and the Singular Cardinals Problem comes from the following lemma:

Lemma 2. Assume that N has the covering property with respect to $M \subseteq N$, and that M satisfies the GCH. Then the following statement holds in N: whenever κ is singular, 2^κ is the minimal cardinal λ so that $\lambda \geq \sup\{2^\alpha \mid \alpha < \kappa\}$, and $cf(\lambda) > \kappa$.

$\vdash$ Let us argue in N. If for some $\gamma < \kappa$, $\gamma \leq \beta < \kappa$ implies $2^\beta = 2^\gamma$, then $2^\kappa = 2^\gamma$, and the Lemma holds in this case. Otherwise, setting $\delta = \sup\{2^\alpha \mid \alpha < \kappa\}$, since $cf(\delta) = cf(\kappa)$, we must show that $2^\kappa = \delta^+$. To do this, we use the easy-to-prove fact that $2^\kappa = \delta^{cf(\kappa)}$. Now by the covering property, every subset of δ of cardinality $cf(\kappa)$ is included in a subset of δ which is in M, having (in N) cardinality $cf(\kappa) + \omega_1$. Thus,

$$2^\kappa \leq \delta^{cf(\kappa)} \leq (2^\delta)^M \cdot 2^{cf(\kappa)+\omega_1}.$$

But $2^{cf(\kappa)+\omega_1} \leq \delta$ by definition of δ, and as M satisfies the GCH, $(2^\delta)^M = (\delta^+)^M \leq \delta^+$. Thus, $2^\kappa = \delta^+$, and we are done. $\dashv$

Thus, the Singular Cardinals Problem would be solved if V had the covering property with respect to some inner model satisfying the GCH. The fascinating fact is that large cardinals have come into play in the recent work of Jensen, Dodd-Jensen, and Mitchell in determining possible cases of covering properties. Jensen first isolated the essence of the covering idea, proving the following beautiful theorem after seeing Silver's GCH result:

Jensen's Covering Theorem: If $0^\#$ does not exist, then V has the covering property with respect to L.

For a proof, see Devlin-Jensen[1975]; a recent proof by Silver using his Machines makes more transparent some of the ideas. This result immediately clarifies the Singular Cardinals Problem: If for some singular κ, 2^κ is not the minimal possible cardinal (with cofinality $> \kappa$, and at least as large as $\sup\{2^\alpha \mid \alpha < \kappa\}$), then $0^\#$ must exist! (With similar arguments, one can also conclude that $a^\#$ exists for every $a \subseteq \omega$.)

Even stronger results follow from the very recent work of Dodd-Jensen(1976). Their initial purpose was to get an analogous covering property result for $L[U]$, where U is a normal ultrafilter over a measurable cardinal κ. But their investigations went deeper, uncovering a new underlying structure which they designated the Core Model K. Intuitively, K is the inner model of set theory generated by iterating the $\#$ operation, i.e. it is the minimal inner model M such that if for any set $x \in M$, $x^\#$ exists (see §7), then $x^\# \in M$. Thus, K is a relative notion, dependent on the ontological commitments one has made in V. For instance, if $0^\#$ not exist, then $K = L$. If there is an inner model with a measurable cardinal, then K has a nice characterization: We start with a given model of form $L[U]$, and in it we can use the ultrafilter to get a descending nested sequence of iterated ultrapowers $\langle M_\alpha \mid \alpha \in OR \rangle$ as in §8; then K is simply $\bigcap\{M_\alpha \mid \alpha \in OR\}$. In any case, K is always a model of the GCH. The main result:

Theorem (Dodd-Jensen): If there does not exist an inner model with a measurable cardinal, then V has the covering property with respect to K.

Hence, if there is a singular cardinal κ such that 2^κ is not the minimal possible cardinal (under the usual constraints), then there is an inner model with a measurable cardinal! Exciting ongoing work of Mitchell shows that the conclusion can even be extended to the existence of many measurable cardinals in a strong sense (i.e. with ultrafilters of high type in the $\triangleleft$ order—see the end of §9).

Naturally, one begins to wonder whether the simple solution we seem to be getting for the Singular Cardinals Problem is a universal phenomenon, or whether if a sufficiently strong large cardinal hypothesis is used, we could get a model in which 2^κ is not the minimal possible value for some singular κ. The latter is the case: Assume that κ is a κ^{++}-supercompact cardinal. By an application of Silver's forcing (§25), there is a forcing extension in which κ is measurable and $2^\kappa > \kappa^+$. Now by Prikry forcing (§23), we can change the cofinality of κ to ω. In this resulting model, κ is a singular strong limit cardinal so that $2^\kappa > \kappa^+$. However, for any singular strong limit cardinal λ, the minimal possible value for 2^λ is λ^+.

The Prikry-Silver result follows a pattern which seems to be very useful for getting consistency results from large cardinals about singular cardinals: In many of these problems it is possible to get the required consistency result for a regular cardinal, say for a measurable cardinal. We can thereupon make this cardinal singular by some variation of Prikry forcing. If one wants to transfer the Prikry-Silver result to more down-to-earth cardinals like ω_ω, one can follow this pattern, as is done in Magidor[1977] where a model is produced in which ω_ω (several other "small" singular cardinals are possible) is a strong limit cardinal which violates the GCH. The scheme is to start with a cardinal κ which is κ^+-supercompact and violates the GCH, and change the cofinality of κ to ω while simultaneously collapsing many cardinals below κ, so that in the resulting model κ is ω_ω. The fact that κ was κ^+-supercompact is extensively used in the proof for showing that 2^κ is not collapsed in the process to κ^+.

One uncompromising version of the Singular Cardinals Problem is to get a singular cardinal to be the first cardinal which violates the GCH. In view of Silver's result, such a cardinal cannot have uncountable cofinality, and the problem is most interesting for the first singular cardinal, ω_ω. Trying to solve this singular cardinals problem by following the pattern described above runs into difficulties because if κ is at least measurable with $2^\kappa > \kappa^+$, then there are unboundedly many $\alpha < \kappa$ such that $2^\alpha > \alpha^+$. Thus, if we follow the collapsing scheme of Magidor[1977], we are left with unboundedly many cardinals below κ which violate the GCH. A more involved approach, starting with the much stronger assumption of the existence of a huge cardinal with a supercompact cardinal below it, can be used to get a model in which $2^{\omega_\omega} > \omega_{\omega+1}$ but $2^{\omega_n} = \omega_{n+1}$ for every $n \in \omega$ (see Magidor[1978]). Whether the proposition that the GCH holds below a singular cardinal which violates it, has stronger consistency strength than the proposition simply that a singular strong limit cardinal violates the GCH, is still open.

§30. Problems Entailing Large Cardinals

We are finally(!) in sight of the end of this paper. In this section we shall survey a few problems which seem to be leading to large cardinal concepts. We shall intentionally concentrate on those problems, usually concerning "small" sets, in which no large cardinal is mentioned specifically in the formulation. Still, for each of these problems there are some arguments that their solution, at least in one of the possible directions, should involve the consistency of the existence of some large cardinals.

Let us first turn to a topic fresh in our memory from the previous section, the Singular Cardinals Problem. In spite of the progress achieved towards determining the possible behavior of the function $\kappa \mapsto 2^\kappa$ for singular κ, a nice clear statement like the Easton result for regular cardinals is still missing. Particularly appealing special cases of this problem are the following:

(A) Is ZFC consistent with "for every κ, $2^\kappa > \kappa^+$" ? For example,

(B) IS ZFC consistent with "for every κ, $2^\kappa = \kappa^{++}$" ?

The large cardinal connections of this class of problems were summarized in §29; however, even for the known results better lower and upper bounds on consistency strength are desirable.

The next class of problems are connected with more modest large cardinal notions. Recall from §5 that a κ-Aronszajn tree is a κ-tree with no κ-branch. A κ-Souslin tree is a κ-Aronszajn tree such that every antichain of the tree (i.e. a set of mutually incompatible elements) has cardinality $< \kappa$. Souslin's Hypothesis for κ (SH_κ) is the assertion that there is no κ-Souslin tree, a weak version of the tree property for κ. SH_{ω_1} is (equivalent to the original) Souslin's Hypothesis, well-known to be independent of ZFC, both with the Continuum Hypothesis, and with its negation (by work of Jensen, Solovay and Tennenbaum; see Solovay-Tennenbaum[1971] and the monograph Devlin-Johnsbraten[1974]).

The consistency problem for SH_κ when $\kappa > \omega_1$ seems to be much more difficult, especially if we want to retain the GCH. To bring matters into focus, we make some remarks which recall and amplify §21. First of all, Jensen[1972] had actually established that in L, weak compactness for κ is equivalent to SH_κ, for regular κ. We are interested in SH_κ for small κ, and the Mitchell-Silver model cited in §21 certainly satisfied SH_{ω_2}, as there were not even any ω_2-Aronszajn trees in that model. However, $2^\omega = \omega_2$ held in that model, and in fact a classical result of Specker[1951] as cited in §5 necessitates something like this: if $2^\omega = \omega_1$, then there is an ω_2-Aronszajn tree. No such result seems available for ω_2-Souslin trees, so the focal problem in this area is to get SH_{ω_2} and the GCH to hold.

This problem has been extensively investigated by Gregory[1976] who established in particular that: If $2^\omega = \omega_1$, $2^{\omega_1} = \omega_2$, and $E^\omega_{\omega_2}$ hold, then SH_{ω_2} is false, i.e. there is an ω_2-Souslin tree. Hence, if we want SH_{ω_2} and the GCH to hold, we need to guarantee the failure of $E^\omega_{\omega_2}$. As pointed out in §21, this necessitates at least the consistency strength of the existence of a Mahlo cardinal, and very likely, of a weakly compact cardinal.

Further light was cast on the general problem by Shelah in 1978 when he established: Con(ZFC & there is a weakly compact cardinal) implies Con(ZFC & $2^\omega = \omega_1$ & SH_{ω_2}). Shelah improved a result of Laver, who started with a measurable cardinal. The model satisfies $2^{\omega_1} > \omega_2$, and so the GCH still has not been achieved. It is an open question whether large cardinals are needed at all to get SH_{ω_2}, if we disregard the GCH considerations. The accepted guess is that the answer is yes.

The consistency problem for SH_{ω_2} is closely connected with the problem of generalizing Martin s Axiom. Martin's Axiom for κ (MA_κ) states: given a ω_1-c.c. notion of forcing P and a family F of dense subsets of P with $|F| < \kappa$, we can find an F-generic filter for P. (Recall that $G \subseteq P$ is an F-generic filter if it is a generic filter in the usual sense, except that it is only required to meet each member of the family F of dense sets.) The consistency of Martin's Axiom is established in Solovay-Tennenbaum[1971], and extensively investigated in Martin-Solovay

[1970]. The axiom is a sort of enumeration principle which generalizes the Continuum Hypothesis and allows many forcing-like constructions. The original raison d'être was the following implication: MA_κ and $\kappa > \omega_1$ implies SH_{ω_1}. A major generalization of Martin's Axiom would result if we can weaken the chain condition requirement on P. Without some other restriction on P, one can easily verify that a direct generalization is outright false, so there are different directions in which one can go. One possibility is the following generalization of MA_{ω_3} : given a ω_2-c.c. notion of forcing P which is ω-closed and a family F of dense subsets of P with $|F| = \omega_2$, there is an F-generic filter for P. The consistency of this axiom is unknown, and if one tries to generalize the usual argument for Martin's Axiom, it seems very likely that the proof would require a large cardinal assumption in the ground model.

The consistency problem for the tree property for κ is more or less settled by Mitchell's construction, when κ is a successor of a regular cardinal. The problem seems to be much more difficult for successors of singular cardinals. The simplest case is $\omega_{\omega+1}$: Is it consistent that there is no $\omega_{\omega+1}$-Aronszajn tree? The consistency of the other direction is settled by Jensen's result that if $V = L$ and κ is not weakly compact, then there is a κ-Aronszajn tree. Jensen in particular showed that : If the GCH and $\Box_\lambda$ holds, then there is a special λ^+-Aronszajn tree. Special κ-Aronszajn trees were defined in §21; the nice thing about such trees is that they are still κ-Aronszajn trees in any extension in which κ is still a cardinal. This suggests the covering property from §29, and one can argue as follows: The Core Model K satisfies both the GCH and $\Box_\lambda$ for all λ. Remember that if there is no inner model with a measurable cardinal, then V has the covering property with respect to K, and so for any singular cardinal λ, $(\lambda^+)^K = \lambda^+$. In particular, there would be a special $\omega_{\omega+1}$-Aronszajn tree in V. Hence, if we want the consistency of "there are no $\omega_{\omega+1}$-Aronszajn trees", then we have to assume at least the consistency of the existence of a measurable cardinal. (Mitchell's recent work on generalizing K indicates that we have to actually assume the consistency of the existence of many measurable cardinals in a strong sense.) Nothing is known beyond these lower bounds on consistency strength.

Recall from §21 that another consequence of $\Box_{\omega_\omega}$ is that $E^\lambda_{\omega_{\omega+1}}$ holds for every $\lambda < \omega_\omega$. Let us briefly consider the follow proposition:

(*) Whenever $S \subseteq \omega_{\omega+1}$ is stationary, there is an $\alpha < \omega_{\omega+1}$ so that $S \cap \alpha$ is stationary in α.

This proposition is a strong negation of $E^\lambda_{\omega_{\omega+1}}$ for cardinals $\lambda < \omega_\omega$. (Since $\omega_{\omega+1}$ is the successor of a singular cardinal, there is no obvious counterexample to (*), and so we do not need to specialize to $E^\lambda_{\kappa^+}$, as in the general case for successors κ^+.) As a $\Box_\lambda$ sequence is preserved through any extension in which λ^+

remains a cardinal, an argument as in the previous paragraph (or see §21) shows that (*) has the consistency strength at least that of the existence of measurable cardinals. The last result cited in §21, due to Shelah, is a relative consistency result for the failure of $\Box_{\omega_\omega}$ (in fact, for $E^\omega_{\omega_{\omega+1}}$), but the question still remains whether the full (*) can be consistent.

The consistency problem for (*) is related to the following problem from algebra: Let us call an Abelian group almost free iff every subgroup of smaller cardinality is a free Abelian group. It is known that for every ω_n, $n \in \omega$, one can find an almost free, but not free, Abelian group of cardinality ω_n. A theorem of Shelah[1975] states that if λ is singular, every almost free Abelian group of cardinality λ is free. So, it turns out that the first unresolved case is $\omega_{\omega+1}$. It is known that: If (*) fails, then there is an almost free, but not free, Abelian group of cardinality $\omega_{\omega+1}$. Thus, the consistency of the existence of such a group is established (by any model of $\Box_{\omega_\omega}$, for instance). The consistency of the proposition that such a group does not exist is unknown, and it is clear that large cardinal assumptions are required.

Let us conclude this section by reiterating some of the problems concerning "small" sets which we discussed in previous sections. They fall into two categories, and are just a sample of the prominent problems in these categories. The problems that we have been discussing in this section so far fall into the first category (I): to establish the consistency of ZFC plus some proposition, when it is clear that such a consistency proof requires large cardinals. The second category (II) involves situations where a relative consistency proof using large cardinals is known, but it is not clear whether they are essential to the proof; the problem is to eliminate these large cardinal assumptions.

I. Are the following propositions consistent with ZFC:

- (Ia) Projective Determinacy (§27).
- (Ib) The ideal of non-stationary sets over ω_1 is ω_2-saturated (§11).
- (Ic) There is a uniform non-(ω,ω_1)-regular ultrafilter over ω_1 (§13).
- (Id) ω_ω is a Rowbottom cardinal (§6).

II. Are large cardinals necessary to establish the consistency of the following propositions:

- (IIa) There is a uniform ultrafilter over ω_ω which is ω_n-indecomposable for some $0 < n < \omega$ (§13).
- (IIb) Every set of reals which is real, ordinal definable is Lebesgue measurable (§18).
- (IIc) Every normal Moore space is metrizable.

This last problem is not discussed in this paper, but it is a prominent one of set theoretic topology. It is cited here because of its topical nature: very recently, work of Nyikos and Kunen established that Con(ZFC & there is a strongly compact cardinal) implies Con(ZFC & every normal Moore space is metrizable).

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Through these pages, we hope that we have accomplished with some clarity and coherence what we set out to do. If that is so, there should be no need to iterate our main themes and motivating ideas, as they have been extensively developed through elaborations and variations. So, we confine ourselves to some afterthoughts that our long trek has left with us.

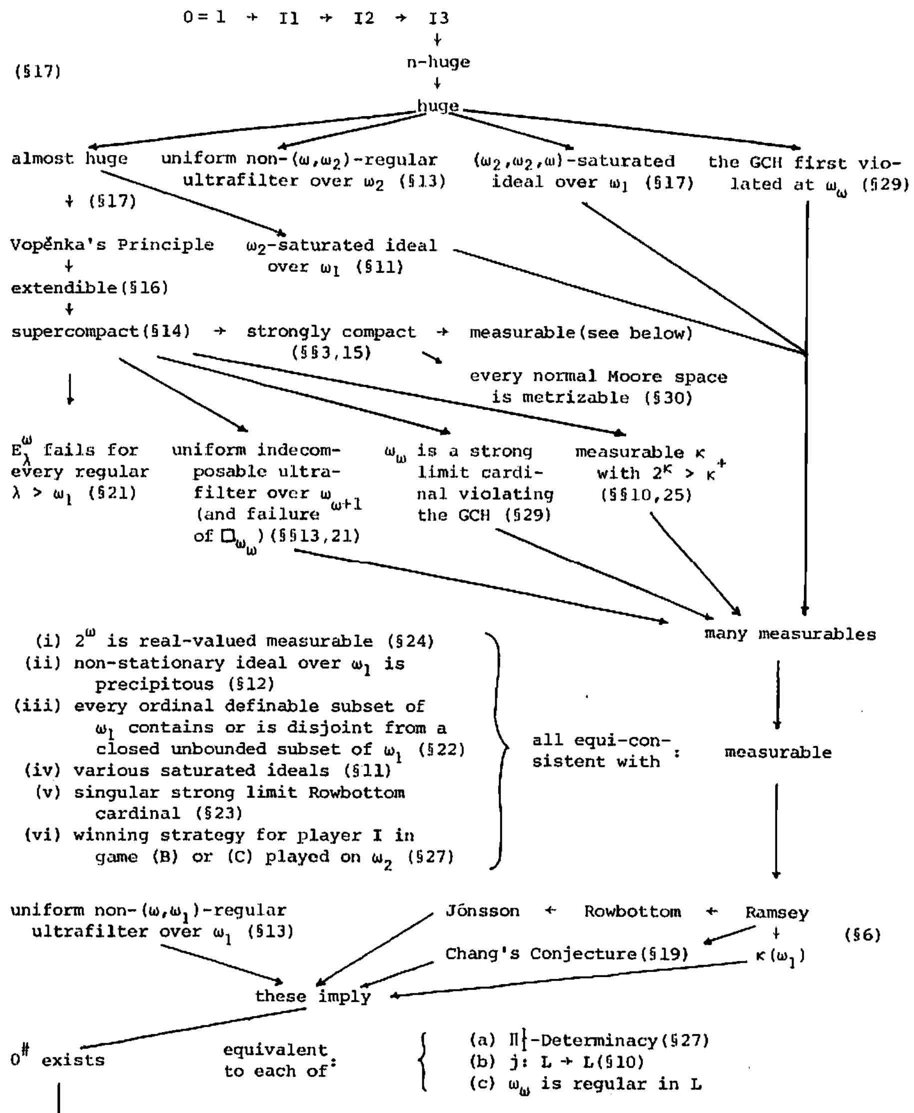
The development of the theory of large cardinals should be regarded as closely integrated with the development of set theory itself. As we have tried to show, the ideas at play are not off to one side, but really in the mainstream of set theory. It is especially through equi-consistency results that we can see how large cardinals encapsulate structural principles which are pivotal to speculations on the nature of sets.

As our edifice grew, we saw how one by one the large cardinals fell into place in a linear hierarchy. This is especially remarkable in view of the ostensibly disparate ideas that motivate their formulation. As remarked by H.Friedman, this hierarchical aspect of the theory of large cardinals is somewhat of a mystery. Are there natural set theoretic propositions whose consistency strength cannot be measured on the scale of large cardinals? In other words, is there a hierarchy of set theoretical principles in another galaxy above ZFC, disjoint and incomparable to our large cardinals? The technological development of forcing for large cardinals has definitely squashed any hope that the large cardinals can decide the power of the Continuum, but perhaps Gödel's hopes can still be realized by new natural principles more intimately connected with the power set operation.

On the other side of the coin, the neat hierarchical structure of the large cardinals and the extensive equi-consistency results that have already been demonstrated to date are strong plausibility arguments for the inevitability of the theory of large cardinals as the natural superstructure on ZFC. Persuasive are the unifying themes that motivate and develop them, which often translate into characterizations of unexpected elegance. Perhaps it is the further cultivation of the ground already shown to be so fertile that will reveal the possible extent of the subtle structure of the set theoretical universe. The Continuum Problem may be left unresolved, but on the other hand recent research has established the remarkable connection of large cardinals to basic problems involving the powers of singular cardinals. Such a fundamental elucidation of structure holds the promise of more to come, and we are confident that what is past is only a prologue.

THE CHART OF CARDINALS

Unless otherwise specified, horizontal arrows indicate direct implications, and downward arrows indicate relative consistency implications.



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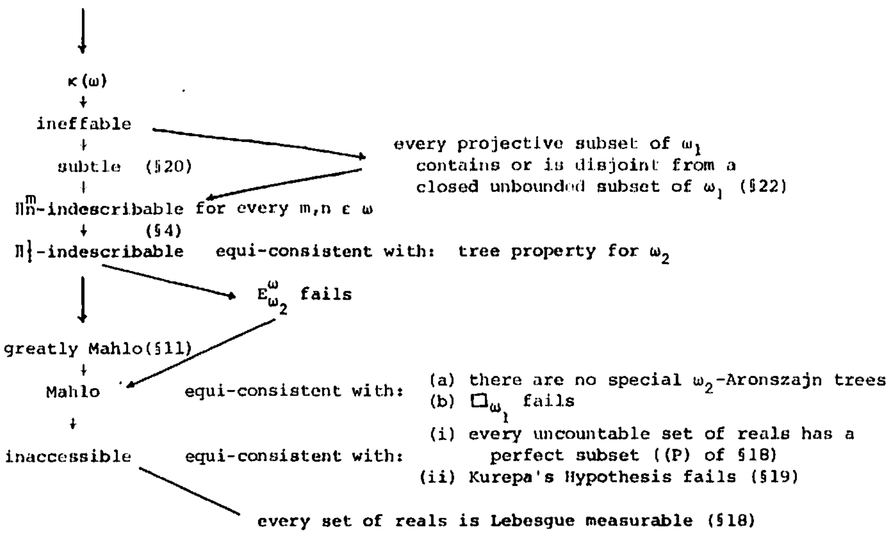
The large cardinals on this page (left column) are compatible with V = L .



BIBLIOGRAPHY

S.Aanderaa
[1973] Inductive definitions and their closure ordinals, in: J.E.Fenstad and P.Hinman,eds., Generalized Recursion Theory (North Holland, Amsterdam) 207-220.

F.G.Abramson, L.A.Harrington, E.M.Kleinberg, and W.S.Zwicker
[1977] Flipping properties: a unifying thread in the theory of large cardinals, Annals Math. Logic 12(1), 37-58.

P.Aczel and W.Richter
[1972] Inductive definitions and analogues of large cardinals, in: Conference in Mathematical Logic, London, 1970, Lecture Notes in Math. 255 (Springer, Berlin) 1-9.
[1974] Inductive definitions and reflecting properties of admissible ordinals, in: J.E.Fenstad and P.Hinman,eds., Generalized Recursion Theory (North Holland, Amsterdam) 301-381.

J.Barwise, M.Kaufmann, and M.Makkai
[1978] Stationary logic, Annals Math. Logic 12(3).

J. Baumgartner
[1975] Ineffability properties of cardinals I, in: Infinite and Finite Sets, Colloquia Mathematica Societatis Janos Bolyai, 10 (dedicated to P.Erdös) (North Holland, Amsterdam) vol.1.
[1976] A new class of order types, Annals Math. Logic 9(3), 187-222.

J. Baumgartner and F.Galvin
(1977) Generalized Erdös cardinals and $0^{\#}$, to appear.

J. Baumgartner, A.Hajnal, and A.Maté
[1975] Weak saturation properties of ideals, in: Infinite and Finite Sets, Colloquia Mathematica Societatis Janos Bolyai, 10 (dedicated to P.Erdös) (North Holland, Amsterdam).

J.Baumgartner, L.Harrington, and E.M.Kleinberg
[1976] Adding a closed unbounded set, J. Sym. Logic 41(2), 481-2.

J.Baumgartner, A.Taylor, and S.Wagon
[1977] On splitting stationary subsets of large cardinals, J. Sym. Logic 42, 203-214.
(1978) Ideals and partitions, to appear.

J.Bell
[1977] Boolean-valued Models and Independence Proofs in Set Theory, Oxford Logic Guides (Clarendon Press, Oxford).

M.Benda and J.Ketonen
[1974] On regularity of ultrafilters, Israel J. of Math. 17, 231-240.

W.Boos
[1975] Lectures on large cardinal axioms, in: Logic Conference, Kiel 1974, Lecture Notes in Math. 499 (Springer, Berlin) 25-88.

L.Bukovsky
[1965] The continuum problem and powers of alephs, Commentationes Math. Univ. Caolinae 6, 181-197.
[1976] Changing cofinality to $\aleph_2$, in: Set Theory and Hierarchy Theory, Lecture Notes in Math. 537 (Springer, Berlin) 37-49.

E.Bull
(1976) Successive large cardinals, to appear, and M.I.T. Ph.D. Thesis.

C.C.Chang
[1971] Sets constructible using $L_{\kappa\kappa}$, in: D.S.Scott,ed., Axiomatic Set Theory, Proc. Symp. Pure Math. 13(1) (Amer. Math. Soc., Providence,R.I.) 1-8.

C.C.Chang and H.J.Keisler
[1973] Model Theory (North Holland, Amsterdam).

G.V.Chudnovsky and D.V.Chudnovsky
[1971] Regularnye i ubyvajusche nepolnye ultrafiltry, Dokl. Akad. Nauk U.S.S.R. 198, 779-782.

W.W.Comfort and S.Negrepontis
[1974] The Theory of Ultrafilters (Springer, Berlin).

Morton Davis
[1964] Infinite games with perfect information, in: Advances in Game Theory, Annals of Math. Studies 52 (Princeton, N.J.) 85-101.

K.J.Devlin
[1973] Some weak versions of large cardinal axioms, Annals Math. Logic 5, 291-325.
[1973a] Aspects of Constructibility, Lecture Notes in Math. 354 (Springer, Berlin).
[1974] Some remarks on changing cofinalities, J. Sym. Logic 39, 27-30.
[1975] Indescribability properties and small large cardinals, in: Logic Conference, Kiel 1974, Lecture Notes in Math. 499 (Springer, Berlin) 89-114.

K.J.Devlin and R.B.Jensen
[1975] Marginalia to a theorem of Silver, in: Logic Conference, Kiel 1974, Lecture Notes in Math. 499 (Springer, Berlin) 115-142.

K.J.Devlin and H.Johnsbraten
[1974] The Souslin Problem, Lecture Notes in Math. 405 (Springer, Berlin).

M.A.Dickmann
[1976] Large Infinitary Languages (North Holland, Amsterdam).

C.DiPrisco
(1976) M.I.T. Ph.D. Thesis.

C.DiPrisco and J.Henle
(1977) On the compactness of $\aleph_1$ and $\aleph_2$, to appear.

C.DiPrisco and W.Zwicker
(1977) Flipping properties and supercompact cardinals, to appear.

T.Dodd and R.B.Jensen
(1976) The core model, circulated notes.

F.R.Drake
[1974] Set Theory (North Holland, Amsterdam).

W.B.Easton
[1964] Powers of regular cardinals, Annals Math. Logic 1, 139-178.

A.Ehrenfeucht and A.Mostowski
[1956] Models of axiomatic theories admitting automorphisms, Fund. Math. 43, 50-68.

P.Erdös and A.Hajnal
[1958] On the structure of set mappings, Acta Math. Acad. Sci. Hung. 9, 111-131.
[1962] Some remarks concerning our paper "On the structure of set mappings", Acta Math. Acad. Sci. Hung. 13, 223-226.
[1966] On a problem of B.Jónsson, Bull. Acad. Polon. Sci. Sér. Sci. Math. Ast. Phys. 14, 61-99.
[1971] Unsolved problems in set theory, in: D.S.Scott,ed., Axiomatic Set Theory, Proc. Symp. Pure Math. 13(1) (Amer. Math. Soc., Providence R.I.) 17-48.

P.Erdös, A.Hajnal, and R.Rado
[1965] Partition relations for cardinal numbers, Acta Math. Acad. Sci. Hung. 16, 93-196.

P.Erdös and R.Rado
[1952] Combinatorial theorems on classifications of subsets of a given set, Proc. London Math. Soc. (3)2, 417-439.
[1956] A partition calculus in set theory, Bull. Amer. Math. Soc. 62, 427-489.

P.Erdös and A.Tarski
[1943] On families of mutually exclusive sets, Ann. Math. 44, 315-429.
[1961] On some problems involving inaccessible cardinals, in: Essays on the Foundations of Mathematics (Magnes Press, Jerusalem) 50-82.

G.Fodor
[1956] Eine Bemerkung zur Theorie der regressiven Funktionen, Acta Sci. Math. (Széged) 17, 139-142.
[1966] On a process concerning inaccessible cardinals I, Acta Sci. Math. (Széged) 27, 111-124.

H.Friedman
[1971] Higher set theory and mathematical practice, Annals Math. Logic 2, 326-357.

L.Fuchs
[1970] Infinite Abelian Groups (Academic Press, London-New York).

H. Gaifman
[1967] Uniform extension operators for models and their applications, in: J.Crossley,ed., Sets, Models and Recursion Theory (North Holland, Amsterdam) 122-155.
[1967a] A generalization of Mahlo's method for obtaining large cardinal numbers, Israel J. Math. 5, 188-200.
[1974] Elementary embeddings of set theory, in: T.J.Jech,ed., Axiomatic Set Theory, Proc. Symp. Pure Math. 13(2) (Amer. Math. Soc., Providence,R.I.) 33-101.

D.Gale and F.M.Stewart
[1953] Infinite games with perfect information, in: Kuhn and Tucker,eds., Contributions to the Theory of Games, vol.3, Annals of Math. Studies 28 (Princeton,N.J.) 245-266.

F.Galvin and A.Hajnal
[1975] Inequalities for cardinal powers, Ann. Math. 101, 491-498.

F.Galvin, T.J.Jech, and M Magidor
(1976) An ideal game, to appear in J. Sym. Logic.

F.Galvin and K.Prikry
(1976) Infinitary Jónsson algebras, to appear.

K.Glöede
[1973] Filters closed under Mahlo's and Gaifman's operation, in: Cambridge Summer School in Mathematical Logic, Lecture Notes in Math. 337 (Springer, Berlin) 495-527.

K.Gödel
[1938] The consistency of the axiom of choice and of the generalized continuum hypothesis, Proc. Natl. Acad. Sci. U.S.A. 24, 556-557.
[1946] Remarks before the Princeton Bicentennial Conference on problems in mathematics, in: M.Davis,ed., The Undecidable (Raven Press, New York) 84-88.
[1947] What is Cantor's continuum problem?, Amer. Math. Monthly 54, 515-525.

J.Green
(1977) Berkeley Ph.D.Thesis.

J.Gregory
[1976] Higher Souslin trees and the generalized continuum hypothesis, J. Sym. Logic 41(3), 663-671.

D.Guaspari
(1973) The largest constructible Π^1_1 set, to appear.

W.P.Hanf
[1964] Incompactness in languages with infinitely long expressions, Fund. Math. 53, 309-324.
On a problem of Erdös and Tarski, Fund. Math. 53, 325-334.

W.P.Hanf and D.S.Scott
[1961] Classifying inaccessible cardinals, Notices Amer. Math. Soc. 8, 445.

L.Harrington
[1973] M.I.T. Ph.D.Thesis.
[1974] The superjump and the first recursively Mahlo ordinal, in: J.E.Fenstad and P.Hinman,eds., Generalized Recursion Theory (North Holland, Amsterdam) 43-52.
(1975) Analytic determinacy and $0^{\#}$, to appear.

F.Hausdorff
[1908] Grundzüge einer Theorie der geordneten Mengen, Math. Annalen 65, 443.

E.Hewitt
[1948] Rings of continuous functions I, Trans. Amer. Math. Soc. 64, 45-99.

T.J.Jech
[1968] ω_1 can be measurable, Israel J. Math. 6, 363-367.
[1971] Trees, J. Sym. Logic 36(1), 1-14.
[1973] Some combinatorial problems concerning uncountable cardinals, Annals Math. Logic 5, 165-198.
(1977) A game theoretic property of Boolean algebras, to appear in the proceedings of Wroclaw 1977 conference.

T.J.Jech, M.Magidor, W.Mitchell, and K.Prikry
(1977) On precipitous ideals, to appear.

T.J.Jech and K Prikry
(1977) Ideals over uncountable sets: applications of almost disjoint functions and generic ultrapowers, to appear.

R.B.Jensen
[1972] The fine structure of the constructible hierarchy, Annals Math. Logic 4, 229-308.
(1975) Coding the universe in a real, circulated notes.

R.B.Jensen and B.Koppelberg
(1977) Regularity of ultrafilters, to appear.

R.B.Jensen and K.Kunen
(1971) Some combinatorial properties of L and V, circulated notes.

R.B.Jensen and R.M.Solovay
[1970] Some applications of almost disjoint sets, in: Y.Bar-Hillel,ed., Mathe-Logic and Foundations of Set Theory (North Holland, Amsterdam) 84-104.

A.Kanamori
[1976] Weakly normal filters and irregular ultrafilters, Trans. Amer. Math. Soc. 220, 393-399.
[1977] Ultrafilters over a measurable cardinal, Annals Math. Logic 11, 315-356.
[1978] On Vopěnka's and related principles, to appear in the proceedings of Wroclaw 1977 conference.

C.Karp
[1964] Languages with Expressions of Infinite Length (North Holland, Amsterdam).

A.Kechris
[1973] Measure and category in effective descriptive set theory, Annnals Math. Logic 5, 337-384.
[1974] On projective ordinals, J. Sym. Logic 39(2), 269-282.
[1975] The theory of countable analytical sets, Trans Amer. Math. Soc. 202, 259-297.
(1977) AD and infinite exponent partition relations, manuscript.

H.J.Keisler
[1971] Model Theory for Infinitary Logic (North Holland, Amsterdam).

H.J.Keisler and A.Tarski
[1964] From accessible to inaccessible cardinals, Fund. Math. 53, 225-308.

J.Ketonen
[1972] Strong compactness and other cardinal sins, Annals Math. Logic 5, 47-76.
[1976] Non-regular ultrafilters and large cardinals, Trans. Amer. Math. Soc. 224, 61-73.

E.M.Kleinberg
[1970] Strong partition properties for infinite cardinals, J. Sym. Logic 35, 410-428.
[1973] Infinitary combinatorics, in: Cambridge Summer School in Mathematical Logic, Lecture Notes in Math. 337 (Springer, Berlin) 361-418.
[1973a] Rowbottom cardinals and Jónsson cardinals are almost the same, J. Sym. Logic 38(3) 423-427.
(1973) The equiconsistency of two large cardinal axioms, to appear in Fund. Math.
[1977] Infinitary Combinatorics and the Axiom of Determinateness, Lecture Notes in Math. 612 (Springer, Berlin).

D.W.Kueker
[1977] Countable approximations and Löwenheim-Skolem theorems, Annals Math. Logic 11(1), 57-103.

K.Kunen
[1968] Inaccessibility properties of cardinals, Stanford Ph.D.Thesis.
[1970] Some applications of iterated ultrapowers in set theory, Annals Math. Logic 1, 179-227.
[1970a] On the compactification of the integers, Notices Amer. Math. Soc. 17,299.
[1971] Elementary embeddings and infinitary combinatorics, J. Sym. Logic 36, 407-413.
[1971a] On the GCH at measurable cardinals, in: R.O.Gandy and C.E.M.Yates,eds., Logic Colloquium '69 (North Holland, Amsterdam) 107-110.
[1973] A model for the negation of the axiom of choice, in: Cambridge Summer School in Mathematical Logic, Lecture Notes in Math. 337 (Springer, Berlin) 489-494.
(1974) Saturated ideals, to appear in J. Sym. Logic.
[1976] Some points in βN, Proc. Cambridge Phil. Soc. 80, 385-398.

K.Kunen and J.B.Paris
[1971] Boolean extensions and measurable cardinals, Annals Math. Logic 2, 359-378.

G.Kurepa
[1935] Ensembles ordonnés et ramifiés, Publ. Math. Univ. Belgrade 4, 1-138.
[1936] L'hypothèse de ramification, Comptes Rendus, Hebdomadaires des Séances de l'Academie des Sciences,Séries A et B, vol.202, 185-7.

R.Laver
(1974) Making the supercompactness of κ indestructible under κ directed closed forcing, to appear.
(1976) A saturation property of ideals and the partition relation $\kappa \to (\kappa,\alpha)^2$.

A.Lévy
[1960] A generalization of Gödel's notion of constructibility, J. Sym. Logic 25, 147-155.
[1965] A hierarchy of formulas in set theory, Mem. Amer. Math. Soc. 57.
[1971] The sizes of the indescribable cardinals, in: D.S.Scott,ed., Axiomatic Set Theory, Proc. Symp. Pure Math. 13(1) (Amer. Math. Soc., Providence, R.I.) 205-218.

A.Lévy and R.M.Solovay
[1967] Measurable cardinals and the continuum hypothesis, Israel J. Math. 5, 234-248.

M.Magidor
[1971] On the role of supercompact and extendible cardinals in logic, Israel J. Math. 10, 147-157.
[1974] Combinatorial characterization of supercompact cardinals, Proc. Amer. Math. Soc. 42(1).
(1975) Changing cofinalities of cardinals, to appear.
[1976] How large is the first strongly compact cardinal?, Annals Math. Logic 10, 33-57.
[1977] On the singular cardinals problem I, Israel J. of Math. 28(1), 1-31.
[1977a] Chang's conjecture and powers of singular cardinals, J. Sym. Logic 42(2), 272-276.
(1977) On the existence of non-regular ultrafilters and cardinality of ultrapowers, to appear.
[1978] On the singular cardinals problem II, to appear in Ann. Math.

P.Mahlo
[1911] Über lineare transfinite Mengen, Ber. Verhandl. Sächs. Akad. Wiss. Leipzig, Math.-Naturw. Kl. 63, 187-225.
[1912] Zur Theorie und Andwendung der ρ_0-Zahlen, Ber. Verhandl. Sächs. Akad. Wiss. Leipzig, Math.-Naturw. Kl. 64, 108-112.
[1913] Zur Theorie und Andwendung der ρ_0-Zahlen II, Ber. Verhandl. Sächs. Akad. Wiss. Leipzig, Math.-Naturw. Kl. 64, 268-282.

R.Mansfield
[1970] Perfect subsets of definable sets of real numbers, Pacific J. Math. 35, 451-457.

D.A.Martin
[1970] Measurable cardinals and analytic games, Fund. Math. 66, 287-291.
(1971) Projective sets and cardinal numbers, to appear.
[1975] Borel Determinacy, Ann. Math. 102, 363-371.

D.A.Martin and R.M.Solovay
[1969] A basis theorem for Σ^1_3 sets of reals, Ann. Math. 89(2), 138-160.
[1970] Internal Cohen extensions, Annals Math. Logic 2, 143-178.

A.R.D.Mathias
[1977] Happy families, Annals Math. Logic 12(1), 59-111.

T.K.Menas
[1974] On strong compactness and supercompactness, Annals Math. Logic 7, 327-359.
[1975] A combinatorial property of $p_\kappa\lambda$, J. Sym. Logic 41(1), 225-233.
[1976] Consistency results concerning supercompactness, Trans. Amer. Math. Soc. 223, 61-91.

A.Miller
[1977] Berkeley Ph.D.Thesis.

W.Mitchell
[1972] Aronszajn trees and the independence of the transfer property, Annals Math. Logic 5, 21-46.
[1974] Sets constructible from sequences of ultrafilters, J. Sym. Logic 39, 57-66.

Y.N.Moschovakis
[1970] Determinacy and prewellorderings of the continuum, in: Y.Bar-Hillel,ed., Mathematical Logic and Foundations of Set Theory (North Holland, Amsterdam) 24-62.
[1973] Analytical definability in a playful universe, in: Suppes et al,eds., Proceedings of Third International Congress for Logic, Methodology and Philosophy of Science, Bucharest 1971 (North Holland, Amsterdam) 77-85.

J.Mycielski
[1964] On the axiom of determinateness, Fund. Math. 53, 205-224.

J.Mycielski and H.Steinhaus
[1962] On a mathematical axiom contradicting the axiom of choice, Bull. Acad. Polon. Sci. Sér. Sci. Math. Ast. Phys. 10, 67-71.

J.Mycielski and S.Swierczkowski
[1964] The Lebesgue measurability and the axiom of determinateness, Fund. Math. 54, 67-71.

J.Mycielski, S.Swierczkowski, and A.Zieba
[1956] On infinite positional games, Bull. Acad. Polon. Sci. Sér. Sci. Math. Ast. Phys. 4, 485-488.

J.Mycielski and A.Zieba
[1955] On infinite positional games, Bull. Acad. Polon. Sci. Sér. Sci. Math. Ast. Phys. 3, 133-136.

J.R.Myhill and D.S.Scott
[1971] Ordinal definability, in: D.S.Scott,ed., Axiomatic Set Theory, Proc. Symp. Pure Math. 13(1) (Amer. Math. Soc., Providence,R.I.) 271-278.

J.B.Paris
[1972] ZF $\vdash$ Σ^0_4 Determinateness, J.Sym. Logic 37, 661-667.

K.Prikry
[1970] Changing measurable into accessible cardinals, Dissertationes Mathematicae (Rozprawy Matematyczne) 68, 5-52.
[1971] On a problem of Gillman and Keisler, Annals Math. Logic 2, 179-187.
[1973] On descendingly complete ultrafilters, in: Cambridge Summer School in Mathematical Logic, Lecture Notes in Math. 337 (Springer, Berlin) 459-488.
(1974) Ideals and powers of cardinals, to appear.

W.N.Reinhardt
[1974] Remarks on reflection principles, large cardinals, and elementary embeddings, in: T.J.Jech,ed., Axiomatic Set Theory, Proc. Symp. Pure Math. 13(2) (Amer. Math. Soc., Providence,R.I.) 189-205.

G.E.Reyes
[1972] $L_{\omega_1\omega}$ is enough: a reduction theorem for some infinitary languages, J. Sym. Logic 37, 705-710.

W.Richter
[1971] Recursively Mahlo ordinals and inductive definitions, in: R.O.Gandy and C.E.M.Yates,eds., Logic Colloquium '69 (North Holland, Amsterdam), 273-288.

F.Rowbottom
[1964] Some strong axioms of infinity incompatible with the axiom of constructibility, Madison Ph.D.Thesis; published as: Annals Math. Logic 3(1971) 1-44.

G.E.Sacks
[1976] Countable admissible ordinals and hyperdegrees, Advances in Math. 20(2), 213-262.

D.S.Scott
[1961] Measurable cardinals and constructible sets, Bull. Acad. Polon. Sci. Sér. Math. Ast. Phys. 7, 145-149.

S.Shelah
[1975] A compactness theorem for singular cardinals, free algebras, Whitehead problem and transversals, Israel J. Math. 21(4), 319-349.
(1977) Jónsson groups, to appear in proceedings of Oxford 1977 conference (North Holland, Amsterdam).

J.R.Shoenfield
[1971] Unramified forcing, in: D.S.Scott,ed., Axiomatic Set Theory, Proc. Symp. Pure Math. 13(1) (Amer. Math. Soc., Providence,R.I.) 357-381.

W.Sierpinski and A.Tarski
[1930] Sur une propriété caractéristique des nombres inaccessibles, Fund. Math. 15, 292-300.

J.H.Silver

[1966] Some applications of model theory in set theory, Berkeley Ph.D.Thesis, published as: Annals Math. Logic 3(1970), 45-110.

[1970] A large cardinal in the constructible universe, Fund. Math. 69, 93-100.

[1971] The consistency of the GCH with the existence of a measurable cardinal, in: D.S.Scott,ed., Axiomatic Set Theory, Proc. Symp. Pure Math. 13(1) (Amer. Math. Soc., Providence,R.I.) 391-395.

[1971a] The independence of Kurepa's conjecture and two-cardinl conjectures in model theory, in: D.S.Scott,ed., Axiomatic Set Theory, Proc. Symp. Pure Math. 13(1) (Amer. Math. Soc., Providence,R.I.) 383-390.

[1971b] Measurable cardinals and Δ^1_3 well-orderings, Ann. Math. 94, 414-446.

[1974] Indecomposable ultrafilters and $0^{\#}$, in: L.Henkin et al,eds., Proceedings of the Tarski Symposium, Proc. Symp. Pure Math. 25 (Amer. Math. Soc., Providence,R.I.).

[1974a] On the singular cardinals problem, in: Proceedings of the International Congress of Mathematicians, Vancouver 1974, vol.1, 265-268.

E.C.Smith and A.Tarski

[1957] Higher degrees of distributivity and completeness in Boolean algebras, Trans. Amer. Math. Soc. 84, 230-257.

R.M.Solovay

[1967] A non-constructible Δ^1_3 set of integers, Trans. Amer. Math. Soc. 127, 58-75.

[1969] The cardinality of Σ^1_2 sets of reals, in: J.J.Bullof,T.C.Holyoke, and S.Wiltahn,eds., Foundations of Mathematics, Symposium papers commemorating the sixtieth birthday of Kurt Gödel (Springer, Berlin) 59-73.

[1970] A model of set theory in which every set of reals is Lebesgue measurable, Ann. Math. 92, 1-56.

[1971] Real-valued measurable cardinals, in: D.S.Scott,ed., Axiomatic Set Theory, Proc. Symp. Pure Math. 13(1) (Amer. Math. Soc., Providence,R.I.) 397-428.

[1974] Strongly compact cardinals and the G.C.H., in: L.Henkin et al,eds., Proceedings of the Tarski Symposium, Proc. Symp. Pure Math. 25 (Amer. Math. Soc., Providence,R.I.) 365-372.

R.M.Solovay, W.N.Reinhardt, and A.Kanamori

[1978] Strong axioms of infinity and elementary embeddings, Annals Math. Logic 13(1), 73-116.

R.M.Solovay and S.Tennenbaum

[1971] Iterated Cohen extensions and Souslin's problem, Ann. Math. 94, 201-245.

E.Specker

[1951] Sur un problème de Sikorski, Colloq. Math. 2, 9-12.

[1957] Zur Axiomatik der Mengenlehre (Fundierungs und Answahl Axiome), Zeitschrift f. math. Logic und Grundlagen der Math. 3, 173-210.

M.Spector

[1978] M.I.T. Ph.D.Thesis.

J.Stern

(1973) The second strongly compact cardinal, notes.

D.H.Stewart

[1966] Bristol M.Sc.Thesis.

A.Tarski

[1939] Ideale in vollständigen Mengenkörpen I, Fund. Math. 32, 45-63.

[1962] Some problems and results relevant to the foundations of set theory, in: E.Nagel, P.Suppes, and A.Tarski,eds., Logic, Methodology, and Philosophy of Science (Stanford, California) 125-135.

A.Taylor

(1977) Regularity properties of ideals and ultrafilters.

S.Ulam

[1930] Zur Masstheorie in der algemeinen Mengenlehre, Fund. Math. 16, 140-150.

[1964] Combinatorial analysis in infinite sets, SIAM Rev., 343-355.

P.Vopěnka and K.Hrbáček

[1966] On strongly measurable cardinals, Bull. Acad. Polon. Sci. Sér. Sci. Math. Ast. Phys. 14, 587-591.

S.Wagon

(1975) The structure of precipitous ideals, to appear.