Some Anisotropic Diffusions

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Abstract

Since the seminal works of Mumford-Shah [16] and Perona-Malik [18], variational and PDE-based methods have found a wide range of applications in Image Processing and have generated steady interest in the mathematical community. The latter is at least partly due to the difficulties one encounters in developing a mathematical theory encopassing equations which are often ill-posed albeit practically very effective. It is this dichotomy between the properties of the practical implementations and the properties of the underlying equations that makes many Imaging Processing models interesting to the mathematical community. A prime example of this is given by anisotropic diffusion as first proposed by Perona and Malik in [18] in order to overcome the limitations of more simpleminded averaging techniques in the context of image denoising. Here an overview is given on the (mainly mathematical) developments in anisotropic diffusions since then.

1 Anisotropic Diffusion and Image Processing

One of the basic important tasks of Image Processing is denoising. Most images are naturally polluted by noise due to intrinsic limitations of acquisition devices and/or the presence of random disturbances in the medium. For theoretical purposes it is often assumed that a gray-scale image u is the result of the superposition of a real informational content u and noise n (assumed to have mean zero and some variance σ)

$$u_0 = u + n\,.$$

The goal of denoising thus becomes to recover u given u_0. A first natural approach would be to perform some sort of local averages which would nullify the noise. The first denoising methods were indeed based on this simple idea (apparently going back to Gabor [15]) and were implemented via a combination

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of local Gaussian averaging and edge detection. Everyday images are characterized by the presence of many edges (sharp interfaces) and, while noise would efficiently be taken care of by such averaging techniques, edges would be blurred thus degrading the underlying image of interest. Edge detection was therefore necessary to undo the blurring introduced in the image by simpe averaging-based denoising. Once it was realized that this kind of averaging techniques amounted to solving a linear diffusive PDE, the door was open to a variety of new models based on nonlinear, and more importantly anisotropic, diffusions which could be thought of as having a built-in edge detector. The Perona-Malik model was one of the first and has come to typify anisotropic diffusions. While the method was proposed at the discrete level, in the literature it has been adopted, almost without exception, in its continuous form

$$\begin{cases} u_t - \operatorname{div}\big(a(|\nabla u|)\nabla u\big) = 0 & \text{in } \Omega \text{ for } t > 0\,, \\ \partial_\nu u = 0 & \text{on } \partial\Omega \text{ for } t > 0\,, \\ u(0,\cdot) = u_0 & \text{in } \Omega\,. \end{cases} \tag{1.1}$$

Hereby the domain $\Omega \subset \mathbb{R}^2$ denotes the image (typically a rectangle), ν the unit outer normal to its boundary, and $a(s)$ a nonlinear diffusion coefficient. Intuitively, the latter has to be chosen in such a way that diffusion be impeded at locations where large gradients are found. These are taken as an indication of the presence of edges. A typical choice is

$$a(s) = \frac{1}{1 + k^2 s^2}\,,\ s \geq 0\,, \tag{1.2}$$

and leads to an ill-posed equation (see [14, 13] for a proof of this in the one-dimensional context). It can be viewed as the gradient flow generated by the energy functional

$$E(u) = \frac{1}{2}\int_\Omega \log\big(1 + k^2|\nabla u|^2\big)\, dx\,,$$

where the ill-posedness is reflected in its convexity properties. Below the threshold value k, the energy functional is convex, while it becomes concave above it. The label anisotropic diffusion is due to the fact that the diffusivity can reverse its sign in normal direction across a hypersurface. This is best seen by computing the divergence in (1.1) and rewriting the Laplacian in local coordinates parallel and normal to level sets of u

$$u_t = \frac{1}{1 + k^2|\nabla u|^2}\Big\{\triangle_{\nabla u^\perp} u + \big[1 - 2\frac{k^2|\nabla u|^2}{1 + k^2|\nabla u|^2}\big]\partial^2_{\nabla u} u\Big\}\,.$$

The mathematical consequences of the ill-posedness cannot be neglected nor avoided since typical practical implementations of (1.1) do lead to remarkably good denoising and deblurring results. The latter are described as "perceptually impressive" in [5], whereas Kychenassamy [14] labels this apparent discrepancy between the properties of the model and those of its implementations as "the Perona-Malik paradox". In order to shed light on this paradox and gain better mathematical understanding of the equations many modifications of the original model (1.1) have been proposed over the years. They will be described in the following sections. It has to be pointed out, however, that a much wider variety of denoising methods have been proposed even within the category of anisotropic diffusion and will not be considered in this paper. The interested reader can find a more comprehensive account of existing methods in the overview article [4]. The rest of the paper is organized as follows. First an overview is given of a variety of modifications of (1.1) which improve on either its mathematical "problems" or on its practical shortcomings. Particular emphasis will be given to a recent modification proposed by the author which has the advantage of providing a mathematically well-posed model preserving the cherished properties of the original Perona-Malik model while overcoming some of its most significant limitations.

2 Regularizations

In this section modified Perona-Malik models will be briefly described that share the central idea of regularization. In the literature two approaches are found. In one, the unkown is (spacially) smoothed out in the nonlinear diffusivity whereas, in the second, it is either relaxed or delayed. One of the first results falling in the first category is due to Catté-Lions-Morel-Coll [5]. They convolute the unknown with a Gaussian inside the nonlinearity to obtain the modified evolution equation

$$u_t - \operatorname{div}\bigl(a(|\nabla u_\sigma|^2)\nabla u\bigr) = 0\,.$$

They can then prove existence and uniqueness of a smooth solution

$$u \in \mathrm{C}\bigl([0,T],\mathrm{L}_2(\Omega)\bigr) \cap \mathrm{L}_2\bigl([0,T],\mathrm{H}^1(\Omega)\bigr)$$

for any $u_0 \in \mathrm{L}_2(\Omega)$. The regularization has the mathematical benefit of making the equation well-posed but the practical inconvenience of introducing blurring. It does therefore sacrifice one of the hallmarks of the original Perona-Malik equation. The optimal choice of the regularization parameter is noise dependent and therefore requires a prori knowledge of the noise level. It has been observed in [22] that the variational problem associated to the Perona-Malik equation has an infinite number of minimizers (piecewise constant images) in some non rigourous sense. In [7] the authors consider the regularized equation

$$\begin{cases} u_t - \sigma\varepsilon^2 \operatorname{div}\bigl(A_\varepsilon(u)\nabla u\bigr) = 0 & \text{in } \Omega \text{ for } t>0\,,\\ u = 0 & \text{on } \partial\Omega \text{ for } t>0\,,\\ u(0,\cdot) = u_0 & \text{in } \Omega\,,\end{cases}$$

and are able to prove that characterstic functions of sets with a sufficient degree of regularity are almost stationary solutions, that is, that they satisfy the steady-state equation up to some order in ε. This was a first attempt to give some mathematical rigor to the intuitive observation later formulated in [22]. They introduce a double regularization through

$$A_\varepsilon(u) = \frac{1}{\varepsilon^2 + |\nabla u_\varepsilon|^2} \begin{bmatrix} |\partial_2 u_\varepsilon|^2 & -\partial_1 u_\varepsilon \partial_2 u_\varepsilon \\ -\partial_1 u_\varepsilon \partial_2 u_\varepsilon & |\partial_1 u_\varepsilon|^2 \end{bmatrix}$$

where u_ε is a regularized version of u obtained by convolution with a smooth kernel.
A decade later Weickert [20, 21] reconsiders the use of truly anisotropic diffusion modifying the equation to

$$u_t - \operatorname{div}\bigl(D(\nabla u_\sigma)\nabla u\bigr) = 0$$

where the unknown is again regularized by a Gaussian kernel and D is a diffusion tensor. He similarly obtains the existence of a unique smooth solution

$$u \in \mathrm{C}\bigl([0,T], \mathrm{L}_2(\Omega)\bigr) \cap \mathrm{H}^1\bigl((0,T), \mathrm{H}^1(\Omega)\bigr)\,,$$

for any $u_0 \in \mathrm{L}_\infty(\Omega)$. The use of a diffusion tensor mitigates somewhat for the blurring introduced by the regularization.
Starting with the non-mathematical contribution of Nitzberg and Shiota [17], relaxation models have also been considered. In [6], the authors consider the relaxed and regularized model

$$\begin{cases} u_t - \operatorname{div}\bigl(D\nabla u\bigr) = 0 \\ \tau D_t + D = F(\nabla u_\sigma) \end{cases}$$

which combines the ideas from [17] and [20]. They obtain existence of a unique solution satisfying

$$\begin{aligned} u &\in \mathrm{L}_2\bigl([0,T], \mathrm{H}^1(\Omega)\bigr) \cap \mathrm{L}_\infty\bigl([0,T], \mathrm{L}_\infty(\Omega)\bigr) \\ D &\in \mathrm{L}_\infty\bigl([0,T], [\mathrm{L}_\infty(\Omega) \cap \mathrm{H}^1(\Omega)]^4\bigr)\,, \end{aligned}$$

for any $u_0 \in \mathrm{L}_\infty(\Omega)$ and $D \in [\mathrm{L}_\infty(\Omega) \cap \mathrm{H}^1(\Omega)]^4$. The main reason for considering this model is to allow for enough equilibria for an initial noisy image to settle down on and thus to avoid the need of introducing a stopping time with the associated problem of its determination. The role of the stopping is played by the relaxation time parameter τ. By making an appropriate choice of F they are able to devise a method which allows for diffusion in "flat" regions in between edges. It should be observed that the dynamic properties of the equation are observed in the limiting case $\sigma = 0$ whereas the mathematical results are derived on the assumption that $\sigma > 0$ (its size is, however, noise independent).
In Belahmidi and in Belahmidi and Chambolle [2, 3] the relaxed model with scalar diffusivity

$$\begin{cases} u_t - \operatorname{div}\bigl(a(v)\nabla u\bigr) = 0 \\ v_t + v = F(|\nabla u|^2) \end{cases}$$

is considered. The existence of unique smooth solutions is established for smooth initial data with the possibility of blowup. On stronger assumptions they prove existence of a global weak solution

$$(u,v) \in \big[\mathrm{H}^1((0,T)\times\Omega)\cap \mathrm{L}_\infty((0,T)\times\Omega)\big]^2 .$$

via an approximating sequence obtained by time discretization

$$\begin{cases} u^{n+1}-u^n = \delta t \operatorname{div}\big(a(v^n)\nabla u^{n+1}\big)\,,\\ v^{n+1}-v^n = \delta t[F(|\nabla u^{n+1}|^2)-v^{n+1}]\,. \end{cases}$$

This model is closer to the original Perona-Malik model and follows more closely the proposal of [17].
Finally a time delayed Perona-Malik equation is considered in [1]. The equation reads

$$\begin{cases} u_t - \nabla\cdot\big(a(\theta * |\nabla u|^2)\nabla u\big) = 0\,,\\ \theta * |\nabla u|^2(t) = \frac{1}{S}\int_{t-S}^{t} |\nabla u(\tau)|^2\,d\tau\,. \end{cases}$$

Local well-posedness of a unique solution

$$u \in \mathrm{L}_p\big([0,T],\mathrm{H}^2_{q,\partial_\nu}(\Omega)\big)\cap \mathrm{H}^1_p\big([0,T),\mathrm{L}_q(\Omega)\big)\,.$$

is established for $u_0 \in \mathrm{H}^2_{q,\partial_\nu}(\Omega) := \big\{ f\in \mathrm{H}^2_q(\Omega)\,\big|\,\partial_\nu f = 0\big\}$ for q large enough. Time-delayed regularizations have the advantage of reflecting a feature common to discrete implementations of Perona-Malik, that is, that the nonlinearity acts on the solution evaluated at the previous time step

$$u^{n+1}-u^n = \delta t \operatorname{div}\big(a(|\nabla u^n|^2)\nabla u^{n+1}\big)$$

in a semi-implicit scheme just as in the above system by Belahmidi and Chambolle. This introduces another layer of regularization beyond that due to the discretization itself of the continuous equation. The author argues that this time-regularization model is affected by blurring to a lesser extent.
All these regularized/relaxed models deliver mathematically tractable equations but, either through the introduction of blurring or through the modification of the dynamical properties of the equation, significantly deviate from the original Perona-Malik model. In Section 3 a novel spatial regularization recently proposed by the author is introduced which is mathematically well-posed and simultanously capable of preserving the widely accepted and observed if not rigorously proven dynamical properties of Perona-Malik (as reported in [22, 14]).

3 Nonlocal Diffusions

In a recent series of papers the author has proposed [12, 11, 10] a novel paradigm for the regularization of the Perona-Malik equation. In contrast to the more standard regularization methods described in the previous section, it only partially smoothes the unkwown in the nonlinearity. This is achieved by the use of fractional derivatives. The equations read

$$\begin{cases} u_t - \operatorname{div}\big(\frac{1}{1+|\nabla^{1-\varepsilon}u|^2}\nabla u\big) = 0 & \text{in } \Omega \text{ for } t > 0\,,\\ u & \text{periodic}\,,\\ u(0,\cdot) = u_0 & \text{in } \Omega\,, \end{cases} \tag{3.1}$$

and

$$\begin{cases} u_t - \frac{1}{1+|(-\triangle)^{1-\varepsilon}u|^2}\triangle u = 0 & \text{in } \Omega \text{ for } t > 0\,,\\ \partial_\nu u = 0 & \text{on } \partial\Omega\,,\\ u(0,\cdot) = u_0 & \text{in } \Omega\,, \end{cases} \tag{3.2}$$

respectively. The amount of regularization can clearly be tuned by means of the parameter $\varepsilon \in (0, 1]$. With no regularization $\varepsilon = 0$ the original Perona-Malik equation is recovered in (3.1), whereas in (3.2) u_x satsifies the Perona-Malik equation in the one-dimensional case if homogeneous Dirichlet conditions are imposed on u. The equations have been proved locally well-posed with weak solutions satisfying

$$u \in \mathrm{L}_p\big([0,T], \mathrm{H}^1_{p,\pi}(\Omega)\big) \cap \mathrm{H}^1_p\big([0,T), \mathrm{H}^{-1}_{p,\pi}(\Omega)\big)$$

and solutions with

$$u \in \mathrm{L}_p\big([0,T], \mathrm{W}^2_{p,\partial_\nu}(\Omega)\big) \cap \mathrm{H}^1_p\big([0,T), \mathrm{L}_p(\Omega)\big)$$

for initial data chosen from $\mathrm{H}^{1-2/p}_{p,\pi}(\Omega)$ and $\mathrm{W}^{2-2/p}_{p,\partial_\nu}(\Omega)$, respectively and for large enough $p > 1$ (depending on the size of $\varepsilon > 0$) in [10, 11]. The subscripts π and ∂_ν are used to indicate spaces of functions satisfying the corresponding boundary condition. The proofs are based on L_p-maximal regularity and exploit the quasilinear nature of the equation if $\varepsilon > 0$.
The new equations are not of variational type anymore but still possess natural Lyapunov functions as follows from

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2\,dx = -\int_\Omega \frac{|\nabla u|^2}{1+|\nabla^{1-\varepsilon}u|^2}\,dx \tag{3.3}$$

and

$$\frac{1}{2}\frac{d}{dt}\int |\nabla u|^2\,dx = -\int_\Omega \frac{|\triangle u|^2}{1+|(-\triangle)^{1-\varepsilon}u|^2}\,dx \tag{3.4}$$

The solutions remain bounded and satisfy

$$\min_\Omega u_0 \le u \le \max_\Omega u_0$$

which is a desirable property for applications to Image Processing where the values of u are confined to the interval $[0, 255]$ for gray-scale images. Global existence can be obtained for $\varepsilon = 0$ and equation (3.1) and for $\varepsilon > 3/4$ and equation

(3.2). This shows that the parameter ε interpolates between the Perona-Malik behavior and pure diffusive behavior. In fact, if a solution exists globally the Lyapunov function will drive it to a trivial steady-state corresponding the average of its initial datum.

The regularization used in (3.1) and (3.2) is much milder than traditional ones. One of the most significant consequences of this fact is that the dynamical behavior of the original Perona-Malik is preserved. Characteristic functions of smooth sets and piecewise linear continuous functions in one dimension are in fact stationary solutions for (3.1) and (3.2), respectively. This has been proved in [11, 10]. The extensive numerical experiments performed in [12] suggest that they are metastable. They thus introduce a time scale over which an intial datum seems to converge to such a stationary solution and, only over a longer time scale, does it converge to a trivial average steady-state. The new equations, in particular (3.1), therefore also prove very effective denoising tools for Image Processing as confirmed in the already mentioned numerical experiments and by the figures in this papers.

It follows that the behavior of solutions is fully explained and determined by the presence of non-trivial steady states and not by the backward nature of the original Perona-Malik equation since, for $\varepsilon > 0$, the equations are of pure forward diffusion type. That reaction-diffusion equations can exhibit highly non-trivial transient behavior has been long known for systems [19] and for other nonlocal diffusions [9]. In these cases, however, the presence of a reaction term plays a crucial role by creating steady-states which compete with diffusion in the determination of the long term fate of solutions. In (3.1) and (3.2) no reaction term is present and the equations represent a novel mechanism engendering nontrivial dynamics in the context of a purely diffusive (and thus well-posed) equation.

4 Open Problems

In spite of the satisfying mathematical properties they enjoy, model equations (3.1) and (3.2) still give rise to interesting mathematical questions. It was pointed out in the previous section that there is a transition from non-trivial behavior to trivial diffusive behavior as ε grows to unity. It is, however, not clear at which critical value the transition occurs. This question seems to be intimately related to that of global existence of smooth solution to smooth initial data. In fact smooth solutions will eventually converge to a trivial steady-state, whereas singularity formation might allow a solution to settle down on a trivial steady-state. The question of global existence, in its turn, is closely related to regularity issues. Let us take (3.2) as an example. Roughly speaking global existence can be established if Hölder regularity of the solution can be proved. As the equation is degenerate, classical estimates of De Giorgi-Nash-Moser type do not apply, nor does the theory developed in [8] for degenerate/singular equations. Latter only applies if the nonlinearity allows for p-Laplacian type estimates. Numerical experiments seem to indicate that the threshold for (3.2) is

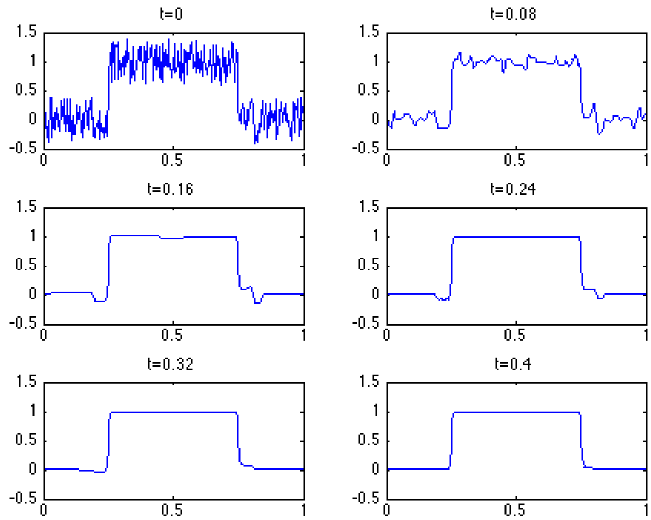

Figure 1: Denoising experiment where noise is superimposed on a simple characteristic function.

found at $\varepsilon = 0.5$. Similarly for equation (3.1). For $\varepsilon = 0.5$, the nonlinearity is in a regime where, at least for local problems, the theory of viscosity solutions should apply and deliver a kind of global solution.

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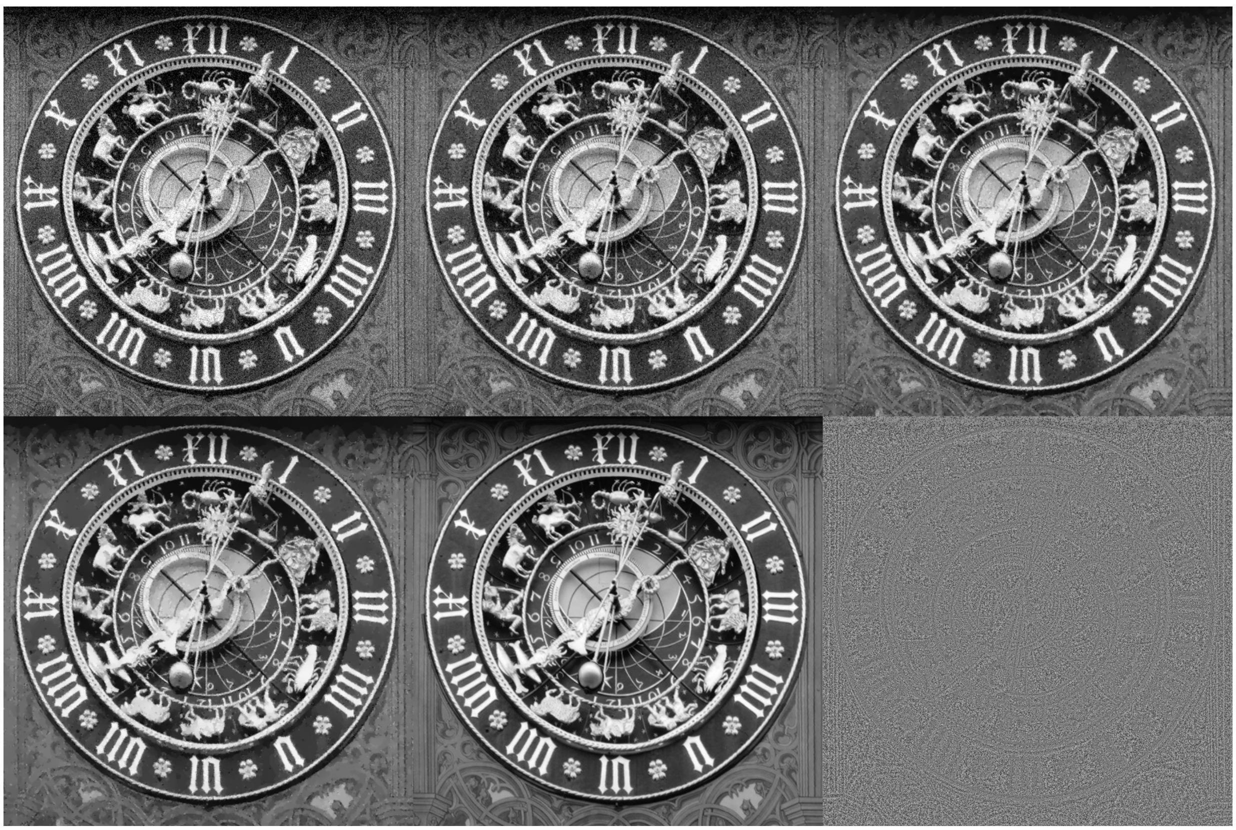

Figure 2: Denoising experiment on a two-dimensional image.

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