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Alexander O. Gelfond

by

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On 7th November 1968 Alexander Osipovič Gelfond passed away.

Born on 24 October 1906 in Petersburg (Leningrad) the son of a physician, he began his studies at Moscow University in 1924. He finished his undergraduate studies in 1927 and the postgraduate studies in 1930. From that time on until the last day of his life he was teaching as professor at Moscow University, holding the Chair of Number Theory. From 1933 on he worked simultaneously at the Steklov Mathematics Institute of the Academy of Sciences of USSR. In 1939 he was elected corresponding member of the same Academy.

His mathematical bequest consists of mathematical papers belonging to several mathematical domains: number theory, theory of functions, differential and integral equations, history of mathematics and others. The most widely known are his works on the theory of transcendental numbers and the theory of analytic functions. These two branches of mathematics were closely related in his investigations. The profound papers, concerning the problem of interpolation of entire functions and establishing the relationship between their growth and the arithmetical properties of their values at algebraic points, created the foundation on which he based his new analytic methods in the theory of transcendental numbers. These methods allowed him to obtain his fundamental results.

It is impossible to give in a short article a detailed report on the whole work of A. O. Gelfond. Therefore in this article our attention will be limited to his more important papers in the theory of transcendental numbers and a short description of the ideas on which his methods are based will be given.

Among the not too numerous methods of the theory of transcendental numbers those of A. O. Gelfond belong to the most important In the last 40 years they were developed and applied as well by himself as by other mathematicians.

At the end of the twenties of our century the theory of transcendental numbers was still very poor. Few facts were definitely settled and very few methods of investigations were in existence. The farther developement of the theory was rendered possible through the introduction by Gelfond (1929–1934) and C. L. Siegel (1929) of new, strong, analytic methods which founded the modern basis of the theory.

The first proof of existence of transcendental numbers we owe to Liouville (1844). He showed that there do not exist "too" good approximations of algebraic numbers by rational fractions. Thus he got a necessary condition for a number to be algebraic and he constructed the first examples of transcendental numbers.

Liouville's theorem supplied a tool for proving the transcendence of a number given by very quickly converging series or by products of rational terms or at least by continued fractions with very quickly increasing partial quotients. However Liouville's method failed to clarify for instance the arithmetical nature of e or π.

The transcendence of e was proved by Ch. Hermite in 1873. Shortly thereafter Lindemann gave in 1882 a proof of transcendence of π, e^{a} and $\ln\beta$ for algebraic $a \neq 0, \beta \neq 0$ and 1.

As early as in the XVIII century Euler took an interest in arithmetical properties of logarithms. In 1748 he proved that if a and b are rational, $\log_a b$ is either rational or transcendental. This theorem of Euler has been put by Hilbert in 1900 in a generalized form as one of his famous 23 problems at the Mathematical Congress in Paris. His seventh problem is: Let a and β be algebraic; is a^{β} transcendental (the trivial exceptions excluding)? Especially, are the numbers $e^{\pi} = i^{-2i}$ and $2^{\sqrt{2}}$ transcendental?

Hilbert added that according to his conviction the solution of the problem is extremely difficult and it will succeed only by a new method.

Until 1929 there were no results connected with this problem. Obviously, the reason for it was the following: In the Hermite–Lindemann method it was decisive that e^{a} is the value for an algebraic argument of the function e^{z} satisfying a simple linear differential equation $y' = y$ with algebraic coefficients, whereas a^{β} is the value of e^{z} for a transcendental argument $\beta\ln a$ or it is a value in an algebraic point of the function a^{z} satisfying a linear differential equation $y' = y\ln a$ with transcendental coefficients.

Thus a new idea for the solution of the problem was needed. That exactly was found by Gelfond in 1929. The idea was related to the problem of growth of an entire function which assumes integer values for integer arguments.

In 1914 G. Polyá proved that if an entire function $f(z)$ assumes integer rational values for all positive integer values of z and satisfies the inequality

$$|f(z)| < c2^{a|z|}, \quad a < 1, \ c > 0\text{—constants},$$

then $f(z)$ is a polynomial. The proof was based on an analysis of properties of coefficients of the development of $f(z)$ into Newton's interpolation series with nodes at the points $1, 2, 3, \ldots$ Gelfond [3] in 1929 proved that if an entire function for integer arguments of the field $Q(i)$ assumes integer values belonging to $Q(i)$ and if

$$|f(z)| \leqslant e^{\gamma|z|^2}, \quad \gamma < \frac{\pi}{2}(1+e^{\frac{164}{\pi}})^{-2},$$

then $f(z)$ is a polynomial. In his solution of this problem he used Newton series with nodes in integer points of $Q(i)$. Towards the end of the same year, in addition, he stated that the same series can be used for a partial solution of Hilbert's seventh problem. He proved namely the

THEOREM. *If $a \neq 0$ and 1, $\beta = i\sqrt{b}$, b — positive rational number, then a^{β} is transcendental.*

We shall illustrate the idea of the proof by the example $e^{\pi} = i^{-2i}$.

Let all integers $x+iy$ of the Gauss field $Q(i)$ be ordered according to increasing modulus. Then for all z

$$e^{\pi z} = A_0 + \sum_{n=1}^{\infty} A_n(z-z_0)\ldots(z-z_{n-1}), \tag{1}$$

$$A_n = \frac{1}{2\pi i}\oint_{C_n} \frac{e^{\pi\zeta}}{(\zeta-z_0)\ldots(\zeta-z_n)}\,d\zeta \tag{2}$$

$$= \sum_{k=0}^{n} \frac{e^{\pi z_k}}{(z_k-z_0)\ldots(z_k-z_{k-1})(z_k-z_{k+1})\ldots(z_k-z_n)}$$

$$= \sum_{k=0}^{n} e^{\pi z_k} t_{nk}^{-1},$$

where C_n is the circle $|\zeta| = R_n$, $R_n > |z_n|$. Let Ω_n be the least common multiple of the numbers t_{nk}, where $1 \leqslant k \leqslant n$. A calculation of the exponents of the prime divisors of t_{nk} and Ω_n makes it possible to obtain the inequalities

$$|\Omega_n| \leqslant e^{n\ln n/2+\gamma_1 n}, \tag{3}$$

$$|\Omega_n t_{nk}^{-1}| \leqslant e^{\gamma_2 n}. \tag{4}$$

Here and in the sequel the numbers $\gamma_1, \gamma_2, \ldots$ are positive constants.

Let $z_k = x_k + iy_k$; then the rational integer x_k satisfies the equality

$$x_k = O(\sqrt{k}).$$

Therefore by (2) and

$$e^{\pi i y_k} = \pm 1$$

there follows

$$A_n \Omega_n = \sum_{s=0}^{s_0} D_s e^{\pi s},$$

where $s_0 = O(\sqrt{n})$ and D_s are integers of the field $Q(i)$; by (4)

$$|D_s| < e^{\gamma_3 n}.$$

It is known that if $\zeta = P(a)$, where a is algebraic and $P(z)$ is a polynomial whose coefficients are integers of $Q(i)$, and if $P(z)$ is not the zero polynomial, then there can be given a lower bound for $|\zeta|$. It depends on a, on the degree and on the height of $P(z)$. If we assume e^π to be algebraic, such a lower bound can be given also for $A_n \Omega_n$. Thus one sees that either

$$A_n \Omega_n = 0$$

or

$$|A_n \Omega_n| > e^{-\gamma_4 n}. \tag{5}$$

On the other hand by the integral representation (2) and by the inequality (3) one gets the estimation

$$|A_n \Omega_n| < e^{-n \ln n/2 + \gamma_5 n}. \tag{6}$$

For sufficiently large n the inequalities (5) and (6) are not consistent; therefore $A_n = 0$ for $n > N$. By (1) we obtain hence that $e^{\pi z}$ is a polynomial. This contradiction completes the proof.

The argument given above may be almost literally repeated when instead of e^π the number $\alpha^{i\sqrt{b}}$ is considered. One needs only to use the equations

$$\alpha^z = A_0 + \sum_{n=1}^{\infty} A_n (z - z_0)(z - z_1) \ldots (z - z_{n-1}), \tag{7}$$

$$A_n = \sum_{k=0}^{n} \frac{\alpha^{z_k}}{(z_k - z_0) \ldots (z_k - z_{k-1})(z_k - z_{k+1}) \ldots (z_k - z_n)} = \sum_{k=0}^{n} \alpha^{z_k} t_{nk}^{-1}. \tag{8}$$

Here the numbers $x + iy\sqrt{b}$, x and y — rational integers, are taken as nodes.

In 1930 R. O. Kuzmin proved that this method may be used also in the case of β being a real quadratic rational. Thus the transcendence of $2^{\sqrt{2}}$ was proved.

Farther attempts to apply the method presented above to the solution of the full Euler–Hilbert problem failed. If $\nu > 2$ is the degree of β, then good estimations for Ω_n have been obtained only in the case when the numbers

$$x_0 + x_1\beta + \ldots + x_{\nu-1}\beta^{\nu-1} \quad (x_0, x_1, \ldots, x_{\nu-1} \text{ — rational integers})$$

were taken as nodes: However, then the coefficients A_n were polynomials with algebraic coefficients of $\nu - 1$ numbers

$$\alpha^\beta, \ \alpha^{\beta^2}, \ \ldots, \ \alpha^{\beta^{\nu-1}}. \tag{i}$$

To get a lower bound for $|A_n|$ it had to be assumed that all these numbers were algebraic. In this way it could be proved only that among numbers (i) one at least is transcendental.

However, this first method of Gelfond created the possibility of establishing many other results, e.g. those of K. Bole, C. L. Siegel, J. Koksma and J. Popken, A. V. Lotocki. Gelfond himself [9] applied the method to a new proof of Lindemann's theorem.

In 1929 C. L. Siegel published a paper in which, taking as an example the examination of arithmetical properties of Bessel functions, he demonstrated a new method in the theory of transcendental numbers. That method, improved afterwards by A. B. Šidlovski, made it possible to prove in many cases the transcendence and algebraic independence in algebraic points of the so called E-functions(1). This method, however, could not be applied to Euler–Hilbert problem because α^z for algebraic α does not belong to the class of E-functions.

Thus, exactly as before, for a full solution of Euler–Hilbert problem at that time a new method was needed. That was given by A. O. Gelfond in 1934. He proved [17], [18] the

THEOREM. *If $\alpha \neq 0$ and 1, $\beta \neq 0$ and 1, are algebraic numbers then $\ln\alpha/\ln\beta$ is either rational or transcendental.*

COROLLARY 1. *If $\alpha \neq 0$ and 1 is algebraic and β is an algebraic irrationality then α^β is transcendental.*

COROLLARY 2. *If α and β are algebraic then $\log_\alpha\beta$ is either rational or transcendental.*

The main features of the new method (which we shall call the second method of Gelfond) will now be demonstrated by presenting the main steps of the proof of the enunciated theorem. Let

$$f(z) = \sum_{k=0}^{q} \sum_{l=0}^{q} C_{kl} \alpha^{kz} \beta^{lz}. \tag{9}$$

(1) Entire functions with algebraic derivatives at the point $z = 0$ which have certain arithmetical properties.

By Dirichlet's box-principle there is proved the existence of integer rational C_{kl} (not all equal to 0) such that the following inequalities

$$(10)\quad |f^{(s)}(t)|\leqslant e^{-\gamma_6 q^2\ln q\cdot\ln^{-1}\ln q},\quad s=0,1,\ldots,r_1-1,\ t=0,1,\ldots,r_2-1,$$

$$(11)\quad r_1=[q^2\ln^{-1}\ln q\cdot\ln^{-1}q],\quad r_2=[\ln^2\ln q],$$

$$(12)\quad |C_{kl}|\leqslant 3^{q^2}$$

hold. Obviously

$$(13)\quad f^{(s)}(t)=\sum_{k=0}^{q}\sum_{l=0}^{q}C_{kl}(k\ln\alpha+l\ln\beta)^s\alpha^{kt}\beta^{lt}$$
$$=\ln^s\beta\sum_{k=0}^{q}\sum_{l=0}^{q}C_{kl}(k\eta+l)\alpha^{kt}\beta^{lt},$$

where

$$\eta=\ln\alpha/\ln\beta.$$

Therefore the numbers $f^{(s)}(t)\ln^{-s}\beta$ are polynomials in η with algebraic coefficients. If we assume that η is algebraic, from (11) and (12) it follows that either $f^{(s)}(t)=0$ or

$$(14)\quad |f^{(s)}(t)|>e^{-\gamma_7 q^2-\gamma_8 qt-\gamma_9 s\ln q}.$$

Using (11) we conclude that for $s<r_1$, $t<r_2$ and $q\geqslant q_1$ the inequalities (10) and (14) are not consistent. Therefore

$$(15)\quad f^{(s)}(t)=0,\quad s=0,1,\ldots,r_1-1,\ t=0,1,\ldots,r_2-1.$$

Thus there are "many" zeros of $f(z)$ in a "not too big" vicinity of $z=0$. By confronting it with the estimation of the modulus of $f(z)$ Gelfond proves that $f(z)$ is "small" in an "fairly big" circle. Indeed, by equations (15)

$$f(z)\prod_{t=0}^{r_2-1}(z-t)^{-r_1}$$

is an entire function. Using the maximum principle and Cauchy's integral formula for the derivatives of an analytic function he proves the inequalities

$$(16)\quad |f^{(s)}(t)|<e^{-\gamma_{10}q^2\ln\ln q}$$

for $s=0,1,\ldots,r_1-1$, $t=0,1,\ldots,[\sqrt{q}]$.

Now, from inequalities (14) and (16) it follows that for $q\geqslant q_2$

$$(17)\quad f^{(s)}(t)=0,\quad s=0,1,\ldots,r_1-1,\ t=0,1,\ldots,[\sqrt{q}].$$

These equations supply the means for proving that for $z=0$ the function $f(z)$ has a zero of a "high" multiplicity. Indeed, applying the maximum principle to the entire function

$$f(z)\{z(z-1)\ldots(z-[\sqrt{q}])\}^{-r_1}$$

and using Cauchy's formula we get the inequality

$$|f^{(s)}(0)|\leqslant e^{-q^{7/3}},\quad s=0,1,\ldots,(q+1)^2-1,\ q\geqslant q_3.$$

Comparing it with inequalities (14) one gets

$$(18)\quad f^{(s)}(0)=\sum_{k=0}^{q}\sum_{l=0}^{q}C_{kl}(k\ln\alpha+l\ln\beta)^s=0,$$
$$s=0,1,\ldots,(q+1)^2-1,\ q\geqslant q_4.$$

This is a linear homogeneous system of equations. Its determinant is not 0 since it is Vandermonde's determinant formed by powers of different numbers $k\ln\alpha+l\ln\beta$ ($\ln\alpha/\ln\beta$ — irrational!). Therefore for all k and l

$$C_{kl}=0,$$

contrarily to the choice of these numbers.

It should be mentioned that a short time afterwards an independant proof of the same theorem was given by Th. Schneider.

The method presented above allows to obtain also an estimation of the measure of transcendence of the numbers $\zeta_1=\alpha^\beta$ and $\zeta_2=\ln\alpha/\ln\beta$; this means inequalities of the type

$$(19)\quad |P(\zeta)|\geqslant\varphi(\zeta;n,H),$$

where $P(z)\neq 0$ is a polynomial with rational integer coefficients, the height and degree of which are not greater than H and n, respectively.

It is known that inequalities of type (19) are closely related to inequalities of type

$$|\zeta-\theta|\geqslant\psi(\zeta;n,H),$$

where θ is an algebraic number the degree and height of which are not greater than n and H, respectively.

Gelfond came back several times to estimations of this type. In 1935 he proved [23] the inequalities

$$(20)\quad \begin{aligned}&|\alpha^\beta-\theta|>H^{-(\ln\ln H)^{5+\varepsilon}},\quad H\geqslant H_1(\alpha,\beta,\varepsilon,n),\\&\left|\frac{\ln\alpha}{\ln\beta}-\theta\right|>e^{-\ln^{5+\varepsilon}H},\quad H\geqslant H_2(\alpha,\beta,\varepsilon,n).\end{aligned}$$

In 1939 he proved [33] that

$$(21)\quad \left|\frac{\ln\alpha}{\ln\beta}-\theta\right|>e^{-\ln^{3+\varepsilon}H},\quad H\geqslant H_3(\alpha,\beta,\varepsilon,n),$$

and in 1949 he established [56] still sharper estimates

$$|\alpha^\beta-\theta|>\exp\left(-\frac{n^3(n+\ln H)}{1+\ln^3 n}\ln^{2+\varepsilon}(n+\ln H)\right),\quad n+\ln H\geqslant H_4(\alpha,\beta,\varepsilon),$$

(22)

$$\left|\frac{\ln\alpha}{\ln\beta}-\theta\right|>\exp\left(-n^2(n+\ln H)^{2+\varepsilon}\right),\quad n+\ln H\geqslant H_5(\alpha,\beta,\varepsilon).$$

By the second method of Gelfond several other important results of the theory of transcendental numbers have been established. For instance G. Ricci, F. Franklin, M. L. Platonov used it with success.

A. O. Gelfond put stress on the importance of effective estimations from below of the moduli of linear forms with algebraic — even only rational – coefficients depending on m logarithms of algebraic numbers. For $m=2$ such estimations result from (20), (21) and (22). For arbitrary m however such estimations have been established for the first time by A. Baker in 1966. Baker proved several inequalities of the type

$$|\beta_1\ln\alpha_1+\ldots+\beta_m\ln\alpha_m|>Ce^{-\ln^k H} \tag{23}$$

where H is equal to the maximum of heights of $\beta_1,\ldots,\beta_m$, $k>m+1$ and C is a constant. For establishing these inequalities Baker used the second method of Gelfond supplemented by some very ingeneous reasoning.

An inequality of type (23) permitted Baker to get an effective estimation stronger than that of Liouville mentioned above and to give an effective form to Thue's theorem on finiteness of the set of solutions of the equation $f(x,y)=N$, where N and the coefficients of the irreducible form f are rational integers and the degree of f is $\geqslant 3$.

The inequalities (20)–(22) of Gelfond were basic for proofs of several number-theoretic theorems. It will be sufficient to mention the papers by B. I. Segal, Yu. V. Linnik, N. G. Čudakov, A. Schinzel, J. W. S. Cassels and G. B. Babaev.

In 1969 N. G. Čudakov showed that these inequalities make it possible to obtain a bound for discriminants of imaginary quadratic fields with one class of ideals. Using inequality (22) Baker got a partial solution of analogous problem in the case of fields with two classes of ideals.

K. Mahler generalized the second Gelfond method to p-adic fields; he proved the p-adic analogue of the theorem on the transcendence of α^β. Gelfond himself [35] obtained the p-adic analogue of inequality (21) in 1940. He proved that, except the trivial cases, the congruence

$$\alpha^n\pm\beta^m\equiv 0\,(\mathrm{mod}\,p^t),\quad t=[\ln^{3+\varepsilon}q],\quad q=\max(|n|,|m|) \tag{24}$$

has for any $\varepsilon>0$ a finite number of solutions in integral rational n and m.

Using that theorem and the inequality (21) Gelfond proved that the equation

$$\alpha^x+\beta^y=\gamma^z,$$

where α,β and γ are real algebraic numbers different from 0 and ±1 and such that at least one of them is not an algebraic unit, has only a finite number of integer rational solutions x,y,z, except in case

$$\alpha=\pm2^{n_1},\quad \beta=\pm2^{n_2},\quad \gamma=\pm2^{n_3},$$

where n_1, n_2 and n_3 are rational.

Effective bounds for possible solutions of inequality (21) and the congruence (24) were given by A. Schinzel in 1967. As a consequence he got several interesting number-theoretic theorems.

In 1949 Gelfond strengthened his second method. This rendered possible to establish several assertions on algebraic independence of numbers. As an important tool he used the following

LEMMA. *Let $a_0>1$, let $\sigma(x)>0$ and $\theta(x)>0$ be monotonically increasing for $x\geqslant x_0$ and $a_0\sigma(x)\geqslant\sigma(x+1)$. If for fixed α and each integer $N\geqslant N_0$ there exists a polynomial $P(x)\not\equiv0$, with integral coefficients, of height H and degree n for which the inequalities*

$$|P(\alpha)|<e^{-\sigma^2(N)\theta(n)},\quad \max(N,\ln H)\leqslant\sigma(N),$$

hold, then α is algebraic.

Notice that the hypothesis of the lemma may be weakened; instead of $\theta(x)$ a sufficiently large constant may be taken. This was indicated by Gelfond himself.

The main features of the third method of Gelfond we shall illustrate by the proof of the following

THEOREM. *If $\alpha\neq0$ and 1 is algebraic and β is a cubic irrational then $\omega_1=\alpha^\beta$ and $\omega_2=\alpha^{\beta^2}$ are algebraically independent.*

Assume to the contrary that an equation $P(\omega_1,\omega_2)=0$ is satisfied where $P(u,v)\not\equiv0$ is an irreducible polynomial whose coefficients are rational integers. We introduce the function

$$f(z)=\sum_{k_1=0}^{q}\sum_{k_2=0}^{q}\sum_{k_3=0}^{q}A_{k_1k_2k_3}e^{(k_1+k_2\beta+k_3\beta^2)\eta z},\quad \eta=\ln\alpha, \tag{25}$$

$$A_{k_1k_2k_3}=\sum_{k_0=0}^{q_1}C_{k_0k_1k_2k_3}\omega_1^{k_0},\quad q_1=[q^{3/2}\ln^{1/4}q].$$

For rational integer t_1, t_2 and t_3 the numbers

$$f_{st_1t_2t_3}=f^{(s)}(t_1+t_2\beta+t_3\beta^2)\eta^{-s}$$

are polynomials in ω_1 and ω_2. Their coefficients are linear forms in $C_{k_0k_1k_2k_3}$ with algebraic coefficients. By Dirichlet's box principle and by equation $P(\omega_1, \omega_2) = 0$ one obtains the existence of rational integer $C_{k_0k_1k_2k_3}$ which are not simultaneously equal to 0, such that the conditions

$$(26)\qquad \begin{gathered} f_{st_1t_2t_3} = f^{(s)}(t_1+t_2\beta+t_3\beta^2) = 0, \qquad |C_{k_0k_1k_2k_3}| < e^{2q^{3/2}\ln^{1/4}q}, \\ 0 \leqslant t_1, t_2, t_3 \leqslant q_0 = [q^{1/2}\ln^{1/4}q], \\ 0 \leqslant s \leqslant s_0 = [\gamma_{11}q^{3/2}\ln^{-1/4}q], \qquad q \geqslant Q_1 \end{gathered}$$

are satisfied.

From (26) it follows the equation

$$(27)\quad f^{(s)}(z) = \frac{-s!}{(2\pi)^2}\oint_{\Gamma}\oint_{\Gamma_1}\prod_{t_1=0}^{q_0}\prod_{t_2=0}^{q_0}\prod_{t_3=0}^{q_0}\left(\frac{\gamma-t_1-t_2\beta-t_3\beta^2}{\zeta-t_1-t_2\beta-t_3\beta^2}\right)^{s_0+1}\frac{f(\zeta)}{(\gamma-z)^{s+1}(\zeta-\gamma)}\,d\zeta\,d\gamma$$

where Γ denotes the circle $|\gamma| = \sqrt{q\ln q}$ and Γ_1 the circle $|\zeta| = q^2$. From (25), (26) and (27) it follows:

$$(28)\qquad \begin{gathered} |f^{(s)}(t_1+t_2\beta+t_3\beta^2)| < e^{-\gamma_{12}q^3\ln q}, \\ 0 \leqslant s \leqslant [4q^{3/2}\ln^{-1/4}q] = s_1, \quad 0 \leqslant t_1, t_2, t_3 \leqslant q_0, \quad q \geqslant Q_2. \end{gathered}$$

Among the numbers $f^{(s)}(t_1+t_2\beta+t_3\beta^2)$ satisfying (28) there are different from 0. One proves it assuming the contrary. Indeed, let all these numbers be $= 0$. Introduce the function

$$(29)\qquad \begin{gathered} F(z) = f(z)A^{-1} = \sum_{k_1=0}^{q}\sum_{k_2=0}^{q}\sum_{k_3=0}^{q}B_{k_1k_2k_3}e^{(k_1+k_2\beta+k_3\beta^2)\eta z}, \\ B_{k_1k_2k_3} = A_{k_1k_2k_3}A^{-1}, \qquad A = \max|A_{k_1k_2k_3}|. \end{gathered}$$

$A \neq 0$ because not all the numbers $C_{k_0k_1k_2k_3}$ are zero and ω_1 is transcendental. By our assumption

$$F^{(s)}(t_1+t_2\beta+t_3\beta^2) = 0, \qquad 0 \leqslant s \leqslant s_1, \qquad 0 \leqslant t_1, t_2, t_3 \leqslant q_0.$$

From (27), writing s_1 instead of s_0 and $F(z)$ instead of $f(z)$, we get the estimation

$$|F^{(s)}(0)| \leqslant e^{-\gamma_{13}q^3\ln q}, \qquad s = 0, 1, \ldots, (q+1)^3-1, \qquad q \geqslant Q_3,$$

and from that and from the equation

$$B_\nu = \sum_{k=0}^{N-1}B_k\sum_{s=0}^{N-1}D_{s\nu}\tau_k^s = \sum_{s=0}^{N-1}D_{s\nu}\varphi^{(s)}(0),$$

where

$$\sum_{s=0}^{N-1}D_{s\nu}z^s = \frac{z(z-1)\ldots(z-\nu+1)(N-1-z)\ldots(\nu+1-z)}{\nu!(N-\nu-1)!},$$

$$\varphi(z) = \sum_{k=0}^{N-1}B_k e^{\tau_k z},$$

we get for all numbers $B_{k_1k_2k_3}$ the inequality

$$|B_{k_1k_2k_3}| < 1, \qquad q > Q_4.$$

This is impossible, since by (29) the modulus of one at least of the numbers $B_{k_1k_2k_3}$ is $= 1$.

Thus there exists among the numbers (28) at least one different from 0. It is a polynomial in ω_1, ω_2 with algebraic coefficients. Multiplying it by polynomials with conjugate coefficients we get a number $P_0(\omega_1, \omega_2)$, where P_0 is a polynomial in ω_1, ω_2 with rational coefficients and

$$(30)\qquad 0 < |P_0(\omega_1, \omega_2)| < e^{-\gamma_{14}q^3\ln q}, \qquad q > Q_5.$$

Now the equation $P(\omega_1, \omega_2) = 0$ renders possible the elimination of ω_2 and eventually we get for any $q > Q_6$ the existence of a polynomial $P_q(z)$ with rational integer coefficients which satisfies the inequalities

$$|P_q(\omega_1)| < e^{-\gamma_{15}q^3\ln q}, \qquad \max(n_q, \ln H_q) < \gamma_{16}q^{3/2}\ln^{1/4}q,$$

where n_q and H_q denote the degree and height of $P_q(z)$ respectively. Gelfond's lemma mentioned above leads to the result: $\omega_1 = a^\beta$ is algebraic. Thus we get a contradiction proving that $P(\omega_1, \omega_2) = 0$ is impossible.

The outline of the proof given above can be refined to get [58] the inequality

$$|P(\omega_1, \omega_2)| > e^{-e^{\sigma^{4+\varepsilon}}}, \qquad \varepsilon > 0,\ \sigma = \max(n, \ln H) > \sigma_0(\varepsilon),\ n = n_1+n_2.$$

Here $P(u, v) \not\equiv 0$ is a polynomial with rational integer coefficients of height H and of degrees n_1 and n_2 in u and v, respectively.

We shall present now some of the main results obtained by Gelfond by his third method.

Denote by R_0 an extension of the field Q of rational numbers by one transcendental number. By R_1 we shall denote an algebraic field of finite order, or a field we get by extending R_0 by a root of a polynomial with coefficients from R_0.

We say that the number ζ_1 is *algebraically representable by the number* ζ_2 if there exists a polynomial $P(u, v)$ with algebraic coefficients for which $P(u, \zeta_2) \not\equiv 0$ and $P(\zeta_1, \zeta_2) = 0$. By this definition all algebraic numbers are algebraically representable by any number. A transcendental number

ζ_1 may be algebraically represented by a number ζ_2 only if ζ_2 is transcendental and in such a case a sufficient and necessary condition is that ζ_1 and ζ_2 be algebraically dependent. It is obvious that the field we get by extending Q by $\zeta_1, \ldots, \zeta_s$ will be a field R_1 if and only if all the numbers $\zeta_1, \ldots, \zeta_s$ can be algebraically represented by one of them.

THEOREM 1. *Let the numbers η_0, η_1, η_2 as well as the numbers $a_0 = 1$, a_1, a_2 be linearly independent over Q and let the inequality*

$$|x_0\eta_0+x_1\eta_1+x_2\eta_2| > e^{-\tau x \ln x}, \quad |x_i| \leqslant x,\ i = 0, 1, 2,$$

where $\tau > 0$ is a constant and x_0, x_1, x_2, $|x_0|+|x_1|+|x_2| > 0$ are rational integers, hold for $x > x'$. Then, the extension of Q generated by 11 numbers $a_1, a_2, e^{\eta_i a_k}$ $(i = 0, 1, 2;\ k = 0, 1, 2)$ cannot be a field R_1.

THEOREM 2. *Let $\eta_0 \neq 0$, η_1/η_0 — irrational, $a_0 = 1$, a_1, a_2 — linearly independent over Q and let for $x > x'$ the following inequality be satisfied:*

$$\left| x_0 + x_1 \frac{\eta_1}{\eta_0} \right| > e^{-\tau x^2 \ln x}, \quad 0 < |x_0|+|x_1| \leqslant x,$$

where $\tau > 0$ is a constant and x_0, x_1 are rational integers. Then, the extension of Q generated by 10 numbers $\eta_0, \eta_1, a_1, a_2, e^{\eta_i a_k}$ $(i = 0, 1;\ k = 0, 1, 2)$ cannot be a field R_1.

We shall draw some conclusions from these theorems. Let $a \neq 0$ and 1 be an algebraic number, and let $\nu \neq 0$ be rational. Then

(i) Not all the numbers $a^{e^{\nu}}, a^{e^{2\nu}}, a^{e^{3\nu}}, a^{e^{4\nu}}$ can be algebraically represented by e; therefore one at least of them is transcendental.

(ii) Not all the numbers $e^{e^{\nu}}, e^{e^{2\nu}}, e^{e^{3\nu}}$ can be algebraically represented by e; therefore one at least of them is transcendental.

(iii) Not all the numbers, $a^{\ln^{\nu} a}, a^{\ln^{2\nu} a}, a^{\ln^{3\nu} a}$ can be algebraically represented by $\ln a$. Therefore one at least of them is transcendental.

(iv) Let β be an algebraic number of degree $\geqslant 3$. Then all the numbers $a^{\beta}, a^{\beta^2}, a^{\beta^3}, a^{\beta^4}$ cannot be algebraically represented by one of them. In the special case when β is a cubic irrationality, a^{β} and a^{β^2} are algebraically independent. Availing himself of the third Gelfond method A. A. Šmelev got a series of analogous results.

In 1962 Gelfond [93] published his fourth, this time "elementary" (this means without using analytic functions), method. It rendered possible a proof of his theorem on the transcendence of a^{β} in the case of real a and β. The only tool he used belonging to mathematical analysis was Rolle's theorem.

We pointed out already the connection of the first three methods of Gelfond with his investigations on functions assuming integral values. He came back to those investigations several times and obtained many important results. We shall mention some of them.

THEOREM 1 (cf. [15]). *Let $g(z)$ be an entire function, β an integer $\neq 0$ and ± 1, $g(\beta^n)$ a rational integer for $n = 1, 2, \ldots$ and let*

$$g(z) = o(e^{\ln^2|z|/4\ln\beta}|z|^{-1/2}), \quad |z| \to +\infty.$$

Then $g(z)$ is a polynomial.

A function $f(z)$ is called *normally integral-valued of degree ν on the set E* if $f(a)$ is an integer of a field K of degree ν for all $a \in E$ and for any $\delta > 0$ there exists $C_0 = C_0(\delta)$ such that for all numbers $f(a)^*$ ($f(a)^*$ denotes any conjugate to $f(a)$),

$$|f(a)^*| < C_0\big(M(|a|)\big)^{1+\delta}, \quad M(r) = \max_{|z|=r} |f(z)|.$$

THEOREM 2 (cf. [69]). *Let $f(z)$ be normally integral-valued on $E = \{a_k+\beta_k\}$, $|a_k| \leqslant |a_{k+1}|$, $|\beta_k| \leqslant |\beta_{k+1}|$, $k = 0, 1, \ldots$ and*

$$N_1(r) = \sum_{|a_k|\leqslant r} 1, \quad N_2(r) = \sum_{|\beta_k|\leqslant r} 1, \quad N(r) = \min\{N_1(r), N_2(r)\}.$$

There exist constants θ, λ, e.g. $\theta > 2+\sqrt{2}$ and $\lambda < (8\nu)^{-1}\ln\{((\theta-1)^2-1)/(2\theta-2)\}$ such that if the inequality

$$\ln M(\theta r) < \lambda N(r)$$

holds, the function $f(z)$ satisfies a functional equation

$$\sum_{k=0}^{m} A_k f(z+\beta_k) = 0, \quad m > 1,$$

where $A_1, \ldots, A_m$ are algebraic numbers not all equal to 0.

We shall give but a very short account of Gelfond's works concerning other branches of number theory.

In paper [71] using the identity

$$\sum_{n=1}^{\infty} \theta(n) \sum_{k=1}^{\infty} f(x^{kn}) = \sum_{n=1}^{\infty} \sum_{p^s|n} \big(1+\theta(p)+\ldots+\theta(p^s)\big) f(x^n),$$

valid for any multiplicative function $\theta(n)$ satisfying the conditions

$$|\theta(n)| \leqslant 1, \quad f(x) = o(x^{\beta}), \quad \beta > 0 - \text{constant},$$

Gelfond proved the following

THEOREM. *If an absolutely multiplicative function $\theta(n)$ assumes only the values $0, 1, -1$ and*

$$\overline{\lim}\, \frac{1}{\sqrt{n}} \left| \sum_{k=1}^{n} \theta(k) \right| = \varrho < \frac{1}{12}, \tag{31}$$

then

$$\sum_{n=1}^{\infty}\frac{\theta(n)}{n}\neq 0.$$

Condition (31) probably cannot be weakened by replacing $\sqrt{n}$ by a more quickly increasing function. Indeed, the Riemann hypothesis implies

$$\Big|\sum_{k=1}^{n}\mu(k)\Big|<n^{\frac{1}{2}+\delta_n},\quad \lim_{n\to\infty}\delta_n=0,$$

and it is well known that

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n}=0.$$

In the same paper Gelfond gave a new, very simple proof that $L(1,\chi)\neq 0$ for real characters.

Some of his papers ([73], [77], [93]) were devoted to an interesting generalization of the notion of arithmetical progression.

Let $a_1,\ldots,a_\nu$ be positive integers, $b_1,\ldots,b_\nu$ non-negative integers. He introduced linear substitutions

$$L_k(x)=a_kx+b_k,\quad a_k+b_k\geqslant 2,\quad k=1,2,\ldots,\nu.$$

The inverse substitution $L_k^{-1}(x)$ can be applied to numbers congruent to b_k modulo a_k. Two positive integers m and n are called *equivalent* if they can be linked by a chain of substitutions

$$L_{n_1}^{\delta_1}[L_{n_2}^{\delta_2}\ldots[L_{n_s}^{\delta_s}(m)]\ldots]=L_{k_1}^{\varepsilon_1}[L_{k_2}^{\varepsilon_2}\ldots[L_{k_q}^{\varepsilon_q}(n)]\ldots],$$

$$\varepsilon_i=\pm 1,\quad \delta_j=\pm 1,\quad 1\leqslant i\leqslant s,\quad 1\leqslant j\leqslant q.$$

He proved some necessary and sufficient conditions for the finiteness of classes. He considered especially the case

$$L_k(x)=mx+t_km+k,\quad 0\leqslant k\leqslant m-1,\quad m\geqslant 2,$$

t_k — integers. In this case the number of classes is equal to

$$q=\sum_{k=0}^{m-1}t_k.$$

Denoting by $N_\nu(x)$ the number of elements of ν-th class ($\nu=1,2,\ldots,q$) which are not greater than x, he found the asymptotic formula

$$N_\nu(x)=x\varphi_\nu\left(\frac{\ln x}{\ln m}\right)+O(1).$$

Here $\varphi_\nu(u)$ is a periodic function of period 1 satisfying the Lipschitz condition. If $t_1=t_2=\ldots=t_\nu=t$, he solved the problem of distribution of primes in the sequence generated by the substitution $L_k(x)=mx+tm+k$. For that purpose he introduced and investigated ζ-function of a class. He got

$$(32)\qquad \sum_{n\leqslant x}\big(N_\nu(n)-N_\nu(n-1)\big)\Lambda(n)=x\varphi_\nu\left(\frac{\ln x}{\ln m}\right)+o(x),$$

where $\Lambda(n)$ is Mangoldt's function. He indicated a way for improving the error term of the last formula.

In [93] he presented one of the simplest variants of the proof that the number of prime numbers in primitive progressions is infinite. He is also an author of a very elementary proof, in A. Selberg's spirit, of Hadamard's theorem for the sequence $1,2,3,\ldots$, for the arithmetical progression and for the sequence of numbers of ν-th class, mentioned above.

By combining A. Selberg's elementary method with simple notions of the theory of functions he arrived at a proof (not elementary, in the usual terminology, but very simple) of

$$(33)\qquad \pi(x,m,l)=\frac{\operatorname{li}x}{\varphi(m)}+O(xe^{-\beta\ln^{1/9}x}).$$

The proof is based on the equation

$$\sum_{n\leqslant x}\frac{1}{n^{1+it}}=\frac{i}{t}x^{-it}+a(t)+O\left(\frac{1}{x}\right),$$

where

$$a(t)=1+\frac{1}{it}+\sum_{n=2}^{\infty}\left\{\frac{1}{it}(n^{-it}-[n-1]^{-it})+n^{-1-it}\right\}.$$

He proved that for $|t|>t_0$

$$a(t)>\frac{1}{300\ln^5 t}.$$

From that he derived the estimations for $\left|\frac{\zeta'}{\zeta}(s)\right|$ and $\left|\frac{L'}{L}(s,\chi)\right|$ and the absence of zeros of $\zeta(s)$ and $L(s,\chi)$ in corresponding domains. Using curvilinear integrals he got (33).

It should be noted that later W. Wirsing and E. Bombieri succeeded to get the error term $O(x\ln^{-A}x)$, $A>0$, by a completely elementary, although very complicated method.

In paper [91] Gelfond derived an approximate functional equation for a class of Dirichlet series.

In papers [52] and [65] he derived very general theorems on distribution of fractional parts of systems of functions.

Denote by L a sequence $y_1, y_2, \ldots, y_k, \ldots$ of numbers monotonically increasing to infinity and let M be a sequence of points $\tau_k(\beta_{1k}, \ldots, \beta_{\nu k})$ everywhere dense in the unit cube of ν-dimensional space. The fractional parts $f_1(y), \ldots, f_\nu(y)$ will be called (φ, M)*-everywhere densely distributed* if the system of inequalities

$$0 \leqslant \{f_i(y)\} - \beta_{ik} \leqslant \varphi(y), \quad i = 1, 2, \ldots, \nu,$$

where $\varphi(y)$ is an arbitrary monotonically decreasing function, $0 \leqslant \varphi(y) \leqslant 1$, $\lim_{y\to\infty} \varphi(y) = 0$, has infinitely many solutions $y \in L$ for each point τ_k. Then the following theorem is true:

THEOREM. *Let the functions* $f_1(t_1, y), \ldots, f_\nu(t_\nu, y)$ *satisfy the conditions*:

(i) $f_i(t_i, y)$ *is defined for all* $y \in L$ *in the interval* $a_i \leqslant t_i \leqslant b_i$, $i = 1, 2, \ldots, \nu$.

(ii) $\dfrac{\partial f_i}{\partial t_i} = f_i'(t_i, y_k)$ *is continuous and non-decreasing in the interval* $[a_i, b_i]$; $f_i'(a_i, y_k) \geqslant 1$ *for all* i *and* k.

(iii) *If* t_i *is fixed the derivative* $f_i'(t_i, y_k) \to \infty$ *for* $k \to \infty$.

Then, for an arbitrarily fixed sequence M *and for arbitrary* $\varepsilon > 0$ *and arbitrarily chosen function* $\varphi(t)$, *in the intervals* $[c_i, d_i]$, $d_i = c_i + \varepsilon$, $c_i \geqslant a_i$, $d_i \leqslant b_i$, *there are points* a_i, $c_i \leqslant a_i \leqslant d_i$, $i = 1, 2, \ldots, \nu$, *for which the fractional parts of functions* $f_1(a_1, y), \ldots, f_\nu(a_\nu, y)$ *are* (φ, M)*-everywhere densely distributed.*

For instance in an arbitrary interval there exist points where the fractional parts of functions

$$ax^s, \quad (1+a)^x, \quad e^{(1+a)^x}, \quad s > 0, \; a > 0, \; x = 1, 2, \ldots$$

are (φ, M)-everywhere densely distributed.

Starting from these results Gelfond obtained conditions for the impossibility of prolonging several lacunary series. Fabry's theorem follows from this theorem in a very simple way.

As a consequence he got from this theorem the following:

If the Dirichlet series

$$f(z) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n z}$$

is convergent in the halfplane $\mathrm{Re}\, z > 0$ *and* λ_n *are monotonically increasing and farther*

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = \infty,$$

$$U(z) = \prod_{m=1}^{\infty} (1 - \lambda_m^{-2} z^2),$$

$$\overline{\lim_{n\to\infty}} |c_n U'(\lambda_n)|^{1/\lambda_n} = 1,$$

then $f(z)$ *cannot be prolonged beyond the imaginary axis.*

Much of his time A. O. Gelfond devoted to teaching work. He lectured at the Moscow University, among other topics, on mathematical analysis, analytic functions, number theory and gave many special courses. For a long period he directed research seminars and seminars for students on number theory and analytic functions. He had a great number of students. Many of them are now known specialists in number theory and analytic functions.

He wrote several books: [24], [67], [68], [69], [93] and [99]. Some of them were translated into different languages.

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Weak forms of Mann's density theorem extended to sets of lattice points

by

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§ 1. Introduction. Let Q_n be the set of all nonzero n-dimensional lattice points with nonnegative integer coordinates. We will use the usual componentwise addition and subtraction of elements of Q_n, and the usual partial ordering: For any x and y in Q_n, $x<y$ if $y-x$ is in Q_n. If S is any subset of Q_n and F is any finite subset of Q_n then $S(F)$ will denote the number of elements in $S\cap F$. For any x in Q_n let $\mathrm{L}x=\{y\in Q_n\colon y\leqslant x\}$. If A and B are subsets of Q_n, $A+B$ will denote the set of all elements of the form $a, b, a+b$, where $a\in A$, $b\in B$, while $A-B$ is the set of all elements of A which are not in B.

A fundamental subset of Q_n or, briefly, a fundamental set, is defined to be any finite nonempty subset R of Q_n such that $x\in R$ implies $\mathrm{L}x\subseteq R$. For any subset A of Q_n Müller [8] has defined the density of A to be the $\operatorname{glb} A(R)/Q_n(R)$, taken over all fundamental sets R. For $n=1$ this is clearly the Schnirelmann density of A.

With this family of fundamental sets and definition of density, several results have been obtained for subsets of Q_n which are analogous to well-known theorems of additive number theory for sets of positive integers. (See [2], [3], [5], [6], [8], [9].) In this note we will discuss the extension of the famous theorem of Mann [7] to Q_n. Using the notation given above, an n-dimensional analogue to Mann's theorem may be stated as follows.

(I) Let A and B be subsets of Q_n, let $C=A+B$, and let R be any fundamental subset of Q_n. Then either $C(R)=Q_n(R)$ or there exists a fundamental set $W\subseteq R$ such that no maximal element of W is in C and $C(R)/Q_n(R)\geqslant [A(W)+B(W)]/Q_n(W)$.

The statement (I) is false for $n>1$, as is shown by the following example for Q_2. (For $n>2$ this example may be embedded in Q_n). Let the fundamental set $R=\mathrm{L}(2,5)\cup\mathrm{L}(5,2)$. In the figure below lattice points of $(A-B)\cap R$ are marked by $\times$, those of $A\cap B\cap R$ by $\bullet$, those