Highly composite numbers

Proceedings of the London Mathematical Society, 2, XIV, 1915, 347 – 409

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This paper, as it appeared orginally, is not complete. Since the *London Mathematical Society* was in some financial difficulty at that time, Ramanujan had to suppress part of what he had written in order to save expense. The unpublished part of the manuscript with annotation has been reproduced: Highly Composite Numbers by Srinivasa Ramanujan, Jean-Louis Nicolas, Guy Robin, *The Ramanujan Journal*, Volume 1, Issue 2, 1997, pp. 119 – 153.

I.
Introduction and Summary of Results

1. The number $d(N)$ of divisors of N varies with extreme irregularity as N tends to infinity, tending itself to infinity or remaining small according to the form of N. In this paper I prove a large number of results which add a good deal to our knowledge of the behaviour of $d(N)$.

It was proved by Dirichlet * that

$$\frac{d(1)+d(2)+d(3)+\cdots+d(N)}{N} = \log N + 2\gamma - 1 + O\left(\frac{1}{\sqrt{N}}\right)\ \dagger$$

where γ is the Eulerian constant. Voronöi ‡ and Landau § have shewn that the error term may be replaced by $O(N^{-\frac{2}{3}+\epsilon})$, or indeed $O(N^{-\frac{2}{3}}\log N)$. It seems not unlikely that the real value of the error is of the form $O(N^{-\frac{3}{4}+\epsilon})$, but this is as yet unproved. Mr. Hardy has, however, shewn recently ¶ that the equation

$$\frac{d(1)+d(2)+d(3)+\cdots d(N)}{N} = \log N + 2\gamma - 1 + o(N^{-\frac{3}{4}})$$

is certainly false. He has also proved that

$$\begin{aligned} d(1)+d(2))+\cdots+d(N-1)+\tfrac{1}{2}d(N)) - N\log N - (2\gamma-1)N - \tfrac{1}{4} \\ = \sqrt{N}\sum_{1}^{\infty}\frac{d(n)}{\sqrt{n}}[H_1\{4\pi\sqrt{(nN)}\} - Y_1\{4\pi\sqrt{(nN)}\}], \end{aligned}$$

where Y_n is the ordinary second solution of Bessel's equation, and

$$H_1(x) = \frac{2}{\pi}\int_1^{\infty}\frac{we^{-xw}dw}{\sqrt{(w^2-1)}};$$

and that the series on the right-hand side is the sum of the series

$$\frac{N^{\frac{1}{4}}}{\pi\sqrt{2}}\sum_{1}^{\infty}\frac{d(n)}{n^{\frac{3}{4}}}\cos\{4\pi\sqrt{(nN)} - \tfrac{1}{4}\pi\},$$

and an absolutely and uniformly convergent series.

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* *Werke*, Vol.2, p.49.

† $f = O(\phi)$ means that a constant exists such that $|f| < K\phi$: $f = o(\phi)$ means that $f/\phi \to 0$.

‡ *Crelle's Journal*, Vol. 126. p. 241.

§ *Göttinger Nachrichten*, 1912.

¶ *Comptes Rendus* May 10, 1915.

The "average" order of $d(N)$) is thus known with considerable accuracy. In this paper I consider, not the average order of $d(N)$), but its maximum order. This problem has been much less studied. It is obvious that

$$d(N) < 2\sqrt{N}.$$

It was shewn by Wigert[*] that

$$d(N) < 2^{\frac{\log N}{\log\log N}(1+\epsilon)} \tag{i}$$

for all positive values of ϵ and all sufficiently large values of N, and that

$$d(N) > 2^{\frac{\log N}{\log\log N}(1-\epsilon)} \tag{ii}$$

for an infinity of values of N. From (i) it follows in particular that

$$d(N) < N^{\delta}$$

for all positive values of δ and all sufficiently large values of N.
Wigert proves (i) by purely elementary reasoning, but uses the "Prime Number Theorem"[†]to prove (ii). This is, however, unnecessary, the inequality (ii) being also capable of elementary proof. In § 5 I shew, by elementary reasoning, that

$$d(N) < 2^{\frac{\log N}{\log\log N}+O\frac{\log N}{(\log\log N)^2}}$$

for all values of N, and

$$d(N) > 2^{\frac{\log N}{\log\log N}+O\frac{\log N}{(\log\log N)^2}}$$

for an infinity of values of N. I also shew later on that, if we assume all known results concerning the distribution of primes, then

$$d(N) < 2^{Li(\log N)+O[\log N e^{-a\sqrt{(\log\log N)}}]}$$

for all values of N, and

$$d(N) > 2^{Li(\log N)+O[\log N e^{-a\sqrt{(\log\log N)}}]}$$

for an infinity of values of N, where a is a positive constant.
I then adopt a different point of view, I define a highly composite number as a number whose number of divisors exceeds that of all its predecessors. Writing such a number in the form

$$N = 2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdots p^{a_p},$$

[*] *Arkiv för Matematik*, Vol. 3, No.18
[†] The theorem that $\pi(x) \sim \frac{x}{\log x}$, $\pi(x)$ being the number of primes not exceeding x.

I prove that

$$a_2 \geq a_3 \geq a_5 \geq \cdots \geq a_p,$$

and that

$$a_p = 1,$$

for all highly composite values of N except 4 and 36.
I then go on to prove that the indices near the beginning form a decreasing sequence in the stricter sense, i.e., that

$$a_2 > a_3 > a_5 > \cdots > a_\lambda,$$

where λ is a certain function of p.
Near the end groups of equal indices may occur, and I prove that there are actually groups of indices equal to

$$1, 2, 3, 4, \ldots, \mu,$$

where μ again is a certain function of p. I also prove that if λ is fairly small in comparison with p, then

$$a_\lambda \log \lambda \sim \frac{\log p}{\log 2};$$

and that the later indices can be assigned with an error of at most unity.
I prove also that two successive highly composite numbers are asymptotically equivalent, i.e., that the ratio of two consecutive such numbers tends to unity. These are the most striking results. More precise ones will be found in the body of the paper. These results give us a fairly accurate idea of the structure of a highly composite number.
I then select from the general aggregate of highly composite numbers a special set which I call "superior highly composite numbers". I determine completely the general form of all such numbers, and I shew how a combination of the idea of a superior highly composite number with the assumption of the truth of the Riemann hypothesis concerning the roots of the ζ-function leads to even more precise results concerning the maximum order of $d(N)$. These results naturally differ from all which precede in that they depend on the truth of a hitherto unproved hypothesis.

II.
Elementary Results concerning the Order of $d(N)$.

2. Let $d(N)$ denote the number of divisors of N, and let

$$N = p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_n^{a_n}, \tag{1}$$

where $p_1, p_2, p_3, \ldots, p_n$ are a given set of n primes. Then

$$d(N) = (1 + a_1)(1 + a_2)(1 + a_3) \cdots (1 + a_n). \tag{2}$$

From (1) we see that

$$
\begin{aligned}
&(1/n)\log(p_1p_2p_3\cdots p_nN)\\
&\quad=(1/n)\{(1+a_1)\log p_1+(1+a_2)\log p_2+\cdots+(1+a_n)\log p_n\}\\
&\quad>\{(1+a_1)(1+a_2)(1+a_3)\cdots(1+a_n)\log p_1\log p_2\cdots\log p_n\}^{1/n}.
\end{aligned}
$$

Hence we have

$$d(N)<\frac{\{(1/n)\log(p_1p_2p_3\cdots p_nN)\}^n}{\log p_1\log p_2\log p_3\cdots\log p_n}, \tag{3}$$

for all values of N.

We shall now consider how near to this limit it is possible to make $d(N)$ by choice of the indices $a_1, a_2, a_3, \ldots, a_n$. Let us suppose that

$$1+a_m=v\frac{\log p_n}{\log p_m}+\epsilon_m(m=1,2,3,\ldots,n), \tag{4}$$

where v is a large integer and $-\frac{1}{2}<\epsilon_m<\frac{1}{2}$. Then, from (4), it is evident that

$$\epsilon_n=0. \tag{5}$$

Hence, by a well-known theorem due to Dirichlet*, it is possible to choose values of v as large as we please and such that

$$|\epsilon_1|<\epsilon,\ |\epsilon_2|<\epsilon,\ |\epsilon_3|<\epsilon,\ldots,|\epsilon_{n-1}|<\epsilon, \tag{6}$$

where $\epsilon\leq v^{-1/(n-1)}$. Now let

$$t=v\log p_n,\quad \delta_m=\epsilon_m\log p_m. \tag{7}$$

Then from (1), (4) and (7), we have

$$\log(p_1p_2p_3\cdots p_nN)=nt+\sum_1^n\delta_m. \tag{8}$$

Similarly, from (2), (4) and (7) we see that

$$
\begin{aligned}
d(N)&=\frac{(t+\delta_1)(t+\delta_2)\cdots(t+\delta_n)}{\log p_1\log p_2\log p_3\cdots\log p_n}\\
&=\frac{t^n\exp\left\{\frac{\sum\delta_m}{t}-\frac{\sum\delta_m^2}{2t^2}+\frac{\sum\delta_m^3}{3t^3}-\cdots\right\}}{\log p_1\log p_2\log p_3\cdots\log p_n}
\end{aligned}
$$

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* *Werke*, Vol. 1, p. 635.

$$= \quad \left(t+\frac{\sum \delta_m}{n}\right)^n \frac{\exp\left\{-\dfrac{n\sum \delta_m^2-(\sum \delta_m)^2}{2nt^2}+\dfrac{n^2\sum \delta_m^3-(\sum \delta_m)^3}{3n^2t^3}-\cdots\right\}}{\log p_1 \log p_2 \log p_3 \cdots \log p_n}$$

$$= \quad \frac{\{(1/n)\log(p_1p_2p_3\cdots p_nN)\}^n}{\log p_1 \log p_2 \cdots \log p_n}$$

$$\left[1-\tfrac{1}{2}(\log N)^{-2}\left\{n^2\sum \delta_m^2 - n\left(\sum \delta_m\right)^2\right\}+\cdots\right], \tag{9}$$

in virtue of (8)). From (6), (7) and (9) it follows that it is possible to choose the indices $a_1, a_2, \ldots, a_n$ so that

$$d(N)=\frac{\{(1/n)\log(p_1p_2p_3\cdots p_nN)\}^n}{\log p_1 \log p_2 \cdots \log p_n}\{1-O(\log N)^{-2n/(n-1)}\}, \tag{10}$$

where the symbol O has its ordinary meaning.

The following examples shew how close an approximation to $d(N)$ may be given by the right-hand side of (3). If

$$N = 2^{72}\cdot 7^{25},$$

then, according to (3), we have

$$d(N) < 1898.00000685\ldots; \tag{11}$$

and as a matter of fact $d(N) = 1898$. Similarly, taking

$$N = 2^{568}\cdot 3^{358},$$

we have, by (3),

$$d(N)) < 204271.000000372\ldots; \tag{12}$$

while the actual value of $d(N)$ is 204271. In a similar manner, when

$$N = 2^{64}\cdot 3^{40}\cdot 5^{27},$$

we have, by (3),

$$d(N) < 74620.00412\ldots; \tag{13}$$

while actually

$$d(N) = 74620.$$

3. Now let us suppose that, while the number n of different prime factors of N remains fixed, the primes p_ν, as well as the indices a_ν, are allowed to vary. It is evident that $d(N)$, considered as a function of N, is greatest when the primes p_ν are the first n primes, say $2, 3, 5, \ldots, p$, where p is the nth prime. It therefore follows from (3) that

$$d(N) < \frac{\{(1/n)\log(2\cdot 3\cdot 5\cdots p\cdot N)\}^n}{\log 2\log 3\log 5\cdots\log p}, \tag{14}$$

and from (10) that it is possible to choose the indices so that

$$d(N) = \frac{\{(1/n)\log(2\cdot 3\cdot 5\cdots p\cdot N)\}^n}{\log 2\log 3\log 5\cdots\log p}\{1 - O(\log N)^{-2n/(n-1)}\}. \tag{15}$$

4. Before we proceed to consider the most general case, in which nothing is known about N, we must prove certain preliminary results. Let $\pi(x)$ denote the number of primes not exceeding x, and let

$$\vartheta(x) = \log 2 + \log 3 + \log 5 + \cdots + \log p,$$

and

$$\varpi(x) = \log 2\cdot\log 3\cdot\log 5\cdots\log p,$$

where p is the largest prime not greater than x; also let $\phi(t)$ be a function of t such that $\phi'(t)$ is continuous between 2 and x. Then

$$\begin{aligned}\int_2^x \pi(t)\phi'(t)dt &= \int_2^3 \phi'(t)dt + 2\int_3^5 \phi'(t)dt + 3\int_5^7 \phi'(t)dt \\ &\quad + 4\int_7^{11}\phi'(t)dt + \cdots + \pi(x)\int_p^x \phi'(t)dt \\ &= \{\phi(3)-\phi(2)\} + 2\{\phi(5)-\phi(3)\} + 3\{\phi(7)-\phi(5)\} \\ &\quad + 4\{\phi(11)-\phi(7)\} + \cdots + \pi(x)\{\phi(x)-\phi(p)\} \\ &= \pi(x))\phi(x) - \{\phi(2)+\phi(3)+\phi(5)+\cdots+\phi(p)\}. \end{aligned} \tag{16}$$

As an example let us suppose that $\phi(t) = \log t$. Then we have

$$\pi(x)\log x - \vartheta(x) = \int_2^x \frac{\pi(t)}{t}dt. \tag{17}$$

Again let us suppose that $\phi(t) = \log\log t$. Then we see that

$$\pi(x)\log\log x - \log\varpi(x) = \int_2^x \frac{\pi(t)}{t\log t}dt. \tag{18}$$

But

$$\int_2^x \frac{\pi(t)}{t\log t}dt = \frac{1}{\log x}\int_2^x \frac{\pi(t)}{t}dt + \int_2^x \left(\frac{1}{u(\log u)^2}\int_2^u \frac{\pi(t)}{t}dt\right) du.$$

Hence we have

$$\begin{aligned}
&\pi(x)\log\left\{\frac{\vartheta(x)}{\pi(x)}\right\} - \log\varpi(x) \\
&\quad = \pi(x)\log\left\{\frac{\vartheta(x)}{\pi(x)\log x}\right\} + \frac{1}{\log x}\int_2^x \frac{\pi(t)}{t}dt + \int_2^x \left(\frac{1}{u(\log u)^2}\int_2^u \frac{\pi(t)}{t}dt\right) du. \qquad (19)
\end{aligned}$$

But

$$\begin{aligned}
\pi(x)\log\left\{\frac{\vartheta(x)}{\pi(x)\log x}\right\} &= \pi(x)\log\left\{1 - \frac{\pi(x)\log x - \vartheta(x)}{\pi(x)\log x}\right\} \\
&= \pi(x)\log\left\{1 - \frac{1}{\pi(x)\log x}\int_2^x \frac{\pi(t)}{t}dt\right\} \\
&< -\frac{1}{\log x}\int_2^x \frac{\pi(t)}{t}dt;
\end{aligned}$$

and so

$$\pi(x)\log\left\{\frac{\vartheta(x)}{\pi(x)\log x}\right\} + \frac{1}{\log x}\int_2^x \frac{\pi(t)}{t}dt < 0. \qquad (20)$$

Again,

$$\begin{aligned}
\pi(x)\log\left\{\frac{\vartheta(x)}{\pi(x)\log x}\right\} &= -\pi(x)\log\left\{1 + \frac{\pi(x)\log x - \vartheta(x)}{\vartheta(x)}\right\} \\
&= -\pi(x)\log\left\{1 + \frac{1}{\vartheta(x)}\int_2^x \frac{\pi(t)}{t}dt\right\} > -\frac{\pi(x)}{\vartheta(x)}\int_2^x \frac{\pi(t)}{t}dt;
\end{aligned}$$

and so

$$\begin{aligned}
\pi(x)\log\left\{\frac{\vartheta(x)}{\pi(x)\log x}\right\} + \frac{1}{\log x}\int_2^x \frac{\pi(t)}{t}dt &> -\frac{\pi(x)\log x - \vartheta(x)}{\vartheta(x)\log x}\int_2^x \frac{\pi(t)}{t}dt \\
&= -\frac{1}{\vartheta(x)\log x}\left\{\int_2^x \frac{\pi(t)}{t}dt\right\}^2. \qquad (21)
\end{aligned}$$

It follows from (19), (20) and (21) that

$$\begin{aligned}\int_2^x \left(\frac{1}{u(\log u)^2}\int_2^u \frac{\pi(t)}{t}\,dt\right)du \quad &> \quad \pi(x)\log\left\{\frac{\vartheta(x)}{\pi(x)}\right\} - \log\varpi(x)\\ &> \quad \int_2^x \left(\frac{1}{u(\log u)^2}\int_2^u \frac{\pi(t)}{t}\,dt\right)du\\ &\qquad -\frac{1}{\vartheta(x)\log x}\left\{\int_2^x \frac{\pi(t)}{t}\,dt\right\}^2.\end{aligned}$$

Now it is easily proved by elementary methods* that

$$\pi(x) = O\left(\frac{x}{\log x}\right), \quad \frac{1}{\vartheta(x)} = O\left(\frac{1}{x}\right);$$

and so

$$\int_2^x \frac{\pi(t)}{t}dt = O\left(\frac{x}{\log x}\right).$$

Hence

$$\int_2^x \left(\frac{1}{u(\log u)^2}\int_2^u \frac{\pi(t)}{t}dt\right)du = \int_2^x O\left\{\frac{1}{(\log u)^3}\right\}du = O\left\{\frac{x}{(\log x)^3}\right\};$$

and

$$\frac{1}{\vartheta(x)\log x}\left\{\int_2^x \frac{\pi(t)}{t}dt\right\}^2 = \frac{1}{\vartheta(x)\log x}O\left\{\frac{x^2}{(\log x)^2}\right\} = O\left\{\frac{x}{(\log x)^3}\right\}.$$

Hence we see that

$$\frac{\{\vartheta(x)/\pi(x)\}^{\pi(x)}}{\varpi(x)} = e^{O[x/(\log x)^3]}. \tag{22}$$

5. We proceed to consider the case in which nothing is known about N. Let

$$N' = 2^{a_1}\cdot 3^{a_2}\cdot 5^{a_3}\cdots p^{a_n}.$$

Then it is evident that $d(N) = d(N')$, and that

$$\vartheta(p) \le \log N' \le \log N. \tag{23}$$

*See Landau, *Handbuch*, pp. 71 *et seq.*

It follows from (3) that

$$\begin{aligned} d(N) &= d(N') < \frac{1}{\varpi(p)} \left\{ \frac{\vartheta(p) + \log N'}{\pi(p)} \right\}^{\pi(p)} \\ &\leq \left\{ 1 + \frac{\log N}{\vartheta(p)} \right\}^{\pi(p)} \frac{\{\vartheta(p)/\pi(p)\}^{\pi(p)}}{\varpi(p)} \\ &= \left\{ 1 + \frac{\log N}{\vartheta(p)} \right\}^{\pi(p)} e^{O[p/(\log p)^3]} = \left\{ 1 + \frac{\log N}{\vartheta(p)} \right\}^{\pi(p)+O[p/(\log p)^3]}, \end{aligned} \tag{24}$$

in virtue of (22) and (23). But from (17) we know that

$$\pi(p) \log p - \vartheta(p) = O\left(\frac{p}{\log p} \right);$$

and so

$$\vartheta(p) = \pi(p)\{\log p + O(1)\} = \pi(p)\{\log \vartheta(p) + O(1)\}.$$

Hence

$$\pi(p) = \vartheta(p) \left\{ \frac{1}{\log \vartheta(p)} + O \frac{1}{\{\log \vartheta(p)\}^2} \right\}. \tag{25}$$

It follows from (24) and (25) that

$$d(N) \leq \left\{ 1 + \frac{\log N}{\vartheta(p)} \right\}^{\frac{\vartheta(p)}{\log \vartheta(p)} + O \frac{\vartheta(p)}{[\log \vartheta(p)]^2}}.$$

Writing t instead of $\vartheta(p)$, we have

$$d(N) \leq \left(1 + \frac{\log N}{t} \right)^{\frac{t}{\log t} + O \frac{t}{(\log t)^2}}; \tag{26}$$

and from (23) we have

$$t \leq \log N. \tag{27}$$

Now, if N is a function of t, the order of the right-hand side of (26), considered as a function of N, is increased when N is decreased in comparison with t, and decreased when N is increased in comparison with t. Thus the most unfavourable hypothesis is that N, considered as a function of t, is as small as is compatible with the relation (27). We may therefore write $\log N$ for t in (26). Hence

$$d(N) < 2^{\frac{\log N}{\log \log N} + O \frac{\log N}{(\log \log N)^2}}, \tag{28}$$

for all values of N*

The inequality (28) has been proved by purely elementary reasoning. We have not assumed, for example, the prime number theorem, expressed by the relation

$$\pi(x) \sim \frac{x}{\log x}\ .^{\dagger}$$

We can also, without assuming this theorem, shew that the right-hand side of (28) is actually the order of $d(N)$ for an infinity of values of N. Let us suppose that

$$N = 2 \cdot 3 \cdot 5 \cdot 7 \cdots p.$$

Then

$$d(N) = 2^{\pi(p)} = 2^{\frac{t}{\log t} + O\frac{t}{(\log t)^2}},$$

in virtue of (25). Since $\log N = \vartheta(p) = t$, we see that

$$d(N) = 2^{\frac{\log N}{\log\log N} + O\frac{\log N}{(\log\log N)^2}},$$

for an infinity of values of N. Hence the maximum order of $d(N)$ is

$$2^{\frac{\log N}{\log\log N} + O\frac{\log N}{(\log\log N)^2}}.$$

III.
The Structure of Highly Composite Numbers.

6. A number N may be said to be a highly composite number, if $d(N') < d(N)$ for all values of N' less than N. It is easy to see from the definition that, if N is highly composite and $d(N') > d(N)$, then there is at least one highly composite number M, such that

$$N < M \le N'. \qquad (29)$$

* If we assume *nothing* about $\pi(x)$, we can shew that

$$d(N) < 2^{\frac{\log N}{\log\log N} + O\frac{\log N \log\log\log N}{(\log\log N)^2}}.$$

If we assume the prime number theorem, and nothing more, we can shew that

$$d(N) < 2^{\frac{\log N}{\log\log N} + [1+O(1)]\frac{\log N}{(\log\log N)^2}}.$$

If we assume that

$$\pi(x) = \frac{x}{\log x} + O\frac{x}{(\log x)^2},$$

we can shew that

$$d(N) < 2^{\frac{\log N}{\log\log N} + \frac{\log N}{(\log\log N)^2} + O\frac{\log N}{(\log\log N)^3}}.$$

† $\phi(x) \sim \Psi(x)$ means that $\phi(x)/\Psi(x) \to 1$ as $x \to \infty$.

if N and N' are consecutive highly composite numbers, then $d(M) \leq d(N)$ for all values of M between N and N'. It is obvious that

$$d(N) < d(2N) \tag{30}$$

for all values of N. It follows from (29) and (30) that, if N is highly composite, then there is at least one highly composite number M such that $N < M \leq 2N$. That is to say, there is at least one highly composite number N, such that

$$x < N \leq 2x, \tag{31}$$

if $x \geq 1$.

7. I do not know of any method for determining consecutive highly composite numbers except by trial. The following table gives the consecutive highly composite values of N, and the corresponding values of $d(N)$ and $dd(N)$, up to $d(N) = 10080$.
The numbers marked with the asterisk in the table are called superior highly composite numbers. Their definition and properties will be found in §§ 32, 33.

$dd(N)$	$d(N)$	N
2	$2 = 2$	$^*2 = 2$
2	$3 = 3$	$4 = 2^2$
3	$4 = 2^2$	$^*6 = 2 \cdot 3$
4	$6 = 2 \cdot 3$	$^*12 = 2^2 \cdot 3$
4	$8 = 2^3$	$24 = 2^3 \cdot 3$
3	$9 = 3^2$	$36 = 2^2 \cdot 3^2$
4	$10 = 2 \cdot 5$	$48 = 2^4 \cdot 3$
6	$12 = 2^2 \cdot 3$	$^*60 = 2^2 \cdot 3 \cdot 5$
5	$16 = 2^4$	$^*120 = 2^3 \cdot 3 \cdot 5$
6	$18 = 2 \cdot 3^2$	$180 = 2^2 \cdot 3^2 \cdot 5$
6	$20 = 2^2 \cdot 5$	$240 = 2^4 \cdot 3 \cdot 5$
8	$24 = 2^3 \cdot 3$	$*360 = 2^3 \cdot 3^2 \cdot 5$
8	$30 = 2 \cdot 3 \cdot 5$	$720 = 2^4 \cdot 3^2 \cdot 5$
6	$32 = 2^5$	$840 = 2^3 \cdot 3 \cdot 5 \cdot 7$
9	$36 = 2^2 \cdot 3^2$	$1260 = 2^2 \cdot 3^2 \cdot 5 \cdot 7$
8	$40 = 2^3 \cdot 5$	$1680 = 2^4 \cdot 3 \cdot 5 \cdot 7$
10	$48 = 2^4 \cdot 3$	$*2520 = 2^3 \cdot 3^2 \cdot 5 \cdot 7$
12	$60 = 2^2 \cdot 3 \cdot 5$	$*5040 = 2^4 \cdot 3^2 \cdot 5 \cdot 7$
7	$64 = 2^6$	$7560 = 2^3 \cdot 3^3 \cdot 5 \cdot 7$
12	$72 = 2^3 \cdot 3^2$	$10080 = 2^5 \cdot 3^2 \cdot 5 \cdot 7$
10	$80 = 2^4 \cdot 5$	$15120 = 2^4 \cdot 3^3 \cdot 5 \cdot 7$
12	$84 = 2^2 \cdot 3 \cdot 7$	$20160 = 2^6 \cdot 3^2 \cdot 5 \cdot 7$
12	$90 = 2 \cdot 3^2 \cdot 5$	$25200 = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7$

$dd(N)$	$d(N)$	N
12	$96 = 2^5 \cdot 3$	$27720 = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
9	$100 = 2^2 \cdot 5^2$	$45360 = 2^4 \cdot 3^4 \cdot 5 \cdot 7$
12	$108 = 2^2 \cdot 3^3$	$50400 = 2^5 \cdot 3^2 \cdot 5^2 \cdot 7$
16	$120 = 2^3 \cdot 3 \cdot 5$	$*55440 = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
8	$128 = 2^7$	$83160 = 2^3 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$
15	$144 = 2^4 \cdot 3^2$	$110880 = 2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
12	$160 = 2^5 \cdot 5$	$166320 = 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$
16	$168 = 2^3 \cdot 3 \cdot 7$	$221760 = 2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
18	$180 = 2^2 \cdot 3^2 \cdot 5$	$277200 = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11$
14	$192 = 2^6 \cdot 3$	$332640 = 2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$
12	$200 = 2^3 \cdot 5^2$	$498960 = 2^4 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11$
16	$216 = 2^3 \cdot 3^3$	$554400 = 2^5 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11$
12	$224 = 2^5 \cdot 7$	$665280 = 2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$
20	$240 = 2^4 \cdot 3 \cdot 5$	$*720720 = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
9	$256 = 2^8$	$1081080 = 2^3 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
18	$288 = 2^5 \cdot 3^2$	$*1441440 = 2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
14	$320 = 2^6 \cdot 5$	$2162160 = 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
20	$336 = 2^4 \cdot 3 \cdot 7$	$2882880 = 2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
24	$360 = 2^3 \cdot 3^2 \cdot 5$	$3603600 = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
16	$384 = 2^7 \cdot 3$	$*4324320 = 2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
15	$400 = 2^4 \cdot 5^2$	$6486480 = 2^4 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
20	$432 = 2^4 \cdot 3^3$	$7207200 = 2^5 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
14	$448 = 2^6 \cdot 7$	$8648640 = 2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
24	$480 = 2^5 \cdot 3 \cdot 5$	$10810800 = 2^4 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
24	$504 = 2^3 \cdot 3^2 \cdot 7$	$14414400 = 2^6 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
10	$512 = 2^9$	$17297280 = 2^7 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
21	$576 = 2^6 \cdot 3^2$	$*21621600 = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
24	$600 = 2^3 \cdot 3 \cdot 5^2$	$32432400 = 2^4 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
16	$640 = 2^7 \cdot 5$	$36756720 = 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
24	$672 = 2^5 \cdot 3 \cdot 7$	$43243200 = 2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
30	$720 = 2^4 \cdot 3^2 \cdot 5$	$61261200 = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
18	$768 = 2^8 \cdot 3$	$73513440 = 2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
18	$800 = 2^5 \cdot 5^2$	$110270160 = 2^4 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
24	$864 = 2^5 \cdot 3^3$	$122522400 = 2^5 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
16	$896 = 2^7 \cdot 7$	$147026880 = 2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
28	$960 = 2^6 \cdot 3 \cdot 5$	$183783600 = 2^4 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
30	$1008 = 2^4 \cdot 3^2 \cdot 7$	$245044800 = 2^6 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
11	$1024 = 2^{10}$	$294053760 = 2^7 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
24	$1152 = 2^7 \cdot 3^2$	$*367567200 = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
30	$1200 = 2^4 \cdot 3 \cdot 5^2$	$551350800 = 2^4 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$

$dd(N)$	$d(N)$	N		
18	$1280 = 2^8 \cdot 5$	698377680	$=$	$2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
28	$1344 = 2^6 \cdot 3 \cdot 7$	735134400	$=$	$2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
36	$1440 = 2^5 \cdot 3^2 \cdot 5$	1102701600	$=$	$2^5 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
20	$1536 = 2^9 \cdot 3$	1396755360	$=$	$2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
21	$1600 = 2^6 \cdot 5^2$	2095133040	$=$	$2^4 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
40	$1680 = 2^4 \cdot 3 \cdot 5 \cdot 7$	2205403200	$=$	$2^6 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
28	$1728 = 2^6 \cdot 3^3$	2327925600	$=$	$2^5 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
18	$1792 = 2^8 \cdot 7$	2793510720	$=$	$2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
32	$1920 = 2^7 \cdot 3 \cdot 5$	3491888400	$=$	$2^4 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
36	$2016 = 2^5 \cdot 3^2 \cdot 7$	4655851200	$=$	$2^6 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
12	$2048 = 2^{11}$	5587021440	$=$	$2^7 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
27	$2304 = 2^8 \cdot 3^2$	$*$6983776800	$=$	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
36	$2400 = 2^5 \cdot 3 \cdot 5^2$	10475665200	$=$	$2^4 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
32	$2688 = 2^7 \cdot 3 \cdot 7$	$*$13967553600	$=$	$2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
42	$2880 = 2^6 \cdot 3^2 \cdot 5$	20951330400	$=$	$2^5 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
22	$3072 = 2^{10} \cdot 3$	27935107200	$=$	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
48	$3360 = 2^5 \cdot 3 \cdot 5 \cdot 7$	41902660800	$=$	$2^6 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
32	$3456 = 2^7 \cdot 3^3$	48886437600	$=$	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
20	$3584 = 2^9 \cdot 7$	64250746560	$=$	$2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
45	$3600 = 2^4 \cdot 3^2 \cdot 5^2$	73329656400	$=$	$2^4 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
36	$3840 = 2^8 \cdot 3 \cdot 5$	80313433200	$=$	$2^4 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
42	$4032 = 2^6 \cdot 3^2 \cdot 7$	97772875200	$=$	$2^6 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
13	$4096 = 2^{12}$	128501483120	$=$	$2^7 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
48	$4320 = 2^5 \cdot 3^3 \cdot 5$	146659312800	$=$	$2^5 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
30	$4608 = 2^9 \cdot 3^2$	160626866400	$=$	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
42	$4800 = 2^6 \cdot 3 \cdot 5^2$	240940299600	$=$	$2^4 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
60	$5040 = 7 \cdot 5 \cdot 3^2 \cdot 2^4$	293318625600	$=$	$2^6 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
36	$5376 = 2^8 \cdot 3 \cdot 7$	*321253732800	$=$	$2^6 \cdot 3^3 \cdot 5^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
48	$5760 = 2^7 \cdot 3^2 \cdot 5$	481880599200	$=$	$2^5 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
24	$6144 = 2^{11} \cdot 3$	642507465600	$=$	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
56	$6720 = 2^6 \cdot 3 \cdot 5 \cdot 7$	963761198400	$=$	$2^6 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
36	$6912 = 2^8 \cdot 3^3$	1124388064800	$=$	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
22	$7168 = 2^{10} \cdot 7$	1606268664000	$=$	$2^6 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
54	$7200 = 2^5 \cdot 3^2 \cdot 5^2$	1686582097200	$=$	$2^4 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
40	$7680 = 2^9 \cdot 3 \cdot 5$	1927522396800	$=$	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
48	$8064 = 2^7 \cdot 3^2 \cdot 7$	$*$2248776129600	$=$	$2^6 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
14	$8192 = 2^{13}$	3212537328000	$=$	$2^7 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
56	$8640 = 2^6 \cdot 3^3 \cdot 5$	3373164194400	$=$	$2^5 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
33	$9216 = 2^{10} \cdot 3^2$	4497552259200	$=$	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
72	$10080 = 2^5 \cdot 3^2 \cdot 5 \cdot 7$	6746328388800	$=$	$2^6 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$

8. Now let us consider what must be the nature of N in order that N should be a highly composite number. In the first place it must be of the form

$$2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdot 7^{a_7} \cdots p_1^{a_{p_1}},$$

where

$$a_2 \geq a_3 \geq a_5 \geq \cdots \geq a_{p_1} \geq 1. \tag{32}$$

This follows at once from the fact that

$$d(\varpi_2^{a_2} \varpi_3^{a_3} \varpi_5^{a_5} \cdots \varpi_{p_1}^{a_{p_1}}) = d(2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdots p_1^{a_{p_1}}),$$

for all prime values of $\varpi_2, \varpi_3, \varpi_5, \ldots, \varpi_{p_1}$.
It follows from the definition that, if N is highly composite and $N' < N$, then $d(N')$ must be less that $d(N)$. For example, $\frac{5}{6}N < N$, and so $d(\frac{5}{6}N) < d(N)$. Hence

$$\left(1 + \frac{1}{a_2}\right)\left(1 + \frac{1}{a_3}\right) > \left(1 + \frac{1}{1 + a_5}\right),$$

provided that N is a multiple of 3.
It is convenient to write

$$a_\lambda = 0 \quad (\lambda > p_1). \tag{33}$$

Thus if N is not a multiple of 5 then a_5 should be considered as 0.
Again, a_{p_1} must be less than or equal to 2 for all values of p_1. For let P_1 be the prime next above p_1. Then it can be shewn that $P_1 < p_1^2$ for all values of p_1.*
Now, if a_{p_1} is greater than 2, let

$$N' = \frac{NP_1}{P_1^2}.$$

Then N' is an integer less than N, and so $d(N') < d(N)$. Hence

$$(1 + a_{p_1}) > 2(a_{p_1} - 1),$$

or

$$3 > a_{p_1},$$

*It can be proved by elementary methods that, if $x \geq 1$, there is at least one prime p such that $x < p \leq 2x$. This result is known as Bertrand's Postulate: for a proof, see Landau, *Handbuch*, p. 89. It follows at once that $P_1 < p_1^2$, if $p_1 > 2$; and the inequality is obviously true when $p_1 = 2$. Some similar results used later in this and the next section may be proved in the same kind of way. It is for some purposes sufficient to know that there is always a prime p such that $x < p < 3x$, and the proof of this is easier than that of Bertrand's Postulate. These inequalities are enough, for example, to shew that

$$\log P_1 = \log p_1 + O(1).$$

which contradicts our hypothesis. Hence

$$a_{p_1} \leq 2, \tag{34}$$

for all values of p_1.
Now let $p_1'', p_1', p_1, P_1, P_1'$ be consecutive primes in ascending order. Then, if $p_1 \geq 5, a_{p_1''}$ must be less than or equal to 4. For, if this were not so, we could suppose that

$$N' = \frac{NP_1}{(p_1'')^3}.$$

But it can easily be shewn that, if $p_1 \geq 5$, then

$$(p_1'')^3 > P_1;$$

and so $N' < N$ and $d(N') < d(N)$. Hence

$$(1 + a_{p_1''}) > 2(a_{p_1''} - 2). \tag{35}$$

But since $a_{p_1''} \geq 5$, it is evident that

$$(1 + a_{p_1''}) \leq 2(a_{p_1''} - 2),$$

which contradicts (35); therefore, if $p_1 \geq 5$, then

$$a_{p_1''} \leq 4. \tag{36}$$

Now let

$$N' = \frac{Np_1'' P_1}{p_1' p_1}.$$

It is easy to verify that, if $5 \leq p_1 \leq 19$, then

$$p_1' p_1 > p_1'' P_1;$$

and so $N' < N$ and $d(N') < d(N)$. Hence

$$(1 + a_{p_1})(1 + a_{p_1'})(1 + a_{p_1''}) > 2a_{p_1} a_{p_1'}(2 + a_{p_1''}),$$

or

$$\left(1 + \frac{1}{a_{p_1}}\right)\left(1 + \frac{1}{a_{p_1'}}\right) > 2\left(1 + \frac{1}{1 + a_{p_1''}}\right),$$

But from (36) we know that $1 + a_{p_1''} \leq 5$. Hence

$$\left(1 + \frac{1}{a_{p_1}}\right)\left(1 + \frac{1}{a_{p_1'}}\right) > 2\tfrac{2}{5}. \tag{37}$$

From this it follows that $a_{p_1} = 1$. For, if $a_{p_1} \geq 2$, then

$$\left(1 + \frac{1}{a_{p_1}}\right)\left(1 + \frac{1}{a_{p'_1}}\right) \leq 2\tfrac{1}{4},$$

in virtue of (32). This contradicts (37). Hence, if $5 \leq p_1 \leq 19$, then

$$a_{p_1} = 1. \tag{38}$$

Next let

$$N' = NP_1P'_1/(p_1p'_1p''_1).$$

It can easily be shewn that, if $p_1 \geq 11$, then

$$P_1P'_1 < p_1p'_1p''_1;$$

and so $N' < N$ and $d(N') < d(N)$. Hence

$$(1 + a_{p_1})(1 + a_{p'_1})(1 + a_{p''_1}) > 4a_{p_1}a_{p'_1}a_{p''_1},$$

or

$$\left(1 + \frac{1}{a_{p_1}}\right)\left(1 + \frac{1}{a_{p'_1}}\right)\left(1 + \frac{1}{a_{p''_1}}\right) > 4. \tag{39}$$

From this we infer that a_{p_1} must be 1. For, if $a_{p_1} \geq 2$, it follows from (32) that

$$\left(1 + \frac{1}{a_{p_1}}\right)\left(1 + \frac{1}{a_{p'_1}}\right)\left(1 + \frac{1}{a_{p''_1}}\right) \leq 3\tfrac{3}{8},$$

which contradicts (39). Hence we see that, if $p_1 \geq 11$, then

$$a_{p_1} = 1. \tag{40}$$

It follows from (38) and (40) that, if $p_1 \geq 5$, then

$$a_{p_1} = 1. \tag{41}$$

But if $p_1 = 2$ or 3, then from (34) it is clear that

$$a_{p_1} = 1 \;\text{ or }\; 2. \tag{42}$$

It follows that $a_{p_1} = 1$ for all highly composite numbers, except for 2^2, and perhaps for certain numbers of the form $2^a \cdot 3^2$. In the latter case $a \geq 2$. It is easy to shew that, if $a \geq 3, 2^a.3^2$ cannot be highly composite. For if we suppose that

$$N' = 2^{a-1} \cdot 3 \cdot 5,$$

then it is evident that $N' < N$ and $d(N') < d(N)$, and so

$$3(1+a) > 4a,$$

or

$$a < 3.$$

Hence it is clear that a cannot have any other value except 2. Moreover we can see by actual trial that 2^2 and $2^2 \cdot 3^2$ are highly composite. *Hence*

$$a_{p_1} = 1 \tag{43}$$

for all highly composite values of N save 4 *and* 36*, when*

$$a_{p_1} = 2.$$

Hereafter when we use this result it is to be understood that 4 and 36 are exceptions.

9. It follows from (32) and (43) that N must be of the form

$$\begin{array}{rc} & 2 \cdot 3 \cdot 5 \cdot 7 \cdots p_1 \\ \times & 2 \cdot 3 \cdot 5 \cdot 7 \cdots p_2 \\ \times & 2 \cdot 3 \cdot 5 \cdots p_3 \\ \times & \ldots, \end{array} \tag{44}$$

where $p_1 > p_2 \geq p_3 \geq p_4 \geq \cdots$ and the number of rows is a_2.
Let P_r be the prime next above p_r, so that

$$\log P_r = \log p_r + O(1), \tag{45}$$

in virtue of Bertrand's Postulate. Then it is evident that

$$a_{p_r} \geq r, \quad a_{P_r} \leq r - 1; \tag{46}$$

and so

$$a_{P_r} \leq a_{p_r} - 1. \tag{47}$$

It is to be understood that

$$a_{P_1} = 0, \tag{48}$$

in virtue of (33).
It is clear from the form of (44) that r can never exceed a_2, and that

$$p_{a_\lambda} = \lambda. \tag{49}$$

10. Now let

$$N' = \frac{N}{\nu}\lambda^{[\log\nu/\log\lambda]*},$$

where $\nu \le p_1$ so that N' is an integer. Then it is evident that $N' < N$ and $d(N') < d(N)$, and so

$$(1+a_\nu)(a+a_\lambda) > a_\nu\left(1 + a_\lambda + \left[\frac{\log\nu}{\log\lambda}\right]\right),$$

or

$$1 + a_\lambda > a_\nu\left[\frac{\log\nu}{\log\lambda}\right]. \tag{50}$$

Since the right-hand side vanishes when $\nu > p_1$, we see that (50) is true for all values of λ and ν †.

Again let

$$N' = N_{\mu\lambda}^{-1-[\log\mu/\log\lambda]},$$

where $[\log\mu/\log\lambda] < a_\lambda$, so that N' is an integer. Then it is evident that $N' < N$ and $d(N') < d(N)$, and so

$$(1+a_\mu)(1+a_\lambda) > (2+a_\mu)\left(a_\lambda - \left[\frac{\log\mu}{\log\lambda}\right]\right). \tag{51}$$

Since the right-hand side is less than or equal to 0 when

$$a_\lambda \le [\log\mu/\log\lambda],$$

we see that (51) is true for all values of λ and μ. From (51) it evidently follows that

$$(1+a_\lambda) < (2+a_\mu)\left[\frac{\log(\lambda\mu)}{\log\lambda}\right]. \tag{52}$$

From (50) and (52) it is clear that

$$a_\nu\left[\frac{\log\nu}{\log\lambda}\right] \le a_\lambda \le a_\mu + (2+a_\mu)\left[\frac{\log\mu}{\log\lambda}\right], \tag{53}$$

for all values of λ, μ and ν.

Now let us suppose that $\nu = p_1$ and $\mu = P_1$, so that $a_\nu = 1$ and $a_\mu = 0$. Then we see that

$$\left[\frac{\log p_1}{\log\lambda}\right] \le a_\lambda \le 2\left[\frac{\log P_1}{\log\lambda}\right], \tag{54}$$

for all values of λ. Thus, for example, we have

$$p_1 = 3, \quad 1 \le a_2 \le 4;$$

* $[x]$ denotes as usual the integral part of x.

† That is to say all prime values of λ and ν, since λ in a_λ is be definition prime.

$$p_1 = 5, \quad 2 \leq a_2 \leq 4;$$
$$p_1 = 7, \quad 2 \leq a_2 \leq 6;$$
$$p_1 = 11, \quad 3 \leq a_2 \leq 6;$$

and so on. It follows from (54) that, if $\lambda \leq p_1$, then

$$a_\lambda \log \lambda = O(\log p_1), \quad a_\lambda \log \lambda \neq o(\log p_1). \tag{55}$$

11. Again let

$$N' = N_\lambda^{[\sqrt{\{(1+a_\lambda+a_\mu)\log\mu/\log\lambda\}}]} \mu^{-1-[\sqrt{\{(1+a_\lambda+a_\mu)\log\lambda/\log\mu\}}]},$$

and let us assume for the moment that

$$a_\mu > \sqrt{\{(1+a_\lambda+a_\mu)\log\lambda/\log\mu\}},$$

in order that N' may be an integer. Then $N' < N$ and $d(N') < d(N)$, and so

$$\begin{aligned}(1+a_\lambda)(1+a_\mu) > \quad & \{1 + a_\lambda + [\sqrt{\{(1+a_\lambda+a_\mu)\log\mu/\log\lambda\}}]\} \\ & \times\{a_\mu - [\sqrt{\{(1+a_\lambda+a_\mu)\log\lambda/\log\mu\}}]\} \\ > \quad & \{a_\lambda + \sqrt{\{(1+a_\lambda+a_\mu)\log\mu/\log\lambda\}}\} \\ & \times\{a_\mu - \sqrt{\{(1+a_\lambda+a_\mu)\log\lambda/\log\mu\}}\}.\end{aligned} \tag{56}$$

It is evident that the right-hand side of (56) becomes negative when

$$a_\mu < \sqrt{\{(1+a_\lambda+a_\mu)\log\lambda/\log\mu\}},$$

while the left-hand side remains positive, and so the result is still true. Hence

$$a_\mu \log\mu - a_\lambda \log\lambda < 2\sqrt{\{(1+a_\lambda+a_\mu)\log\lambda\log\mu\}}, \tag{57}$$

for all values of λ and μ. Interchanging λ and μ in (57), we obtain

$$a_\lambda \log\lambda - a_\mu \log\mu < 2\sqrt{\{(1+a_\lambda+a_\mu)\log\lambda\log\mu\}}. \tag{58}$$

From (57) and (58) it evidently follows that

$$|a_\lambda \log\lambda - a_\mu \log\mu| < 2\sqrt{\{(1+a_\lambda+a_\mu)\log\lambda\log\mu\}}, \tag{59}$$

for all values of λ and μ. It follows from this and (55) that, if λ and μ are neither greater than p_1, then

$$a_\lambda \log\lambda - a_\mu \log\mu = O\sqrt{\{\log p_1 \log(\lambda\mu)\}}, \tag{60}$$

and so that, if $\log\lambda = o(\log p_1)$, then

$$a_2 \log 2 \sim a_3 \log 3 \sim a_5 \log 5 \sim \cdots \sim a_\lambda \log \lambda. \tag{61}$$

12. It can easily be shewn by elementary algebra that, if x, y, m and n are not negative, and if

$$|x - y| < 2\sqrt{(mx + ny + mn)},$$

then

$$\left.\begin{array}{l} |\sqrt{(x+n)} - \sqrt{(y+m)}| < \sqrt{(m+n)}; \\ |\sqrt{(x+n)} - \sqrt{(m+n)}| < \sqrt{(y+m)}. \end{array}\right\} \tag{62}$$

From (62) and (59) it follows that

$$|\sqrt{\{(1+a_\lambda)\log\lambda\}} - \sqrt{\{(1+a_\mu)\log\mu\}}| < \sqrt{\{\log(\lambda\mu)\}}, \tag{63}$$

and

$$|\sqrt{\{(1+a_\lambda)\log\lambda\}} - \sqrt{\{\log(\lambda\mu)\}}| < \sqrt{\{(1+a_\mu)\log\mu\}}, \tag{64}$$

for all values of λ and μ. If, in particular, we put $\mu = 2$ in (63), we obtain

$$\begin{aligned} \sqrt{\{(1+a_2)\log 2\}} - \sqrt{\{\log(2\lambda)\}} &< \sqrt{\{(1+a_\lambda)\log\lambda\}} \\ &< \sqrt{\{(1+a_2)\log 2\}} + \sqrt{\{\log(2\lambda)\}}, \end{aligned} \tag{65}$$

for all values of λ. Again, from (63), we have

$$(1+a_\lambda)\log\lambda < (\sqrt{\{(1+a_\nu)\log\nu\}} + \sqrt{\{\log(\lambda\nu)\}})^2,$$

or

$$a_\lambda \log\lambda < (1+a_\nu)\log\nu + \log\nu + 2\sqrt{\{(1+a_\nu)\log\nu\log(\lambda\nu)\}}. \tag{66}$$

Now let us suppose that $\lambda \leq \mu$. Then, from (66), it follows that

$$\begin{aligned} a_\lambda\log\lambda + \log\mu &< (1+a_\nu)\log\nu + \log(\mu\nu) + 2\sqrt{\{(1+a_\nu)\log\nu\log(\lambda\nu)\}} \\ &\leq (1+a_\nu)\log\nu + \log(\mu\nu) + 2\sqrt{\{(1+a_\nu)\log\nu\log(\mu\nu)\}} \\ &= \{\sqrt{\{(1+a_\nu)\log\nu\}} + \sqrt{\log(\mu\nu)}\}^2, \end{aligned} \tag{67}$$

with the condition that $\lambda \leq \mu$. Similarly we can shew that

$$a_\lambda\log\lambda + \log\mu > \{\sqrt{\{(1+a_\nu)\log\nu\}} - \sqrt{\log(\mu\nu)}\}^2, \tag{67'}$$

with the condition that $\lambda \le \mu$.

13. Now let

$$N' = \frac{N}{\lambda} 2^{[\log\lambda/\{\pi(\mu)\log 2\}]} 3^{[\log\lambda/\{\pi(\mu)\log 3\}]} \cdots \mu^{[\log\lambda/\{\pi(\mu)\log\mu\}]},$$

where $\pi(\mu)\log\mu < \log\lambda \le \log p_1$. Then it is evident that N' is an integer less than N, and so $d(N') < d(N)$. Hence

$$\begin{aligned} &\left(1+\frac{1}{a_\lambda}\right)(1+a_2)(1+a_3)(1+a_5)\cdots(1+a_\mu) \\ &\quad > \left\{a_2 + \frac{\log\lambda}{\pi(\mu)\log 2}\right\}\left\{a_3 + \frac{\log\lambda}{\pi(\mu)\log 3}\right\}\cdots\left\{a_\mu + \frac{\log\lambda}{\mu(\mu)\log\mu}\right\}; \end{aligned}$$

that is

$$\begin{aligned} &\left\{a_2\log 2 + \frac{\log\lambda}{\pi(\mu)}\right\}\left\{a_3\log 3 + \frac{\log\lambda}{\pi(\mu)}\right\}\cdots\left\{a_\mu\log\mu + \frac{\log\lambda}{\pi(\mu)}\right\} \\ &\quad < \left(1+\frac{1}{a_\lambda}\right)(a_2\log 2 + \log 2)(a_3\log 3 + \log 3)\cdots(a_\mu\log\mu + \log\mu) \\ &\quad \le \left(1+\frac{1}{a_\lambda}\right)(a_2\log 2 + \log\mu)(a_3\log 3 + \log\mu)\cdots(a_\mu\log\mu + \log\mu). \end{aligned}$$

In other words

$$\begin{aligned} &\left(1+\frac{1}{a_\lambda}\right) \\ &\quad > \left\{1 + \frac{\frac{\log\lambda}{\pi(\mu)} - \log\mu}{a_2\log 2 + \log\mu}\right\}\left\{1 + \frac{\frac{\log\lambda}{\pi(\mu)} - \log\mu}{a_3\log 3 + \log\mu}\right\}\cdots\left\{1 + \frac{\frac{\log\lambda}{\pi(\mu)} - \log\mu}{a_\mu\log\mu + \log\mu}\right\} \\ &\quad > \left\{1 + \frac{\frac{\log\lambda}{\pi(\mu)} - \log\mu}{\{\sqrt{\{(1+a_\nu)\log\nu\}} + \sqrt{\log(\mu\nu)}\}^2}\right\}^{\pi(\mu)}, \end{aligned} \tag{68}$$

w here ν is any prime, in virtue of (67). From (68) it follows that

$$\sqrt{\{(1+a_\nu)\log\nu\}} + \sqrt{\log(\mu\nu)} > \sqrt{\left\{\frac{\frac{\log\lambda}{\pi(\mu)} - \log\mu}{\left(1+\frac{1}{a_\lambda}\right)^{1/\pi(\mu)} - 1}\right\}}, \tag{69}$$

provided that $\pi(\mu)\log\mu < \log\lambda \le \log p_1$.

14. Again let

$$N' = N\lambda 2^{-1-[\log\lambda/\{\pi(\mu)\log 2\}]} 3^{-1-[\log\lambda/\{\pi(\mu)\log 3\}]} \cdots \mu^{-1-[\log\lambda/\{\pi(\mu)\log\mu\}]},$$

where $\mu \le p_1$ and $\lambda > \mu$. Let us assume for the moment that

$$a_\kappa \log \kappa > \frac{\log \lambda}{\pi(\mu)},$$

for all values of κ less than or equal to μ, so that N' is an integer. Then, by arguments similar to those of the previous section, we can shew that

$$\frac{1+a_\lambda}{2+a_\lambda} > \left\{1 - \frac{\frac{\log\lambda}{\pi(\mu)} + \log\mu}{\{\sqrt{\{(1+a_\nu)\log\nu\}} - \sqrt{\log(\mu\nu)}\}^2}\right\}^{\pi(\mu)}. \tag{70}$$

From this it follows that

$$|\sqrt{\{(1+a_\nu)\log\nu\}} - \sqrt{\log(\mu\nu)}| < \sqrt{\left\{\frac{\frac{\log\lambda}{\pi(\mu)} + \log\mu}{1 - \left(\frac{1+a_\lambda}{2+a_\lambda}\right)^{1/\pi(\mu)}}\right\}}, \tag{71}$$

provided that $\mu \le p_1$ and $\mu < \lambda$. The condition that

$$a_\kappa \log \kappa > \{\log\lambda/\pi(\mu)\}$$

is unnecessary because we know from (67′) that

$$|\sqrt{\{(1+a_\nu)\log\nu\}} - \sqrt{\log(\mu\nu)}| < \sqrt{(a_\kappa\log\kappa + \log\mu)} \le \sqrt{\left\{\frac{\log\lambda}{\pi(\mu)} + \log\mu\right\}}, \tag{72}$$

when

$$a_\kappa \log\kappa \le \{\log\lambda/\pi(\mu)\},$$

and the last term in (72) is evidently less than the right-hand side of (71).

15. We shall consider in this and the following sections some important deductions from the preceding formulæ. Putting $\nu = 2$ in (69) and (71), we obtain

$$\sqrt{\{(1+a_2)\log 2\}} > \sqrt{\left\{\frac{\frac{\log\lambda}{\pi(\mu)} - \log\mu}{\left(1+\frac{1}{a_\lambda}\right)^{1/\pi(\mu)} - 1}\right\}} - \sqrt{\log(2\mu)}, \tag{73}$$

provided that $\pi(\mu)\log\mu < \log\lambda \le \log p_1$, and

$$\sqrt{\{(1+a_2)\log 2\}} < \sqrt{\left\{\frac{\frac{\log\lambda}{\pi(\mu)} + \log\mu}{1 - \left(\frac{1+a_\lambda}{2+a_\lambda}\right)^{1/\pi(\mu)}}\right\}} + \sqrt{\log(2\mu)}, \tag{74}$$

provided that $\mu \leq p_1$, and $\mu < \lambda$. Now supposing that $\lambda = p_1$ in (73), and $\lambda = P_1$ in (74), we obtain

$$\sqrt{\{(1+a_2)\log 2\}} > \sqrt{\left\{\frac{\frac{\log p_1}{\pi(\mu)} - \log\mu}{2^{1/\pi(\mu)} - 1}\right\}} - \sqrt{\log(2\mu)}, \tag{75}$$

provided that $\pi(\mu)\log\mu < \log p_1$, and

$$\sqrt{\{(1+a_2)\log 2\}} < \sqrt{\left\{\frac{\frac{\log P_1}{\pi(\mu)} + \log\mu}{1 - 2^{-1/\pi(\mu)}}\right\}} + \sqrt{\log(2\mu)}, \tag{76}$$

provided that $\mu \leq p_1$. In (75) and (76) μ can be so chosen as to obtain the best possible inequality for a_2. If p_1 is too small, we may abandon this result in favour of

$$\left[\frac{\log p_1}{\log 2}\right] \leq a_2 \leq 2\left[\frac{\log P_1}{\log 2}\right], \tag{77}$$

which is obtained from (54) by putting $\lambda = 2$.
After having obtained in this way what information we can about a_2, we may use (73) and (74) to obtain information about a_λ. Here also we have to choose μ so as to obtain the best possible inequality for a_λ. But if λ is too small we may, instead of this, use

$$\begin{aligned}\sqrt{\{(1+a_2)\log 2\}} - \sqrt{\log(2\lambda)} &< \sqrt{\{(1+a_\lambda) log \lambda\}} \\ &< \sqrt{\{(1+a_2)\log 2\}} + \sqrt{\log(2\lambda)},\end{aligned} \tag{78}$$

which is obtained by putting $\mu = 2$ in (63).

16. Now let us consider the order of a_2. From (73) it is evident that, if $\pi(\mu)\log\mu < \log\lambda \leq \log p_1$, then

$$(1+a_2)\log 2 + \log(2\mu) + 2\sqrt{\{(1+a_2)\log 2\log(2\mu)\}} > \frac{\frac{\log\lambda}{\pi(\mu)} - \log\mu}{\left(1+\frac{1}{a_\lambda}\right)^{1/\pi(\mu)} - 1} \tag{79}$$

But we know that for positive values of x,

$$\frac{1}{e^x - 1} = \frac{1}{x} + O(1), \quad \frac{1}{e^x - 1} = O\left(\frac{1}{x}\right).$$

Hence

$$\frac{\log\lambda}{\pi(\mu)}\frac{1}{\left(1+\frac{1}{a_\lambda}\right)^{1/\pi(\mu)} - 1} = \frac{\log\lambda}{\pi(\mu)}\left\{\frac{\pi(\mu)}{\log\left(1+\frac{1}{a_\lambda}\right)} + O(1)\right\}$$

$$= \frac{\log \lambda}{\log\left(1+\frac{1}{a_\lambda}\right)} + O\left\{\frac{\log \lambda}{\pi(\mu)}\right\};$$

and

$$\frac{\log \mu}{\left(1+\frac{1}{a_\lambda}\right)^{1/\pi(\mu)} - 1} = O\left\{\frac{\pi(\mu)\log \mu}{\log\left(1+\frac{1}{a_\lambda}\right)}\right\} = O(\mu a_\lambda).$$

Again from (55) we know that $a_2 = O(\log p_1)$. Hence (79) may be written as

$$a_2 \log 2 + O\sqrt{(\log p_1 \log \mu) + O(\log \mu)}$$
$$\geq \frac{\log \lambda}{\log\left(1+\frac{1}{a_\lambda}\right)} + O\left\{\frac{\log \lambda}{\pi(\mu)}\right\} + O(\mu a_\lambda). \qquad (80)$$

But

$$\log \mu = O(\mu a_\lambda),$$
$$\mu a_\lambda = \frac{\mu}{\log \lambda} \cdot a_\lambda \log \lambda = O\left(\frac{\mu \log p_1}{\log \lambda}\right),$$
$$\frac{\log \lambda}{\pi(\mu)} = O\left\{\frac{\log \lambda \log \mu}{\mu}\right\}.$$

Again

$$\frac{\log \lambda \log \mu}{\mu} + \frac{\mu \log p_1}{\log \lambda} > 2\sqrt{(\log p_1 \log \mu)};$$

and so

$$\sqrt{(\log p_1 \log \mu)} = O\left(\frac{\log \lambda \log \mu}{\mu} + \frac{\mu \log p_1}{\log \lambda}\right).$$

Hence (80) may be replaced by

$$a_2 \log 2 \geq \frac{\log \lambda}{\log\left(1+\frac{1}{a_\lambda}\right)} + O\left(\frac{\log \lambda \log \mu}{\mu} + \frac{\mu \log p_1}{\log \lambda}\right), \qquad (81)$$

provided that $\pi(\mu)\log \mu < \log \lambda \leq \log p_1$. Similarly, from (74), we can shew that

$$a_2 \log 2 \leq \frac{\log \lambda}{\log\left(1+\frac{1}{1+a_\lambda}\right)} + O\left(\frac{\log \lambda \log \mu}{\mu} + \frac{\mu \log p_1}{\log \lambda}\right), \qquad (82)$$

provided that $\mu \leq p_1$ and $\mu < \lambda$. Now supposing that $\lambda = p_1$ in (81), and $\lambda = P_1$ in (82), and also that

$$\mu = O\sqrt{(\log p_1 \log\log p_1)}, \mu \neq o\sqrt{(\log p_1 \log\log p_1)},^{*}$$

[*] $f \neq o(\phi)$ is to be understood as meaning that $|f| > K\phi$, where K is a constant, and $f \neq O(\phi)$ as meaning that $|f|/\phi \to \infty$. They are *not* the mere nagations of $f = o(\phi)$ and $f = O(\phi)$.

we obtain

$$\left.\begin{aligned} a_2 \log 2 &\geq \tfrac{\log p_1}{\log 2} + O\sqrt{(\log p_1 \log\log p_1)}, \\ a_2 \log 2 &\leq \tfrac{\log p_1}{\log 2} + O\sqrt{(\log p_1 \log\log p_1)}. \end{aligned}\right\} \tag{83}$$

From (83) it evidently follows that

$$a_2 \log 2 = \frac{\log p_1}{\log 2} + O\sqrt{(\log p_1 \log\log p_1)}. \tag{84}$$

And it follows from this and (60) that if $\lambda \leq p_1$ then

$$a_\lambda \log \lambda = \frac{\log p_1}{\log 2} + O\{\sqrt{\log p_1 \log \lambda)} + \sqrt{(\log p_1 \log\log p_1)}\}. \tag{85}$$

Hence, if $\log \lambda = o(\log p_1)$, we have

$$a_2 \log 2 \sim a_3 \log 3 \sim a_5 \log 5 \sim \cdots \sim a_\lambda \log \lambda \sim \frac{\log p_1}{\log 2}. \tag{86}$$

17. The relations (86) give us information about the order of a_λ when λ is sufficiently small compared to p_1, in fact, when λ is of the form p_1^ϵ, where $\epsilon \to 0$. Such values of λ constitute but a small part of its total range of variation, and it is clear that further formulæ must be proved before we can gain an adequate idea of the general behaviour of a_λ. From (81), (82) and (84) it follows that

$$\left.\begin{aligned} \frac{\log \lambda}{\log\left(1 + \frac{1}{a_\lambda}\right)} &\leq \frac{\log p_1}{\log 2} + O\left\{\frac{\log \lambda \log \mu}{\mu} + \frac{\mu \log p_1}{\log \lambda} + \sqrt{(\log p_1 \log\log p_1)}\right\}, \\ \frac{\log \lambda}{\log\left(1 + \frac{1}{1+a_\lambda}\right)} &\geq \frac{\log p_1}{\log 2} + O\left\{\frac{\log \lambda \log \mu}{\mu} + \frac{\mu \log p_1}{\log \lambda} + \sqrt{(\log p_1 \log\log p_1)}\right\}, \end{aligned}\right\} \tag{87}$$

provided that $\pi(\mu) \log \mu \log \lambda \leq \log p_1$. From this we can easily shew that if

$$\pi(\mu) \log \mu \log \lambda \leq \log p_1$$

then

$$\left.\begin{aligned} a_\lambda &\leq (2^{\log \lambda / \log p_1} - 1)^{-1} + O\left\{\frac{\log \mu}{\mu} + \frac{\mu \log p_1}{(\log \lambda)^2} + \frac{\sqrt{(\log p_1 \log\log p_1)}}{\log \lambda}\right\}, \\ a_\lambda &\geq (2^{\log \lambda / \log p_1} - 1)^{-1} - 1 + O\left\{\frac{\log \mu}{\mu} + \frac{\mu \log p_1}{(\log \lambda)^2} + \frac{\sqrt{(\log p_1 \log\log p_1)}}{\log \lambda}\right\}. \end{aligned}\right\} \tag{88}$$

Now let us suppose that

$$\log\lambda \neq o\sqrt{\left(\frac{\log p_1}{\log\log p_1}\right)}.$$

Then we can choose μ so that

$$\mu = O\left\{\log\lambda\sqrt{\left(\frac{\log\log p_1}{\log p_1}\right)}\right\},$$

$$\mu \neq o\left\{\log\lambda\sqrt{\left(\frac{\log\log p_1}{\log p_1}\right)}\right\}.$$

Now it is clear that $\log\mu = O(\log\log p_1)$, and so

$$\frac{\log\mu}{\mu} = O\left(\frac{\log\log p_1}{\mu}\right) = O\left\{\frac{\sqrt{(\log p_1 \log\log p_1)}}{\log\lambda}\right\};$$

and

$$\frac{\mu\log p_1}{(\log\lambda)^2} = O\left\{\frac{\sqrt{(\log p_1 \log\log p_1)}}{\log\lambda}\right\}.$$

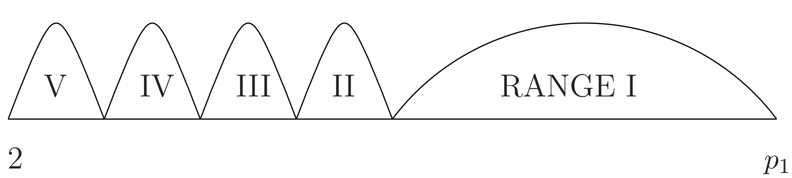
From this and (88) it follows that, if

$$\log\lambda \neq o\sqrt{\left(\frac{\log p_1}{\log\log p_1}\right)},$$

then

$$\left.\begin{aligned} a_\lambda &\leq (2^{\log\lambda/\log p_1} - 1)^{-1} + O\left\{\frac{\sqrt{(\log p_1 \log\log p_1)}}{\log\lambda}\right\}, \\ a_\lambda &\geq (2^{\log\lambda/\log p_1} - 1)^{-1} + O\left\{\frac{\sqrt{(\log p_1 \log\log p_1)}}{\log\lambda}\right\}. \end{aligned}\right\} \tag{89}$$

Now we shall divide that primes from 2 to p_1 into five ranges thus



$$(lp_1)^\kappa \quad e\{\kappa(lp)^{\frac{1}{3}}\}e\left\{\kappa\left(\frac{lp}{l_2p}\right)^{\frac{1}{2}}\right\}e\{\kappa(lpl_2p)^{\frac{1}{2}}\}$$

We shall use the inequalities (89) to specify the behavior of a_λ in ranges I and II, and the formula (85) in ranges IV and V. Range III we shall deal with differently, by a different choice of μ in the inequalities (88). We can easily see that each result in the following sections gives the most information in its particular range.

18. Range I:

$$\log\lambda \neq O\sqrt{(\log p_1 \log\log p_1)}.^*$$

Let

$$\Lambda = [(2^{\log\lambda/\log p_1} - 1)^{-1}],$$

and let

$$(2^{\log\lambda/\log p_1} - 1)^{-1} + \epsilon_\lambda,$$

where $-\frac{1}{2} < \epsilon_\lambda < \frac{1}{2}$, be an integer, so that

$$(2^{\log\lambda/\log p_1} - 1)^{-1} = \Lambda + 1 - \epsilon_\lambda \tag{90}$$

when $\epsilon_\lambda > 0$, and

$$(2^{\log\lambda/\log p_1} - 1)^{-1} = \Lambda - \epsilon_\lambda \tag{91}$$

when $\epsilon_\lambda < 0$. By our supposition we have

$$\frac{\sqrt{(\log p_1 \log\log p_1)}}{\log\lambda} = o(1). \tag{92}$$

First let us consider the case in which

$$\epsilon_\lambda \neq O\left\{\frac{\sqrt{(\log p_1 \log\log p_1)}}{\log\lambda}\right\},$$

so that

$$\frac{\sqrt{(\log p_1 \log\log p_1)}}{\log\lambda} = o(\epsilon_\lambda). \tag{93}$$

It follows from (89), (90), and (93) that, if $\epsilon_\lambda > 0$, then

$$\left.\begin{aligned} a_\lambda &\leq \Lambda + 1 - \epsilon_\lambda + o(\epsilon_\lambda), \\ a_\lambda &\geq \Lambda - \epsilon_\lambda + o(\epsilon_\lambda). \end{aligned}\right\} \tag{94}$$

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*We can with a little trouble replace all equations of the type $f = O(\phi)$ which occur by inequalities of the type $|f| < K\phi$, with definite numerical constants. This would enable us to extend all the different ranges a little. For example, an equation true for

$$\log\lambda \neq O\sqrt{(\log p_1)}$$

would be replaced by an inequality true for $\log\lambda > K\sqrt{(\log p_1)}$, where K is definite constant, and similarly $\log\lambda = o\sqrt{(\log p_1)}$ would be replaced by $\log\lambda < k\sqrt{(\log p_1)}$.

Since $0 < \epsilon_\lambda < \frac{1}{2}$, and a_λ and Λ are integers, it follows from (94) that

$$a_\lambda \leq \Lambda, \quad a_\lambda > \Lambda - 1. \tag{95}$$

Hence

$$a_\lambda = \Lambda. \tag{96}$$

Similarly from (89) (91) and (93) we see that, if $\epsilon_\lambda < 0$, then

$$\left.\begin{aligned} a_\lambda &\leq \Lambda - \epsilon_\lambda + o(\epsilon_\lambda), \\ a_\lambda &\geq \Lambda - 1 - \epsilon_\lambda + o(\epsilon_\lambda). \end{aligned}\right\} \tag{97}$$

Since $-\frac{1}{2} < \epsilon_\lambda < 0$, it follows from (97) that the inequalities (95), and therefore the equation (96), still hold. Hence (96) holds whenever

$$\epsilon_\lambda \neq O\left\{\frac{\sqrt{(\log p_1 \log\log p_1)}}{\log \lambda}\right\}. \tag{98}$$

In particular it holds whenever

$$\epsilon_\lambda \neq 0(1), \tag{99}$$

Now let us consider the case in which

$$\epsilon_\lambda = 0\left\{\frac{\sqrt{(\log p_1 \log\log p_1)}}{\log \lambda}\right\}, \tag{100}$$

so that $\epsilon_\lambda = o(1)$, in virtue of (92). It follows from this and (89) and (90) that, if $\epsilon_\lambda > 0$, then

$$\left.\begin{aligned} a_\lambda &\leq \Lambda + 1 + o(1), \\ a_\lambda &\geq \Lambda + o(1). \end{aligned}\right\} \tag{101}$$

Hence

$$a_\lambda \leq \Lambda + 1, \quad a_\lambda \geq \Lambda;$$

and so

$$a_\lambda = \Lambda \ \text{ or } \ \Lambda + 1. \tag{102}$$

Similarly from (89), (91), and (100), we see that, if $\epsilon_\lambda < 0$, then

$$\left.\begin{aligned} a_\lambda &\leq \Lambda + o(1), \\ a_\lambda &\geq \Lambda - 1 + o(1). \end{aligned}\right\} \tag{103}$$

Hence

$$a_\lambda \le \Lambda, \quad a_\lambda \ge \Lambda - 1;$$

and so

$$a_\lambda = \Lambda \quad \text{or} \quad \Lambda - 1. \tag{104}$$

For example, let us suppose that it is required to find a_λ when $\lambda \sim p_1^{\frac{1}{8}}$. We have

$$(2^{\log\lambda/\log p_1} - 1)^{-1} = (2^{1/8} - 1)^{-1} + o(1) = 11.048\cdots + o(1).$$

It is evident that $\Lambda = 11$ and $\epsilon_\lambda \ne o(1)$. Hence $a_\lambda = 11$.

19. The results in the previous section may be rewritten with slight modifications, in order that the transition of a_λ from one value to another may be more clearly expressed. Let

$$\lambda = p_1^{\frac{\log(1+1/x)}{\log 2}}, \tag{105}$$

and let $x + \epsilon_x$, where $-\frac{1}{2} < \epsilon_x < \frac{1}{2}$, be an integer. Then the range of x which we are now considering is

$$x = o\sqrt{\left(\frac{\log p_1}{\log\log p_1}\right)}, \tag{106}$$

and the results of the previous section may be stated as follows. If

$$\epsilon_x \ne O\left\{x\sqrt{\left(\frac{\log\log p_1}{\log p_1}\right)}\right\}, \tag{107}$$

then

$$a_\lambda = [x]. \tag{108}$$

As a particular case of this we have

$$a_\lambda = [x],$$

when $\epsilon_x \ne o(1)$. But if

$$\epsilon_x = O\left\{x\sqrt{\left(\frac{\log\log p_1}{\log p_1}\right)}\right\}, \tag{109}$$

then when $\epsilon_x > 0$

$$a_\lambda = [x] \quad \text{or} \quad [x+1]; \tag{110}$$

and when $\epsilon_x < 0$

$$a_\lambda = [x] \quad \text{or} \quad [x-1]. \tag{110$'$}$$

20. Range II:

$$\left.\begin{aligned} &\log\lambda = O\sqrt{(\log p_1 \log\log p_1)}, \\ &\log\lambda \neq o\sqrt{\left(\frac{\log p_1}{\log\log p_1}\right)}. \end{aligned}\right\}$$

From (89) it follows that

$$a_\lambda = (2^{\log\lambda/\log p_1} - 1)^{-1} + O\left\{\frac{\sqrt{(\log p_1 \log\log p_1)}}{\log\lambda}\right\}. \tag{111}$$

But

$$(2^{\log\lambda/\log p_1} - 1)^{-1} = \frac{\log p_1}{\log 2\log\lambda} + O(1).$$

Hence

$$a_\lambda \log\lambda = \frac{\log p_1}{\log 2} + O\sqrt{(\log p_1 \log\log p_1)}. \tag{112}$$

As an example we may suppose that

$$\lambda \sim e^{\sqrt{(\log p_1)}}.$$

Then from (112) it follows that

$$a_\lambda = \frac{\sqrt{(\log p_1)}}{\log 2} + O\sqrt{(\log\log p_1)}.$$

21. Range III:

$$\left.\begin{aligned} &\log\lambda = O\sqrt{\left(\frac{\log p_1}{\log\log p_1}\right)}, \\ &\log\lambda \neq o(\log p_1)^{\frac{1}{3}}. \end{aligned}\right\}$$

Let us suppose that $\mu = O(1)$ in (88). Then we see that

$$a_\lambda = \frac{\log p_1}{\log 2\log\lambda} + O(1) + O\left\{\frac{\log\mu}{\mu} + \frac{\mu\log p_1}{(\log\lambda)^2} + \frac{\sqrt{(\log p_1\log\log p_1)}}{\log\lambda}\right\}, \tag{113}$$

or

$$a_\lambda \log\lambda = \frac{\log p_1}{\log 2} + O\left\{\frac{\log\mu\log\lambda}{\mu} + \frac{\mu\log p_1}{\log\lambda} + \sqrt{(\log p_1 \log\log p_1)}\right\}. \tag{114}$$

Now

$$\frac{\log\mu\log\lambda}{\mu} = O(\log\lambda) = o\left(\frac{\log p_1}{\log\lambda}\right),$$

$$\frac{\mu\log p_1}{\log\lambda} = O\left(\frac{\log p_1}{\log\lambda}\right),$$

$$\sqrt{(\log p_1 \log\log p_1)} = O\left(\frac{\log p_1}{\log\lambda}\right).$$

Hence

$$a_\lambda \log\lambda = \frac{\log p_1}{\log 2} + O\left(\frac{\log p_1}{\log\lambda}\right). \tag{115}$$

For example, when

$$\lambda \sim e^{(\log p_1)^{\frac{3}{8}}},$$

we have

$$a_\lambda = \frac{(\log p_1)^{\frac{5}{8}}}{\log 2} + O(\log p_1)^{\frac{1}{4}}.$$

22. Range IV:

$$\left.\begin{aligned} &\log\lambda = O(\log p_1)^{\frac{1}{3}}, \\ &\log\lambda \neq o(\log\log p_1). \end{aligned}\right\}$$

In this case it follows from (85) that

$$a_\lambda \log\lambda = \frac{\log p_1}{\log 2} + O\sqrt{(\log p_1 \log\lambda)}. \tag{116}$$

As an example in this range, when we suppose that

$$\lambda \sim e^{(\log p_1)^{\frac{1}{4}}},$$

we obtain from (116)

$$a_\lambda = \frac{(\log p_1)^{\frac{3}{4}}}{\log 2} + O(\log p_1)^{\frac{3}{8}}.$$

23. Range V: $\log\lambda = O(\log\log p_1)$.

From (85) it follows that

$$a_\lambda \log \lambda = \frac{\log p_1}{\log 2} + O\sqrt{(\log p_1 \log\log p_1)}. \tag{117}$$

For example, we may suppose that

$$\lambda \sim e^{\sqrt{(\log\log p_1)}}.$$

Then

$$a_\lambda = \frac{\log p_1}{\log 2\sqrt{(\log\log p_1)}} + O\sqrt{(\log p_1)}.$$

24. Let λ' be the prime next below λ, so that $\lambda' \leq \lambda - 1$. Then it follows from (63) that

$$\sqrt{\{(1+a_{\lambda'})\log\lambda'\}} - \sqrt{\{(1+a_\lambda)\log\lambda\}} > -\sqrt{\log(\lambda\lambda')}. \tag{118}$$

Hence

$$\sqrt{\{(1+a_{\lambda'}\log(\lambda-1)\}} - \sqrt{\{(1+a_\lambda)\log\lambda\}} > -\sqrt{\{2\log\lambda\}}. \tag{119}$$

But

$$\log(\lambda-1) < \log\lambda - \frac{1}{\lambda} < \log\lambda\left(1 - \frac{1}{2\lambda\log\lambda}\right)^2;$$

and so (119) may be replaced by

$$\sqrt{(1+a_{\lambda'})} - \sqrt{(1+a_\lambda)} > \frac{\sqrt{(1+a_{\lambda'})}}{2\lambda\log\lambda} - \sqrt{2}. \tag{120}$$

But from (54) we know that

$$1 + a_{\lambda'} \geq 1 + \left[\frac{\log p_1}{\log\lambda'}\right] > \frac{\log p_1}{\log\lambda'} > \frac{\log p_1}{\log\lambda}.$$

From this and (120) it follows that

$$\sqrt{(1+a_{\lambda'})} - \sqrt{(1+a_\lambda)} > \frac{\sqrt{(\log p_1)}}{2\lambda(\log\lambda)^{\frac{3}{2}}} - \sqrt{2}. \tag{121}$$

Now let us suppose that $\lambda^2(\log\lambda)^3 < \frac{1}{8}\log p_1$. Then, from (121), we have

$$\sqrt{(1+a_{\lambda'})} - \sqrt{(1+a_\lambda)} > 0,$$

or

$$a_{\lambda'} > a_\lambda. \tag{122}$$

From (122) it follows that, if $\lambda^2(\log\lambda)^3 < \frac{1}{8}\log p_1$, then

$$a_2 > a_3 > a_5 > a_7 > \cdots > a_\lambda. \tag{123}$$

In other words, *in a large highly composite number*

$$2^{a_2}\cdot 3^{a_3}\cdot 5^{a_5}\cdot 7^{a_7}\cdots p_1,$$

the indices comparatively near the beginning form a decreasing sequence in the strict sense which forbids equality. Later on groups of equal indices will in general occur.
To sum up, we have obtained fairly accurate information about a_λ for all possible values of λ. The range I is by far the most extensive, and throughout this range a_λ is known with an error never exceeding 1. The formulæ (86) hold throughout a range which includes all the remaining ranges II - V, and a considerable part of I as well, while we have obtained more precise formulæ for each individual range II-V.

25. Now let us consider the nature of p_r. It is evident that r cannot exceed a_2; i.e., r cannot exceed

$$\frac{\log p_1}{(\log 2)^2} + O\sqrt{(\log p_1 \log\log p_1)}. \tag{124}$$

From (55) it evidently follows that

$$\left.\begin{aligned} a_{p_r}\log p_r &= O(\log p_1),\\ a_{p_r}\log p_r &\neq o(\log p_1);\end{aligned}\right\} \tag{125}$$

$$\left.\begin{aligned} (1+a_{P_r})\log p_r &= O(\log p_1),\\ (1+a_{P_r})\log p_r &\neq o(\log p_1).\end{aligned}\right\} \tag{126}$$

But from (46) we know that

$$\left.\begin{aligned} a_{p_r}\log p_r &\geq r\log p_r,\\ (1+a_{P_r})\log p_r &\leq r\log p_r.\end{aligned}\right\} \tag{127}$$

From (125) - (127) it follows that

$$\left.\begin{aligned} r\log p_r &= O(\log p_1),\\ r\log p_r &\neq o(\log p_1);\end{aligned}\right\} \tag{128}$$

and

$$\left.\begin{aligned} a_{p_r} &= O(r),\\ a_{p_r} &\neq o(r).\end{aligned}\right\} \tag{129}$$

26. Supposing that $\lambda = p_r$ in (81) and $\lambda = P_r$ in (82), and remembering (128), we see that, if $r\mu = o(\log p_1)$, then

$$\log\left(1 + \frac{1}{a_{p_r}}\right) \geq \frac{\log p_r}{a_2 \log 2}\left\{1 + O\left(\frac{\log \mu}{r\mu} + \frac{r\mu}{\log p_1}\right)\right\}, \tag{130}$$

and

$$\log\left(1 + \frac{1}{a + a_{P_r}}\right) \leq \frac{\log P_r}{a_2 \log 2}\left\{1 + O\left(\frac{\log \mu}{r\mu} + \frac{r\mu}{\log p_1}\right)\right\}. \tag{131}$$

But, from (47), we have

$$\log\left(1 + \frac{1}{a_{p_r}}\right) \leq \log\left(1 + \frac{1}{1 + a_{P_r}}\right).$$

Also we know that

$$\log P_r = \log p_r + O(1) = \log p_r \left\{1 + O\left(\frac{1}{\log p_r}\right)\right\} = \log p_r \left\{1 + O\left(\frac{r}{\log p_1}\right)\right\}.$$

Hence (131) may be replaced by

$$\log\left(1 + \frac{1}{a_{p_r}}\right) \leq \frac{\log p_r}{a_2 \log 2}\left\{1 + O\left(\frac{\log \mu}{r\mu} + \frac{r\mu}{\log p_1}\right)\right\}. \tag{132}$$

From (130) and (132) it is evident that

$$\log\left(1 + \frac{1}{a_{p_r}}\right) = \frac{\log p_r}{a_1 \log 2}\left\{1 + O\left(\frac{\log \mu}{r\mu} + \frac{r\mu}{\log p_1}\right)\right\}. \tag{133}$$

In a similar manner

$$\log\left(1 + \frac{1}{1 + a_{P_r}}\right) = \frac{\log p_r}{a_2 \log 2}\left\{1 + O\left(\frac{\log \mu}{r\mu} + \frac{r\mu}{\log p_1}\right)\right\}. \tag{134}$$

Now supposing that

$$\left.\begin{aligned} r\mu &= o(\log p_1), \\ r\mu &\neq O(\log \mu), \end{aligned}\right\} \tag{135}$$

and dividing (134) by (133), we have

$$\frac{\log\left(1 + \frac{1}{1+a_{P_r}}\right)}{\log\left(1 + \frac{1}{a_{p_r}}\right)} = 1 + O\left(\frac{\log \mu}{r\mu} + \frac{r\mu}{\log p_1}\right),$$

or

$$1 + \frac{1}{1 + a_{P_r}} = 1 + \frac{1}{a_{p_r}} + O\left\{\left(\frac{\log \mu}{r\mu} + \frac{r\mu}{\log p_1}\right)/a_{p_r}\right\},$$

that is

$$\frac{1}{1+a_{P_r}} = \frac{1}{a_{p_r}}\left\{1 + O\left(\frac{\log\mu}{r\mu} + \frac{r\mu}{\log p_1}\right)\right\}.$$

Hence

$$a_{p_r} = a_{P_r} + 1 + O\left(\frac{\log\mu}{\mu} + \frac{r^2\mu}{\log p_1}\right), \tag{136}$$

in virtue of (129). But $a_{P_r} \leq r - 1$, and so

$$a_{p_r} \leq r + O\left(\frac{\log\mu}{\mu} + \frac{r^2\mu}{\log p_1}\right). \tag{137}$$

But we know that $a_{p_r} \geq r$. Hence it is clear that

$$a_{p_r} = r + O\left(\frac{\log\mu}{\mu} + \frac{r^2\mu}{\log p_1}\right). \tag{138}$$

From this and (136) it follows that

$$a_{P_r} = r - 1 + O\left(\frac{\log\mu}{\mu} + \frac{r^2\mu}{\log p_1}\right), \tag{139}$$

provided that the conditions (135) are satisfied.
Now let us suppose that $r = o\sqrt{(\log p_1)}$. Then we can choose μ such that $r^2\mu = o(\log p_1)$ and $\mu \neq O(1)$. Consequently we have

$$\frac{\log\mu}{\mu} = o(1), \quad \frac{r^2\mu}{\log p_1} = o(1);$$

and so it follows from (138) and (139) that

$$a_{p_r} = 1 + a_{P_r} = r, \tag{140}$$

provided that $r = o\sqrt{(\log p_1)}$. From this it is clear that, if $r = o\sqrt{(\log p_1)}$, then

$$p_1 > p_2 > p_3 > p_4 > \cdots > p_r. \tag{141}$$

In other words, *in a large highly composite number*

$$2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdots p_1,$$

the indices comparatively near the end form a sequence of the type

$$\ldots 5 \ldots 4 \ldots 3 \ldots 2 \ldots 1.$$

Near the beginning gaps in the indices will in general occur.

Again, let us suppose that $r = o(\log p_1), r \neq o\sqrt{(\log p_1)}$, and $\mu = O(1)$ in (138) and (139). Then we see that

$$\left.\begin{aligned} a_{p_r} &= r + O\left(\frac{r^2}{\log p_1}\right), \\ a_{P_r} &= r + O\left(\frac{r^2}{\log p_1}\right); \end{aligned}\right\} \tag{142}$$

provided that $r = o(\log p_1)$ and $r \neq o\sqrt{(\log p_1)}$. But when $r \neq o(\log p_1)$, we shall use the general result, viz.,

$$\left.\begin{aligned} a_{p_r} &= O(r), \quad a_{p_r} \neq o(r), \\ a_{P_r} &= O(r), \quad a_{P_r} \neq o(r), \end{aligned}\right\} \tag{143}$$

which is true for all values of r except 1.

27. It follows from (87) and (128) that

$$\left.\begin{aligned} \frac{\log p_r}{\log\left(1 + \frac{1}{a_{p_r}}\right)} &\leq \frac{\log p_1}{\log 2} + O\left\{\frac{\log p_1 \log \mu}{r\mu} + r\mu + \sqrt{(\log p_1 \log\log p_1)}\right\}, \\ \frac{\log P_r}{\log\left(1 + \frac{1}{1+a_{P_r}}\right)} &\geq \frac{\log p_1}{\log 2} + O\left\{\frac{\log p_1 \log \mu}{r\mu} + r\mu + \sqrt{(\log p_1 \log\log p_1)}\right\}, \end{aligned}\right\} \tag{144}$$

with the condition that $r\mu = o(\log p_1)$. From this it can easily be shewn, by arguments similar to those used in the beginning of the previous section, that

$$\frac{\log p_r}{\log(1 + 1/r)} = \frac{\log p_1}{\log 2} + O\left\{\frac{\log p_1 \log \mu}{r\mu} + r\mu + \sqrt{(\log p_1 \log\log p_1)}\right\}, \tag{145}$$

provided that $r\mu = o(\log p_1)$.
Now let us suppose that $r = o(\log p_1)$; then we can choose μ such that

$$\mu = o\left(\frac{\log p_1}{r}\right), \quad \mu \neq O(1).$$

Consequently $r\mu = o(\log p_1)$ and $\log \mu = o(\mu)$, and so

$$\frac{\log p_1 \log \mu}{r\mu} = o(\log p_1).$$

From these relations and (145) it follows that, if $r = o(\log p_1)$, then

$$\frac{\log p_r}{\log(1 + 1/r)} \sim \frac{\log p_1}{\log 2}; \tag{146}$$

that is to say that, if $r = o(\log p_1)$, *then*

$$\frac{\log p_1}{\log 2} \sim \frac{\log p_2}{\log(1+\frac{1}{2})} \sim \frac{\log p_3}{\log(1+\frac{1}{3})} \sim \cdots \sim \frac{\log p_r}{\log(1+1/r)}. \tag{147}$$

Again let us suppose that $r = O\sqrt{(\log p_1 \log\log p_1)}$ in (145). Then it is possible to choose μ such that

$$\left.\begin{array}{l} r\mu = O\sqrt{(\log p_1 \log\log p_1)}, \\ r\mu \neq o\sqrt{(\log p_1 \log\log p_1)}. \end{array}\right\} \tag{148}$$

It is evident that $\log\mu = O(\log\log p_1)$, and so

$$\frac{\log p_1 \log\mu}{r\mu} = O\left(\frac{\log p_1 \log\log p_1}{r\mu}\right) = O\sqrt{(\log p_1 \log\log p_1)},$$

in virtue of (148). Hence

$$\frac{\log p_r}{\log(1+1/r)} = \frac{\log p_1}{\log 2} + O\sqrt{(\log p_1 \log\log p_1)}, \tag{149}$$

provided that

$$r = O\sqrt{(\log p_1 \log\log p_1)}.$$

Now let us suppose that $r = o(\log p_1), r \neq o\sqrt{(\log p_1 \log\log p_1)}$ and $\mu = O(1)$, in (145). Then it is evident that

$$\log p_1 = O(r^2), \quad \sqrt{(\log p_1 \log\log p_1} = O(r),$$

and

$$\frac{\log p_1 \log\mu}{r\mu} = O\left(\frac{\log p_1)}{r}\right) = O(r).$$

Hence we see that

$$\frac{\log p_r}{\log(1+1/r)} = \frac{\log p_1}{\log 2} + O(r), \tag{150}$$

if

$$r = o(\log p_1), \quad r \neq o\sqrt{(\log p_1 \log\log p_1)}.$$

But, if $r \neq o(\log p_1)$, we see from (128) that

$$\left.\begin{array}{l} \dfrac{\log p_r}{\log(1+1/r)} = O(\log p_1), \\ \dfrac{\log p_r}{\log(1+1/r)} \neq o(\log p_1). \end{array}\right\} \tag{151}$$

From (150) and (151) it follows that, if $r \neq o\sqrt{(\log p_1 \log\log p_1)}$, then

$$\frac{\log p_r}{\log(1+1/r)} = \frac{\log p_1}{\log 2} + O(r); \tag{152}$$

and from (149) and (152) that, if $r = o(\log p_1)$, then

$$\frac{\log p_r}{\log(1+1/r)} \sim \frac{\log p_1}{\log 2},$$

in agreement with (147). This result will, in general, fail for the largest possible values of r, which are of order $\log p_1$.

It must be remembered that all the results involving p_1 may be written in terms of N, since $p_1 = O(\log N)$ and $p_1 \neq o(\log N)$, and consequently

$$\log p_1 = \log\log N + O(1). \tag{153}$$

28. We shall now prove that successive highly composite numbers are asymptotically equivalent. Let m and n be any two positive integers which are prime to each other, such that

$$\log mn = o(\log p_1) = o(\log\log N); \tag{154}$$

and let

$$\frac{m}{n} = 2^{\delta_2} \cdot 3^{\delta_3} \cdot 5^{\delta_5} \cdots \wp^{\delta_\wp}. \tag{155}$$

Then it is evident that

$$mn = 2^{|\delta_2|} \cdot 3^{|\delta_3|} \cdot 5^{|\delta_5|} \cdots \wp^{|\delta_\wp|}. \tag{156}$$

Hence

$$\delta_\lambda \log\lambda = O(\log mn) = o(\log p_1) = o(a_\lambda \log\lambda); \tag{157}$$

so that $\delta_\lambda = o(a_\lambda)$.

Now

$$d\left(\frac{m}{n}N\right) = d(N)\left(1 + \frac{\delta_2}{1+a_2}\right)\left(1 + \frac{\delta_3}{1+a_3}\right)\cdots\left(1 + \frac{\delta_\wp}{1+a_\wp}\right). \tag{158}$$

But, from (60), we know that

$$a_\lambda \log\lambda = a_2 \log 2 + O\sqrt{(\log p_1 \log\lambda)}.$$

Hence

$$1 + \frac{\delta_\lambda}{1+a_\lambda} \quad = \quad 1 + \frac{\delta_\lambda \log\lambda}{a_2 \log 2} + O\left\{|\delta_\lambda|\left(\frac{\log\lambda}{\log p_1}\right)^{\frac{3}{2}}\right\}$$

$$
\begin{aligned}
&= 1+\frac{\delta_\lambda \log\lambda}{a_2\log 2}+O\left\{|\delta_\lambda|\frac{\log\lambda}{\log p_1}\sqrt{\left(\frac{\log\wp}{\log p_1}\right)}\right\}\\
&= \exp\left\{\frac{\delta_\lambda \log\lambda}{a_2\log 2}+O\frac{|\delta_\lambda|\log\lambda}{\log p_1}\sqrt{\left(\frac{\log\wp}{\log p_1}\right)}+O\left(\frac{\delta_\lambda\log\lambda}{\log p_1}\right)^2\right\}\\
&= \exp\left\{\frac{\delta_\lambda \log\lambda}{a_2\log 2}+O\frac{|\delta_\lambda|\log\lambda}{\log p_1}\sqrt{\left(\frac{\log mn}{\log p_1}\right)}\right\}. \qquad (159)
\end{aligned}
$$

It follows from (155), (156), (158) and (159) that

$$
\begin{aligned}
d\left(\frac{m}{n}N\right) &= d(N)\exp\left\{\frac{\delta_2\log 2+\delta_3\log 3+\cdots+\delta_\wp\log\wp}{a_2\log 2}\right.\\
&\qquad\left.+O\frac{|\delta_2|\log 2+|\delta_3|\log 3+\cdots+|\delta_\wp|\log\wp}{\log p_1}\sqrt{\left(\frac{\log mn}{\log p_1}\right)}\right\}\\
&= d(N)e^{\frac{\log(m/n)}{a_2\log 2}+O\left(\frac{\log mn}{\log p_1}\right)^{\frac{3}{2}}}\\
&= d(N)e^{\frac{1}{a_2\log 2}\left\{\log\frac{m}{n}+O\log mn\sqrt{\left(\frac{\log mn}{\log p_1}\right)}\right\}}. \qquad (160)
\end{aligned}
$$

Putting $m=n+1$, we see that, if

$$\log n = o(\log p_1) = o(\log\log N),$$

then

$$
\begin{aligned}
d\left\{N\left(1+\frac{1}{n}\right)\right\} &= d(N)e^{\frac{1}{a_2\log 2}\left\{\log\left(1+\frac{1}{n}\right)+O\left(\log n\sqrt{\frac{\log n}{\log p_1}}\right)\right\}}\\
&= d(N)\left(1+\frac{1}{n}\right)^{\frac{1+O\left\{n\log n\sqrt{\left(\frac{\log n}{\log\log N}\right)}\right\}}{a_2\log 2}}. \qquad (161)
\end{aligned}
$$

Now it is possible to choose n such that

$$n(\log n)^{\frac{3}{2}} \neq o\sqrt{(\log\log N)},$$

and

$$1+O\left\{n\log n\sqrt{\left(\frac{\log n}{\log\log N}\right)}\right\}>0;$$

that is to say

$$d\left\{N\left(1+\frac{1}{n}\right)\right\}>d(N). \qquad (162)$$

From this and (29) it follows that, if N *is a highly composite number, then the next highly composite number is of the form*

$$N + O\left\{\frac{N(\log\log\log N)^{\frac{3}{2}}}{\sqrt{(\log\log N)}}\right\} \tag{163}$$

Hence the ratio of two consecutive highly composite numbers tends to unity.
It follows from (163) that the number of highly composite numbers not exceeding x is not of the form

$$o\left\{\frac{\log x\sqrt{(\log\log x)}}{(\log\log\log x)^{\frac{3}{2}}}\right\}.$$

29. Now let us consider the nature of $d(N)$ for highly composite values of N. From (44) we see that

$$d(N) = 2^{\pi(p_1)-\pi(p_2)} \cdot 3^{\pi(p_2)-\pi(p_3)} \cdot 4^{\pi(p_3)-\pi(p_4)} \cdots (1 + a_2). \tag{164}$$

From this it follows that

$$d(N) = 2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdots \varpi^{a_\varpi}, \tag{165}$$

where ϖ is the largest prime not exceeding $1 + a_2$; and

$$\alpha_\lambda = \pi(p_{\lambda-1}) + O(p_\lambda). \tag{166}$$

It also follows that, if $\wp_1, \wp_2, \wp_3, \ldots, \wp_\lambda$ are a given set of primes, then a number $\bar{\mu}$ can be found such that the equation

$$d(N) = \wp_1^{\beta_1} \cdot \wp_2^{\beta_2} \cdot \wp_3^{\beta_3} \cdots \wp_\mu^{\beta_\mu} \cdots \wp_\lambda^{\beta_\lambda}$$

is impossible if N is a highly composite number and $\beta_\mu > \bar{\mu}$. We may state this roughly by saying that as N (a highly composite number) tends to infinity, then, not merely in N itself, but also in $d(N)$, the number of prime factors, as well as the indices, must tend to infinity. In particular such an equation as

$$d(N) = k \cdot 2^m, \tag{167}$$

where k is fixed, becomes impossible when m exceeds a certain limit depending on k.
It is easily seen from (153), (164), and (165) that

$$\left.\begin{array}{l} \varpi = O(a_2) = O(\log p_1) = O(\log\log N) = O\{\log\log d(N)\}, \\ \varpi \neq o(a_2) = o(\log p_1) = o(\log\log N) = o\{\log\log d(N)\}. \end{array}\right\} * \tag{168}$$

It follows from (147) that if $\lambda = o(\log p_1)$ then

$$\frac{\log \alpha_2}{\log(1-\frac{1}{2})} \sim \frac{\log \alpha_3}{\log(1-\frac{1}{3})} \sim \frac{\log \alpha_5}{\log(1-\frac{1}{5})} \sim \cdots \frac{\log \alpha_\lambda}{\log(1-\frac{1}{\lambda})}. \tag{169}$$

Similarly, from (149), it follows that if $\lambda = O\sqrt{(\log p_1 \log\log p_1)}$ then

$$\frac{\log(1+\alpha_\lambda)}{\log(1-1/\lambda)} = -\frac{\log p_1}{\log 2} + O\sqrt{(\log p_1 \log\log p_1)}. \tag{170}$$

Again, from (152), we see that if $\lambda \neq o\sqrt{(\log p_1 \log\log p_1)}$ then

$$\frac{\log(1+\alpha_\lambda)}{\log(1-1/\lambda)} = -\frac{\log p_1}{\log 2} + O(\lambda). \tag{171}$$

In the left-hand side we cannot write α_λ instead of $1+\alpha_\lambda$, as α_λ may be zero for a few values of λ.

From (165) and (170) we can shew that

$$\log d(N) = \alpha_2 \log 2 + O(\alpha_3), \quad \log d(N) \neq \alpha_2 \log 2 + o(\alpha_3);$$

and so

$$\log d(N) = \alpha_2 \log 2 + e^{\frac{\log \frac{3}{2}}{\log 2} \log p_1 + O\sqrt{(\log p_1 \log\log p_1)}} \tag{172}$$

But from (163) we see that

$$\log\log d(N) = \log p_1 + O(\log\log p_1).$$

From this and (172) it follows that

$$a_2 \log 2 = \log d(N) - \{\log d(N)\}^{\frac{\log \frac{3}{2}}{\log 2} + O\sqrt{\left\{\frac{\log\log\log d(N)}{\log\log d(N)}\right\}}}. \tag{173}$$

30. Now we shall consider the order of $dd(N)$ for highly composite values of N. It follows from (165) that

$$\log dd(N) = \log(1+\alpha_2) + \log(1+\alpha_3) + \cdots + \log(1+\alpha_\varpi). \tag{174}$$

Now let $\lambda, \lambda', \lambda'', \ldots$ be consecutive primes in ascending order, and let

$$\lambda = O\sqrt{(\log p_1 \log\log p_1)},$$

*More precisely $\varpi \sim a_2$. But this involves the assumption that two consecutive primes are asymptotically equivalent. This follows at once from the prime number theorem. It appears probable that such a result cannot really be as deep as the prime number theorem, but nobody has succeeded up to now in proving it by elementary reasoning.

$$\lambda \neq o\sqrt{(\log p_1 \log\log p_1)}.$$

Then, from (174), we have

$$\begin{aligned} \log dd(N) &= \log(1+\alpha_2)+\log(1+\alpha_3)+\cdots+\log(1+\alpha_\lambda) \\ &+ \log(1+\alpha_{\lambda'})+\log(1+a_{\lambda''})+\cdots+\log(1+\alpha_\varpi). \end{aligned} \tag{175}$$

But, from (170), we have

$$\begin{aligned} \log(1+\alpha_2) &+ \log(1+\alpha_3)+\cdots+\log(1+\alpha_\lambda) \\ = &- \frac{\log p_1}{\log 2}\log\left\{(1-\tfrac{1}{2})(1-\tfrac{1}{3})(1-\tfrac{1}{5})\cdots\left(1-\frac{1}{\lambda}\right)\right\} \\ &+ O\sqrt{(\log p_1\log\log p_1)}\log\left\{(1-\tfrac{1}{2})(1-\tfrac{1}{3})\cdots\left(1-\frac{1}{\lambda}\right)\right\}. \end{aligned} \tag{176}$$

It can be shewn, without assuming the prime number theorem*, that

$$-\log\left\{(1-\tfrac{1}{2})(1-\tfrac{1}{3})(1-\tfrac{1}{5})\cdots\left(1-\frac{1}{p}\right)\right\} = \log\log p+\gamma+O\left(\frac{1}{\log p}\right), \tag{177}$$

where γ is the Eulerian constant. Hence

$$\log\left\{(1-\tfrac{1}{2}(1-\tfrac{1}{3})(1-\tfrac{1}{5})\cdots\left(1-\frac{1}{p}\right)\right\} = O(\log\log p).$$

From this and (176) it follows that

$$\begin{aligned} \log(1+\alpha_2) &+ \log(1+\alpha_3)+\cdots+\log(1+\alpha_\lambda) \\ = &- \frac{\log p_1}{\log 2}\log\left\{(1-\tfrac{1}{2})(1-\tfrac{1}{3})\cdots\left(1-\frac{1}{\lambda}\right)\right\} \\ &+ O\{\sqrt{(\log p_1\log\log p_1)}\log\log\lambda\} \\ = &- \frac{\log p_1}{\log 2}\log\left\{(1-\tfrac{1}{2})(1-\tfrac{1}{3})\cdots\left(1-\frac{1}{\lambda}\right)\right\} \\ &+ O\{\sqrt{(\log p_1\log\log p_1)}\log\log\log p_1\}. \end{aligned} \tag{178}$$

Again, from (152), we see that

$$\begin{aligned} &\log(1+\alpha_{\lambda'})+\log(1+\alpha_{\lambda''})+\cdots+\log(1+\alpha_\varpi) \\ &= -\frac{\log p_1}{\log 2}\log\left\{\left(1-\frac{1}{\lambda'}\right)\left(1-\frac{1}{\lambda''}\right)\cdots\left(1-\frac{1}{\varpi}\right)\right\} \end{aligned}$$

*See Landau, *Handbuch*, p.139.

$$
\begin{aligned}
&+O\left\{\lambda'\log\left(1-\frac{1}{\lambda'}\right)+\lambda''\log\left(1-\frac{1}{\lambda''}\right)+\cdots+\varpi\log\left(1-\frac{1}{\varpi}\right)\right\}\\
&=-\frac{\log p_1}{\log 2}\log\left\{\left(1-\frac{1}{\lambda'}\right)\left(1-\frac{1}{\lambda'}\right)\cdots\left(1-\frac{1}{\varpi}\right)\right\}+O\{\pi(\varpi)-\pi(\lambda)\}\\
&=-\frac{\log p_1}{\log 2}\left\{\left(1-\frac{1}{\lambda'}\right)\left(1-\frac{1}{\lambda''}\right)\cdots\left(1-\frac{1}{\varpi}\right)\right\}+O\left(\frac{\log p_1}{\log\log p_1}\right).
\end{aligned}\tag{179}
$$

From (175), (178) and (179) it follows that

$$
\begin{aligned}
\log dd(N) &= -\frac{\log p_1}{\log 2}\log\left\{(1-\tfrac{1}{2})(1-\tfrac{1}{3})\cdots\left(1-\frac{1}{\lambda}\right)\right\}\\
&\quad +O\{\sqrt{(\log p_1\log\log p_1)}\log\log\log p_1\}\\
&\quad -\frac{\log p_1}{\log 2}\log\left\{\left(1-\frac{1}{\lambda'}\right)\left(1-\frac{1}{\lambda''}\right)\cdots\left(1-\frac{1}{\varpi}\right)\right\}+O\left(\frac{\log p_1}{\log\log p_1}\right)\\
&= -\frac{\log p_1}{\log 2}\log\left\{(1-\tfrac{1}{2})(1-\tfrac{1}{3})\cdots\left(1-\frac{1}{\varpi}\right)\right\}+O\left(\frac{\log p_1}{\log\log p_1}\right)\\
&= \frac{\log p_1}{\log 2}\left\{\log\log\varpi+\gamma+O\left(\frac{1}{\log\varpi}\right)\right\}+O\left(\frac{\log p_1}{\log\log p_1}\right)\\
&= \frac{\log p_1}{\log 2}\left\{\log\log\log p_1+\gamma+O\left(\frac{1}{\log\log p_1}\right)\right\}+O\left(\frac{\log p_1}{\log\log p_1}\right)\\
&= \frac{\log\log N}{\log 2}\left\{\log\log\log\log N+\gamma+O\left(\frac{1}{\log\log\log N}\right)\right\},
\end{aligned}\tag{180}
$$

in virtue of (177), (168), and (163). Hence, if N is a highly composite number, then

$$
dd(N)=(\log N)^{\frac{1}{\log 2}\left\{\log\log\log\log N+\gamma+O\left(\frac{1}{\log\log\log N}\right)\right\}}. \tag{181}
$$

31. It may be interesting to note that, as far as the table is constructed,

$$
2,2^2,2^3,\ldots,2^{13},\quad 3,3\cdot 2,3\cdot 2^2,\ldots,3\cdot 2^{11},\quad 5\cdot 2,5\cdot 2^2,\ldots,5\cdot 2^8,
$$
$$
7\cdot 2^5,7\cdot 2^6,\ldots,7\cdot 2^{10},\quad 9,9\cdot 2,9\cdot 2^2,\ldots,9\cdot 2^{10},
$$

and so on, occur as values of $d(N)$. But we know from § 29 that $k\cdot 2^m$ cannot be the value of $d(N)$ for sufficiently large values of m; and so numbers of the form $k\cdot 2^m$ which occur as the value of $d(N)$ in the table must disappear sooner or later when the table is extended. Thus numbers of the form $5\cdot 2^m$ have begun to disappear in the table itself. The powers of 2 disappear at any rate from 2^{18} onwards. The least number having 2^{18} divisors is

$$
2^7\cdot 3^3\cdot 5^3\cdot 7\cdot 11\cdot 13\cdots 41\cdot 43,
$$

while the smaller number, viz.,

$$2^8 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdots 41$$

has a larger number of divisors. viz. $135 \cdot 2^{11}$. The numbers of the form $7 \cdot 2^m$ disappear at least from $7 \cdot 2^{13}$ onwards. The least number having $7 \cdot 2^{13}$ divisors is

$$2^6 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13 \cdots 31 \cdot 37,$$

while the smaller number, viz.

$$2^9 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdots 31$$

has a larger number of divisors, viz. $225 \cdot 2^8$.

IV
Superior Highly Composite Numbers

32. A number N may be said to be a superior highly composite number if there is a positive number ϵ such that

$$\frac{d(N)}{N^\epsilon} \geq \frac{d(N')}{(N')^\epsilon}, \tag{182}$$

for all values of N' less that N, and

$$\frac{d(N)}{N^\epsilon} > \frac{d(N')}{(N')^\epsilon} \tag{183}$$

for all values of N' greater that N.
All superior highly composite numbers are also highly composite. For, if $N' < N$, it follows from (182) that

$$d(N) \geq d(N') \left(\frac{N}{N'}\right)^\epsilon > d(N');$$

and so N is highly composite.

33. Now let us consider what must be the nature of N in order that it should be a superior highly composite number. In the first place it must be of the form

$$2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdots p_1^a, \tag{184}$$

or of the form

$$\begin{array}{rl} & 2 \cdot 3 \cdot 5 \cdot 7 \cdots p_1 \\ \times & 2 \cdot 3 \cdot 5 \cdot 7 \cdots p_2 \end{array}$$

$$\begin{array}{ll} \times & 2 \cdot 3 \cdot 5 \cdots p_3 \\ \times & \ldots\ldots: \end{array}$$

i.e. it must satisfy the conditions for a highly composite number. Now let

$$N' = N/\lambda,$$

where $\lambda \leq p_1$. Then from (182) it follows that

$$\frac{1+a_\lambda}{\lambda^{\epsilon a_\lambda}} \geq \frac{a_\lambda}{\lambda^{\epsilon(a_\lambda - 1)}},$$

or

$$\lambda^\epsilon \leq \left(1 + \frac{1}{a_\lambda}\right). \tag{185}$$

Again let

$$N' - N_\lambda.$$

Then, from (183), we see that

$$\frac{1+a_\lambda}{\lambda^{\epsilon a_\lambda}} > \frac{2+a_\lambda}{\lambda^{\epsilon(a_\lambda + 1)}},$$

or

$$\lambda^\epsilon > \left(1 + \frac{1}{1+a_\lambda}\right). \tag{186}$$

Now supposing that $\lambda = p_1$ in (185) and $\lambda = P_1$ in (186), we obtain

$$\frac{\log 2}{\log P_1} < \epsilon \leq \frac{\log 2}{\log p_1}. \tag{187}$$

Now let us suppose that $\epsilon = 1/x$. Then, from (187), we have

$$p_1 \leq 2^x < P_1. \tag{188}$$

That is, p_1 is the largest prime not exceeding 2^x. It follows from (185) that

$$a_\lambda \leq (\lambda^{1/x} - 1)^{-1}. \tag{189}$$

Similarly, from (186),

$$a_\lambda > (\lambda^{1/x} - 1)^{-1} - 1. \tag{190}$$

From (189) and (190) it is clear that

$$a_\lambda = [\lambda^{1/x} - 1)^{-1}]. \tag{191}$$

Hence N is of the form

$$2^{[(2^{1/x}-1)^{-1}]} \cdot 3^{[(3^{1/x}-1)^{-1}]} \cdot 5^{[(5^{1/x}-1)^{-1}]} \dots p_1, \tag{192}$$

where p_1 is the largest prime not exceeding 2^x.

34. Now let us suppose that $\lambda = p_r$ in (189). Then

$$a_{p_r} \leq (p_r^{1/x} - 1)^{-1}.$$

But we know that $r \leq a_{pr}$. Hence

$$r \leq (p_r^{1/x} - 1)^{-1},$$

or

$$p_r \leq \left(1 + \frac{1}{r}\right)^x. \tag{193}$$

Similarly by supposing that $\lambda = P_r$ in (190), we see that

$$a_{P_r} > (P_r^{1/x} - 1)^{-1} - 1.$$

But we know that $r - 1 \geq ap_r$. Hence

$$r > (P_r^{1/x} - 1)^{-1}$$

or

$$P_r > \left(1 + \frac{1}{r}\right)^x. \tag{194}$$

From (193) and (194) it is clear that p_r is the largest prime not exceeding $(1+1/r)^x$. Hence N is of the form

$$\begin{array}{ll} & 2 \cdot 3 \cdot 5 \cdot 7 \cdots p_1 \\ \times & 2 \cdot 3 \cdot 5 \cdot 7 \cdots p_2 \\ \times & 2 \cdot 3 \cdot 5 \cdots p_3 \\ \times & \dots\dots\dots, \end{array} \tag{195}$$

where p_1 is the largest prime not greater than $2^x, p_2$ is the largest prime not greater than $\left(\frac{3}{2}\right)^x$, and so on. In other words N is of the form

$$e^{\vartheta(2^x)+\vartheta(\frac{3}{2})^x+\vartheta(\frac{4}{3})^x+\cdots}, \tag{196}$$

and $d(N)$ is of the form

$$2^{\pi(2^x)} \cdot (\tfrac{3}{2})^{\pi(\frac{3}{2})^x} \cdot (\tfrac{4}{3})^{\pi(\frac{4}{3})^x} \cdots. \tag{197}$$

Thus to every value of x not less than 1 correspondence one, and only one, value of N.

35. Since

$$\frac{d(N)}{N^{1/x}} \geq \frac{d(N')}{(N')^{1/x}},$$

for all values of N', it follows from (196) and (197) that

$$d(N) \leq N^{1/x} \frac{2^{\pi(2^x)}}{e^{(1/x)\vartheta(2^x)}} \frac{(\frac{3}{2})^{\pi(\frac{3}{2})^x}}{e^{(1/x)\vartheta(\frac{3}{2})^x}} \frac{(\frac{4}{3})^{\pi(\frac{4}{3})^x}}{e^{(1/x)\vartheta(\frac{4}{3})^x}} \cdots, \tag{198}$$

for all values of N and x; and $d(N)$ is equal to the right-hand side when

$$N = e^{\vartheta(2^x)+\vartheta(\frac{3}{2})^x+\vartheta(\frac{4}{3})^x+\cdots}. \tag{199}$$

Thus, for example, putting $x = 2$, 3, 4 in (198), we obtain

$$\left.\begin{aligned} &d(N) \leq \sqrt{(3N)}, \\ &d(N) \leq 8(3N/35)^{\frac{1}{3}}, \\ &d(N) \leq 96(3N/50050)^{\frac{1}{4}}, \end{aligned}\right\} \tag{200}$$

for all values of N; and $d(N) = \sqrt{(3N)}$ when $N = 2^2 \cdot 3; d(N) = 8(3N/35)^{\frac{1}{3}}$ when $N = 2^3 \cdot 3^2 \cdot 5 \cdot 7; d(N) = 96(3N/50050)^{\frac{1}{4}}$ when

$$N = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13.$$

36. M and N are consecutive superior highly composite numbers if there are no superior highly composite numbers between M and N.

From (195) and (196) it is easily seen that, if M and N are any two superior highly composite numbers, and if $M > N$, then M is a multiple of N; and also that, if M and N are two consecutive superior highly composite numbers, and if $M > N$, then M/N is a prime number. From this it follows that consecutive superior highly composite numbers are of the form

$$\pi_1, \quad \pi_1\pi_2, \quad \pi_1\pi_2\pi_3, \quad \pi_1\pi_2\pi_3\pi_4, \quad \ldots, \tag{201}$$

where $\pi_1, \pi_2, \pi_3 \ldots$ are primes. In order to determine $\pi_1, \pi_2, \ldots$ we proceed as follows. Let x_1' be the smallest value of x such that $[2^x]$ is prime x_2' the smallest value of x such that $[\left(\frac{3}{2}\right)^x]$ is prime, and so on; and let $x_1, x_2, \ldots$ be the numbers $x_1', x_2' \ldots$ arranged in order of magnitude. Then π_n is the prime corresponding to x_n, and

$$N = \pi_1\pi_2\pi_3 \cdots \pi_n, \tag{202}$$

if $x_n \leq x < x_{n+1}$.

37. From the preceding results we see that the number of superior highly composite numbers not exceeding

$$e^{\vartheta(2^x)+\vartheta(\frac{3}{2})^x+\vartheta(\frac{4}{3})^x+\cdots} \tag{203}$$

is

$$\pi(2^x)+\pi(\tfrac{3}{2})^x+\pi(\tfrac{4}{3})^x+\cdots.$$

In other words if $x_n \le x < x_{n+1}$ then

$$n=\pi(2^x)+\pi(\tfrac{3}{2})^x+\pi(\tfrac{4}{3})^x+\cdots. \tag{204}$$

It follows from (192) and (202) that, of the primes $\pi_1, \pi_2, \pi_3, \ldots, \pi_n$, the number of primes which are equal to a given prime ϖ is equal to

$$[(\varpi^{1/x}-1)^{-1}]. \tag{205}$$

Further, the greatest of the primes $\pi_1, \pi_2, \pi_3 \ldots, \pi_n$ is the largest prime not greater than 2^x, and is asymptotically equivalent to the natural nth prime, in virtue of (204).

The following table gives the values of π_n and x_n for the first 50 values of n, that is till x_n reaches very nearly 7.

$\pi_1 = 2$	$x_1 = \frac{\log 2}{\log 2} = 1$
$\pi_2 = 3$	$x_2 = \frac{\log 3}{\log 2} = 1.5849\cdots$
$\pi_3 = 2$	$x_3 = \frac{\log 2}{\log(\frac{3}{2})} = 1.7095\cdots$
$\pi_4 = 5$	$x_4 = \frac{\log 5}{\log 2} = 2.3219\ldots$
$\pi_5 = 2$	$x_5 = \frac{\log 2}{\log(\frac{4}{3})} = 2.4094\ldots$

$$\pi_6 = 3 \qquad x_6 = \frac{\log 3}{\log(\frac{3}{2})} = 2.7095\ldots$$

$$\pi_7 = 7 \qquad x_7 = \frac{\log 7}{\log 2} = 2.8073\ldots$$

$$\pi_8 = 2 \qquad x_8 = \frac{\log 2}{\log(\frac{5}{4})} = 3.1062\ldots$$

$$\pi_9 = 11 \qquad x_9 = \frac{\log 11}{\log 2} = 3.4594\ldots$$

$$\pi_{10} = 13 \qquad x_{10} = \frac{\log 13}{\log 2} = 3.7004\ldots$$

$$\pi_{11} = 2 \qquad x_{11} = \frac{\log 2}{\log(\frac{6}{5})} = 3.8017\ldots$$

$$\pi_{12} = 3 \qquad x_{12} = \frac{\log 3}{\log(\frac{4}{3})} = 3.8188\ldots$$

$$\pi_{13} = 5 \qquad x_{13} = \frac{\log 5}{\log(\frac{3}{2})} = 3.9693\cdots$$

$$\pi_{14} = 17 \qquad x_{14} = \frac{\log 17}{\log 2} = 4.0874\ldots$$

$$\pi_{15} = 19 \qquad x_{15} = \frac{\log 19}{\log 2} = 4.2479\ldots$$

$$\pi_{16} = 2 \qquad x_{16} = \frac{\log 2}{\log(\frac{7}{6})} = 4.4965\ldots$$

$$\pi_{17} = 23 \qquad x_{17} = \frac{\log 23}{\log 2} = 4.5235\ldots$$

$$\pi_{18} = 7 \qquad x_{18} = \frac{\log 7}{\log(\frac{3}{2})} = 4.7992\ldots$$

$$\pi_{19} = 29 \qquad x_{19} = \frac{\log 29}{\log 2} = 4.8579\ldots$$

$$\pi_{20} = 3 \qquad x_{20} = \frac{\log 3}{\log(\frac{5}{4})} = 4.9233\ldots$$

$$\pi_{21} = 31 \qquad x_{21} = \frac{\log 31}{\log 2} = 4.9541\ldots$$

$$\pi_{22} = 2 \qquad x_{22} = \frac{\log 2}{\log(\frac{8}{7})} = 5.1908\ldots$$

$$\pi_{23} = 37 \qquad x_{23} = \frac{\log 37}{\log 2} = 5.2094\ldots$$

$$\pi_{24} = 41 \qquad x_{24} = \frac{\log 41}{\log 2} = 5.3575\ldots$$

$$\pi_{25} = 43 \qquad x_{25} = \frac{\log 43}{\log 2} = 5.4262\ldots$$

$$\pi_{26} = 47 \qquad x_{26} = \frac{\log 47}{\log 2} = 5.5545\ldots$$

$$\pi_{27} = 5 \qquad x_{27} = \tfrac{\log 5}{\log(\frac{4}{3})} = 5.5945\ldots$$

$$\pi_{28} = 53 \qquad x_{28} = \tfrac{\log 53}{\log 2} = 5.7279\ldots$$

$$\pi_{29} = 59 \qquad x_{29} = \tfrac{\log 59}{\log 2} = 5.8826\ldots$$

$$\pi_{30} = 2 \qquad x_{30} = \tfrac{\log 2}{\log(\frac{9}{8})} = 5.8849\ldots$$

$$\pi_{31} = 11 \qquad x_{31} = \tfrac{\log 11}{\log(\frac{3}{2})} = 5.9139\ldots$$

$$\pi_{32} = 61 \qquad x_{32} = \tfrac{\log 61}{\log 2} = 5.9307\ldots$$

$$\pi_{33} = 3 \qquad x_{33} = \tfrac{\log 3}{\log(\frac{6}{5})} = 6.0256\ldots$$

$$\pi_{34} = 67 \qquad x_{34} = \tfrac{\log 67}{\log 2} = 6.0660\ldots$$

$$\pi_{35} = 71 \qquad x_{35} = \tfrac{\log 71}{\log 2} = 6.1497\ldots$$

$$\pi_{36} = 73 \qquad x_{36} = \tfrac{\log 73}{\log 2} = 6.1898\ldots$$

$$\pi_{37} = 79 \qquad x_{37} = \tfrac{\log 79}{\log 2} = 6.3037\ldots$$

$$\pi_{38} = 13 \qquad x_{38} = \tfrac{\log 13}{\log(\frac{3}{2})} = 6.3259\ldots$$

$$\pi_{39} = 83 \qquad x_{39} = \tfrac{\log 83}{\log 2} = 6.3750\ldots$$

$$\pi_{40} = 89 \qquad x_{40} = \tfrac{\log 89}{\log 2} = 6.4757\ldots$$

$$\pi_{41} = 2 \qquad x_{41} = \tfrac{\log 2}{\log(\frac{10}{9})} = 6.5790\ldots$$

$$\pi_{42} = 97 \qquad x_{42} = \tfrac{\log 97}{\log 2} = 6.5999\ldots$$

$$\pi_{43} = 101 \qquad x_{43} = \tfrac{\log 101}{\log 2} = 6.6582\ldots$$

$$\pi_{44} = 103 \qquad x_{44} = \tfrac{\log 103}{\log 2} = 6.6724\ldots$$

$$\pi_{45} = 107 \qquad x_{45} = \tfrac{\log 107}{\log 2} = 6.7414\ldots$$

$$\pi_{46} = 7 \qquad x_{46} = \tfrac{\log 7}{\log(\frac{4}{3})} = 6.7641\ldots$$

$$\pi_{47} = 109 \qquad x_{47} = \tfrac{\log 109}{\log 2} = 6.7681\ldots$$

$$\pi_{48} = 113 \qquad x_{48} = \tfrac{\log 113}{\log 2} = 6.8201\ldots$$

$$\pi_{49} = 17 \qquad x_{49} = \tfrac{\log 17}{\log(\frac{3}{2})} = 6.9875\ldots$$

$$\pi_{50} = 127 \qquad x_{50} = \tfrac{\log 127}{\log 2} = 6.9886\ldots$$

38. It follows from (17) and (198) that $\log d(N) \leq F(x)$, where

$$F(x) = \frac{1}{x}\log N + \frac{1}{x}\left\{\int_2^{2^x} \frac{\pi(t)}{t}dt + \int_2^{(\frac{3}{2})^x} \frac{\pi(t)}{t}dt + \int_2^{(\frac{4}{3})^x} \frac{\pi(t)}{t}dt + \cdots\right\}, \tag{206}$$

for all values of N and x. In order to obtain the best possible upper limit for $\log d(N)$, we must choose x so as to make the right-hand side a minimum.

The function $F(x)$ is obviously continuous unless $(1 + 1r)^x = p$, where r is a positive integer and p a prime. It is easily seen to be continuous even then, and so continuous without exception. Also

$$\begin{aligned} F'(x) = -\frac{1}{x^2}\log N - \frac{1}{x^2}&\left\{\int_2^{2^x} \frac{\pi(t)}{t}dt + \int_2^{(\frac{3}{2})^x} \frac{\pi(t)}{t}dt + \cdots\right\} \\ &+\frac{1}{x}\{\pi(2^x)\log 2 + \pi(\tfrac{3}{2})^x \log\tfrac{3}{2} + \cdots\} \\ &= \frac{1}{x^2}\{\vartheta(2^x) + \vartheta(\tfrac{3}{2})^x + \vartheta(\tfrac{4}{3})^x + \cdots - \log N\}, \end{aligned} \tag{207}$$

unless $(1 + 1/r)^x = p$, in virtue of (17).

Thus we see that $F(x)$ is continuous, and $F'(x)$ exists and is continuous except at certain isolated points. The sign of $F'(x)$, where it exists, is that of

$$\vartheta(2^x) + \vartheta(\tfrac{3}{2})^x + \vartheta(\tfrac{4}{3})^x + \cdots - \log N,$$

and
$$\vartheta(2^x) + \vartheta(\tfrac{3}{2})^x + \vartheta(\tfrac{4}{3})^x + \cdots,$$
is a monotonic function. Thus $F'(x)$ is first negative and then positive, changing sign once only, and so $F(x)$ has a unique minimum. Thus $F(x)$ is a minimum when x is a function of N defined by the inequalities

$$\vartheta(2^v) + \vartheta(\tfrac{3}{2})^v + \vartheta(\tfrac{4}{3})^v + \cdots \left\{\begin{array}{l} < \log N (y < x) \\ > \log N (y > x) \end{array}\right\}. \tag{208}$$

Now let $D(N)$ be a function of N such that

$$D(N) = 2^{\pi(2^x)}(\tfrac{3}{2})^{\pi(\frac{3}{2})^x}(\tfrac{4}{3})^{\pi(\frac{4}{3})^x}\cdots, \tag{209}$$

where x is the function of N defined by the inequalities (208). Then, from (198), we see that

$$d(N) \leq D(N), \tag{210}$$

for all values of N; and $d(N) = D(N)$ for all superior highly composite values of N. Hence $D(N)$ is the maximum order of $d(N)$. In other words, $d(N)$ will attain its maximum order when N is a superior highly composite number.

V.
Application to the Order of $d(N)$.

39. The most precise result known concerning the distribution of the prime numbers is that

$$\left.\begin{aligned} \pi(x) &= Li(x) + O(xe^{-a\sqrt{\log x}}), \\ \vartheta(x) &= x + O(xe^{-a\sqrt{\log x}}), \end{aligned}\right\} \tag{211}$$

where

$$Li(x) = \int^x \frac{dt}{\log t}$$

and a is a positive constant.
In order to find the maximum order of $d(N)$ we have merely to determine the order of $D(N)$ from the equations (208) and (209). Now, from (208), we have

$$\log N = \vartheta(2^x) + O(\tfrac{3}{2})^x = \vartheta(2^x) + o(2^{2x/3});$$

and so

$$\vartheta(2^x) = \log N + o(\log N)^{\frac{2}{3}}; \tag{212}$$

and similarly from (209) we have

$$\pi(2^x) = \frac{\log D(N)}{\log 2} + o(\log N)^{\frac{2}{3}}. \tag{213}$$

It follows from (211) - (213) that the maximum order of $d(N)$ is

$$2^{Li(\log N) + O[\log N e^{-a\sqrt{(\log\log N)}}]}. \tag{214}$$

It does not seen to be possible to obtain an upper limit for $d(N)$ notably more precise than (214) without assuming results concerning the distribution of primes which depend on hitherto unproved properties of the Riemann ζ-function.

40. We shall now assume that the "Riemann hypothesis" concerning the ζ-function is true, i.e., that all the complex roots of $\zeta(s)$ have their real part equal to $\frac{1}{2}$. Then it is known that

$$\vartheta(x) = x - \sqrt{x} - \sum \frac{x^\rho}{\rho} + O(x^{\frac{1}{3}}), \tag{215}$$

where ρ is a complex root of $\zeta(s)$, and that

$$\pi(x) = Li(x) - \tfrac{1}{2}Li(\sqrt{x}) - \sum Li(x^\rho) + O(x^{\frac{1}{3}})$$

$$= Li(x) - \frac{\sqrt{x}}{\log x} - \frac{2\sqrt{x}}{(\log x)^2} - \frac{1}{\log x}\sum \frac{x^\rho}{\rho} - \frac{1}{(\log x)^2}\sum \frac{x^\rho}{\rho^2} + O\left\{\frac{\sqrt{x}}{(\log x)^3}\right\}, \quad (216)$$

since $\sum \frac{x^\rho}{\rho^k}$ is absolutely convergent when $k > 1$. Also it is known that

$$\sum \frac{x^\rho}{\rho} = O\{\sqrt{x(\log x)^2}\}; \quad (217)$$

and so

$$\vartheta(x) - x = O\{\sqrt{x(\log x)^2}\}. \quad (218)$$

From (215) and (216) it is clear that

$$\pi(x) = Li(x) + \frac{\vartheta(x) - x}{\log x} - R(x) + O\left\{\frac{\sqrt{x}}{\log x)^3}\right\}, \quad (219)$$

where

$$R(x) = \frac{2\sqrt{x} + \sum \frac{x^\rho}{\rho^2}}{(\log x)^2} \quad (220)$$

But it follows from Taylor's theorem and (218) that

$$Li\vartheta(x) - Li(x) = \frac{\vartheta(x) - x}{\log x} + O(\log x)^2, \quad (221)$$

and from (219) and (221) it follows that

$$\pi(x) = Li\vartheta(x) - R(x) + O\left\{\frac{\sqrt{x}}{(\log x)^3}\right\}. \quad (222)$$

41. It follows from the functional equation satisfied by $\zeta(s)$, viz.,

$$(2\pi)^{-s}\Gamma(s)\zeta(s)\cos\tfrac{1}{2}\pi s = \tfrac{1}{2}\zeta(1-s), \quad (223)$$

that

$$(1-s)\pi^{-\frac{1}{4}\sqrt{s}}\Gamma\left(\frac{1+\sqrt{s}}{4}\right)\zeta\left(\frac{1+\sqrt{s}}{2}\right)$$

is an integral function of s whose apparent order is less than 1, and hence is equal to

$$\Gamma(\tfrac{1}{4})\zeta(\tfrac{1}{2})\prod\left\{1 - \tfrac{s}{(2\rho-1)^2}\right\},$$

[where ρ runs through the complex roots of $\zeta(s)$ whose imaginary parts are positive]. From this we can easily deduce that

$$s(1+s)\pi^{-\frac{1+s}{2}}\Gamma\left(\frac{1+s}{2}\right)\zeta(1+s) = \prod\left(1 + \frac{s}{\rho}\right), \quad (224)$$

[where ρ now runs through all the roots]. Subtracting 1 from both sides, dividing the result by s, and then making $s \to 0$, we obtain

$$\sum \frac{1}{\rho} = 1 + \tfrac{1}{2}(\gamma - \log 4\pi), \tag{225}$$

where γ is the Eulerian constant. Hence we see that

$$\begin{aligned}\left|\sum \frac{x^\rho}{\rho^2}\right| \le \sum \left|\frac{x^\rho}{\rho^2}\right| = \sqrt{x}\sum \frac{1}{\rho(1-\rho)} = \sqrt{x}\sum\left(\frac{1}{\rho} + \frac{1}{1-\rho}\right)& \\ = 2\sqrt{x}\sum \frac{1}{\rho} = \sqrt{x}(2 + \gamma - \log 4\pi).& \end{aligned} \tag{226}$$

It follows from (220) and (226) that

$$(\log 4\pi - \gamma)\sqrt{x} \le R(x)(\log x)^2 \le (4 + \gamma - \log 4\pi)\sqrt{x}. \tag{227}$$

It can easily be verified that

$$\left.\begin{aligned} \log 4\pi - \gamma = 1.954, \\ 4 + \gamma - \log 4\pi = 2.046 \end{aligned}\right\} \tag{228}$$

approximately.

42. Now

$$R(x) = \frac{2\sqrt{x} + S(x)}{(\log x)^2},$$

where

$$S(x) = \sum \frac{x^\rho}{\rho^2};$$

so that, considering $R(x)$ as a function of a continuous variable, we have

$$\begin{aligned} R'(x) &= \frac{1}{\sqrt{x}(\log x)^2} - \frac{4\sqrt{x} + 2S(x)}{x(\log x)^3} + \frac{S'(x)}{(\log x)^2} \\ &= \frac{S'(x)}{(\log x)^2} + O\left\{\frac{1}{\sqrt{x}(\log x)^2}\right\}, \end{aligned}$$

for all values of x for which $S(x)$ possesses a differential coefficient.
Now the derived series of $S(x)$, viz.,

$$\bar{S}(x) = \frac{1}{x}\sum \frac{x^\rho}{\rho},$$

is uniformly convergent throughout any interval of positive values of x which does not include any value of x of the form $x = p^m$; and $S(x)$ is continuous for all values of x. It follows that

$$S(x_1) - S(x_2) = \int_{x_1}^{x_2} \bar{S}(x)dx,$$

for all positive values of x_1 and x_2, and that $S(x)$ possesses a derivative

$$S'(x) = \bar{S}(x),$$

whenever x is not of the form p^m. Also

$$\bar{S}(x) = O\left\{\frac{(\log x)^2}{\sqrt{x}}\right\}.$$

Hence

$$R(x+h) = R(x) + \int_x^{x+h} O\left(\frac{1}{\sqrt{t}}\right) dt = R(x) + O\left(\frac{h}{\sqrt{x}}\right). \tag{229}$$

43. Now

$$\begin{aligned} \log N &= \vartheta(2^x) + \vartheta(\tfrac{3}{2})^x + O(\tfrac{4}{3})^x \\ &= \vartheta(2^x) + (\tfrac{3}{2})^x + O\{x^2(\tfrac{3}{2})^{\frac{1}{2}x}\} + O(\tfrac{4}{3})^x \\ &= \vartheta(2^x) + (\tfrac{3}{2})^x + O(2^{5x/12}). \end{aligned}$$

Similarly $\log D(N) = \log 2 \cdot \pi(2^x) + \log(\frac{3}{2})Li(\frac{3}{2})^x + O(2^{5x/12})$. Writing X for 2^x, we have

$$\left.\begin{aligned} &\log N = \vartheta(X) + X^{\log(\frac{3}{2})/\log 2} + O\{(\log N)^{\frac{5}{12}}\}; \\ &\log D(N) = \log 2.\pi(X) + \log(\tfrac{3}{2})Li\{X^{\log(\frac{3}{2})/\log 2}\} + O(X^{\frac{5}{12}}. \end{aligned}\right\} \tag{230}$$

It follows that

$$\log N = X + O[X^{\log(\frac{3}{2}/\log 2}];$$

and so

$$X = \log N + O[(\log N)^{\log(\frac{3}{2})/\log 2}]. \tag{231}$$

Again, from (230) and (231), it follows that

$$\log N = \vartheta(X) + (\log N)^{\log(\frac{3}{2})/\log 2} + O\{(\log N)^{\frac{5}{12}}\}, \tag{232}$$

and

$$\begin{aligned} \log D(N) &= \log 2.\pi(X) + \log(\tfrac{3}{2})Li(\log N)^{\log(\frac{3}{2})/\log 2} + O\{(\log N)^{\frac{5}{12}}\} \\ &= \log 2\left\{Li\vartheta(X) - R(X) + O\left[\frac{\sqrt{X}}{(\log X)^3}\right]\right\} \\ &\qquad + \log(\tfrac{3}{2})Li\{(\log N)^{\log(\frac{3}{2})/\log 2}\} + O\{(\log N)^{\frac{5}{12}}\}, \end{aligned} \tag{233}$$

in virtue of (222). From (231) and (233) it evidently follows that

$$\begin{aligned}\log D(N) &= \log 2 \cdot Li\vartheta(X) - \log 2 \cdot R(X) + \log(\tfrac{3}{2}) Li\{(\log N)^{\log(\frac{3}{2})/\log 2}\} \\ &\quad + O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\} \\ &= \log 2 \cdot Li\{\log N - (\log N)^{\log(\frac{3}{2})/\log 2} + O(\log N)^{\frac{5}{12}}\} \\ &\quad - \log 2 \cdot R\{\log N + O(\log N)^{\log(\frac{3}{2}/\log 2}\} \\ &\quad + \log(\tfrac{3}{2}) Li\{(\log N)^{\log(\frac{3}{2})/\log 2}\} + O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}, \qquad (234)\end{aligned}$$

in virtue of (231) and (232). But

$$\begin{aligned}&Li\{\log N - (\log N)^{|log(\frac{3}{2})/\log 2}\} + O(\log N)^{\frac{5}{12}}\} \\ &= Li(\log N) - \frac{(\log N)^{\log(\frac{3}{2})/\log 2}}{\log\log N} + O\left\{\frac{(\log N)^{\frac{5}{12}}}{\log\log N}\right\} + O\left\{\frac{(\log N)^{\{2\log(\frac{3}{2})/\log 2\}-1}}{(\log\log N)^2}\right\} \\ &= Li(\log N) - \frac{(\log N)^{\log(\frac{3}{2})/\log 2}}{\log\log N} + O(\log N)^{\frac{5}{12}};\end{aligned}$$

and

$$\begin{aligned}R\{\log N + O(\log N)^{\log(\frac{3}{2})/\log 2}\} &= R(\log N) + O\{(\log N)^{\{\log(\frac{3}{2})/\log 2\} - \frac{1}{2}}\} \\ &= R(\log N) + O(\log N)^{\frac{1}{10}},\end{aligned}$$

in virtue of (229). Hence (234) may be replaced by

$$\begin{aligned}\log D(N) &= \log 2 \cdot Li(\log N) + \log(\tfrac{3}{2}) Li\{(\log n)^{\log(\frac{3}{2})\log 2}\} \\ &- \log 2 \frac{(\log N)^{\log(\frac{3}{2})/\log 2}}{\log\log N} - \log 2 . R(\log N) + O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}. \qquad (235)\end{aligned}$$

That is to say the maximum order of $d(N)$ is

$$2^{Li(\log N) + \phi(N)}, \qquad (236)$$

where

$$\begin{aligned}\phi(N) &= \frac{\log(\frac{3}{2})}{\log 2} Li\{(\log N)^{\log(\frac{3}{2})/\log 2}\} - \frac{(\log N)^{\log(\frac{3}{2})/\log 2}}{\log\log N} - R(\log N) \\ &\quad + O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}.\end{aligned}$$

This order is actually attained for an infinity of values of N.

44. We can now find the order of the number of superior highly composite numbers not exceeding a given number N. Let N' be the smallest superior highly composite number greater than N, and let

$$N' = e^{\vartheta(2^x)+\vartheta(\frac{3}{2})^x+\vartheta(\frac{4}{3})^x+\cdots}.$$

Then, from § 37, we know that

$$2N \leq N' \leq 2^x N, \tag{237}$$

so that $N' = 0(N \log N)$; and also that the number of superior highly composite numbers not exceeding N' is

$$n = \pi(2^x) + \pi(\tfrac{3}{2})^x + \pi(\tfrac{4}{3})^x + \cdots.$$

By arguments similar to those of the previous section we can shew that

$$\begin{aligned} n &= Li(\log N) + Li(\log N)^{\log(\frac{3}{2})/\log 2} - \frac{(\log N)^{\log(\frac{3}{2})/\log 2}}{\log\log N} - R(\log N) \\ &\quad + O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}. \end{aligned} \tag{238}$$

It is easy to see from § 37 that, if the largest superior highly composite number not exceeding N is

$$2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdots p^{a_p},$$

then the number of superior highly composite numbers not exceeding N is the sum of all the indices, viz,

$$a_2 + a_3 + a_5 + \cdots + a_p$$

45. Proceeding as in § 28, we can shew that, if N is a superior highly composite number and m and n are any two positive integers such that [n is a divisor of N, and]

$$\log mn = o(\log\log N),$$

then

$$d\left(\frac{m}{n}N\right) = d(N) 2^{\frac{\log(m/n)}{\log\log N} + o\left(\frac{\log mn}{\log\log N}\right)^2}. \tag{239}$$

From this we can easily shew that the next highly composite number is of the form

$$N + O\left\{\frac{N(\log\log\log N)^2}{\log\log N}\right\}. \tag{240}$$

Again, let us S and S be any two consecutive superior highly composite numbers, and let

$$S = e^{\vartheta(2^x)+\vartheta(\frac{3}{2})+\vartheta(\frac{4}{3})^x+\cdots}$$

Then it follows from § 35 that

$$d(N) < \left(\frac{N}{S}\right)^{1/x} d(S), \tag{241}$$

for all values of N except S and s'. Now, if S be the nth superior highly composite number, so that

$$x_n \leq x < x_{n+1},$$

where x_n is the same as in § 36, we see that

$$d(N) < \left(\frac{N}{S}\right)^{1/x_n} d(S), \tag{241$'$}$$

for all values of N except S and s'. If N is S or S', then the inequality becomes an equality. It follows from § 36 that $d(S) \leq 2d(S')$. Hence, if N be highly composite and $S' < N < S$, so that $dS' < d(N) < d(S)$, then

$$\tfrac{1}{2}d(S) < d(N) < d(S'), \quad d(S') < d(N) < 2d(S').$$

From this it is easy to see that the order (236) is actually attained by $d(N)$ whenever N is a highly composite number. But it may also be attained when N is not a highly composite number. For example, if

$$N = (2 \cdot 3 \cdot 5 \cdots p_1) \times (2 \cdot 3 \cdot 5 \cdots p_2),$$

where p_1 is the largest prime not greater than 2^x, and p_2 the largest prime not greater than $(\frac{3}{2})^x$, it is easily seen that $d(N)$ attains the order (236): and N is not highly composite.

VI.
Special Forms of N.

46. In §§-38 we have indirectly solved the following problem: to find the relations which must hold between $x_1, x_2, x_3, \ldots$ in order that

$$2^{\pi(x_1)} \cdot (\tfrac{3}{2})^{x(x_2)} \cdot (\tfrac{4}{3})^{\pi(x_3)} \cdots$$

may be a maximum, when it is given that

$$\vartheta(x_1) + \vartheta(x_2) + \vartheta(x_3) + \cdots$$

is a fixed number. The relations which we obtained are

$$\frac{\log 2}{\log x_1} = \frac{\log(\frac{3}{2})}{\log x_2} = \frac{\log(\frac{4}{3})}{\log x_3} = \cdots.$$

This suggests the following more general problem. If N is an integer of the form

$$e^{c_1\vartheta(x_1) + c_2\vartheta(x_2) + c_3\vartheta(x_3) + \cdots}, \tag{242}$$

where $c_1, c_2, c_3, \ldots$ are any given positive integers, it is required to find the nature of N, that is to say the relations which hold between $x_1, x_2, x_3, \ldots$, when $d(N)$ is of maximum order. From (242) we see that

$$d(N) = (1+c_1)^{\pi(x_1)} \left(\frac{1+c_1+c_2}{1+c_1}\right)^{\pi(x_2)} \left(\frac{1+c_1+c_2+c_3}{1+c_1+c_2}\right)^{\pi(x_3)} \tag{242$'$}$$

If we define the "superior" numbers of the class (242) by the inequalities

$$\frac{d(N)}{N^{\epsilon}} \geq \frac{d(N')}{(N')^{\epsilon}},$$

for all values of N' less than N, and

$$\frac{d(N)}{N^{\epsilon}} > \frac{d(N')}{(N')^{\epsilon}},$$

for all values of N' greater than N, N and N' in the two inequalities being of the form (242), and proceed as in § 33, we can shew that

$$d(N) \leq N^{1/x} \frac{(1+c_1)^{\pi\{(1+c_1)^{x/c_1}\}}}{e^{(c_1/x)\vartheta\{(1+c_1)^{x/c_1}\}}} \frac{\left(\frac{1+c_1+c_2}{1+c_1}\right)^{\pi\left\{\left(\frac{1+c_1+c_2}{1+c_1}\right)^{x/c_2}\right\}}}{e^{(c_2/x)\vartheta\left\{\left(\frac{1+c_1+c_2}{1+c_1}\right)^{x/c_2}\right\}}} \cdots, \tag{243}$$

for all values of x, and for all values of N of the form (242). From we can shew, by arguments similar to those of § 38, that N must be of the form

$$e^{c_1\vartheta\{(1+c_1)^{x/c_1}\}+c_2\vartheta\left\{\left(\frac{1+c_1+c_2}{1+c_1}\right)^{x/c_2}\right\}+c_3\vartheta\left\{\left(\frac{1+c_1+c_2+c_3}{1+c_1+c_2}\right)^{x/c_3}\right\}+\cdots}, \tag{244}$$

and $d(N)$ of the form

$$(1+c_1)^{\pi\left\{(1+c_1)^{\frac{x}{c_1}}\right\}} \left(\frac{1+c_1+c_2}{1+c_1}\right)^{\pi\left\{\left(\frac{1+c_1+c_2}{1+c_1}\right)^{\frac{x}{c_2}}\right\}} \left(\frac{1+c_1+c_2+c_3}{1+c_1+c_2}\right)^{\pi\left\{\left(\frac{1+c_1+c_2+c_3}{1+c_1+c_2}\right)^{\frac{x}{c_3}}\right\}} \tag{244$'$}$$

From (244) and (244') we can find the maximum order of $d(N)$, as in § 43.

47. We shall now consider the order of $d(N)$ for some special forms of N. The simplest case is that in which N is of the form

$$2 \cdot 3 \cdot 5 \cdot 7 \cdots p;$$

so that

$$\log N = \vartheta(p),$$

and

$$d(N) = 2^{\pi(p)}.$$

It is easy to shew that

$$d(N) = 2^{Li(\log N) - R(\log N) + O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}}. \tag{245}$$

In this case $d(N)$ is exactly a power of 2, and this naturally suggests the question: what is the maximum order of $d(N)$ when $d(N)$ is exactly a power of 2?

It is evident that, if $d(N)$ is a power of 2, the indices of the prime divisors of N cannot be any other numbers except 1,3,7,15,31, ...; an so in order that $d(N)$ should be of maximum order, N must be of the form

$$e^{\vartheta(x_1)+2\vartheta(x_2)+4\vartheta(x_3)+8\vartheta(x_4)+\cdots},$$

and $d(N)$ of the form

$$2^{\pi(x_1)+\pi(x_2)+\pi(x_3)+\cdots}.$$

It follows from § 46 that, in order that $d(N)$ should be of maximum order N, must be of the form

$$e^{\vartheta(x)+2\vartheta(\sqrt{x})+4\vartheta(x^{\frac{1}{4}})+8\vartheta(x^{\frac{1}{8}})+\cdots}, \tag{246}$$

and $d(N)$ of the form

$$2^{\pi(x)+\pi(\sqrt{x})+\pi(x^{\frac{1}{4}})+\pi(x^{\frac{1}{8}})+\cdots}. \tag{247}$$

Hence the maximum order of $d(N)$ can easily be shewn to be

$$2^{Li(\log N)+\frac{4\sqrt{(\log N)}}{(\log\log N)^2}-R(\log N)+O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}}. \tag{248}$$

It is easily seen from (246) that the least number having 2^n divisors is

$$2 \cdot 3 \cdot 4 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 16 \cdot 17 \cdot 19 \cdot 23 \cdot 25 \cdot 29 \cdots \quad \text{to } n \text{ factors}, \tag{249}$$

where 2, 3, 4, 5, 7, ... are the natural primes, their squares, fourth powers and so on, arranged according to order of magnitude.

48. We have seen that the last indices of the prime divisors of N must be 1, if $d(N)$ is of maximum order. Now we shall consider the maximum order of $d(N)$ when the indices of the prime divisors of N are never less than an integer n. In the first place, in order that $d(N)$ should be of maximum order, N must be of the form

$$e^{n\vartheta(x_1)+\vartheta(x_2)+\vartheta(x_3)+\cdots},$$

and $d(N)$ of the form

$$(1+n)^{\pi(x_1)}\left(\frac{2+n}{1+n}\right)^{\pi(x_2)}\left(\frac{3+n}{2+n}\right)^{\pi(x_3)}\cdots.$$

It follows from §46 that N must be of the form

$$e^{n\vartheta\{(1+n)^{x/n}\}+\vartheta\{\left(\frac{2+n}{1+n}\right)^{x}\}+\vartheta\{\left(\frac{3+n}{2+n}\right)^{x}\}+\cdots}, \tag{250}$$

and $d(N)$ of the form

$$(1+n)^{\pi\{(1+n)^{x/n}\}}\left(\frac{2+n}{1+n}\right)^{\pi\{\left(\frac{1+n}{1+n}\right)^{x}\}}\left(\frac{3+n}{2+n}\right)^{\pi\{\left(\frac{3+n}{2+n}\right)^{x}\}}\cdots \tag{251}$$

Then, by arguments similar to those of § 43, we can shew that the maximum order of $d(N)$ is

$$(n+1)^{Li\{(1/n)\log N\}+\phi(N)}, \tag{252}$$

where

$$\begin{aligned}\phi(N) \quad = \quad & \left\{\frac{\log(n+2)}{\log(n+1)}-1\right\} Li\left\{\left(\frac{1}{n}\log N\right)^{n\frac{\log(n+2)}{\log(n-1)}-n}\right\} \\ & -\frac{\left(\frac{1}{n}\log N\right)^{n\frac{\log(n+2)}{\log(n-1)}-n}}{n\log\left(\frac{1}{n}\log N\right)}-R\left(\frac{1}{n}\log N\right)+)\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}.\end{aligned}$$

if $n \geq 3$, it is easy to verify that

$$n\frac{\log(n+2)}{\log(n+1)}-n<\tfrac{1}{2};$$

and so (252) reduces to

$$(n+1)^{Li\{(1/n)\log N\}-R\{(1/n)\log N\}+O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}}, \tag{253}$$

provided that $n \geq 3$.

49. Let us next consider the maximum order of $d(N)$ when N is a perfect nth power. In order that $d(N)$ should be of maximum order, N must be of the form

$$e^{n\vartheta(x_2)+n\vartheta(x_2)+n\vartheta(x_3)+\cdots}$$

and $d(N)$ of the form

$$(1+n)^{\pi(x_1)}\left(\frac{1+2n}{1+n}\right)^{\pi(x_2)}\left(\frac{1+3n}{1+2n}\right)^{\pi(x_3)}\cdots.$$

It follows from § 46 that N must be of the form

$$e^{n\vartheta\{(1+n)^x\}+n\vartheta\left\{\left(\frac{1+2n}{1+n}\right)^x\right\}+n\vartheta\left\{\left(\frac{1+3n}{1+2n}\right)^x\right\}+\cdots}, \tag{254}$$

and $d(N)$ of the form

$$(1+n)^{pi\{(1+n)^x\}}\left(\frac{1+2n}{1+n}\right)^{\pi\left\{\left(\frac{1+2n}{1+n}\right)^x\right\}}\left(\frac{1+3n}{1+2n}\right)^{\pi\left\{\left(\frac{1+3n}{1+2n}\right)^x\right\}}\cdots \tag{255}$$

Hence we can shew that the maximum order of $d(N)$ is

$$(n+1)^{Li\{(1/n)\log N\}-R\{(1/n)\log N\}+O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}}, \tag{256}$$

provided that $n>1$.

50. Let $l(N)$ denote the least common multiple of the first N natural numbers. Then it can easily be shewn that

$$l(N)=2^{[\log N\log 2]}\cdot 3^{[\log N/\log 3]}\cdot 5^{[\log N/\log 5]}\cdots p, \tag{257}$$

where p is the largest prime not greater than N. From this we can shew that

$$l(N)=e^{\vartheta(N)+\vartheta(\sqrt{N})+\vartheta(N^{\frac{1}{3}})+\vartheta(N^{\frac{1}{4}})+\cdots}; \tag{258}$$

and so

$$d\{l(N)\}=2^{\pi(N)}\left(\tfrac{3}{2}\right)^{\pi(N)}\left(\tfrac{4}{3}\right)^{\pi(N^{\frac{1}{3}})}\cdots. \tag{259}$$

From (258) and (259) we can shew that, if N is of the form $l(M)$, then

$$d(N)=2^{Li(\log N)+\phi(N)}, \tag{260}$$

where

$$\phi(N)=\frac{\log(\frac{9}{8})}{\log 2}\frac{\sqrt{(\log N)}}{\log\log N}+\frac{4\log(\frac{3}{2})}{(\log\log N)^2}-R(\log N)+O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}.$$

It follows from (258) that

$$l(N) = e^{N+O\{\sqrt{N(\log)^2}\}}; \tag{261}$$

and from (259) that

$$d\{l(N)\} = 2^{Li(n)+O(\sqrt{N}\log N)}. \tag{262}$$

51. Finally, we shall consider the number of divisors of $N!$ It is easily seen that

$$N! = 2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdots p^{a_p}, \tag{263}$$

where p is the largest prime not greater than N, and

$$a_\lambda = \left[\frac{N}{\lambda}\right] + \left[\frac{N}{\lambda^2}\right] + \left[\frac{N}{\lambda^3}\right] + \cdots.$$

It is evident that the primes greater than $\frac{1}{2}N$ and not exceeding N appear once in $N!$, the primes greater than $\frac{1}{3}N$ and not exceeding $\frac{1}{2}N$ appear twice, and so on up to those greater than $N/[\sqrt{N}]$ and not exceeding $N/(\sqrt{N}] - 1)$, appearing $[\sqrt{N}] - 1$ times*. The indices of the smaller primes cannot be specified so simply. Hence it is clear that

$$N! = e^{\vartheta(N)+\vartheta(\frac{1}{2}N)+\vartheta(\frac{1}{3}N)+\cdots+\vartheta\left(\frac{N}{[\sqrt{N}]-1}\right)} \times 2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdots \varpi^{a_\varpi}, \tag{264}$$

where ϖ is the largest prime not greater than $\sqrt{N}$, and

$$a_\lambda - 1 + [\sqrt{N}] = \left[\frac{N}{\lambda}\right] + \left[\frac{N}{\lambda^2}\right] + \left[\frac{N}{\lambda^3}\right] + \cdots.$$

From (264) we see that

$$\begin{aligned} d(N!) &= 2^{\pi(N)}(\tfrac{3}{2})^{\pi(\frac{1}{2}N)}(\tfrac{4}{3})^{\pi(\frac{1}{3}N)}\cdots \text{ to } [\sqrt{N}] - 1 \text{ factors} \\ &\quad \times e^{O\{\log(1+a_2)+\log(1+a_3)+\cdots+\log(1+a_\varpi)\}} \\ &= 2^{\pi(N)}(\tfrac{3}{2})^{\pi(\frac{1}{2}N)}(\tfrac{4}{3})^{\pi(\frac{1}{3}N)}\cdots \text{ to } [\sqrt{N}] - 1 \text{ factors} \\ &\quad \times e^{O\{\varpi\log(1+a_2)\}} \\ &= 2^{Li(N)}(\tfrac{3}{2})^{Li(\frac{1}{2}N)}(\tfrac{4}{3})^{Li(\frac{1}{3}N)}\cdots \text{ to } [\sqrt{N}] \text{ factors} \\ &\quad \times e^{O(\sqrt{N}\log N)}. \end{aligned} \tag{265}$$

Since

$$Li(N) = \frac{N}{\log N} + O\left\{\frac{N}{(\log N)^2}\right\},$$

*Strictly speaking, this is true only when $N \geq 4$.

we see that

$$d(N!) = C^{\frac{N}{\log N} + O\left\{\frac{N}{(\log N)^2}\right\}}, \tag{266}$$

where

$$C = (1+1)^1 (1+\tfrac{1}{2})^{\frac{1}{2}} (1+\tfrac{1}{3})^{\frac{1}{3}} (1+\tfrac{1}{4})^{\frac{1}{4}} \ldots.$$

From this we can easily deduce that, if N is of the form $M!$, then

$$d(N) = C^{\frac{\log N}{(\log\log N)^2} + \frac{2\log N \log\log\log N}{(\log\log N)^3} + O\left\{\frac{\log N}{(\log\log N)^3}\right\}}, \tag{267}$$

where C is the same constant as in (266).

52. It is interesting in this connection to shew how, by considering numbers of certain special forms, we can obtain lower limits for the maximum orders of the iterated functions $dd(n)$ and $ddd(n)$. By supposing that

$$N = 2^{2-1} \cdot 3^{3-1} \cdots p^{p-1},$$

we can shew that

$$dd(n) > 4^{\frac{\sqrt{(2\log n)}}{\log\log n}} \tag{268}$$

for an infinity of values of n. By supposing that

$$N = 2^{2^{a_2}-1} \cdot 3^{3^{a_3}-1} \cdots p^{p^{a_p}-1},$$

where

$$a_\lambda = \left[\frac{\log p}{\log \lambda}\right] - 1$$

we can shew that

$$ddd(n) > (\log n)^{\log\log\log\log n} \tag{269}$$

for an infinity of values of n.