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The Passage of Fast Electrons and the Theory of Cosmic Showers

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Introduction

It is well known that according to relativistic quantum mechanics, electrons and positrons with energy large compared with their rest mass have a very large probability when passing through the field of a nucleus of losing a large fraction of their energy in one process by emitting radiation. Hard quanta have a correspondingly large probability of creating electron pairs. Until recently it was believed that the direct measurements of Anderson and Neddermeyer on the energy loss of fast electrons showed that though this energy loss by radiation existed, it was much smaller for energies greater than about 10^8 e-volts than that theoretically predicted, and it was therefore assumed that the present quantum mechanics began to fail for energies greater than about this value. More recent experiments by Anderson and Neddermeyer (1936) have, however, led them to revise their former conclusions, and their new and more accurate experiments show that up to energies of 300 million e-volts (the highest energies measured in their experiments) and probably higher, the experimentally measured energy loss of fast electrons is in agreement with that predicted theoretically. In fact, one may say that at the moment there are no *direct* measurements of energy loss by fast electrons which conclusively prove a breakdown of the theory. This is particularly satisfactory, inasmuch as the theoretical reasons for expecting a breakdown of the theoretical formulae at energies greater than about 137 mc^2, namely the neglect of the classical "radius" of the electron, have been shown by v. Weizsäcker (1934) and Williams (1934) to be unfounded. Under these circumstances, and in view of the experimental evidence mentioned above, it is reasonable *as a working hypothesis* to assume the theoretical formulae for energy loss and pair creation to be valid for all energies, however high, and to work out the consequences which result from them.

It is our aim to deduce results which can be compared directly with cosmic ray experiments and which will then allow one to decide whether or not the theory fails for extremely high energies, and in the latter case, at what point the failure begins.

A number of facts immediately present themselves which seem to be incompatible with the large observed energy losses mentioned above if we assume that the primary cosmic ray particles are electrons or positrons, for example, Regener's measurements of the absorption of cosmic radiation in the atmosphere. The theoretically calculated average "range" of an electron or quantum of 10^{12} e-volts is only of the order of 2 km. of air, 2 m. of water and 4 cm. of lead, whereas in fact electrons are observed at sea-level which appear to have penetrated the atmosphere a thickness of 8 km. The chance that an electron of 10^{12} e-volts should penetrate to sea-level and retain an energy $> 10^8$ e-volts is only about 10^{-5}. We shall try to show in this paper that these difficulties are only apparent, and that more careful considerations reveal that the above-mentioned experiments are quite reconcilable with the present theory and observations of energy loss.

The reason for this is the following. Although it is true that the theory predicts that after travelling a certain small distance a fast electron will have lost all its energy by emitting radiation, this energy is not subdivided into a large number of soft quanta, but on the contrary divided between a very few quanta, each of which has an energy comparable with the initial energy of the electron. Moreover, for the very high energies with which we are concerned in cosmic radiation the quanta move in very nearly the same direction as the original electron. Each of these quanta creates an electron pair after travelling another short distance, and these particles again travel in very nearly the same direction as the original light quantum. These electrons and positrons can then again emit radiation quanta by colliding with nuclei, and these quanta will again create pairs, and so on, there being no limit to the number of steps possible. In particular, there is quite a large probability that an electron emits a quantum which takes nearly its whole energy, this quantum then creating a pair, of which the electron has nearly the whole energy of the quantum, the positron being left with very little. This electron will then travel in nearly the same direction and have nearly the same energy as the original electron, and will behave in the same way as the original electron would have done if it had lost only a very small amount of its energy.

But this case is exceptional. Normally a certain amount of subdivision of the energy will take place. It is, however, important to realize that the greatest effective extent by which the energy of an electron can be reduced in one collision is a half, since in any other case either the resulting quantum or the electron itself after the collision must have an energy more than half the initial energy of the electron, and this energy is available for further transformations. The same applies to the creation of a pair by a quantum. Thus the effective average "loss" of energy which occurs in each process is less than a half. Moreover, the effect of the subdivision is roughly to double the number of particles at each step.

The result of these successive processes is therefore twofold:

Firstly, after passing through a plate of some heavy substance of a suitable thickness we can show that the original electron may emerge accompanied by a large number of electrons of large energy, which would all appear to come from some small region in the plate. Such a phenomenon would resemble the showers observed in experiments on cosmic radiation, and we shall show that under certain circumstances as many as 1000 or more positive and negative electrons may emerge from the plate.*

Moreover, the number of electrons with energy above some arbitrary large value which are found at a certain thickness below the top of a layer of substance due to the impact of a homogeneous beam of electrons on the top of the layer is given by a curve which has a very close resemblance to the well-known curves found by Rossi, and to Regener's ionization curve in the atmosphere.

Secondly, it can be shown that the effective "absorption coefficient" calculated from the tail end of these curves has a value which is much less than the smallest absorption coefficient for hard γ-radiation. Similar curves have been wrongly used to prove the existence of a radiation much more penetrating than any possible theoretically.

We may describe the theory which is put forth in this paper as the normal quantum theory of showers, inasmuch as it depends only on describing the interaction of matter and radiation by Dirac's relativistic wave equation, and the quantum theory of radiation. The limits of our theory are therefore the limits of relativistic quantum mechanics. The number of particles which emerge from a plate of given thickness depends on the ratio of the initial energy of the electron or positron creating the shower to the minimum energy of the particles considered in the shower, and increases as this ratio increases. Showers in which more than 1000 *fast* particles with energy greater than 10^7 e-volts are ejected only occur when this ratio is about 10,000 which necessitates energies of 10^{11} e-volts for the particle

* The idea that cosmic ray showers could be explained in this way had already been expressed in 1934 by L. Nordheim in a conversation with one of us (H.), but owing to the ill-founded suspicion in which the theory was then held, it did not seem worth while carrying out any calculations. Mr Carmichael had also pointed out in a conversation with one of us (B.) that one could explain the showers by successive processes of multiplication.

creating the shower. If, therefore, it is shown experimentally that the theory of energy loss fails for energies greater than about 10^{10} e-volts, our explanation of large showers also fails. We should emphasize, however, that such a failure would not invalidate our theory for showers of less than say ten particles, since these can be produced by electrons in the region where we know from experiment that the theory is valid.

Further, in our theory a large shower of more than 100 particles requires thicknesses of at least 1 cm. of lead. If we find that such showers are produced *frequently* in lead plates of less than 1 cm., it would be evidence of the inadequacy of our theory. In both the above cases we should have to fall back on a completely new theory of the showers, which introduced a new interaction between the particles themselves and also, possibly, light. An elegant theory of this type has been put forward by Heisenberg (1936), and though it is still in a tentative form, it may be that the explanation of the largest showers may have to be referred to it.

1—Problem and Approximations

The question we wish to answer is this: *Given an electron which enters a thick layer of matter with an energy E_0, what is the number of electrons with energy greater than E found at any given point below the top of the layer?*

Since we are interested only in electrons with energy greater than 10^7–10^8 e-volts, and no electrons or quanta with less energy will appear in our calculations, a number of approximations become possible.

Firstly, ionization and the Compton-effect can be neglected entirely. The moment an electron becomes so slow that its rate of energy loss by ionization equals its rate of energy loss by radiation, we consider it to be "stopped". Since it then loses its energy within a very short range by ionization and little energy is available for the type of transformations we consider here, it need not be considered further in our calculation. The limit at which this happens is roughly 10^7 e-volts in lead, and $1{\cdot}5 \times 10^8$ e-volts in air and water.

Secondly, as has been stated already, all particles or quanta which result in the above two processes and have an energy greater than 10^7 e-volts make a very small angle with the particle or quantum producing them, so that to a very good approximation the whole problem can be treated as a *one-dimensional* one. The question of angular spread will be considered in more detail in § 5.

Even so, the calculation can only be carried out by making certain approximations, and we now proceed to state and discuss the various assumptions which are involved.

The differential effective "cross-section" for the emission of a quantum with energy between $k \equiv h\nu$ and $k+dk$ by an electron or positron of energy E_0 may be written in the form

$$\Phi_k d\left(\frac{k}{E_0}\right) = \bar{\Phi} F\left(\frac{k}{E_0}\right)\frac{E_0}{k} d\left(\frac{k}{E_0}\right), \tag{1}$$

where $\bar{\Phi}$ is a constant which only depends on the material in question and is given by

$$\bar{\Phi} = \frac{Z^2}{137}\left(\frac{e^2}{mc^2}\right)^2, \tag{2}$$

(Z = atomic number). The function F has been calculated theoretically by Bethe and Heitler (1934) (cf. also Heitler 1936, p. 170). We have to replace the exact function by suitable approximations in two cases.

(a) In order to calculate the *straggling*, i.e. the probability for various energy losses of the electron after having travelled a certain distance, we assume $F(k/E_0)$ to be of the form

$$F\left(\frac{k}{E_0}\right) = \frac{k}{E_0}\frac{a}{\log E_0/(E_0-k)}, \tag{3}$$

where a is a constant and has the numerical value

$$a = \begin{cases} 20 \text{ in lead,} \\ 23 \text{ in air and water.} \end{cases} \tag{4}$$

With this assumption the probability that an electron with energy E_0 after travelling a certain distance λ has an energy between E and $E+dE$ is (Heitler 1936, p. 225)

$$w(b\lambda,\eta)\,d\eta = \frac{e^{-\eta}\,\eta^{b\lambda-1}}{\Gamma(b\lambda)}\,d\eta, \tag{5}$$

where
$$\eta = \log E_0/E \tag{5'}$$
and
$$b = a\bar{\Phi}\sigma; \tag{5''}$$

σ is the number of nuclei per cubic centimetre of the substance. Hence the probability of the electron having an energy greater than E after travelling a distance λ is

$$W(b\lambda,y) \equiv \int_0^y w(b\lambda,\eta)\,d\eta = \int_0^y \frac{e^{-\eta}\,\eta^{b\lambda-1}}{\Gamma(b\lambda)}\,d\eta \equiv \frac{\Gamma(b\lambda,y)}{\Gamma(b\lambda)}, \tag{6}$$

where $\Gamma(b\lambda, y)$ is the "incomplete gamma-function". (See the appendix.)

(b) In calculating the *number of quanta* between k and $k+dk$ emitted by an electron of energy E_0 we assume $F(k/E_0)$ to be constant:

$$F\left(\frac{k}{E_0}\right) = a', \tag{7}$$

where a' is another constant with a value different from a.

We determine a' so that the average rates of energy loss of the electron according to assumptions (a) and (b) are the same. Thus

$$\frac{1}{\bar{\Phi}}\int_0^{E_0} k\,\Phi_k \frac{dk}{E_0} = E_0 a\int_0^{\infty}(1-e^{-\eta})\,e^{-\eta}\frac{d\eta}{\eta} = E_0 a\log 2 = E_0 a'$$

or

$$a' = a\log 2. \tag{8}$$

(c) The differential effective cross-section for the *creation* by a light quantum k of a *pair*, the positron of which has an energy between E_+ and $E_+ + dE_+$, may be written in the form

$$\Phi_{E_+} dE_+ = \bar{\Phi}\,G(E_+, k)\,d\left(\frac{E_+}{k}\right). \tag{9}$$

The function $G(E_+, k)$ has also been calculated theoretically (Heitler 1936, p. 199, 215). The result is that the cross-section for pair production is almost independent of the distribution of energy between the two pair electrons. Thus we put

$$G(E_+, k) = G(k), \tag{10}$$

where $G(k)$ depends only on k, but not on E_+/k.

(d) The *total* effective *cross-section* for the *creation* of a *pair* by the quantum k then becomes

$$\Phi_{pair} = \int_0^k \Phi_{E_+} dE_+ = \bar{\Phi}\,G(k). \tag{10'}$$

Now Φ_{pair} as calculated using the exact function $G(E_+, k)$ increases very slowly with the energy and becomes constant at very high energies. (As k increases from 100 to 1000 mc^2 $G(k)$ only increases by about 20 %.) We shall therefore further assume that

$$G(k) = \text{const.} = G\,(\text{say}). \tag{11}$$

With this assumption the probability of a quantum travelling a distance λ is

$$e^{-\bar{\Phi}G\sigma\lambda} = e^{-\tau\lambda}, \quad \tau = G\bar{\Phi}\sigma, \tag{11'}$$

where the *absorption coefficient* τ is now to be considered as constant. G has the values

$$G \sim \begin{cases} 11\cdot 5 \text{ for lead,} \\ 14 \text{ for air and water.} \end{cases} \tag{12}$$

The above formulae (1), (6), (7), (8), (10′), (11), (12) give the probabilities for all the fundamental processes occurring in our calculations. The nature of the material (atomic number Z and density σ) occurs only in the constants b and τ. Both depend upon Z and σ in the same way (being nearly proportional to $Z^2\sigma$) and are independent of the energy on our assumptions. More strictly, both constants increase very slowly with the energy, but again in almost the same way. The ratio τ/b is therefore almost exactly a universal constant independent of the material and the energy. With the values (4) for a and (12) for G we obtain

$$G/a = \tau/b \equiv \alpha \cong 0{\cdot}6. \tag{13}$$

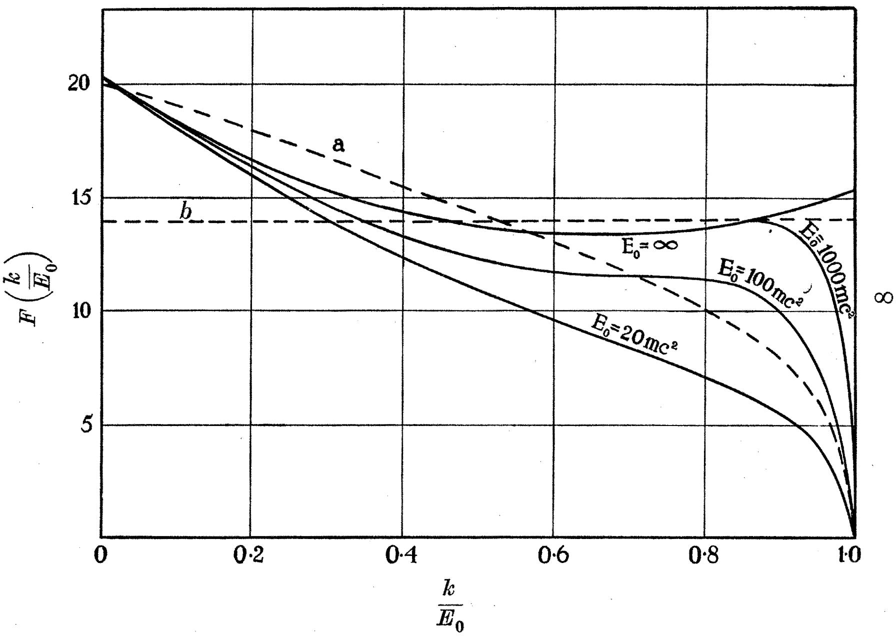

FIG. 1—Pb

Furthermore the thickness of our layer λ occurs only in the form $b\lambda$ or $\tau\lambda$, b and τ both having the dimensions cm.$^{-1}$. It will therefore be convenient to introduce a new measure of length

$$l = b\lambda \tag{14}$$

(l is a pure number), where the unit length $l = 1$ corresponds to a thickness λ_0 (in cm.) $= 1/b = 1/a\overline{\Phi}\sigma$. Provided all lengths are measured in these units our results will be the *same for all materials*. α represents the absorption coefficient for light quanta in these units λ_0^{-1}.

The assumption that α is constant does not lead to great inaccuracy. In one special case we have carried out our calculations with a slightly different value of α and find that our results do not depend critically on the value of α.

The unit thickness $\lambda_0 = 1/b$ introduced by (14) corresponds to

$$0{\cdot}40 \text{ cm. Pb}; \ 1{\cdot}4 \text{ cm. Fe}; \ 7{\cdot}8 \text{ cm. Al}; \ 34 \text{ cm. } H_2O; \ 275 \text{ m. air.*} \qquad (15)$$

With the four assumptions (*a*)–(*d*) made above, it is possible to carry out the calculation. All the approximations are correct to within an accuracy of 20–30 %, and we do not aim at more in the present paper. Fig. 1 gives the exact form of the cross-section for the emission of light quanta and the forms corresponding to assumptions (*a*) and (*b*) (dotted curves). It shows that except when $E_0 - k \ll E_0$ the approximations are accurate to well within 20–30 %. The region $E_0 - k \ll E_0$ where this is not so is small and does not play any vital role in our calculations. Similar remarks are true of the creation of pairs by quanta. The inaccuracies introduced by the various approximations tend to some extent to compensate one another, so that our final results should be accurate to about 30 % or less.

2—Calculations

Let us suppose that we have a homogeneous beam of positrons of energy E_0 falling on the top of a layer of substance. Let $f_+(l, E)$ be the number of positrons whose energy is greater than E at a point l (in the units (15)) below the surface. We include in this number the primary positrons as well as the secondaries, tertiaries, etc. Similarly, let $f_-(l, E)$ be the number of negative electrons with energy greater than E at a point l below the surface. Finally, let $h(l, k)\,dk$ be the number of quanta whose energy lies between k and $k+dk$ also at a point l below the surface.

Quanta with energy k can be emitted by all electrons with energy $>k$. Then by (1), (7) and (14) the number of quanta with energy between k and $k+dk$ emitted by positrons in travelling the distance $dl = b\,d\lambda$ is

$$a'\overline{\Phi}\sigma f_+(l, k)\frac{dk}{k}d\lambda = \log 2 f_+(l, k)\frac{dk}{k}dl.$$

A similar equation holds for the emission of light quanta by the negative electrons.

The number of quanta with an energy between k and $k+dk$ transformed into *pairs* in the same distance dl is, according to (10′), (11′),

$$\tau h(l, k)\,dk\,d\lambda = \alpha h(l, k)\,dk\,dl.$$

The "equation of continuity" for the number of light quanta then becomes

$$\frac{\partial h(l, k)}{\partial l} = \frac{\log 2}{k}\{f_+(l, k) + f_-(l, k)\} - \alpha h(l, k).$$

* Normal pressure, temperature 15° C.

The solution of this equation is

$$h(l,k) = \frac{\log 2}{k} e^{-\alpha l} \int_0^l e^{+\alpha l'} \{f_+(l',k) + f_-(l',k)\} \, dl' + h(0,k) \, e^{-\alpha l}. \tag{16}$$

Here $h(0, k)$ is just a "constant" of integration, which gives the distribution of quanta k which fall on the surface of the layer. In our particular problem $h(0,k)$ is zero.

By (9), (11), (13) the number of positrons in the energy range dE' produced in the distance dl' by quanta in the energy range dk is

$$\alpha \frac{dE'}{k} h(l',k) \, dk \, dl'.$$

Positrons of this energy can be created by all quanta with energies $k \geqslant E'$. Thus, the total number of such positrons produced in dl' is

$$\alpha dE' \, dl' \int_{E'}^{\infty} \frac{h(l',k)}{k} dk \equiv H(l', E') \, dE' \, dl', \text{ say.}$$

In travelling further the positrons lose energy. The probability that a positron after having travelled a certain distance has an energy greater than E is given by (6). Thus the total number of positrons with energies greater than E at the point l below the surface which have been created with any initial energy $E' > E$ at any point $l' < l$ is

$$\int_0^l dl' \int_E^{\infty} W\left(l - l', \log \frac{E'}{E}\right) H(l', E') \, dE'$$
$$= \alpha \int_0^l dl' \int_E^{\infty} dE' \int_{E'}^{\infty} dk \frac{h(l',k)}{k} W\left(l - l', \log \frac{E'}{E}\right).$$

A further contribution to the number of positrons at the point l arises from the primary positrons themselves. Their number (normalized for one incident positron) is given by (6) with $y = \log(E_0/E)$, where E_0 is the primary energy of the positron. Thus the total number of positrons with energies $> E$ at the point l is given by

$$f_+(l,E) = W\left(l, \log \frac{E_0}{E}\right) + \alpha \int_0^l dl' \int_E^{\infty} dE' \int_{E'}^{\infty} dk \frac{h(l',k)}{k} W\left(l - l', \log \frac{E'}{E}\right). \tag{17}$$

The number of negative electrons created in each energy range is equal to the number of created positrons, and further, no negative electrons are assumed to enter the surface of the layer. Thus $f_-(0, E)$ is zero and hence

$$f_-(l,E) = f_+(l,E) - W\left(l, \log \frac{E_0}{E}\right). \tag{18}$$

It is convenient to introduce instead of the energies E, k the logarithmic variables defined by

$$y = \log(E_0/E); \quad y_k = \log(E_0/k); \quad y' = \log(E_0/E'). \tag{19}$$

We simply write then $f(l, y)$ instead of $f(l, E)$.

Using (18) equation (16) becomes

$$h(l,k)\,dk = \log 2\,dy_k\,e^{-\alpha l}\int_0^l e^{\alpha l'}\{W(l',y_k)+2f_-(l',y_k)\}\,dl'. \tag{20}$$

If we introduce (20) into (17) and use (18) we get finally (denoting the integration variable in (20) by l'')

$$f_-(l,y) = \alpha\log 2$$
$$\times\int_0^l dl' e^{-\alpha l'}\int_0^{l'} dl'' e^{\alpha l''}\int_0^y dy' e^{-y'}\int_0^{y'} dy_k e^{y_k} W(l-l', y-y')\{W(l'',y_k)+2f_-(l'',y_k)\}.$$

Interchanging the order of integration over y' and y_k the integration over y' can be carried out according to equation (38), which is proved in the appendix. Writing $l-l'$ for l' we obtain

$$f_-(l,y)$$
$$=\alpha\log 2\,e^{-\alpha l}\int_0^l dl'\int_0^{l-l'} dl''\,e^{\alpha(l'+l'')}\int_0^y dy_k\,W(l'+1, y-y_k)\{W(l'',y_k)+2f_-(l'',y_k)\}. \tag{21}$$

This is the final *integral equation* we have to solve.

The solution of this equation can be written in the form of a series

$$f_-(l,y) = \sum_{n=1}^{\infty} f_n(l,y), \tag{22}$$

where

$$f_n(l,y) = \frac{(2\alpha\log 2)^n}{2}e^{-\alpha l}\int_0^l dl'\int_0^y dy' e^{\alpha l'}\frac{l'^n(l-l')^{n-1}(y-y')^{n-1}}{n!\,(n-1)!\,(n-1)!}W(l'+n, y'). \tag{23}$$

To prove this we introduce (22), (23) into the right-hand side of (21), and consider only the nth term $2f_n(l'', y_k)$.

We get by a suitable change of the notation of the integration variables

$$\frac{(2\alpha\log 2)^{n+1}}{2}e^{-\alpha l}\int_0^l dl'\,e^{\alpha l'}\int_0^{l-l'} dl''\int_0^{l''} dl''' e^{\alpha l'''}\frac{l'''^n(l''-l''')^{n-1}}{n!\,(n-1)!}$$
$$\times\left\{\int_0^y dy'\,W(l'+1, y-y')\int_0^{y'}\frac{(y'-y'')^{n-1}}{(n-1)!}W(l'''+n, y'')\,dy''\right\}. \tag{24}$$

Using the relation (39) which is proved in the appendix, the expression in curly brackets becomes

$$\int_0^y \frac{(y-y')^n}{n!}W(l'+l'''+n+1, y')\,dy'.$$

We now change the order of the l'' and l''' integrations in (24), so that the l'' integration is to be carried out first and extends from l''' to $l-l'$, and the l''' integration is to be carried out afterwards, and extends from 0 to $l-l'$. The l'' integration can then be performed without any difficulty, and (24) reduces to

$$\frac{(2\alpha\log 2)^{n+1}}{2}e^{-\alpha l}\int_0^l dl'\int_0^{l-l'}dl'''\,e^{\alpha(l'+l''')}\frac{l'''^n(l-l'-l''')^n}{n!\,n!}\int_0^y\frac{(y-y')^n}{n!}\times W(l'+l'''+n+1,y')\,dy'.$$

Treating l' and $l'+l'''$ as variables instead of l' and l''', one integration (over l''') can again be carried out. Writing l' instead of $l'+l'''$ we easily obtain

$$\frac{(2\alpha\log 2)^{n+1}}{2}e^{-\alpha l}\int_0^l dl'\,e^{\alpha l'}\frac{l'^{n+1}(l-l')^n}{(n+1)!\,n!}\int_0^y\frac{(y-y')^n}{n!}\times W(l'+n+1,y')\,dy'=f_{n+1}(l,y).$$

Thus the second term in the curly brackets on the right-hand side of (21) is just equal to f_{n+1}. It may similarly be shown using equation (37) of the appendix that the term which results from $W(l'',y_k)$ on the right-hand side of (21) is just equal to $f_1(l,y)$. Thus inserting the full series (22) in (21) we obtain

$$\sum_1^\infty f_n=f_1+\sum_1^\infty f_{n+1}$$

which is obviously true. It therefore follows that (22), (23) is the solution of the integral equation (21).

The different terms $f_n(l,y)$ in the series (22) have a direct physical meaning, and, indeed, the nth term represents the electrons produced in the nth step, i.e. with n intermediary γ-quanta. Suppose we put f equal to zero in the right-hand side of (21), which is equivalent to assuming that the secondary electrons produce no further electrons. Then we would get for the left-hand side just $f_1(l,y)$. Thus $f_1(l,y)$ represents the number of secondaries. Putting $f=f_1$ in the right-hand side of (21), we get a further term $f_2(l,y)$ added to the left-hand side of (21). Hence f_2 represents the electrons produced by the secondary electrons, in other words, the tertiary electrons. Similarly f_n represents the electrons produced in the nth step. The factor 2 which multiplies $f_-(l,y)$ on the right-hand side of (21) is due to the fact that in the first and later steps there are just as many positrons as electrons.

It is not possible to evaluate the integral (23) exactly in any convenient form, nor have we succeeded in summing the series (22). We therefore carry out the y' integration approximately. Writing

$$R_n(l',y)\equiv\int_0^y\frac{(y-y')^{n-1}}{(n-1)!}W(l'+n,y')\,dy' \tag{25}$$

we obtain after an easy transformation by the definition of W (equation (6))

$$R_n = \frac{1}{n!\,\Gamma(l'+n)}\int_0^y e^{-t}\,t^{l'+n-1}(y-t)^n\,dt. \tag{26}$$

This result is exact. (26) can be integrated exactly in the form of a series, but for our purpose it is more convenient to express the result approximately in a closed form. An inspection of (26) shows that the integrand has a sharp maximum somewhere between 0 and y. We may write the integral in the form

$$\int_0^y e^{F(t)}\,dt,$$

and expand the exponent thus $F(t) = F(y_0) - \frac{1}{2}F''(y_0)\,t^2$, keeping only the first two terms of the expansion. y_0 is the value of t at the maximum of the integrand. Our approximation is equivalent to replacing the actual curve representing the integrand by a Gaussian error curve of suitable height and width. The limits of integration may then be extended from 0 to ∞ without appreciable error, and using Stirling's expansion for the gamma-function, (26) becomes

$$R_n(l', y) = e^{l'+2n-1-y_0}\left(\frac{y_0}{l'+n-1}\right)^{l'+n-1}\left(\frac{y-y_0}{n}\right)^n \frac{1}{\sqrt{2\pi n(l'+n-1)\,F_0''}}, \tag{27}$$

where $$y_0 = \frac{y+l'+2n-1}{2} - \tfrac{1}{2}\sqrt{(y+l'+2n-1)^2 - 4y(l'+n-1)},$$

$$F_0'' = \frac{l'+n-1}{y_0^2} + \frac{n}{(y-y_0)^2}. \tag{27'}$$

We have checked (27) by comparing it with the exact value of (26) in certain special cases, and find that it gives very good results.

The integration over l' in (23) can be carried out in the same way. The result, however, is very complicated. The further stages of the calculation have, therefore, been carried out numerically, and the l' integration has been performed graphically.

3—Results

The results* of our calculations are given in Table I. The figures give the average number $\bar{N}$† of electrons or positrons with energies larger than E produced by one primary electron (or positron) with energy E_0. l is the thickness of the layer in the units given by (15), and $y = \log(E_0/E)$.

* Similar results seem to have been obtained by Oppenheimer (1936).

† $\bar{N}$ is identical with $f_-(l, y)$. For the purpose of the following sections this notation is more convenient.

In fig. 2 we have plotted the curve for $y = 5$. The dotted curve is the ordinary straggling curve for the primary particle ($n = 0$) giving the probability (equation (6)) that the primary particle has an energy $> E$ after

TABLE I*

$y \backslash l$	2	3	5	7	10	13	17	22	30	40
3	0·67	0·9	0·82	0·42	0·11	0·025	4×10^{-3}	$3·4 \times 10^{-4}$	6×10^{-6}	—
5	2·5	4	6	5·9	3·4	1·3	0·28	0·03	$7·5 \times 10^{-4}$	9×10^{-6}
7	8	13	25	34	37	23	8	1·5	0·07	$1·5 \times 10^{-3}$
10	15	40	125	270	565	670	450	165	16	0·6

* The figures for $l = 2$ and 40 and some of the figures for $y = 7$ are extrapolated.

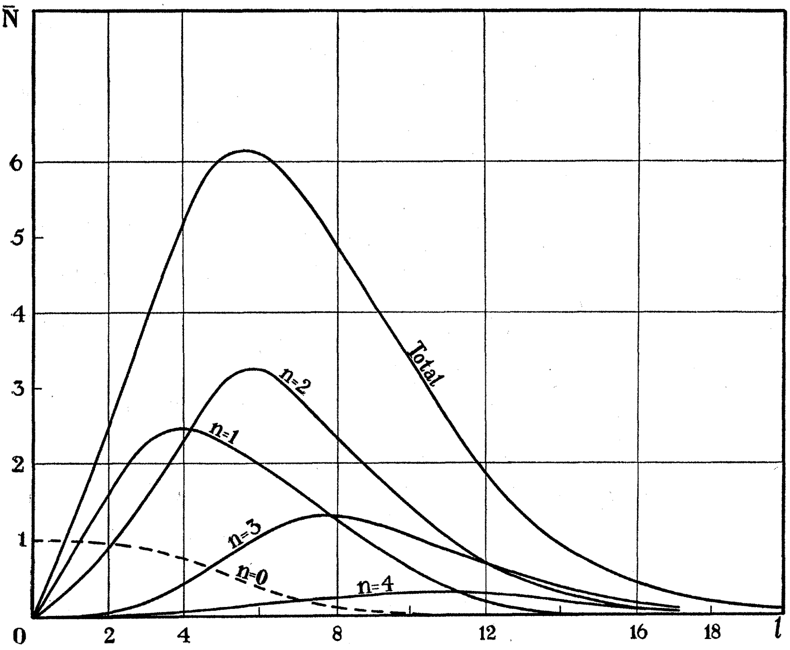

FIG. 2—$y = 5$.

having travelled a distance $\lambda = l/b$, i.e. the mean number of primaries with this energy. We have also plotted the various contributions $f_n(l, y)$ in (22). The curves marked $n = 1, 2, 3 \ldots$ represent the number of secondaries, tertiary, ... electrons produced by the primary. The sum of all these (except $n = 0$) is the curve marked total. As it is seen from the graph the number of secondaries is, except for very small l, far larger than the number of primaries ($n = 0$). The series $\sum_n f_n$ converges very well, the contributions arising from stages where $n > 4$ can be entirely neglected. The curve has a maximum for l between 5 and 6. The value of the mean number of particles $\bar{N}$ is then about 6, showing that on the average at these thicknesses we should expect six positive and six negative electrons.

In fig. 3 we have plotted the total number of electrons or positrons for $y = 3, 7, 10$ as a function of l. To bring the three curves on the same figure, a different scale has been used for the ordinates of each curve. The straggling curve for the primary particle is represented by the dotted line for the case

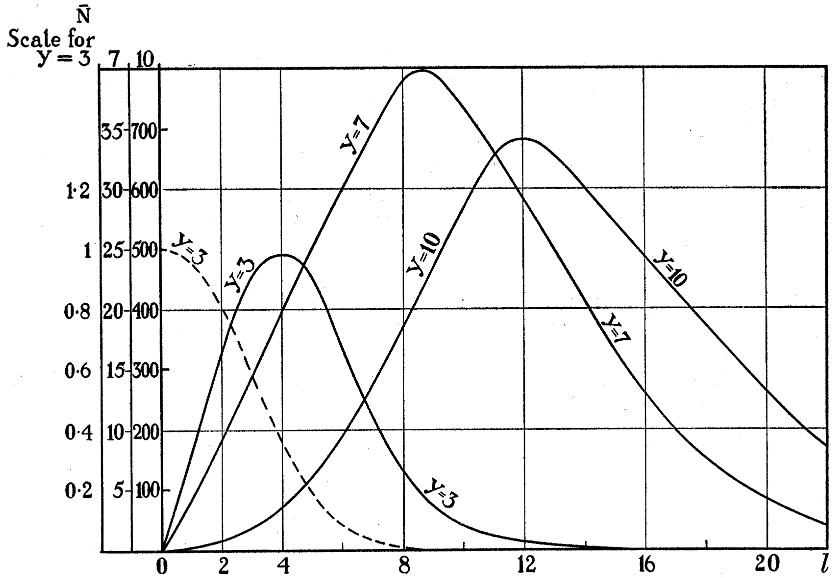

FIG. 3

$y = 3$. In this case the number of secondaries is much greater than the number of primaries only for large thicknesses l. For $y = 7$ and 10, the straggling curves for the primary particles are too small to be drawn conveniently on the corresponding scales.

In fig. 4 we have plotted the logarithm to the base ten of the number of secondaries $\bar{N}$ as a function of y for various values of l. This graph allows us to interpolate roughly for values of y other than those given in our Table. The envelope of these curves represents the logarithm to the base ten of the maximum number of electrons produced for each y. It is almost exactly a straight line, at least for values of y between 2 and 13. From the slope of this straight line we find that the maximum number of electrons or positrons produced as a function of E_0/E is well represented by the formula

$$\bar{N}_m = 0{\cdot}062\,(E_0/E)^{0{\cdot}93}. \qquad (28)$$

The thickness l_m at which the maximum number is produced shifts very slowly to larger values with increasing E_0/E. For $E_0/E = 20(y = 3)$ $l_m = 4$, and for $E_0/E = 22{,}000(y = 10)$ $l_m = 12$.

For a given E the number of secondaries increases very rapidly with E_0. If, for instance, a primary electron with an energy $E_0 = 2 \times 10^{11}$ e-volts passes through a lead plate 5 cm. thick (corresponding to $l = 12$, according to (15)), as many as 600 positrons and 600 electrons with energies larger than 10^7 e-volts ($y = 10$) emerge from the lower surface of the plate. We call this group of particles a *shower*.

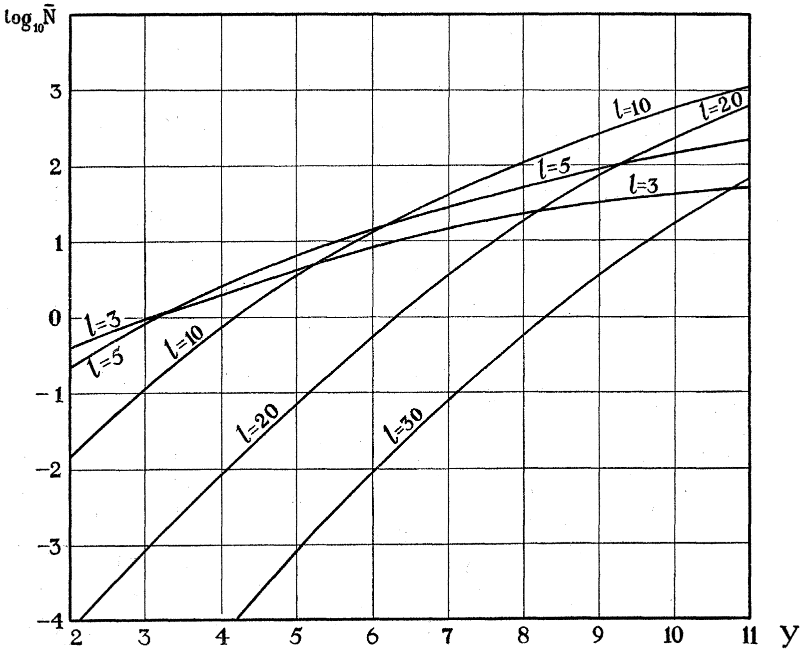

FIG. 4

It is clear that such a shower will also be accompanied by a certain number of hard light quanta. Their energy distribution can easily be calculated from (16). The number of hard quanta is of the same order of magnitude as the number of electrons.

4—Fluctuations

The function $f_-(l, y)$ gives the average number of secondary electrons or positrons which emerge after a thickness l with energy greater than $E = E_0 \exp(-y)$. The actual number may vary very considerably from the average, and we now proceed to investigate the fluctuations further. The number of positrons between E and $E+dE$ corresponding to y lying between y and $y+dy$ is

$$\frac{\partial f(l,y)}{\partial y}\,dy = p(y)\,dy, \text{ say.} \tag{29}$$

We now consider the whole range of y from zero to infinity divided into a number of very small finite sections dy such that (29) may be considered very small compared to unity in each. $p(y)\,dy$ has to be interpreted as the probability of finding a particle in the particular section dy. The probability of finding N particles one in each of the sections $dy_{\alpha_1}, dy_{\alpha_2}, \ldots, dy_{\alpha_N}$, and none in the others is then

$$p(y_{\alpha_1})\,dy_{\alpha_1}\,p(y_{\alpha_2})\,dy_{\alpha_2}\ldots p(y_{\alpha_N})\,dy_{\alpha_N}\prod_{\nu}[1-p(y_{\alpha_\nu})\,dy_{\alpha_\nu}],$$

where the last factor is the product of $1-p(y)\,dy$ for all the sections between 0 and y excepting the N sections $dy_{\alpha_1}\ldots dy_{\alpha_N}$. In the limit when the numbers of sections dy is increased to infinity, this product may be written as usual in the form

$$e^{-\int_0^y p(y)\,dy} = e^{-f_-(l,y)}. \tag{30}$$

We henceforth denote by $\bar{N}\equiv f_-$ the *average* number of electrons or positrons (not counting the primary) which emerge from a plate of thickness l, with energies greater than $E = E_0\exp(-y)$. We use N to denote the *actual* number of electrons or positrons which emerge in any particular case. The total probability of finding N particles anywhere between 0 and y is therefore as usual

$$w_{\bar{N}}(N) = \frac{\left(\int_0^y p(y)dy\right)^N}{N!}e^{-\bar{N}} = \frac{e^{-\bar{N}}\bar{N}^N}{N!}. \tag{31}$$

It is obvious that the sum of (31) over all integral values of N from 0 to ∞, i.e. the probability of finding any number of particles between 0 and ∞ in the range 0 to y is just unity, as it must be. Thus the fluctuations are independent of the values of l and y separately, and only depend on $\bar{N}$, the average number of particles. Of course, if l be kept fixed, the value of $\bar{N}$ depends on the value of y, or if y be kept fixed, on the value of l. The probability of finding a number of particles between R and S is therefore (according to (36) of the appendix)

$$P(R,S,\bar{N}) = \sum_{N=R}^{S} e^{-\bar{N}}\frac{\bar{N}^N}{N!} = W(R,\bar{N}) - W(S+1,\bar{N}). \tag{31'}$$

In Table II we give the probability of finding 0, 1, between 2 to 5, between 6 to 10, and between 11 and ∞ electrons or positrons for different values of $\bar{N}$, the average number of particles expected. The table shows that the fluctuations are large.

TABLE II

N \ $\bar{N}$	0·1	0·5	1·0	5·0	20	100
0	0·905	0·607	0·368	0·0067	0	0
1	0·0905	0·304	0·368	0·034	0	0
2– 5	0·0045	0·089	0·263	0·576	0	0
6–10	0	0	0·001	0·368	0·002	0
11–∞	0	0	0	0·015	0·998	1·000

It can easily be seen that formula (31) is also valid for the total number of particles (positrons + electrons), $\bar{N}$ being then twice the average number of positrons as given in Table I.

5—ANGULAR SPREAD

So far, as has already been stated, we have considered the whole problem as a one-dimensional one. This is a good approximation in most cases. We now wish to consider roughly the angular spread of the particles found after a given thickness l.

If we denote by E the energy of an electron, positron or light quantum produced in some elementary process, then the mean angle which the direction of such an electron, positron or light quantum makes with the direction of the original electron, positron or light quantum producing it is of the order mc^2/E and is to this approximation independent of any other factor in the process.

Let us now consider an electron found at a depth l with an energy E, and let us suppose that the "genealogy" of this electron consists of $2n$ stages in each of which it was a particle or quantum of energy

$$E_1, E_2, \ldots, E_r, \ldots, E_{2n} = E.$$

The factor 2 is due to the fact that we have also to count the intermediate light quanta as one stage. In each process its mean angular deflexion is of the order mc^2/E_r so that if all the angular deflexions add up, the maximum angle which the direction of the electron E makes with the direction of the original particle which entered the upper surface of the material and started the whole chain is of the order

$$\frac{mc^2}{E_1}+\frac{mc^2}{E_2}+\ldots+\frac{mc^2}{E_r}+\ldots+\frac{mc^2}{E}.$$

Since $E_1 > E_2 > \ldots > E$ the maximum total deflexion is of an order less than $2n\,mc^2/E$. If we consider the individual deflexions to be at random, as they are, the mean deflexion would then be less than of the order

$$\theta \sim \sqrt{2n}\,\frac{mc^2}{E}. \tag{32}$$

Here $2n$ is the number of stages in the "genealogy" of the particular particle we have considered. The mean value of n is just the suffix of the term $f_n(l, y)$ in the series (22) which gives the greatest contribution to $f_-(l, y)$ for the particular values of l and y under consideration, and is hence a function of l and y.

The way the term which gives the greatest contribution depends on l for $y = 5$ can be seen from fig. 2. The dependence is similar for all values of y. The larger the values of l and y, the larger is the value of n. Taking $y = 10$, $l = 20$, for example, we find that the greatest contribution comes from about $n = 5$, and a finite but small contribution still comes from $n = 8$. Thus the mean spread for this case is of the order $\sqrt{10}mc^2/E$ and the maximum spread is less than of the order $16mc^2/E$. Taking $E_0 \sim 2 \times 10^{11}$ e-volts, $E \sim 10^7$ e-volts ($y = 10$) these quantities are respectively about 9° and 46°. Naturally the particles which emerge at 46° about the path of the original particle are very few indeed, and have the lowest energy considered (E). Most particles in this particular case appear within a cone whose half angle is about 9° about the path of the original particle.*

The particles with energies higher than the minimum energy E considered have a smaller spread than the slow particles. Thus in a shower the fast particles are concentrated more in the centre of the shower whilst the particles emerging at larger angles are those of small energies. This seems to be in agreement with most of the cloud chamber photographs taken of showers.

It appears from these considerations that our treatment of the problem as a one-dimensional one is in general well founded.

6—Shower Creation by γ-rays

We now consider very briefly what happens when very hard γ-rays pass through matter. It is clear that we will get phenomena very like those produced by electrons, since after travelling a certain distance, corresponding to the mean "range" of the photon, and defined as the reciprocal

* This seems to be in qualitative agreement with the remarks of Ehrenberg (1936) on the spread of particles in a shower.

of the absorption coefficient, the photon will create a pair. Each particle of the pair will then behave as we have already investigated.

The number of secondaries produced by a primary *photon* can be calculated in the same way as for an electron. We shall not do this here since in experiments on cosmic radiation the energy distribution of the quanta falling on the plate is in general not known. For large thicknesses and large values of y we may get a rough idea of the effect of a quantum k_0 thus: After travelling a mean distance $1/\alpha$ (in the units (15)) the quantum gives birth to a pair, and in most of the cases, the energy is distributed more or less equally between the particles of the pair. Each particle will produce $f_-(l', y')$ electrons at a distance l' below the point of its creation, y' being roughly $\log(k_0/2E)$. Thus, the number of electrons or positrons g with energy greater than E found at a depth l below the top due to a γ-ray of energy k_0 is very roughly

$$g(l, y) = 2f_-\left(l - \frac{1}{\alpha}, y - \log 2\right), \tag{33}$$

where $y = \log(k_0/E)$ and f_- is given in § 3. This formula is only valid for large l and y. Hence we have to shift all curves of figs. 2 and 3 to the right by a distance $1/\alpha = 1{\cdot}67$ (according to (13)), multiply the ordinate by 2, and use instead of the proper value of y, $y - \log 2$.

7—Comparison with Experiments

It is not our purpose here to give a detailed discussion of all cosmic ray phenomena. This is in any case impossible, since in order to apply our calculations to any actual experiment, we should have to know the actual energy distribution of the electrons incident on the top of the layer under consideration. This distribution is in general not known. Even where it is, for example for electrons falling on a plate at sea-level, our predictions would be vitiated by the fact that the electrons falling on the plate are also accompanied by γ-radiation, and this has as important an effect as the electrons. Thus all we can hope to do is to give a qualitative description of the phenomena, and to show how our calculations explain some of the most characteristic features of the cosmic radiation.

A—Showers—The curves represented in figs. 2 and 3 very much resemble *Rossi's* (1934) well-known *transition curve*. What is measured here is the number of triple coincidences between three counters placed below a lead plate as a function of the thickness of the plate. There is, however, no exact correlation between the Rossi curve and our calculated curves. The number of triple coincidences depends on the number of groups of N particles appearing simultaneously, and the probability of a group of N particles producing a triple coincidence. The latter probability is higher the larger the value of N, and this value increases with increasing y for a given thickness. Moreover, in the experimental disposition of the counters, the angles at which the particles would have to appear to cause a triple coincidence is fairly large. This therefore favours slowly emerging particles, or larger values of y for a given E_0, since then the angular spread is larger (§ 5). Thus the experimental conditions have a bias in favour of large values of y. On the other hand, there are more relatively slow particles (i.e. with energy less than 10^9 e-volts, say) falling on the top of the plate than fast ones, so that for a given energy of the particles emerging at the bottom of the plate, the primary energy distribution favours small values of y. However, as figs. 2 and 3 show, the curves for all y have a similar shape, and the maximum shifts only slowly to the right with increasing y. It is for this reason that the resulting Rossi curve still has the same general shape as the curves of figs. 2 and 3.

Assuming 10^7 e-volts as the lowest energy with which particles emerge from the lead plate (for smaller energies the electrons are stopped within a very short range because of the energy loss by ionization, cf. § 1) and a mean primary energy between 10^8 and 10^9 e-volts, the mean value of y lies between 3 and 5. We find according to § 3, that the maximum of the Rossi curves should lie between $l = 4$ and 5 corresponding (by equation (15)) to 1·6–2 cm. of lead, as is in fact observed. The average number of particles to be expected at the maximum lies between 3 (including the primary) and 12.*

On the other hand the Rossi curves seem to have a very long tail at large thicknesses. It is possible that this tail may be due to primaries with very high energy (10^{11} e-volts, say) which though few in number produce a large number of secondaries at large thicknesses.

According to § 3 an electron with sufficiently high energy can produce more than a thousand secondaries in traversing a lead plate of 5 cm. or more. It follows that in heavy materials even large showers and Hoffmann bursts are not incompatible with existing theories, but are rather just what we should expect if present quantum mechanics continued to be valid up to energies of 10^{11} e-volts.

Our calculations also give the *energy distribution* among the particles of

* It had been suggested some time ago and worked out in detail by one of us (Bhabha 1933) that it was possible to explain the main features of the Rossi curve by assuming an intermediate shower producing radiation. Our present theory is a justification of this picture, inasmuch as it shows that for the smallest showers only one, or at most two, intermediate steps are important.

a shower. For a thickness l and primary energy E_0 we know $\bar{N}(l, y)$, the average number of electrons or positrons whose energy is greater than $E = E_0 \exp(-y)$ so that $\bar{N}(l, y_1) - \bar{N}(l, y_2)$ gives the mean number of electrons in a shower with energies lying between $E_2 = E_0 \exp(-y_2)$ and $E_1 = E_0 \exp(-y_1)$. The energy distribution can be taken directly from fig. 4. For a shower emerging from a 4 cm. lead plate ($l = 10$) caused by a primary electron of 2×10^{10} e-volts we find, for instance, the following distribution:

0·2 particle with $E > 10^9$ e-volts.

4 particles with E between 2×10^8 and 10^9 e-volts.

22 particles with E between 5×10^7 and 2×10^8 e-volts.

120 particles with E between 1×10^7 and 5×10^7 e-volts.

Most of the shower particles emerge with small energies.

For a given thickness of plate, the larger showers are in general associated with larger values of y. But as we can see from figs. 2 and 3, the larger the value of y, the larger is the value of l at which the maximum of the Rossi curve lies. Thus our theory allows of a prediction which it should be possible to establish or disprove experimentally. The maximum of the Rossi curve for larger showers should lie at greater thicknesses than that of the Rossi curve for smaller showers, and with increasing size of the showers under consideration, the maximum should very slowly continue to shift to greater thicknesses.*

B—The Regener curve at High Altitudes—Pfotzer and Regener (1935) and Pfotzer (1936) have measured the number of single cosmic ray particles as a function of the altitude above sea-level. The curve obtained by them is very similar to an ordinary transition curve. It has a maximum at a pressure of 8 cm. Hg corresponding to a depth of $l = 3$ in our units (cf. equation (15)) below the top of the atmosphere. If the number of primaries entering the atmosphere is normalized to 100 the number of particles at the maximum is about 250. (In Pfotzer's measurements the number of primaries was 115.)

* We are indebted to Mr Carmichael for informing us that his experiments show indications of the predicted shift of the maximum. We are also indebted to him for drawing our attention to a note by Bøggild (1936), where the same shift is mentioned.

The well-known transition effects are also a direct consequence of our theory. For *the lower limit* of energy down to which the multiplication processes considered here are effective is much higher in lighter elements (smaller Z) than in heavy ones (10 m.e-volts in lead, 150 m.e-volts in air), so that for the same primary energy the shower produced in a given thickness of some light element will be much smaller than one produced in *the equivalent thickness* (defined by (15)) of some heavy element, which leads at once to transition effects.

These observations can easily be explained by our theory. We obtain a maximum at the right thickness for $y = 2{\cdot}5$–3. Assuming that the minimum energy of the electrons observed in air is $E = 1{\cdot}5 \times 10^8$ e-volts (cf. § 1) the primary electrons would have an energy of about 2–3×10^9 e-volts. For $y = 3$, the number of secondaries at the maximum is just about 1 positive and 1 negative electron (see fig. 3) and counting also the primary electrons we obtain a total number of particles at the maximum of $\sim 2{\cdot}4$ times the number of particles entering the atmosphere, which is just what is observed.*

Furthermore, 3×10^9 e-volts is about the minimum energy required by an electron in order to reach the top of the atmosphere at a latitude of 50° owing to the magnetic field of the earth. At the equator this minimum energy is very much higher (3×10^{10} e-volts). Thus most of the primary electrons (and their secondaries) observed at a latitude of 50° could not appear at the equator. This agrees very well with observations by Clay (1934) who found that the ionization at high altitudes near the equator amounts to only a few per cent of the value found at a latitude of 50°.

Thus we believe it highly probable that at least a large fraction of the primary cosmic rays are positrons or electrons with energies of about 3×10^9 e-volts and that *electrons of this energy still behave in accordance with the theory.*

At smaller altitudes Regener's curve decreases gradually, the number of particles observed at sea-level amounting still to about 5 % of the number of primaries.† As we shall show in the following section, they can be explained if we assume primary electrons with an energy of the order 10^{12} e-volts.

C—Penetrating Power of Electrons—The original calculations of Bethe and one of us (H.) showed that the average range of an electron of even 10^{10} e-volts energy was only 1·5 km. in air. Since cosmic ray electrons are observed in great numbers at sea-level (8 km. air) it was supposed that this proved a breakdown of the theory for electrons of such high energy. The attribution of these electrons to primary protons meets with the difficulty that heavy particles do not produce a sufficient number of fast secondary electrons.

Our calculations show, however, that the use of this fact to prove a

* This explanation of the maximum of Regener's curve has also been given by Nordheim in a letter to one of us (H.).

† The Regener curve has also a minor hump at a pressure of 30 cm. Hg corresponding to $l = 12$. This hump could easily be explained by a suitable choice of the primary energy spectrum. It is not our aim to give here a complete analysis of the Regener curve. This will perhaps be done in a later paper.

breakdown of the theory was incorrect. Though it is true that no electron of any reasonable energy has any chance of penetrating the whole atmosphere, there is a large probability that one of its secondaries arrives at the bottom of the atmosphere with a comparatively high energy. In our units the thickness of the atmosphere is $l = 29$. Assuming again $1 \cdot 5 \times 10^8$ e-volts for the minimum energy of the secondary electrons in air we find from § 3 that the probability P for an electron of primary energy E_0 making its effects felt at sea-level by producing one secondary electron, is:

E_0	3×10^9	$1 \cdot 5 \times 10^{11}$	3×10^{12} e-volts.
P	2×10^{-3}	20	4000 %

The figures are considerably greater than a straightforward consideration of the energy loss would indicate, as the probability for the original electron reaching sea-level and having an energy greater than $1 \cdot 5 \times 10^8$ e-volts is for any reasonable primary energy practically equal to zero.

In order to explain that, according to Regener, 5 % of the number of primary electrons are observed at sea-level we would, for example, only have to assume that about 0·1 % of the primary electrons had an energy of the order 3×10^{12} e-volts. These fast electrons will then just produce the right number of secondaries at sea-level. Hence, by assuming a suitable energy spectrum for the primary electrons we should have no difficulty in explaining the Regener curve. We must emphasize that we make these remarks tentatively, and our main purpose is only to show that there is no real discrepancy between theory and the absorption curve in the atmosphere.*

It is usual in discussing absorption curves to calculate an *effective absorption coefficient* as if the decrease took place exponentially at that point. If, in the same way, we define $\rho = -d \log \overline{N}/dl$ for each y as the absorption coefficient, ρ depends of course on l (at the maximum of our curves $\rho = 0$). We find for $l = 17$ (corresponding to a pressure of 45 cm. Hg

* From the above figures it follows, however, that the latitude effect at sea-level (amounting to ~ 14 % of the total ionization) cannot, on our theory, be attributed to primary electrons. To explain it we would have to assume that about 1 % of the primary particles are protons with energies of the order 3×10^9 e-volts. On the other hand at an altitude of 4000 m. at least a substantial fraction of the latitude effect is due to primary electrons of this energy.

It is clear from what we have said that no such thing as a sharply defined range can be attributed to electrons of any large energy. The explanation of the cut-off of the latitude effect at sea-level and latitudes of 50° as due to the stopping of slower particles in the atmosphere must be taken with the greatest caution and reserve. Indeed, the fact that at high altitudes the cut-off seems to take place at about the same latitude, if established, would make such an explanation untenable (Cosyns 1936).

in the atmosphere) and $l = 29$ (sea-level) the values given in Table III. The units are λ_0^{-1}.

TABLE III

l \ y	3	5	7	10
17	0·49	0·41	0·31	0·16
29	0·5	0·45	0·39	0·29

The absorption coefficient calculated from the observed absorption curve in the atmosphere at a depth of 50 cm. Hg below the top is about 0·2 in the above units. This indicates that the electrons observed at this altitude are mainly produced by primaries with $y \cong 9$ corresponding to a primary energy E_0 of the order 10^{12} e-volts. Although the observed value 0·2 is three times as small as the smallest absorption coefficient for γ-rays (in our units 0·6) there is no difficulty in explaining this value of ρ on the basis of this theory.

We must mention in this connexion, however, that the absorption coefficient calculated from Regener's curve under 100 m. of water has a value which is about a hundred times smaller even than the values quoted above.* We should also remark that Rossi and others have performed experiments in which single particles appear to cause triple coincidences in three counters placed in a plane and separated by 100 cm. of lead. This thickness corresponds to $l = 250$, and the chance that an electron should make its effects felt through such a large thickness is negligible on our theory with any reasonable value of E_0. We do not believe it possible to explain such a low value of the absorption coefficient on the lines of this paper. We must conclude, either that the extremely hard radiation which penetrates 250 m. of water consists of particles of protonic mass, or that the quantum theory breaks down for radiation of the highest energies if it consists of electrons. We should like to emphasize, however, that the former possibility does not carry with it necessarily the existence of negative protons yet, since the rough equality in the number of positive and negative particles which is known to exist up to high energies has not been shown to persist at great depths of water, or below these great thicknesses of lead. Indeed, according to Blackett and Brode (1936) "particles with energy greater than 10^{10} e-volts seem to be mainly positive". These may well be the protons in question.

* A similarly small absorption coefficient has been measured by Auger, Ehrenfest and Leprince-Ringuet (1936) for the "hard component" of the cosmic radiation found at Jungfraujoch after large thicknesses of lead.

It is a pleasure to express our thanks to Professor N. Bohr for his interest in this work and one of us (B.) in particular would like to thank him for the pleasant time spent at his institute.

We should also like to express our thanks to Dr Nordheim with whom we have discussed by correspondence most of the questions treated in this paper. We are also greatly indebted to Miss Littleton of the Physics Department of Bristol University who has carried out many of the numerical calculations.

One of us (H.) would like to express his gratitude to the Academic Assistance Council for the grant of a research fellowship during the tenure of which this work has been performed.

Mathematical Appendix

We now proceed to prove some properties of the function $W(l, y)$ which are required in our calculations.*

W is defined by

$$W(l,y) = \frac{1}{\Gamma(l)}\int_0^y e^{-\eta}\,\eta^{l-1}\,d\eta. \tag{34}$$

Integrating by parts we obtain the recurrence formula

$$W(l,y) = \frac{e^{-y}\,y^l}{\Gamma(l+1)} + W(l+1,y). \tag{35}$$

For integral and positive l, W may be reduced to

$$W(l,y) = 1 - e^{-y}\sum_{\nu=0}^{l-1}\frac{y^\nu}{\nu!}. \tag{36}$$

We first prove the relation

$$\int_0^y W(x,y-y')\,W(\xi,y')\,dy' = \int_0^y W(x+\xi,y')\,dy', \tag{37}$$

where x and ξ are not necessarily integral. Using (34) the left-hand side of (37) becomes

$$\frac{1}{\Gamma(x)\,\Gamma(\xi)}\int_0^y dy'\int_0^{y-y'} e^{-\eta}\,\eta^{x-1}\,d\eta\int_0^{y'} e^{-u}\,u^{\xi-1}\,du,$$

or interchanging the order of the y' and η integrations

$$\frac{1}{\Gamma(x)\,\Gamma(\xi)}\int_0^y e^{-\eta}\,\eta^{x-1}\,d\eta\int_0^{y-\eta} dy'\int_0^{y'} e^{-u}\,u^{\xi-1}\,du,$$

* The function W has been tabulated to 7 figures by Pearson, "Tables of the incomplete Γ-function, for values of l from 0 to 50." It there appears under a different notation. Cf. also Jahnke-Emde, "Tables of Functions," Leipzig, 1933.

and further interchanging the order of the y' and u integrations

$$\frac{1}{\Gamma(x)\,\Gamma(\xi)}\int_0^y e^{-\eta}\,\eta^{x-1}d\eta\int_0^{y-\eta} e^{-u}\,u^{\xi-1}\,du\,.\int_u^{y-\eta} dy'$$
$$=\int_0^y d\eta\int_0^{y-\eta}\frac{e^{-(u+\eta)}\,(y-u-\eta)}{\Gamma(x)\,\Gamma(\xi)}\,\eta^{x-1}\,u^{\xi-1}\,du.$$

This is of the same form as Dirichlet's integral. Introducing $y' = \eta+u$ and η as new variables we obtain at once

$$\frac{1}{\Gamma(x+\xi)}\int_0^y e^{-y'}(y-y')\,y'^{x+\xi-1}\,dy' = \int_0^y \frac{e^{-y'}\,y'^{x+\xi-1}}{\Gamma(x+\xi)}\,dy'\int_0^{y-y'} dt$$
$$=\int_0^y W(x+\xi,t)\,dt,$$

which proves (37).

We now prove that

$$\int_0^y e^{y'}\,W(l,y')\,dy' = e^y\,W(l+1,y). \tag{38}$$

By (34) the left-hand side is equal to

$$\frac{1}{\Gamma(l)}\int_0^y e^{y'}dy'\int_0^{y'} e^{-\eta}\,\eta^{l-1}\,d\eta = \int_0^y \frac{e^{-\eta}\,\eta^{l-1}}{\Gamma(l)}\,(e^y-e^\eta)\,d\eta$$
$$= e^y\,W(l,y)-\frac{y^l}{\Gamma(l+1)} = e^y\,W(l+1,y),$$

by (35).

We finally wish to establish that

$$\int_0^y W(x,y-y')\,dy'\int_0^{y'}\frac{(y'-y'')^{n-1}}{\Gamma(n)}\,W(\xi,y'')\,dy'' = \int_0^y \frac{(y-y')^n}{\Gamma(n+1)}\,W(x+\xi,y')\,dy'. \tag{39}$$

Writing $y-y'$ for y' the left-hand side becomes

$$\int_0^y W(x,y')\,dy'\int_0^{y-y'}\frac{(y-y'-y'')^{n-1}}{\Gamma(n)}\,W(\xi,y'')\,dy''.$$

If we introduce the variable $u = y'+y''$ instead of y'' in this integral and then interchange the order of the y' and u integrations, we obtain, according to (37)

$$\int_0^y \frac{(y-u)^{n-1}}{\Gamma(n)}\,du\int_0^u W(x,y')\,W(\xi,u-y')\,dy'$$
$$= \int_0^y \frac{(y-u)^{n-1}}{\Gamma(n)}\,du\int_0^u W(x+\xi,y')\,dy'.$$

Interchanging again the order of integration we finally obtain

$$\int_0^y W(x+\xi,y')\,dy'\int_{y'}^y \frac{(y-u)^{n-1}}{\Gamma(n)}\,du = \int_0^y \frac{(y-y')^n}{\Gamma(n+1)}\,W(x+\xi,y')\,dy',$$

which proves (39).

Summary

We have used relativistic quantum mechanics to calculate, subject to some simplifying assumptions (§ 1), the number of secondary positive and negative electrons produced by a fast primary electron with energy E_0 passing through a layer of matter of thickness l. The process in question is the following: The primary electron in the field of a nucleus has a large probability of emitting a hard light quantum which then creates a pair. The pair electrons emit light quanta again which create pairs and so on.

The results are represented in § 3. The number of secondaries increases rapidly with E_0. If an electron of 10^{11} e-volts passes through a lead plate of 5 cm. thickness the number of particles emerging from the plate amounts to 1000 or more. Thus showers can be explained by the ordinary quantum theory.

The fluctuations of the number of particles (§ 4), the angular spread (§ 5) and energy distribution (§ 7 A) of the showers and the shower production by light quanta (§ 6) are discussed briefly.

The comparison with experiments (§ 7) shows that Rossi's transition curve and Regener's absorption curve in the atmosphere (§ 7 B) can be understood on this theory. The penetrating power of fast electrons (§ 7 C) appears to be very much greater than a straightforward consideration of the energy loss would indicate. The absorption coefficient of the radiation found at a depth of 100 m. of water cannot, however, be understood on the basis of this theory if this radiation is due to primary electrons.

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