SOME POTENTIAL PROBLEMS ARISING IN THE THEORY OF AXIAL TURBOMACHINES

Thesis by

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SUMMARY

Some potential problems raised by the trailing-vortex theory of axial turbomachines are solved in the cases of cylindrical semi-infinite and infinite helicoidal vortices and of trailing vortices in a cone. The analysis is carried out for the cylindrical doubly infinite case and the dynamical problems are set up.

The results are in a form where further applications to the physical problems may be undertaken and actual computations worked out.

It is hoped that this work will be completed in the future.

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I. INTRODUCTION

The central problems of the theory of axial-flow turbomachines subject to treatment by the methods of non-viscous, incompressible fluid theory are essentially the following two:

The "inverse problem". Given: blade loading, fluid state far upstream, rotation speed of the machine. Find: the velocity field, blade shape, distribution of energy.

The "direct problem". Given: blade shape, rotation speed, fluid state far upstream. Find: velocity field, blade loading, distribution of energy.

Because of the great geometrical and therefore analytical complication of the problem, as formulated in three dimensions, two different methods of reduction to two dimensions have been developed, answering to different and complementary needs:

1. The airfoil lattice theory has been applied in each isolated "infinitesimal" cylindrical layer, neglecting their interaction.
2. The actual shape of the blades has been neglected, and the shed vorticity assumed distributed continuously downstream of the blade row.

References: Meyer[1]; Marble[2]. - The purpose of the following treatment is to formulate a vortex theory of axial flow turbomachines, based on the principles of Prandtl's wing theory and Joukowski's screw propeller theory, and to solve a few of the potential problems thus formulated.

The axial motion of the incoming fluid is assumed uniform, as well as the rotation. The problem is linearized by assuming that the vorticity is carried downstream along the undisturbed streamlines, (and not by its own induced velocity). Each blade being represented by a radial bound vortex of strength varying with radial distance, the shed vortices fill a screw surface of constant pitch. Such a vortex sheet is highly unstable, and it is uninteresting to attempt to refine the treatment by assuming the trailing vortices to be other than strictly helicoidal. But in the case of blades imparting a solid body rotation to the fluid one may compute the pitch of the helix on the basis of the angular velocity downstream (which makes our theory applicable to the stator as well).

In the first chapter, the method of expansion of the potential ϕ in a series of harmonics (Kawada[3]; Florine[4]) is used to derive the influence function of an infinite helicoidal vortex between two cylinders; the series expansion for ϕ_ς is approximated by an easily summable series, and ϕ_ς is finally given by an integral, the kernel of which reproduces Florine's result for screw propellers with an additional term representing the influence of the boundaries.

This investigation gives an insight into the behavior of axial and tangential velocities at a blade and permits one to set up the dynamical relations between circulation, thrust, torque and the induced velocities; it does not give, however, any indication as to radial induced velocity near the blade; this last is fortunately of little interest in the study of the interaction between successive rows of blades because one could not anyway take account of its variation around the axis. A mean value, such as determined by Marble[2] as function of radius and distance from blade row gives a sufficient estimate.- In the second chapter, the behavior of the velocity near the blade is investigated by rigorous and then by approximate methods.- Finally the third chapter gives a preliminary mathematical investigation of the three-dimensional field of velocities induced by a trailing vortex between two cones.

In no case does the treatment go beyond mathematical generalities and it will have to be completed by a numerical study of a few significant cases for a comparison with previous theories, and a check on all the successive approximations.

It is to be remarked that many aspects and methods of this theory are closely related to the study of the electromagnetic field due to a loose helicoidal conducting wire. See H. Lamb[5] P. Jacottet[6] who limit themselves to the case where the coordinate z does not enter explicitly in the results. However the methods also apply when the current propagates along the wire, and to more complicated cases.

CHAPTER I

TWO-DIMENSIONAL PROBLEMS FOR CYLINDERS

2. Velocities in the Trefftz plane: Green's function.

Simple symmetry considerations show that, similarly to what happens in wing theory, the axial and tangential velocities induced on the blades, and on the bisecting lines of these blades, by the shed vortices are exactly equal to one-half of the velocities induced by a doubly infinite vortex. There lies the interest of the following investigation.

Biot-Savart Law cannot be conveniently used, because there are no well-defined images in a cylinder. One applies therefore the method of separation of variables to Laplace's equation to find directly the perturbation velocity potential ϕ, as the solution of this equation which satisfies the required condition of continuity, and one-valuedness or many-valuedness and the boundary conditions.

A return vortex is added along the axis to make easier the passage to three-dimensional considerations. (This axial vortex is then part of the shed vortex system).

The purely formal operations which follow will be justified later on when the actual series development on which they had been performed will be obtained.

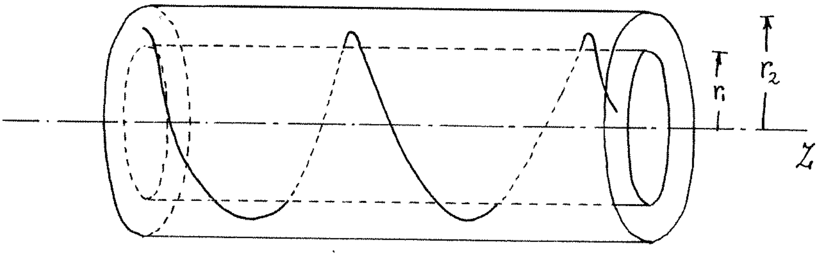
Notations p = number of blades

$\frac{\gamma(\mu)}{p}$ = strength of each shed vortex filament

v = advance velocity

ω = angular velocity

z, θ, r cylindrical coordinates fixed with respect to the blade row



If the new variables z; $\zeta = \theta - \frac{\omega z}{v}$; $\mu = \frac{\omega r}{v}$ are introduced into the problem, the equations of the helicoidal vortices are

$$\zeta = \frac{2\pi}{p} k \qquad (k = 0, 1, \dots p-1) \quad ; \quad \mu = \mu_0$$

the equations of the cylindrical boundaries are $\mu = \mu_1$; $\mu = \mu_2$, and the potential ϕ is the solution of Laplace's equation, depending only on the geometrical position of the point relative to the helices. That is: ϕ depends on μ and ζ , but is independent of z.

The equation for the velocity potential is :

$$\Delta' \phi = \phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} + \frac{1}{r^2}\phi_{\theta\theta} = 0$$

or in terms of the new variables :

$$\Delta \phi = \phi_{\mu\mu} + \frac{1}{\mu}\phi_\mu + \left(1 + \frac{1}{\mu^2}\right)\phi_{\zeta\zeta} = 0 \tag{2.1}$$

The boundary and periodicity conditions are: <u>For</u> $\mu_0 \leq \mu \leq \mu_2$, ϕ is a single valued, continuous function of (μ, ζ) odd and periodic in ζ, with period $\frac{2\pi}{p}$, its μ-derivative $\frac{\partial \phi}{\partial \mu} = \frac{\partial \phi}{\partial r}\frac{v}{\omega} = u_r \frac{v}{\omega}$ must vanish for $\mu = \mu_2$. <u>For</u> $\mu_1 < \mu < \mu_0$, ϕ will be a multiple-valued function of ζ , increasing by γ/p when ζ increases by $2\pi/p$. Its normal μ-derivative vanishes for $\mu = \mu_1$, <u>For</u> $\mu = \mu_0$, $\zeta \neq \frac{2\pi}{p}k$, ϕ and ϕ_μ must be continuous in μ . - Separating the variables, let $\phi(\mu, \zeta) = M(\mu) Z(\zeta)$.

Then

$$M''Z + \frac{1}{\mu} M'Z + \left(1 + \frac{1}{\mu^2}\right) MZ'' = 0$$

and $\dfrac{M'' + M'/\mu}{(1 + 1/\mu^2)M} = -\dfrac{Z''}{Z} = n^2$ must be independent of μ and ζ

$Z'' + n^2 Z = 0$ gives $Z = A\zeta + B$ if $n = 0$

$Z = A\cos n\zeta + B\sin n\zeta$ if $n \neq 0$; from periodicity conditions, n must then be an integer multiple of p : $n = mp$

$M'' + \dfrac{M'}{\mu} - n^2\left(1 + \dfrac{1}{\mu^2}\right)M = 0$ gives $M = A\log\mu + B$ if $n = 0$

$M = A\, I_n(n\mu) + B\, K_n(n\mu)$ if $n \neq 0$,

where I_n, K_n are the Bessel functions of purely imaginary argument. (See Watson(7).)

Keep only the terms which satisfy our periodicity conditions. The solution is then, with undeterminate coefficients

$$\delta\phi_0 = \sum_{m=1}^{\infty} \sin mp\zeta \left\{ a_m K_{mp}(mp\mu) + b_m I_{mp}(mp\mu) \right\} \qquad \underline{\mu > \mu_0} \tag{2.2a}$$

$$\delta\phi_i = \sum_{m=1}^{\infty} \sin mp\zeta \left\{ a'_m K_{mp}(mp\mu) + b'_m I_{mp}(mp\mu) \right\} + \frac{\gamma}{2\pi}\left(\zeta - \frac{\pi}{p}\right) \qquad \underline{\mu < \mu_0} \tag{2.2b}$$

Develop: $\dfrac{1}{2}\left(\zeta - \dfrac{\pi}{p}\right) = -\displaystyle\sum_{m=1}^{\infty} \frac{1}{mp} \sin mp\zeta$ and write that the four boundary conditions are satisfied identically in ζ ; this determines the four coefficients of each harmonic (see Appendix p.35 for details).

If $n = mp$ $(m = 1, 2 \dots \infty)$ and expressions such as $I'(1)$ stand for $\dfrac{d}{d\mu} I_n(n\mu)\Big|_{\mu=\mu_1}$ one obtains the expressions:

$$\delta\phi_0 = -\frac{\gamma(\mu_0)\mu_0}{\pi} \sum_{m=1}^{\infty} \frac{[I'(2)K - K'(2)I][K'(0)I'(1) - K'(1)I'(0)]}{K'(2)I'(1) - K'(1)I'(2)} \frac{\sin mp\zeta}{mp} \tag{2.3a}$$

$$\delta\phi_i = \frac{\gamma(\mu_0)\mu_0}{\pi} \sum_{m=1}^{\infty} \frac{[I'(1)K - K'(1)I][K'(2)I'(0) - K'(1)I'(0)]}{K'(2)I'(1) - K'(1)I'(2)} \frac{\sin mp\zeta}{mp} + \frac{\gamma(\mu_0)}{2\pi}\left(\zeta - \frac{\pi}{p}\right) \tag{2.3b}$$

To obtain the actual linearized velocities, integrate these contributions along the blade, with for $\gamma(\mu_0)$ the (unknown) distribution of vorticity along the blades $= \left.\frac{\partial\Gamma(\mu)}{\partial\mu}\right|_{\mu=\mu_0}$ (i.e. the sum of vorticities around all blades):

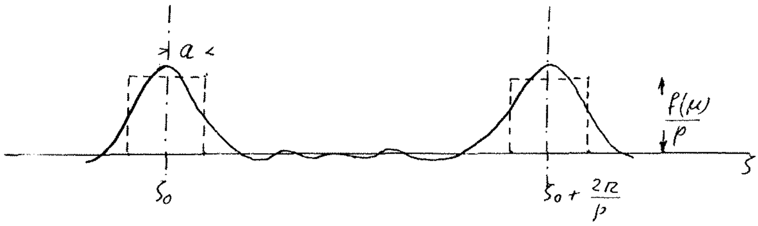
$$\phi(\mu) = \int_{\mu_1}^{\mu} \delta\phi_0[\mu, \mu_0/\mu_1, \mu_2] \frac{\partial\Gamma}{\partial\mu_0} d\mu_0 + \int_{\mu}^{\mu_2} \delta\phi_i[\mu, \mu_0/\mu_1, \mu_2] \frac{\partial\Gamma}{\partial\mu_0} d\mu_0 \tag{2.4}$$

There is of course no question of handling these whole infinite series. Two methods have been considered to obtain more tractable expressions: a "vorticity wake" method, and a method of approximate estimation of the successive coefficients by terms of a directly summable series.

3. "Vorticity wake" method of approximation

Ackeret[8] points out that a sharp trailing vortex is anyway an idealized concept, and that the vorticity is actually distributed in a wake of finite "width". This width might for instance be assumed $= a$, a constant along the blade, and the following scheme, intermediate between the strict vortex approach, and Marble's theory[4], might be adopted:

Writing $\quad \gamma_0 = \frac{fa}{\pi}\ ; \ \gamma_q = \frac{2f}{\pi pq}\cos pqa \quad ; \ f(\mu_0) = \frac{\partial\Gamma'}{\partial\mu_0}$



Let $$\gamma(\mu_0) = \gamma_0(\mu_0) + \sum_{q=1}^{\infty} \gamma_q(\mu_0) \cos pq(\zeta - \zeta_0) \tag{3.1}$$

Five terms of the series give an excellent approximation. The corresponding Bessel functions are found in tables, and one has to integrate numerically:

$$\int_{\mu_1}^{\mu} \frac{\partial \Gamma'}{\partial \mu_0} K'_{mp}(mp\mu_0)\, d\mu_0 \qquad ; \qquad \int_{\mu}^{\mu_2} \frac{\partial \Gamma'}{\partial \mu_0} K'_{mp}(mp\mu_0)\, d\mu_0$$

(and same thing for I'_{mp}) for m = 1,2,3,4,5 only. This procedure is extremely convenient when a distribution of Γ' is preassigned; and it may then be used to bridge the gap between the present theory and the lattice theory. However it is not suitable for handling any inverse problem.

4. Summation of the series for ϕ_ζ with the aid of Nicholson's approximation on I_n, K_n functions.

The essential components of reduced velocity will be the tangential velocity, and the velocity perpendicular to the screw surface. Both are linked to ϕ_ζ . By differentiation of the expression for ϕ , the following series is obtained:

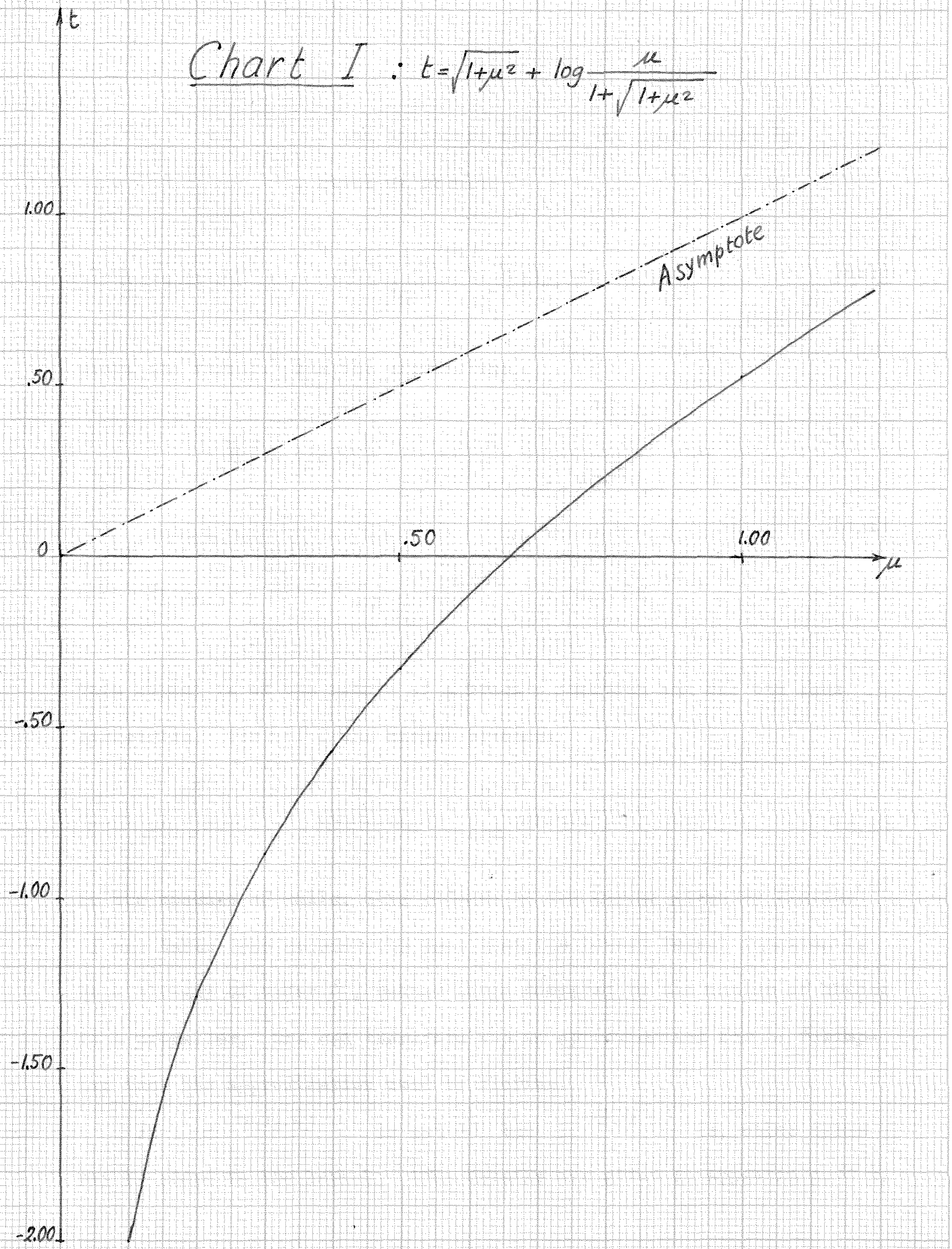
$$\pi\phi_\zeta = -\int_{\mu_1}^{\mu} \frac{\partial\Gamma}{\partial\mu_0}\mu_0\, d\mu_0 \sum_{m=1}^{\infty} \cos mp\zeta \left\{ \frac{I'(2)K - K'(2)I}{K'(2)I'(1) - K'(1)I'(2)} I'(1)K'(0) - \frac{I'(2)K' - K'(2)I}{K'(2)I'(1) - I'(2)K'(1)} K'(1)I'(0) \right\}$$

$$+ \int_{\mu}^{\mu_2} \frac{\partial\Gamma}{\partial\mu_0}\mu_0\, d\mu_0 \left[\frac{1}{2} + \mu_0 \sum_{m=1}^{\infty} \cos mp\zeta \left\{ \frac{I'(1)K - K'(1)I}{K'(2)I'(1) - K'(1)I'(2)} K'(2)I'(0) - \frac{I'(1)K - K'(1)I}{K'(2)I'(1) - I'(2)K'(1)} I'(2)K'(0) \right\} \right] \tag{4.1}$$

The lowest order for the Bessel functions in ϕ_ζ is already p. Therefore the approximate formula derived by Nicholson[9] can be used, and gives errors which are very reasonable (see Florine[4]; Annexe 2). By a series of integrations by parts (details of which are given in Appendix p. 36) the following result is derived:

$$\pi\phi_\zeta = \frac{\Gamma - \Gamma_2}{2} + \sum_{m=1}^{\infty} \frac{\cos mp\zeta}{(1+\mu^2)^{1/4}} \left\{ \int_{t_1}^{t} \frac{\partial[\Gamma_0(1+\mu_0^2)^{1/4}]}{\partial t_0} dt_0 \, \frac{\cosh mp(t-t_2)\, \sinh mp(t_0-t_1)}{\sinh mp(t_2-t_1)} + \int_{t}^{t_2} \frac{\partial[\Gamma_0(1+\mu_0^2)^{1/4}]}{\partial t_0} dt_0 \, \frac{\cosh mp(t-t_1)\, \sinh mp(t_0-t_2)}{\sinh mp(t_2-t_1)} \right\} \tag{4.2}$$

where $\quad t(\mu) = \sqrt{1+\mu^2} + \log \dfrac{\mu}{1+\sqrt{1+\mu^2}} \quad$ is tabulated in Chart I and Γ_0, Γ_2 are the total circulations at $\mu = \mu_0, \mu_2$. Γ_2 can be different from 0 (contrary to what is true for propellers) because for the purpose of this theory, the tips of the blades are assumed to join the outer wall. The next step is to write the trigonometric and hyperbolic functions as sums of exponentials. One obtains in this way a geometric series (which shows a posteriori that the manipulations on series were justified). The summation of this series, if one can assume that


t
Chart I : $t = \sqrt{1+\mu^2} + \log \frac{\mu}{1+\sqrt{1+\mu^2}}$
Asymptote
1.00
.50
0
-.50
-1.00
-1.50
-2.00
.50
1.00
μ

$e^{-2p(t_2-t_1)} \ll 1$, and therefore replace the denominator by $\frac{1}{2} e^{p(t_2-t_1)}$, gives as result:

$$(2\pi\phi_\zeta - \Gamma_2)(1+\mu^2)^{1/4} = \int_{t_1}^{t_2} \frac{\partial[\Gamma_0(1+\mu_0^2)^{1/4}]}{\partial t_0} \mathcal{K}(t_0, t/t_1, t_2)\, dt_0 \tag{4.3}$$

where the kernel $\mathcal{K}$ has the expression:

$$\mathcal{K}(t_0, t/t_1, t_2) = \frac{1 - e^{2p(t_0-t)}}{e^{2p(t_0-t_1)} - 2\cos p\zeta\, e^{p(t_0-t_1)} + 1} + \frac{e^{-2p(t+t_0-2t_2)} - 1}{e^{-2p(t+t_0-2t_2)} - 2\cos p\zeta\, e^{-p(t+t_0-2t_2)} + 1} - \frac{e^{2p(t+t_0-2t_1)} - 1}{e^{2p(t+t_0-2t_1)} - 2\cos p\zeta\, e^{p(t+t_0-2t_1)} + 1} \tag{4.4}$$

This kernel is of the nature of a Green's function and has the same symmetries. For $\zeta = 0$, the kernel becomes:

$$\frac{\mathcal{K}}{2} = \frac{e^{pt}}{e^{pt} - e^{pt_0}} + \frac{e^{p(t+t_0-2t_2)}}{1 - e^{p(t+t_0-2t_2)}} - \frac{e^{-p(t+t_0-2t_1)}}{1 - e^{-p(t+t_0-2t_1)}} \tag{4.5}$$

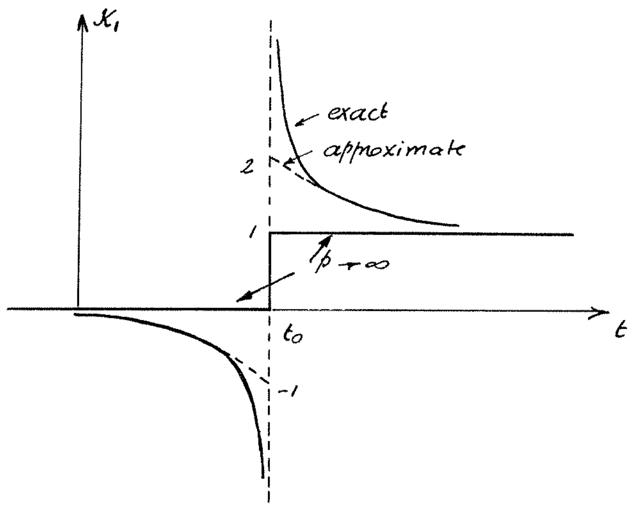
In the absence of walls, this reduces to the kernel given by Florine(4), if one takes into account the fact that the lowest Bessel function is here already of order p , rather large compared to the number of blades in a propeller. One can therefore use an approximation for I_n, K_n simpler than the approximation used by Florine.

If $e^{-2p(t_2-t_1)}$ is not $\ll 1$ but $e^{-4p(t_2-t_1)}$ is, other terms may be added to the kernel. (The details are given in Appendix p. 39).

<u>Behavior of this kernel, especially for p large</u>

Consider the case $\zeta = 0$, i.e. to the velocity on the blade.

$\mathcal{K}_1 = \dfrac{e^{pt}}{e^{pt} - e^{pt_0}}$ has a pole of order 1 for $\mu = \mu_0$, where it behaves like $\dfrac{1}{p(t-t_0)}$. The two branches, for $t > t_0$, where $\mathcal{K}_1 > 0$, and $\to +1$ as $t \to \infty$, i.e. $\mu \to +\infty$; and $t < t_0$, where $\mathcal{K}_1 < 0$, and $\to -0$ as $t \to -\infty$, i.e. $\mu \to 0$, are geometrically identical.



For $p \to \infty$, $\mathcal{K}_1$ tends toward the step function, uniformly for $|t - t_0| > \varepsilon$. If $p \gg 1$, and $\dfrac{\partial \Gamma_0}{\partial t_0}$ is very "smooth", i.e. its second derivative very small, one may neglect the infinities in taking the Cauchy principal value, i.e. integrate from t_1 to $t_0 - \varepsilon$ and from $t_0 + \varepsilon$ to t_2 . But for $|t - t_0| > \varepsilon$ and for $p \gg 1$, this kernel can be excellently approximated by:

$$\mathcal{K}_1 = \begin{cases} 1 + e^{p(t-t_0)} & \text{for } t > t_0 \\ -e^{-p(t-t_0)} & \text{for } t < t_0 \end{cases}$$

The rest $\mathcal{K}_2$ of the kernel is always regular, the value is large $\left(= e^{2p(t_2 - t_1)}\right)$ only where $t = t_0 = t_1$ or $t = t_0 = t_2$; otherwise it is small, of the order of

$$e^{p(t + t_0 - 2t_2)} - e^{-p(t + t_0 - 2t_1)}$$

The result obtained for $p \gg 1$ means simply that when ϕ is independent of ζ, the vorticity in layers which envlop μ_0 is the only one to act: this is well-known in electromagnetic theory.

If $\zeta \neq 0$ the kernel is a regular function of t (or μ)

5. Formulation of the dynamic problems of the rotor in terms of the circulation function

So far the circulation function $\frac{\Gamma}{p}$ was assumed known around each blade, idealized as a bound radial vortex. This function $\Gamma(\mu)$ has now to be related to the actual geometrical properties of the blades, and to the forces acting on the blades. This leads to a series of integro-differential relations yielding the equations appropriate for each problem. Assume that the variation of the blade shape with r is slow enough to permit to neglect altogether its reaction on axial and tangential induced velocities through the radial velocity.

The axial and tangential velocity in Trefftz plane are $u_a = -\frac{\omega}{v}\phi_\zeta$; $u_t = \frac{\phi_\zeta}{r} = \frac{\omega}{v}\frac{\phi_\zeta}{\mu}$. The resultant of the velocities at the blade is $W = \frac{1}{2}\phi_\zeta \frac{\omega}{v}\frac{\sqrt{1+\mu^2}}{\mu}$ perpendicular to the resultant $v\sqrt{1+\mu^2}$ of v and ωr. Call $c(\mu)$ the chord, $m(\mu)$ the two-dimensional-lift-curve slope, $\beta(\mu)$ the angle of the zero-lift curve of the blade with the axis; Γ/μ being as before the circulation around any individual blade, the "strip theory" gives the relation :

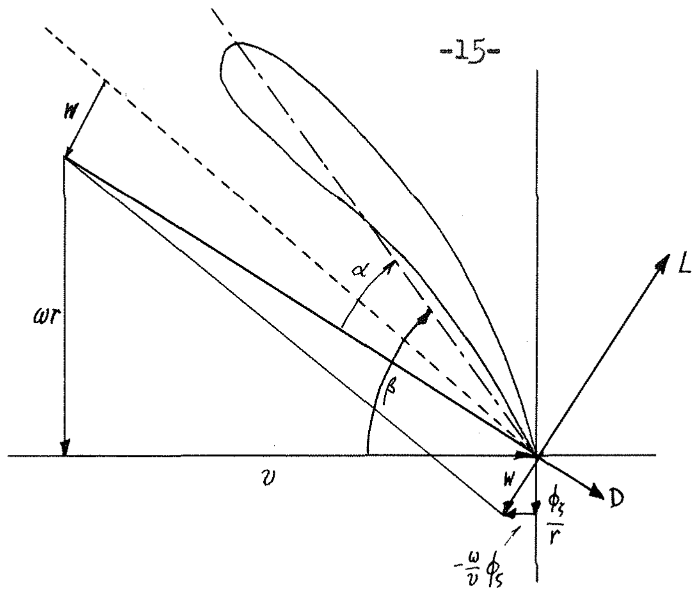
$$\frac{\Gamma(\mu)}{p} = \frac{1}{2}C_L v\sqrt{1+\mu^2}\,c = \frac{1}{2}m(\mu)c(\mu)\left[(\beta - \tan^{-1}\mu)\,v\sqrt{1+\mu^2} - W\right]$$

or

$$\frac{\Gamma(\mu)}{p} = \frac{(\beta - \tan^{-1}\mu)\sqrt{1+\mu^2}\,v}{2}\,m(\mu)c(\mu) - \frac{1}{4v\mu}\,m(\mu)c(\mu)\,\omega\sqrt{1+\mu^2}\,\phi_\zeta\Big|_{\zeta_0=0} \tag{5.1}$$

This equation, linear in $\Gamma(\mu)$, relates $\Gamma(\mu)$ to the geometry of the blade, to ω and v.

The lift and drag formulas written for the blade strip are :



$$\frac{\omega}{v}\frac{dL}{d\mu} = \frac{dL}{dr} = \frac{1}{2} m(\mu) c(\mu)\rho\left(\beta(\mu) - \tan^{-1}\mu - \frac{W}{v\sqrt{1+\mu^2}}\right) v^2(1+\mu^2)$$

$$\frac{\omega}{v}\frac{dD}{d\mu} = \frac{dD}{dr} = \frac{1}{2} c(\mu)\rho v^2(1+\mu^2) C_{D_0} + \frac{dL}{d\mu}\frac{\omega}{v}\frac{W}{v\sqrt{1+\mu^2}}$$

and, since the thrust and the torque are given by

$$\frac{\omega}{v}\frac{dT}{d\mu} = \frac{dL}{dr}\frac{\mu}{\sqrt{1+\mu^2}} + \frac{dD}{dr}\frac{1}{\sqrt{1+\mu^2}}$$

$$\frac{\omega}{v}\frac{dQ}{d\mu} = -\frac{dL}{dr}\frac{1}{\sqrt{1+\mu^2}} + \frac{dD}{dr}\frac{\mu}{\sqrt{1+\mu^2}}$$

we obtain the following equalities

$$\frac{\omega}{v}\frac{dT}{d\mu} = \frac{1}{2} m(\mu) c(\mu)(\beta - \tan^{-1}\mu)\rho v^2\mu\sqrt{1+\mu^2} + \frac{1}{2} c(\mu)\rho v^2\sqrt{1+\mu^2}\, C_{D_0} +$$

$$+ \frac{1}{2} m(\mu) c(\mu)(\beta - \tan^{-1}\mu - 1)\frac{\omega}{\mu}\rho(1+\mu^2)\phi_s - \frac{1}{2v^2\mu^2} m(\mu) c(\mu)\omega^2(1+\mu^2)\rho\phi_s^2 \qquad (5.2)$$

$$\frac{\omega}{v}\frac{dQ}{d\mu} = -\frac{1}{2} m(\mu) c(\mu)(\beta - \tan^{-1}\mu)\rho v^2\sqrt{1+\mu^2} + \frac{1}{2}\rho v^2\mu\sqrt{1+\mu^2}\, c(\mu)\, C_{D_0} +$$

$$+ \frac{1}{2} m(\mu) c(\mu)\left(\beta - \tan^{-1}\mu + \frac{1}{\mu^2}\right)\omega\rho(1+\mu^2)\phi_s - \frac{1}{2v^2\mu} m(\mu) c(\mu)\omega^2(1+\mu^2)\rho\phi_s^2 \qquad (5.3)$$

These relations are unfortunately not linear and the second order terms do not seem a priori much smaller than the first order ones.

All the preceding relations permit the formulation of the following dynamical problem:

<u>1) Find the circulation which would induce a given tangential velocity in Trefftz plane.</u>

$$\int_{t_1}^{t_2} \frac{\partial C_0}{\partial t_0} K \, dt_0 = (2\pi \phi_s - \Gamma_2)(1+\mu^2)^{1/4} = \left(\frac{2\pi v \mu}{\omega} u_t - \Gamma_2\right)(1+\mu^2)^{1/4}$$

where $C(\mu) = \Gamma(\mu)(1+\mu^2)^{1/4}$. This will be solved by successive approximations on the kernel, Γ_2 being given directly as $\int_0^{2\pi} u_t(\mu_2)\mu_2 \, d\zeta$. First keep only $K_1 = \frac{e^{pt}}{e^{pt} - e^{pt_0}}$ and neglect the rest K_2 of the kernel K. Pass to the coordinate θ defined by

$$e^{pt} = A + B\cos\theta \quad ; \quad A = \frac{1}{2}(e^{pt_1} + e^{pt_2}) \quad ; \quad B = \frac{1}{2}(e^{pt_1} - e^{pt_2}) \quad ; \quad E = -\frac{A}{B}$$

Florine (4: Annexe 4) shows that the equation becomes

$$\int_0^{\pi} \frac{\partial C_0}{\partial \theta_0} \frac{d\theta_0}{\cos\theta_0 - \cos\theta} = \left(\frac{2\pi v\mu}{\omega} u_t - \Gamma_2\right)(1+\mu^2)^{1/4} \frac{e^{pt_2} - e^{pt_1}}{2e^{pt}} = Q(\theta) \qquad (5.4)$$

and its solution is

$$C(\theta) = \frac{\sin\theta}{\pi^2} \int_0^{\pi} \frac{\int_{\theta}^{\theta'} Q(\theta'') \frac{\sin\theta'' \, d\theta''}{E - \cos\theta''}}{\cos\theta - \cos\theta'} \, d\theta' \qquad (5.5)$$

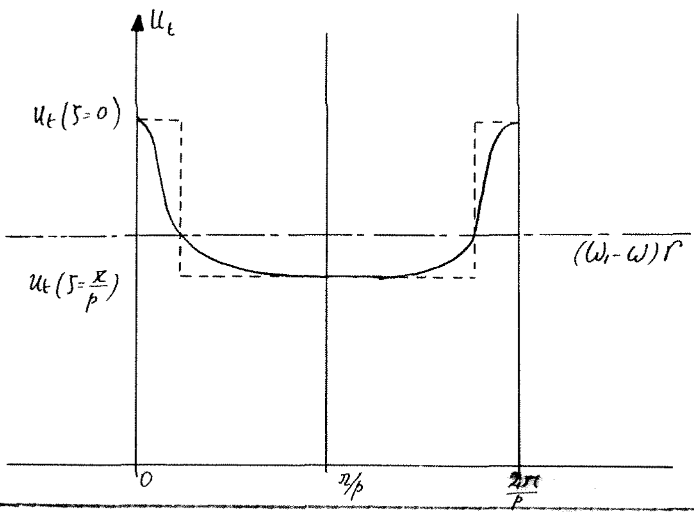
This expression was obtained by looking for a Fourier series expansion of $C(\theta)$, and Florine gives other, equivalent, forms. To C can be added an arbitrary function $A + B(\theta)$ which gives the value zero to the integral, but serves to obtain the desired values for Γ_1 and Γ_2 .

The value of $C(\theta)$ is substituted into K_2 and the corresponding correction to $Q(\theta)$ gives by inversion of the Fredholm equation (5.4) a correction to $C(\theta)$.

A particular case is investigated in more detail: the solid body rotation blades: the rotation far downstream is a solid body rotation of velocity ω_1. There is a priori a question as to the practicality of such a flow: it is of course not realisable rigorously, and ω_1 will be only the average of angular rotation, defined such that at a z far downstream:

$$\Gamma(r) = \int_0^{2\pi} r\, u_t \, d\zeta = 2\pi r\,(\omega_1 - \omega)\, r \tag{5.6}$$

therefore the problem is actually again a simple "inverse problem" leading to no integral equation, but to an integral. Now in this integral, one shall be much closer to reality if ω is replaced by ω_1 in the definition of μ *. To obtain reasonably close values of u_t as a function of ζ, one may compute u_t only for $\zeta = 0$ and $\zeta = \frac{\pi}{p}$, and then interpolate by smoothing the step curve built on these two values which gives the average $(\omega_1 - \omega)\,\Gamma$.



The most important question remains to find the distribution of u_t with z, near the blade;

*This will also permit one to treat in the same theory the case of a stator inducing a solid body rotation.

this is left to the second chapter.

2) Given the blade geometry, find $\Gamma(\mu)$.

Relation (5.1) is the integral equation for Γ, once we agree on a certain Γ_2. It is a Fredholm equation of second kind and there is little hope in finding an exact solution numerically tractable. For p large, the following approximate procedure is suggested:

a) Neglect the $\mathcal{K}_2$ part of K and assimilate $\mathcal{K}_1$ to the step function it becomes for $p \to \infty$. Then

$$\Gamma^{(0)}(\mu)\left[\frac{1}{p} + \frac{m(\mu)c(\mu)\omega}{4\pi v}\frac{\sqrt{1+\mu^2}}{\mu}\right] = \frac{(\beta(\mu) - \tan^{-1}\mu)\sqrt{1+\mu^2}\, m(\mu)c(\mu)\, v}{2}$$

$\Gamma^{(0)}$ is the 0-order approximation of Γ.

b) Write the equation as $\int_{t_1}^{t} \frac{\partial C_0^{(1)}}{\partial t_0} \mathcal{K}_1\, dt_0 = S(t)$. $S(t)$ is obtained by replacing Γ by $\Gamma^{(0)}$ in $\frac{\Gamma(\mu)}{p}$ and $\int \frac{\partial C_0}{\partial t_0} \mathcal{K}_2\, dt_0$. This is of Florine's type and gives $\Gamma^{(1)}$.

To estimate the importance of terms where the approximate value is used, compare $\frac{1}{p}$ to $\frac{mc\sqrt{1+\mu^2}}{4\pi r}$. Take as typical values: $m \sim 6$; $\frac{c}{r} \sim \frac{1}{5}$; $\mu \sim 2$; $p \sim 30$. Then the first expression is roughly 1/6 of the second. The approximation procedure appears as justified.

c) Repeat procedure b) with $\Gamma^{(1)}$ to get $\Gamma^{(2)}$.

3) Functioning off design.

If $c(\mu)$ and $\beta(\mu)$ have been designed to give a certain distribution of velocity at a given $\frac{v}{\omega}$, which modification in u_t and Γ will be the consequence of a change of $\frac{v}{\omega}$. The following treatment applied to eq. (5.1) is suggested: The change in Γ is first of all neglected in Γ/p and the new value of ϕ_s is deduced by equalling the right hand side of both old and new (5.1). This gives a Florine's type equation, for the new Γ.

This new Γ is placed in the left hand side of (5.1) and the equation solved for a better value of u_t, and eventually of Γ by another operation.

4) Determine the shape of blades giving minimum wake less (Betz's problem.

This problem is not as crucial as in propeller theory, because of the presence of downstream stages, the energy is not definitely lost. However, the present method gives to it elegant solution.

Betz's theory requires that

$$u_a = -\frac{w}{2}\,\frac{\mu^2}{1+\mu^2} = -\phi_s\,\frac{\omega}{v}$$

with w const. This leads to:

$$\frac{2\Gamma(u)}{p\,m(u)\,c(\mu)} = \text{angle of attack } \alpha(\mu) - \frac{w}{2v}\,\frac{\mu}{1+\mu^2} \qquad (5.7)$$

and if Γ is determined by (say) the condition of solid body rotation downstream, this is an all solved equation for $\alpha(\mu)$ as a function of w, ω_1 (the rotation downstream) and $c(\mu)$.

It may be noted that S. Goldstein's classical approach to this problem in propellers allows itself to be applied with little difficulty to the case of a turbomachine.

CHAPTER II

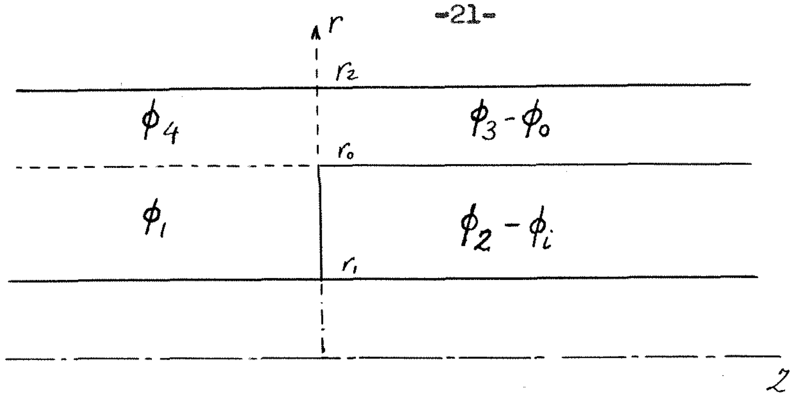
BEHAVIOR OF THE TANGENTIAL INDUCED VELOCITY IN THE NEIGHBORHOOD OF THE BLADES.

The velocity on a blade and on a blade bisector induced by the shed vortices was seen to be one-half of the velocity at infinity. However this result leaves aside the velocity induced by the bound vortices. Section 6 gives a derivation of the total induced velocity (infinite on the blade); the problem is enormously more complicated than in 1, and an approximate study is desirable.

It would be tempting, because of the exponential nature in of the correction terms, to try to fit at best an exponential "decay" coefficient, the same for all ζ 's. This however is impossible, because the circulation must remain constant along the z-axis for $z>0$. In §7 these questions are discussed in detail

6. Velocities induced by p helicoidal semi-infinite vortex filaments, starting at $z=0$.

Consider the potential field composed of the solution of (2.3) for $z>0$ and 0 for $z<0$. This field satisfies Laplace's equation in three variables, the boundary conditions, continuity conditions for $\mu=\mu_0$, and periodicity conditions. However it is discontinuous for $z=0$. Therefore add to it a solution of Laplace's equation in three dimensions, continuous in μ, satisfying the B.C. for $\mu=\mu_1, \mu_2$, everywhere one-valued and periodic in ζ of period $\frac{2\pi}{p}$; and having for $z=0$ precisely as jump the ϕ_o or ϕ_i given by (2.3). This solution will have four different analytic expressions.



In coordinates μ, θ and $z = \frac{\omega z'}{v}$ (where z' is the natural coordinate) the equation is:

$$\phi_{\mu\mu} + \frac{1}{\mu}\phi_\mu + \frac{1}{\mu^2}\phi_{\theta\theta} + \phi_{zz} = 0$$

In the harmonic of order mp in ζ, the variables are separated :

$$\phi^{(m)} = e^{imp\theta} M(\mu) Z(z)$$

for $m = 0$, take $\phi = MZ$, because ϕ must be one-valued.

Differentiate:

$$ZM'' + \frac{ZM'}{\mu} - \frac{Zm^2p^2M}{\mu^2} + MZ'' = 0$$

$$M'' + \frac{M'}{\mu} - \frac{Mm^2p^2}{\mu^2} + k^2M = 0 \; ; \; Z'' - k^2Z = 0$$

where k is some fixed number to be determined. This gives as solutions

$$M = \begin{cases} J_{mp}(k\mu) \\ Y_{mp}(k\mu) \end{cases} \qquad Z = \begin{cases} e^{kz} \\ e^{-kz} \end{cases}$$

Write

$$\begin{aligned} \phi_1 &= \sum_{m\neq 0}{}' e^{imp\theta} \sum_k e^{kz}\left\{F'_{mp}(k)\, J_{mp}(k\mu) + G'_{mp}(k)\, Y_{mp}(k\mu)\right\} \\ &\qquad + e^{-kz}\left\{P'_{mp}(k)\, J_{mp}(k\mu) + Q'_{mp}(k)\, Y_{mp}(k\mu)\right\} \\ \phi_4 &= \sum_{m\neq 0}{}' e^{imp\theta} \sum_k e^{kz}\left\{F^4_{mp}(k)\, J_{mp}(k\mu) + G^4_{mp}(k)\, Y_{mp}(k\mu)\right\} \\ &\qquad + e^{-kz}\left\{P^4_{mp}(k)\, J_{mp}(k\mu) + Q^4_{mp}(k)\, Y_{mp}(k\mu)\right\} \end{aligned} \tag{6.1}$$

and analogous expressions for $\phi_2 - \phi_i$; $\phi_3 - \phi_0$. These must be continuous for μ_0 and their derivation must vanish for μ_1 and μ_2 for any z . However this condition is not realizable for any value of k. The calculations given in Appendix p.41 show that if non-trivial solutions for $F's$ and $G's$ are to be found, k must satisfy the following characteristic equation:

$$J'_{mp}(k\mu_2)\, Y'_{mp}(k\mu_1) - J'_{mp}(k\mu_1)\, Y'_{mp}(k\mu_2) = 0 \tag{6.2}$$

Call k_m^α the characteristic values. Meyer (1:Anhang) indicates a series of methods for their approximate determination. For instance for $p = 10$, $m = 1$, $\mu_2 = 1$, $\mu_1 = .8$, the first root is already 11.089, the second 19.35 and their orders of magnitude increase proportionally to m , and to α.

The solutions $\phi_1, \phi_2, \ldots$ may now be expressed in double series of these Bessel functions. Considering that all these functions must vanish when $|z| \to \infty$, the four series are

$$\phi_1 = \sum_{\substack{m \neq 0 \\ m=-\infty}}^{\infty} e^{imp\theta} \sum_{\alpha=1}^{\infty} A_m^\alpha\, e^{k_m^\alpha z} \left(J'(1)Y - JY'(1)\right)\left(J'(2)Y'(0) - J'(0)Y'(2)\right)$$

$$\phi_2 = \sum e^{imp\theta} \sum A_m^\alpha\, e^{k_m^\alpha z} \left(J'(2)Y - JY'(2)\right)\left(J'(1)Y'(0) - J'(0)Y'(1)\right)$$

$$\phi_2 - \phi_i = \sum e^{imp\theta} \sum B_m^\alpha\, e^{-k_m^\alpha z} \left(J'(1)Y - JY'(1)\right)\left(J'(2)Y'(0) - J'(0)Y'(2)\right)$$

$$\phi_3 - \phi_0 = \sum e^{imp\theta} \sum B_m^\alpha\, e^{-k_m^\alpha z} \left(J'(2)Y - JY'(2)\right)\left(J'(1)Y'(0) - J'(0)Y'(1)\right) \tag{6.3}$$

The last operation is to determine the coefficients A_m B_m by the condition of smooth joining for $z=0$. First write the continuity of the derivative

$$\sum_{\alpha=1}^{\infty} k_\alpha^m \left(A_m^\alpha + B_m^\alpha\right) \mathcal{B}_m^\alpha(\mu) = 0$$

where $$\mathcal{B}_m^\alpha = \begin{cases} [J'(1)Y - JY'(1)][J'(2)Y'(0) - J'(0)Y'(2)] & \mu_1 < \mu < \mu_0 \\ [J'(2)Y - JY'(2)][J'(1)Y'(0) - J'(0)Y'(1)] & \mu_0 < \mu < \mu_2 \end{cases}$$

is precisely by definition of k_α^m one of the set of eigensolutions of Bessel equation, with vanishing derivatives for $\mu = \mu_1, \mu_2$; they are complete and orthogonal. Therefore:

$$B_m^\alpha = -A_m^\alpha$$

The last condition is continuity at $z=0$; write $a_m\, b_m,\, a'_m\, b'_m$ for the known coefficients of (2.2), that is:

$$\begin{aligned} \phi_o &= -2i \sum_{m=1}^{\infty} \left(e^{imp\zeta} - e^{-imp\zeta}\right)\left\{a_m K_{mp}(mp\mu) + b_m I_{mp}(mp\mu)\right\} \\ \phi_i &= -2i \sum_{m=1}^{\infty} \left(e^{imp\zeta} - e^{-imp\zeta}\right)\left\{a'_m K_{mp}(mp\mu) + b_m I_{mp}(mp\mu) - \frac{\gamma}{\pi mp}\right\} \end{aligned} \tag{6.4}$$

For $z=0$, $\zeta=\theta$; and therefore $A_0^\alpha = 0$, and for $m \neq 0$,

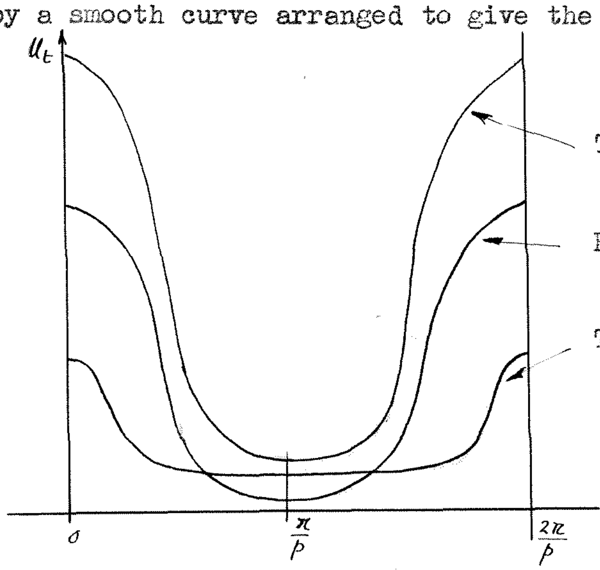
$$\sum_{\alpha=1}^{\infty} A_m^\alpha \mathcal{B}_m^\alpha = \begin{cases} -i\left\{a'_m K_{mp}(mp\mu) + b'_m I_{mp}(mp\mu)\right\} + \dfrac{i\gamma}{\pi mp} \\ -i\left\{a_m K_{mp}(mp\mu) + b_m I_{mp}(mp\mu)\right\} \end{cases}$$

which finally determines A_m^α's as coefficients of the development in generalized Fourier series in the eigenfunctions $\mathcal{B}_m^\alpha$ $\left[\frac{\partial}{\partial\mu}\mathcal{B}_m^\alpha = 0 \text{ for } \mu = \mu_1, \mu_2\right]$ of another function F with $\frac{\partial}{\partial\mu}F = 0$ at $\mu = \mu_1, \mu_2$.

There is of course no question of actually computing all these coefficients. However given the orders of magnitude of k_m^α the computation of the first coefficients would give excellent precision, as soon as z leaves a close neighborhood of 0, which is physically of no interest.

7. Approximate solution.

Following the idea already used in §5, part 1, investigate separately the behavior of ϕ_ζ for $\zeta = 0$ and $\zeta = \frac{\pi}{p}$, and then fair in between by a smooth curve arranged to give the correct average for ϕ_ζ. The



contributions of the bound and trailing vortices are investigated separately for the case of solid body rotation blades.

Trailing vortex. In this case introduce a single exponential decay same for all ζ's, and for this purpose use the average of ϕ, say $\bar{\Phi}$ which gives the correct circulation at $z = \infty$. At $z = 0$, the corresponding contribution to velocity will be $\frac{\Gamma(\mu)}{2}$. The following is a very rough approximation which has however the merit of great simplicity.

Thus $\bar{\Phi}$ is defined by $\quad u_t = \frac{1}{r}\bar{\Phi}_\theta = (\omega_1 - \omega) r$

That is $\quad \bar{\Phi} = (\omega_1 - \omega)\frac{v^2\mu^2}{\omega^2}\theta = H\mu^2\theta \qquad H = (\omega_1 - \omega)\frac{v^2}{\omega^2}$

One wishes to have:

$$\phi = \Phi\left(1 - \frac{1}{2} e^{-\lambda z}\right) \qquad z > 0$$

$$\phi = \Phi\left(\frac{1}{2} e^{\lambda z}\right) \qquad z < 0$$

Apply Laplace's operator Δ to the exponential term only:

$$\Delta\phi = \left(3\theta + \frac{v^2}{\omega^2}\lambda^2\mu^2\theta\right) H \frac{1}{2} e^{\lambda z}$$

It is impossible to have $\Delta\phi \equiv 0$. Therefore the mean weighted square only will be minimized:

$$\Lambda = \int_{\mu_1}^{\mu_2}\int_0^{\infty}\int_0^{2\pi}\left[\left(3 + \frac{v^2}{\omega^2}\lambda^2\mu^2\right)^2\theta^2 e^{2\lambda z}\right] z^2 \, d\theta \, dz \, d\mu$$

where z^2 has for purpose to favor large values of z. Integrate in z and θ,

$$\Lambda = \frac{1}{\lambda^2}\int_{\mu_1}^{\mu_2}\left(3 + \frac{v^2}{\omega^2}\lambda^2\mu^2\right)^2 d\mu$$

$$= \frac{9(\mu_2 - \mu_1)}{\lambda^2} + 2\left(\mu_2^3 - \mu_1^3\right)\frac{v^2}{\omega^2} + \frac{\lambda^2}{5}\left(\mu_2^5 - \mu_1^5\right)\frac{v^4}{\omega^4}$$

Differentiate with respect to λ:

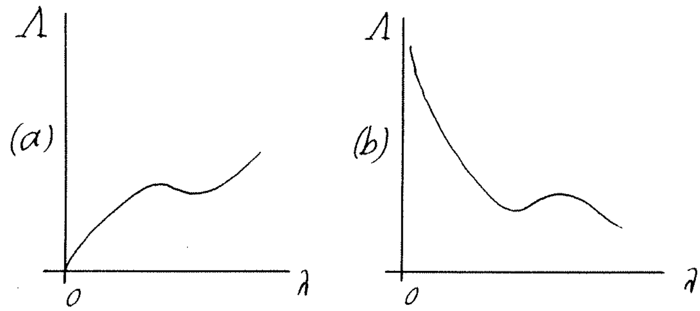
$$\frac{d\Lambda}{d\lambda} = -\frac{18(\mu_2 - \mu_1)}{\lambda^3} + \frac{2\lambda}{5}\left(\lambda_2^5 - \mu_1^5\right)\frac{v^4}{\omega^4} = 0$$

This gives $$\lambda = \left(\frac{45(r_2 - r_1)}{r_2^5 - r_1^5}\right)^{1/4} \qquad (7.1)$$

This value of λ is seen to be independent of $\frac{v}{\omega}$.

It may be noted that the procedure is also applicable for different forms of $\bar{\Phi}$.

A more refined approach would result if we keep for $\bar{\Phi}$ the actual expression for the velocity potential far downstream. The expression for λ depends considerably on the weighting factor (z^2 in our case). For a reasonable result, this weighting factor should give for Λ an expression which is infinite for $\lambda = 0$ and $\lambda = \infty$. This in particular limits the possible exponents of a factor of type z^α. An incorrect choice of α would result in a variation of type



(a) or (b) or more complicated, with more than one root in λ for $\frac{d\Lambda}{d\lambda} = 0$, and other phenomena. These facts make difficult the critical examination of the whole approximation procedure.

<u>Bound vortex.</u> As an approximation, for z small, consider the action of the two nearest vortices only. The tangential component is:

$$\frac{\Gamma(\mu)}{2\pi} \left(\frac{z}{z^2 + r^2\theta^2} + \frac{z}{z^2 + r^2\left(\frac{2\pi}{p} - \theta\right)^2} \right) \qquad (7.2)$$

with the assumption, usual in the study of the vorticities near a wing, that the velocity can be considered as induced by only the immediately close section of the bound vortex.

This part of the induced velocity will give another contribution to the total circulation around the axis: this contribution is $\Gamma/2$ for $z=0$.

CHAPTER III

TRAILING VORTICES IN A CONICAL TURBOMACHINE

A great simplification resulted in Chapter I from the possibility in the case of a cylindrical turbomachine of assuming in a first study that the field is essentially independent of the axial coordinate. Chapter II gave a refinement of this theory to take the axial coordinate into account.

Such a two-step procedure is unfortunately impossible in the case of a conical turbomachine and a mathematical investigation must take three coordinates into account even if the vortex is assumed "infinite". The following section gives only a preliminary mathematical investigation of the influence function.

8. Preliminary study of the influence function.

Let the polar coordinates be r', φ', θ' fixed with respect to the blade row, the walls of the machine being $\varphi' = \varphi_1, \varphi_2$. From the origin flows an incompressible source of strength Q; the emitted fluid rotates like a solid body with angular velocity ω .

From the circle $r' = r_0$, $\varphi' = \varphi_0$ are shed p vortices of strength γ/p , they are continued by the bound vortices $r' = r_0$, $0 < \varphi' < \varphi_0$ and the axis $\varphi = 0$, $r > r_0$. Neglect all self-induced displacements. The differential equation of the shed vortices is:

$$\frac{d\theta}{\omega} = \frac{4\pi r^2}{Q} dr \qquad \text{Therefore} \qquad \theta' - \theta_0 = \frac{4\pi\omega}{3Q} r^3 = K r^3 \qquad \text{where } K = \frac{4\pi\omega}{3Q}$$

Let the origin of θ' be defined in such a fashion that the p vortices have the equation:

$$\zeta = \theta' - Kr^3 = \frac{2\pi}{p} k \qquad k = 0, 1, \dots p-1 \tag{8.1}$$

The potential V of induced velocities is a solution of Laplace's equation

$$V_{r'r'} + \frac{2V_{r'}}{r'} + \frac{1}{r'^2 \sin^2\varphi'} V_{\theta'\theta'} + \frac{1}{r'^2 \sin\varphi'} \frac{\partial}{\partial\varphi'}(\sin\varphi' V_{\varphi'}) = 0 \tag{8.2}$$

satisfying the following boundary and other conditions:

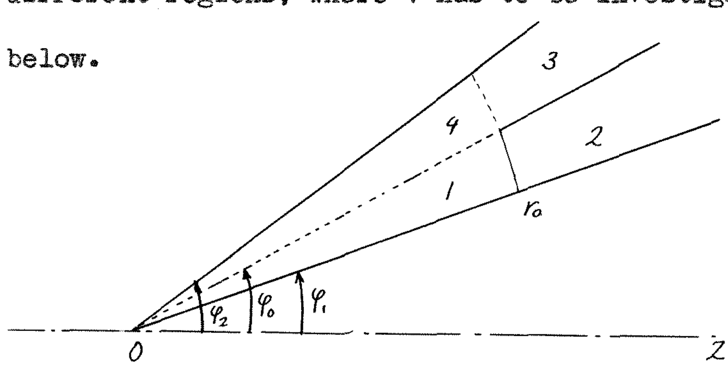
For $r' < r_0$ it is a finite, one-valued, smooth function of $r'\varphi'\theta'$; its φ-derivative vanishes for $\varphi = \varphi_1, \varphi_2$

For $r' > r_0$ it must be infinite for $\varphi = \varphi_0$, $\zeta = \frac{2\pi}{p} k$, on any of the vortices; it is otherwise smooth and finite; its φ-derivative vanishes for $\varphi = \varphi_1, \varphi_2$; for $\varphi > \varphi_0$, it is one-valued, for $\varphi < \varphi_0$, it increases by γ/p whenever θ' increases by $2\pi/p$.

Define

$$\begin{cases} \zeta = \theta' - Kr^3 \\ \mu = \cos\varphi' \\ r' = r \end{cases}$$

Four different regions, where V has to be investigated separately, are shown below.



The solutions in 1, and 4 are called ϕ_1 ϕ_4 . The solutions in 2 and 3 are divided into two parts: ϕ_2 ϕ_3 satisfying the same conditions as ϕ_1 ϕ_4 and ϕ_0 ϕ_c taking care of the circulation. Separating the variables, define R and M through

$$V = \sum_{m=-\infty}^{+\infty} e^{imp\theta} R(r)\, M(\mu)$$

The equations obtained for R and M are classical. They give, if we introduce a number n which is such that $n(n+1)$ is a real ,

$$R = r^n \quad or \quad r^{-(n+1)}$$

$$M = \begin{cases} P_n^{mp}(\mu) \quad or \quad Q_n^{mp}(\mu) & \text{if } n \text{ is not an integer or if } n \text{ is an integer} \geqslant mp \\ P_n^{-mp}(\mu) \quad or \quad Q_n^{mp}(\mu) & \text{if } n \text{ is an integer} < mp \end{cases}$$

(notations of Hobson: $P_n^{-m}(x) = \lim\limits_{\ell \to m} \dfrac{(n-\ell)!}{(n+\ell)!} P_n^{\ell}(x)$)

<u>Case of ϕ_1, ϕ_2, ϕ_3, ϕ_4</u>

In ϕ_1 , ϕ_4 are to be kept only the terms in r^n $(n \geqslant 0)$

In ϕ_2 , ϕ_3 are to be kept only the terms in $r^{-(n+1)}$ $(n+1 \geqslant 0$

One expects n not to be an integer. Therefore write for instance:

$$\begin{aligned} \phi_1 &= \sum_{m=-\infty}^{\infty} e^{imp\theta} \sum_n r^n \left(A_{nm}^1 P_n^{mp}(\mu) + B_{nm}^1 Q_n^{mp}(\mu) \right) \\ \phi_4 &= \sum_{m=-\infty}^{\infty} e^{imp\theta} \sum_n r^n \left(A_{nm}^4 P_n^{mp}(\mu) + B_{nm}^4 Q_n^{mp}(\mu) \right) \end{aligned} \tag{8.3}$$

The boundary conditions and the continuity at μ_0 lead to four homogeneous linear equations in A, B's for every m , and n. These equations are compatible only if

$$\frac{d}{d\mu} P_n^{mp}(\mu_1) \frac{d}{d\mu} Q_n^{mp}(\mu_2) - \frac{d}{d\mu} P_n^{mp}(\mu_2) \frac{d}{d\mu} Q_n^{mp}(\mu_1) = 0 \tag{8.4}$$

The same equation being also obtained from ϕ_2 ϕ_3 ; and the coefficients are then defined only in their relative values. The absolute values will be determined later by the conditions of continuity at $r = r_0$.

Case of ϕ_o, ϕ_i.

One wishes to have these as functions of ζ, r, μ, so as to have the required divergence for $\zeta=0$, $\mu=\mu_o$. Therefore write the solutions as:

$$\phi_o = \sum_{m=-\infty}^{\infty} e^{imp\zeta} \sum_n e^{impKr^3} \left(\alpha r^n + \beta r^{-(n+1)}\right)\left(\gamma P_n^{\pm mp}(\mu) + \delta Q_n^{mp}(\mu)\right)$$

$$\phi_i = \sum_{m=-\infty}^{\infty} e^{imp\zeta} \sum_n e^{impKr^3} \left(\alpha' r^n + \beta' r^{-(n+1)}\right)\left(\gamma' P_n^{\pm mp}(\mu) + \delta' Q_n^{mp}(\mu)\right) + \frac{\gamma}{2\pi}\left(\zeta + Kr^3 - \frac{\pi}{p}\right)$$

The domain of variation of n and the various coefficients are to be determined. For this purpose write the circulation term as

$$\frac{\gamma}{2\pi}\left(\zeta - \frac{\pi}{p}\right) + \frac{\gamma K r^3}{2\pi} = -\frac{\gamma}{\pi}\sum_{m\neq 0}{}' \frac{e^{imp\zeta}}{2imp} + \frac{\gamma K r^3}{2\pi}$$

- Case of $m=0$; one needs

$$\phi_o^o = \sum \left(A r^{-(n+1)} + B r^n\right)\left(C P_n(\mu) + D Q_n(\mu)\right)$$

$$\phi_i^o = \sum \left(A' r^{-(n+1)} + B' r^n\right)\left(C' P_n(\mu) + D' Q_n(\mu)\right) + \frac{\gamma K}{2\pi} r^3$$

for any $n\neq 3$, one finds again the homogeneous conditions studied for ϕ_2, ϕ_3 : the corresponding terms have already been included in these parts of the potential.

for $n=3$, a non-homogeneous set of equations is obtained which gives:

$$\phi_o^o = -\frac{\gamma K}{2\pi}\left(1-\mu_o^2\right) \frac{[PQ'(2) - QP'(2)][P'(0)Q'(1) - Q'(0)P'(1)]}{P'(2)Q'(1) - Q'(2)P'(1)} r^3$$

$$\phi_i^o = \frac{\gamma K}{2\pi}\left(1-\mu_o^2\right) \frac{[Q'(1)P - P'(1)Q][P'(0)Q'(2) - Q'(0)P'(2)]}{P'(2)Q'(1) - Q'(2)P'(1)} r^3 + \frac{\gamma K}{2\pi} r^3 \qquad (8.5)$$

where $Q'(2)$ for instance stands for $\left.\frac{d}{d\mu} Q_3(\mu)\right|_{\mu=\mu_2}$

- Case of $m \neq 0$, one wants the boundary conditions to be satisfied by:

$$e^{-impKr^3}\phi_0^m = \sum_n \left(\alpha r^{-(n+1)} + r^n\right)\left(A_{nm}^o P_n^{\pm mp} + B_{nm}^o Q_n^{mp}\right)$$

$$e^{-impKr^3}\phi_i^m = \sum_n \left(\alpha r^{-(n+1)} + r^n\right)\left(A_{nm}^i P_n^{\pm mp} + B_{nm}^i Q_n^{mp}\right) + \frac{i\gamma}{2\pi mp} e^{-impKr^3}$$

The solution will have to be smooth at μ_0 and its μ-derivative to vanish at $\mu_1 \mu_2$ for any r. For this purpose, develop ~~s~~ again the circulation term in series:

$$\frac{i\gamma}{2\pi mp} e^{-impKr^3} = \frac{i\gamma}{2\pi mp} + \frac{\gamma K}{2\pi} r^3 - \frac{i\gamma K^2 mp r^6}{4\pi} + \cdots$$

$$= e_0^m + e_3^m r^3 + e_6^m r^6 + \cdots$$

By identification one obtains 1) that the coefficients α of $r^{-(n+1)}$ are zero - 2) that the coefficients A_{mn}^o, B_{nm}^o, A_{nm}^i, B_{nm}^i of $P_n^{\pm mp}$, Q_n^{mp} in ϕ_0^m, ϕ_i^m must for each m,n satisfy a set of four non-homogeneous linear equations. Remember that:

$$P_n^m \frac{dQ_n^m}{d\mu} - Q_n^m \frac{dP_n^m}{d\mu} = \frac{2^{2m}}{1-\mu^2} \frac{\Gamma\left(\frac{n+m+1}{2}\right)\Gamma\left(\frac{n+m+2}{2}\right)}{\Gamma\left(\frac{n-m+1}{2}\right)\Gamma\left(\frac{n-m+2}{2}\right)}$$

$$= \frac{\Gamma_n^m}{1-\mu^2} \quad \text{(say)}$$

(Magnus-Oberhettinger, p. 83)

Then the final result is:

$$\phi_o^m = -\sum_{q=0}^{\infty} e^{impKr^3} \frac{e_{3q}^m}{\Gamma_{3q}^{mp}} r^{3q} \frac{[PQ'(2)-QP'(2)][P'(0)Q'(1)-Q'(0)P'(1)]}{P'(2)Q'(1)-P'(1)Q'(2)} (1-\mu_o^2)$$

$$\phi_i^m = \sum e^{impKr^3} \frac{e_{3q}^m}{\Gamma_{3q}^{mp}} r^{3q} \frac{[PQ'(1)-QP'(1)][P'(0)Q'(2)-Q'(0)P'(2)]}{P'(2)Q'(1)-P'(1)Q'(2)} (1-\mu_o^2) + \frac{i\gamma}{2\pi mp} \tag{8.6}$$

where $P'(2)$ stands for $\left.\frac{d}{d\mu} P_{3q}^{\pm mp}\right|_{\mu=\mu_2}$

$$\begin{cases} + \text{ if } & 3q \geq mp \\ - \text{ if } & 3q < mp \end{cases}$$

The complete formal solution of the problem requires as a final step the development of the above function taken for $r=r_o$ in a Fourier series of the eigen-functions corresponding to the different solutions of the characteristic equation, for fixed m. The result would have to be integrated over μ_o. The corresponding integrals will not be written down explicitly.

APPENDIX

EXPLICIT DETAILS OF DERIVATIONS

Page (6) The four boundary conditions are:

$$a_m K_0 + b_m I_0 - a'_m K_0 - b'_m I_0 = -\frac{\gamma}{\pi m p}$$

$$a_m K'_0 + b_m I'_0 - a'_m K'_0 - b'_m I'_0 = 0$$

$$a_m K'_2 + b_m I'_2 = 0$$

$$- a'_m K'_1 - b'_m I'_1 = 0$$

Where $K'_1 = \frac{d}{d\mu} K_{mp}(mp\mu)\Big/_{\mu=\mu_1}$

for instance

The determinant of the system is

$$\Delta = \begin{vmatrix} K_0 & I_0 & -K_0 & -I_0 \\ K'_0 & I'_0 & -K'_0 & -I'_0 \\ K'_2 & I'_2 & 0 & 0 \\ 0 & 0 & -K'_1 & -I'_1 \end{vmatrix} = \begin{vmatrix} K_0 & I_0 & 0 & 0 \\ K'_0 & I'_0 & 0 & 0 \\ K'_2 & I'_2 & K'_2 & I'_2 \\ 0 & 0 & -K'_1 & -I'_1 \end{vmatrix} = -\begin{vmatrix} K_0 & I_0 \\ K'_0 & I'_0 \end{vmatrix} \times \begin{vmatrix} K'_2 & I'_2 \\ K'_1 & I'_1 \end{vmatrix}$$

$$\Delta = -\frac{K'_2 I'_1 - K'_1 I'_2}{\mu}$$

and

$$\Delta \frac{\pi m p}{\gamma} a_m = I'_2 \left(K'_0 I'_1 - K'_1 I'_0 \right)$$

$$\Delta \frac{\pi m p}{\gamma} b_m = -K'_2 \left(K'_0 I'_1 - K'_1 I'_0 \right)$$

$$\Delta \frac{\pi m p}{\gamma} a'_m = I'_1 \left(K'_0 I'_2 - K'_2 I'_0 \right)$$

$$\Delta \frac{\pi m p}{\gamma} b'_m = -K'_1 \left(K'_0 I'_2 - K'_2 I'_0 \right)$$

Page (9)

By integration by parts, we obtain

$$\int_{\mu_1}^{\mu} \frac{\partial \Gamma}{\partial \mu_0} \mu_0 \frac{dK_{mp}}{d\mu_0} d\mu_0 = \Gamma_\mu \frac{dK_{mp}}{d\mu} - \Gamma_1 \mu_1 \frac{dK_{mp}}{d\mu_1} - p^2 m^2 \int_{\mu_1}^{\mu} \Gamma_0 \left(\mu_0 + \frac{1}{\mu_0}\right) K_{mp}(mp\mu_0) d\mu_0$$

$$\int_{\mu}^{\mu_2} \frac{\partial \Gamma}{\partial \mu_0} \mu_0 \frac{dK_{mp}}{d\mu_0} d\mu_0 = -\Gamma_\mu \frac{dK_{mp}}{d\mu} + \Gamma_2 \mu_2 \frac{dK_{mp}}{d\mu_2} - p^2 m^2 \int_{\mu}^{\mu_2} \Gamma_0 \left(\mu_0 + \frac{1}{\mu_0}\right) K_{mp}(mp\mu_0) d\mu_0$$

and same formulas are valid with $I_n(\mu)$ replacing $K_n(\mu)$.

The integrated terms summed over m yield:

$$\sum_{m=1}^{\infty} \frac{\cos mp\zeta(-\Gamma\mu)}{K_2' I_1' - K_1' I_2'} \left\{ (I_2' K - K_2' I)(I_1' K' - K_1' I') + (I_1' K - K_1' I)(K_2' I' - K' I_2') \right\}$$

$$= -\sum_{m=1}^{\infty} \cos pm\zeta\, \Gamma(\mu)\, \mu$$

And the remaining terms yield:

$$\pi \phi_\zeta = \frac{\Gamma_2 - \Gamma}{2} + \sum_{m=1}^{\infty} \cos pm\zeta \left\{ -\Gamma + \frac{p^2 m^2}{K_2' I_1' - K_1' I_2'} \times \right.$$

$$\times \left[\int_{\mu_1}^{\mu} \Gamma_0 \left(\mu_0 + \frac{1}{\mu_0}\right) (I_2' K - K_2' I)(I_1' K_0 - K_1' I_0) d\mu_0 \right.$$

$$\left. \left. + \int_{\mu}^{\mu_2} \Gamma_0 \left(\mu_0 + \frac{1}{\mu_0}\right) (I_1' K - K_1' I)(I_2' K_0 - K_2' I_0) d\mu_0 \right] \right\}$$

Nicholson's approximation is (see 9)

$$K_n(\eta\mu) \sim \frac{e^{-nt(\mu)}}{\sqrt{\frac{2n}{\pi}}\;(1+\mu^2)^{1/4}} \qquad t = \sqrt{1+\mu^2} + \log\frac{\mu}{1+\sqrt{1+\mu^2}}$$

$$I_n(\eta\mu) \sim \frac{e^{nt(\mu)}}{\sqrt{2n\pi}\;(1+\mu^2)^{1/4}}$$

By one of the recurrence formulas on functions I, K we obtain then for I'_n and K'_n the following approximations:

$$\frac{d}{d\mu} K_n(\eta\mu) \sim -\frac{1}{2}\frac{e^{-nt(\mu)}}{\sqrt{\frac{2n}{\pi}}\;(1+\mu^2)^{1/4}}\left(e^{-t(\mu)} + e^{+t(\mu)}\right)$$

$$\frac{d}{d\mu} I_n(\eta\mu) \sim \frac{1}{2}\frac{e^{nt(\mu)}}{\sqrt{2n\pi}\;(1+\mu^2)^{1/4}}\left(e^{-t(\mu)} + e^{+t(\mu)}\right)$$

and therefore:

$$K'_i I'_j - K'_j I'_i \sim \frac{1}{4}\sinh\left[n(t_i - t_j)\right]\frac{\left(e^{-t(\mu_i)} + e^{t(\mu_i)}\right)\left(e^{-t(\mu_j)} + e^{t(\mu_j)}\right)}{(1+\mu_i^2)^{1/4}\,(1+\mu_j^2)^{1/4}}$$

$$K I'_i - K'_i I \sim \frac{1}{2}\cosh\left[n(t - t_i)\right]\frac{e^{-t(\mu_i)} + e^{t(\mu_i)}}{(1+\mu^2)^{1/4}\,(1+\mu_i^2)^{1/4}}$$

It results from an oral communication of R. S. Phillips that precise upper and lower bounds can be found for the ratios I'/I and K'/K. The first integration by parts has left the derivatives only in terms of type $I'(1)$ or $K'(1)$, and they enter to the same degree in both numerator and denominator. Therefore the additional uncertainty on $I'K'$ (coming from the arbitrariness of the choice of recurrence formula giving I', K') is immaterial insofar as it is translated by a factor independent of t, and very close to 1 for p not small.

Substitute into ϕ_ς : the expressions derived for the Bessel function combinations:

$$-\pi\phi_\varsigma = \frac{\Gamma_1 - \Gamma_2}{2} + \sum_{m=1}^{\infty} \cos pm\varsigma \Big\{ \Gamma -$$

$$- pm \int_{\mu_1}^{\mu} \Gamma_0 \frac{(1+\mu_0^2)^{3/4}}{\mu_0} \frac{\cosh mp(t-t_2)\cosh mp(t_0-t_1)}{(1+\mu^2)^{1/4}\sinh mp(t_2-t_1)} d\mu_0$$

$$- pm \int_{\mu}^{\mu_2} \Gamma_0 \frac{(1+\mu_0^2)^{3/4}}{\mu_0} \frac{\cosh mp(t-t_1)\cosh mp(t_0-t_2)}{(1+\mu^2)^{1/4}\sinh mp(t_2-t_1)} d\mu_0 \Big\}$$

Switch to variable t:

$$dt = \frac{(1+\mu^2)^{1/2}}{\mu} d\mu \quad ; \quad \frac{(1+\mu_0^2)^{3/4}}{\mu_0} d\mu_0 = (1+\mu_0^2)^{1/4} dt_0$$

and integrate by parts again

$$\int_{t_1}^{t} [\Gamma_0 (1+\mu_0^2)^{1/4}]\, pm \cosh pm(t_0-t_1)\, dt_0 =$$

$$\Gamma (1+\mu^2)^{1/4} \sinh mp(t-t_1) - \int_{t_1}^{t} \frac{\partial[\Gamma_0(1+\mu_0^2)^{1/4}]}{\partial t_0} \sinh pm(t_0-t_1)\, dt_0$$

and similarly for the other integrals, and the final result is:

$$-\pi\phi_\varsigma = \frac{\Gamma_1 - \Gamma_2}{2} + \sum_{m=1}^{\infty} \cos mp\varsigma \frac{1}{(1+\mu^2)^{1/4}} \times$$

$$\Big\{ \int_{t_1}^{t} \frac{\partial[\Gamma_0(1+\mu_0^2)^{1/4}]}{\partial t_0} dt_0 \frac{\cosh pm(t-t_2)\sinh mp(t_0-t_1)}{\sinh mp(t_2-t_1)}$$

$$+ \int_{t}^{t_2} \frac{\partial[\Gamma_0(1+\mu_0^2)^{1/4}]}{\partial t_0} dt_0 \frac{\cosh pm(t-t_1)\sinh mp(t_0-t_2)}{\sinh mp(t_2-t_1)} \Big\}$$

Page (11) Summation of the series on the kernel

Develop the cosh in sums of exponentials and neglect $e^{-2p(t_2-t_1)} \ll 1$

$$-2\pi\phi_\zeta = \Gamma - \Gamma_2 + \frac{1}{2(1+\mu^2)^{1/4}}\left[\int_{t_1}^{t} \frac{\partial[\Gamma_0(1+\mu_0^2)^{1/4}]}{\partial t_0}\, dt_0 \sum_{m=1}^{\infty}\left\{ e^{mp(t+t_0-2t_2+i\zeta)} + \right.\right.$$

$$+ e^{mp(t_0-t+i\zeta)} - e^{mp(t-t_0+2t_1-2t_2+i\zeta)} - e^{mp(-t+2t_1-t_0+i\zeta)} + e^{mp(t+t_0-2t_2-i\zeta)} +$$

$$\left. + e^{mp(t_0-t-i\zeta)} - e^{mp(t-t_0+2t_1-2t_2-i\zeta)} - e^{mp(-t-t_0+2t_1-i\zeta)} \right\}$$

$$+ \int_{t}^{t_2} \frac{\partial[\Gamma_0(1+\mu_0^2)^{1/4}]}{\partial t_0}\, dt_0 \sum_{m=1}^{\infty}\left\{ e^{mp(t+t_0-2t_2+i\zeta)} \right.$$

$$+ e^{mp(-t+t_0+2t_1-2t_2+i\zeta)} - e^{mp(t-t_0+i\zeta)} - e^{mp(-t-t_0+2t_1+i\zeta)} + e^{mp(t+t_0-2t_2-i\zeta)}$$

$$\left.\left. + e^{mp(-t+t_0+2t_1-2t_2-i\zeta)} - e^{mp(t-t_0-i\zeta)} - e^{mp(-t-t_0+2t_1-i\zeta)} \right\}\right]$$

All the moduli of the geometric series are < 1. Dropping the terms in $e^{-2p(t_2-t_1)}$ which still subsist, and summing,

$$-2\pi\phi_\zeta = \Gamma - \Gamma_2 + \frac{1}{2(1+\mu^2)^{1/4}}\left[\int_{t_1}^{t_2} \frac{\partial[\Gamma_0(1+\mu_0^2)^{1/4}]}{\partial t_0}\, dt_0 \left\{ \frac{e^{p(t+t_0-2t_2+i\zeta)}}{1-e^{p(t+t_0-2t_2+i\zeta)}} \right.\right.$$

$$\left. - \frac{e^{-p(t+t_0-2t_1-i\zeta)}}{1-e^{-p(t+t_0-2t_1-i\zeta)}} \right\} + \int_{t_1}^{t} \frac{\partial[\,]}{\partial t_0}\, dt_0\, \frac{e^{p(t_0-t+i\zeta)}}{1-e^{p(t_0-t+i\zeta)}} - \int_{t}^{t_2} \frac{\partial[\,]}{\partial t_0}\, dt_0\, \frac{e^{p(-t_0+t-i\zeta)}}{1-e^{p(-t_0+t-i\zeta)}} \Bigg]$$

(+ same terms with $-\zeta$ instead of $+\zeta$.)

Now the sum of the last two terms of the second line equals

$$-\int_{t_1}^{t_2} \frac{\partial[\,]}{\partial t_0}\, dt_0\, \frac{e^{-p(t_0-t+i\zeta)}}{1-e^{-p(t_0-t+i\zeta)}} - \Gamma(1+\mu^2)^{1/4}$$

The Γ cancels the initial Γ. Finally <u>the first approximation kernel is</u>

$$\mathcal{K} = \frac{1}{1-e^{p(t_0-t+i\zeta)}} + \frac{1}{1-e^{p(t_0-t-i\zeta)}} - \frac{1}{1-e^{-p(t+t_0-2t_2+i\zeta)}}$$

$$- \frac{1}{1-e^{-p(t+t_0-2t_2-i\zeta)}} + \frac{1}{1-e^{p(t+t_0-2t_1+i\zeta)}} + \frac{1}{1-e^{p(t+t_0-2t_1-i\zeta)}}$$

Now $$\frac{1}{Ae^{ip\zeta}-1} + \frac{1}{Ae^{-ip\zeta}-1} = \frac{A^2-1}{A^2+1-2A\cos p\zeta} - 1$$

For $\zeta=0$ $$\frac{1}{A-1} + \frac{1}{A-1} = \frac{2}{A-1}$$

For $\zeta=\frac{\pi}{p}$ $$\frac{1}{-A-1} + \frac{1}{-A-1} = -\frac{2}{A+1}$$

And the kernel becomes: (the 1 drops out by integration)

$$\mathcal{K} = -\frac{1+e^{2p(t_0-t)}}{e^{2p(t_0-t)}+1-2\cos p\zeta\, e^{p(t_0-t)}} + \frac{e^{-2p(t+t_0-2t_2)}-1}{e^{-2p(t+t_0-2t_2)}+1-2\cos p\zeta\, e^{-p(t+t_0-2t_2)}}$$

$$-\frac{e^{2p(t+t_0-2t_1)}-1}{e^{2p(t+t_0-2t_1)}+1-2\cos p\zeta\, e^{p(t+t_0-2t_1)}}$$

For $\zeta=0$ $$\frac{\mathcal{K}}{2} = \frac{1}{1-e^{p(t_0-t)}} + \frac{e^{p(t+t_0-2t_2)}}{1-e^{p(t+t_0-2t_2)}} - \frac{e^{-p(t+t_0-2t_1)}}{1-e^{-p(t+t_0-2t_1)}}$$

or with terms of order $e^{-2p(t_2-t_1)}$ dropping out:

$$\frac{\mathcal{K}}{2} = \frac{e^{pt}}{e^{pt}-e^{pt_0}} + \frac{e^{p(t+t_0-2t_2)}-e^{-p(t+t_0-2t_1)}}{1-e^{p(t+t_0-2t_2)}-e^{-p(t+t_0-2t_1)}}$$

If one may neglect $e^{-4p(t_2-t_1)}$ but not $e^{-2p(t_2-t_1)}$ one must add to the kernel for $\mathfrak{J}$, the terms:

$$\frac{e^{-2p(t_2-t_1)}\left[e^{p(t+t_0-2t_2)}-e^{-p(t+t_0-2t_1)}\right]}{1-e^{-2p(t_2-t_1)}\left[e^{p(t+t_0-2t_2)}+e^{-p(t+t_0-2t_1)}\right]} - \frac{e^{-2p(t_2-t_1)}\left[e^{p(t-t_0)}-e^{-p(t-t_0)}\right]}{1-e^{-2p(t_2-t_1)}\left[e^{p(t-t_0)}+e^{-p(t-t_0)}\right]}$$

Page (22) The Boundary equations are

$$F^1_{mp} J_0 + G^1_{mp} Y_0 - F^4_{mp} J_0 - G^4_{mp} Y_0 = 0$$

$$F^1_{mp} J_0' + G^1_{mp} Y_0' - F^4_{mp} J_0' - G^4_{mp} Y_0' = 0$$

$$F^1_{mp} J_1' + G^1_{mp} Y_1' = 0$$

$$- F^4_{mp} J_2' - G^4_{mp} Y_2' = 0$$

where $Y_1' = \frac{d}{d\mu} Y_{mp}(k\mu)\Big/_{\mu=\mu_1}$

The determinant is

$$\begin{vmatrix} J_0 & Y_0 & -J_0 & -Y_0 \\ J_0' & Y_0' & -J_0' & -Y_0' \\ J_1' & Y_1' & 0 & 0 \\ 0 & 0 & -J_2' & -Y_2' \end{vmatrix} = \begin{vmatrix} J_0 & Y_0 & 0 & 0 \\ J_0' & Y_0' & 0 & 0 \\ J_1' & Y_1' & J_1' & Y_1' \\ 0 & 0 & -J_2' & -Y_2' \end{vmatrix} = -\frac{1}{\mu_0}\left(J_2' Y_1' - J_1' Y_2' \right)$$

and if it is 0, the coefficients are:

$$F^1_{mp} = Y_1' \left(J_0' Y_2' - J_2' Y_0' \right)$$

$$G^1_{mp} = -J_1' \left(J_0' Y_2' - J_2' Y_0' \right)$$

$$F^4_{mp} = -Y_2' \left(J_0' Y_1' - J_1' Y_0' \right)$$

$$G^4_{mp} = J_2' \left(J_0' Y_1' - J_1' Y_0' \right)$$

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