RIEMANN ZEROS QUANTUM CHAOS FUNCTIONAL DETERMINANTS RIEMANN ZEROS AND A TRACE FORMULAE

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ABSTRACT: We study the relation between the Guzwiller Trace for a dynamical system and the Riemann-Weil trace formula for the Riemann zeros, using the Bohr-Sommerfeld quantization condition, the WKB rules and the fractional calculus we obtain a method to define implicitly a potential $f^{-1}(x)$ for a Hamiltonian in one dimension, we also apply this method to define a Hamiltonian whose energies are the square of the Riemann zeros (imaginary part) $E_n = \gamma_n^2$, also we show that for big 'x' the potential is very close to an exponential function.

In this paper and for simplicity we use units so $2m = 1 = \hbar$

- *Keywords:* = Riemann Hypothesis, WKB semiclassical approximation, Gutzwiller trace formula, Bohr-Sommerfeld quantization,exponential potential.

1. RIEMANN ZEROS AND TRACE FORMULAE

Given a Hamiltonian in one dimension plus boundary conditions on the real line $[0,\infty)$

$$Hy(x) = E_n y(x) = -\frac{\hbar^2}{2m}\frac{d^2 y(x)}{dx^2} + f(x)y(x) \qquad y(0) = 0 = y(\infty) \qquad (1.1)$$

Can we recover the potential $f(x)$ from spectral data ?, for example if we knew the Eigenvalue staircase of the problem (1.1) $N(E) = \sum_{n=0}^{\infty} H(E - E_n)$, then we could use the Bohr-Sommerfeld quantization condition, see [9] for our problem as

$$\begin{aligned} 2\pi\hbar\left(n + \frac{1}{2}\right) &= \iint_{R^2} dxdpH\left(E - H(qx,p)\right) = 2\sqrt{2m}\int_0^{a=a(E)} \sqrt{E_n - V(x)}dx = \\ & 2\sqrt{2m}\int_0^E \sqrt{E_n - x}\,\frac{df^{-1}}{dx} = \sqrt{2m}\sqrt{\pi}D_x^{-\frac{1}{2}} f^{-1}(x) \end{aligned} \qquad (1.2)$$

The number 'a' is a turning point where the momentum is $p=0$ so $f(a)=E$
The idea of the Borh-Sommerfeld quantization condition (1.2) is the following, we equate the smooth part of the spectral staircase to an integer plus 1/2

$$\langle N(E)\rangle = \frac{1}{2\pi\hbar}\iint_{R^2} dxdpH\left(E-H(qx,p)\right) = n+\frac{1}{2} \qquad H(x)=\begin{cases}1 & \text{x>0}\\ 0 & \text{x<0}\end{cases} \quad (1.3)$$

Here , inside (1.2) we have used the definitions of the fractional derivative and integral of order ½. [10] (for fractional calculus)

$$\frac{d^{\frac{1}{2}}f(x)}{dx^{\frac{1}{2}}} = \frac{1}{\Gamma(1/2)}\frac{d}{dx}\int_0^x \frac{dtf(t)}{\sqrt{x-t}} \qquad \frac{d^{-\frac{1}{2}}f(x)}{dx^{-\frac{1}{2}}} = \frac{1}{\Gamma(1/2)}\int_0^x dt\frac{f(t)}{\sqrt{x-t}} \quad (1.4)$$

Also for our Hamiltonian we have imposed boundary conditions on the half line $[0,\infty)$ so the Eigenfunctions $Hy_n(x)=E_n y(x)$ satisfy the boundary conditions $y_n(0)=0=y_n(\infty)$.

From (1.4) we obtain that the inverse of the potencial can be described implicitly in terms of the half-derivative of smooth part of the Eigenvalue staircase as the function $f^{-1}(x)=\sqrt{\frac{2\pi\hbar^2}{m}}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\langle N(x)\rangle$.

This result (4) can be improved with the aid of the Gutzwiller's trace formula for the density of states [6] , formula (1.42) valid (it is assumed trough all the paper) in the limit $\hbar \to 0$ for the Planck's constant.

$$\rho(E)=\langle\rho(E)\rangle+\frac{1}{\pi\hbar}\sum_{\gamma_p}\sum_{k=1}^{\infty}\frac{A_{\gamma_p} cos\left(\frac{k}{\hbar}S_{\gamma_p}(E)-\frac{\pi}{2}k\mu_{\gamma_p}\right)}{\left|\det\left(M_{\gamma_p}^k-1\right)\right|^{\frac{1}{2}}}+O(\hbar) \qquad E=\frac{p^2}{2m} \qquad (1.5)$$

With $S_\gamma(E)=\int_C pdx=\sqrt{2mE}=\sqrt{2m}pl_\gamma$ being the action over the closed orbit for the momentum, l_γ is the length of the closed orbit , μ_{γ_p} is a Maslov index and $\det\left(M_{\gamma_p}^k-1\right)$ is the determinant of the Monodromy Matrix, A_{γ_p} (see [6] for further references) are constants related to the orbits. Equation (1.5) is a better expression to evaluate the Eigenvalue staircase (by integration) since $\frac{dN(x)}{dx}=\rho(x)$, also from expression (1.5) we can obtain a trace formula

$$\sum_{n=0}^{\infty} h(p_n) = \int_0^{\infty} dp \langle \rho(p) \rangle h(p) + \frac{1}{\hbar} \sum_{\gamma_p} \sum_{k=1}^{\infty} \frac{A_{\gamma_p} F_{\gamma_p}^k [h] \left(\frac{kS(E)_{\gamma p}}{\hbar} - k\frac{\mu_{\gamma p}}{2} \right)}{\left| \det\left(M_{\gamma_p}^k - 1 \right) \right|^{\frac{1}{2}}} + O(\hbar) \quad (1.6)$$

$$F_{\gamma_p}^k [h](u) = \frac{1}{\pi} \int_0^{\infty} dp h(p) \cos\left(\frac{kp}{\hbar} - \frac{\pi}{2} \mu_{\gamma p} \right) \quad (1.7)$$

Since the energy is related to the momentum of the particle by $E = p^2$, then we must choose and even function of the momentum $h(p) = h(-p)$ so this test function may be also defined for negative 'p' , in both cases the trace formulae (1.5) and (1.6) are real for real values of the Energy.

For the EXACT Eigenvalue staircase the half derivative can be evaluated formally as $f^{-1}(x) = \sqrt{\frac{2\hbar^2}{m}} \sum_{n=0}^{\infty} \frac{H(E - p_n^2)}{\sqrt{E - p_n^2}}$ if we insert this function inside (1.6)

$$f^{-1}(x) = \sqrt{\frac{2\hbar^2 \pi}{m}} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \langle N(x) \rangle + \sqrt{\frac{2\pi}{m}} \sum_{\gamma_p} \sum_{k=1}^{\infty} \frac{A_{\gamma_p} J_0 \left(k \frac{S(x)_{\gamma p}}{\hbar} - \frac{\pi}{2} k \mu_{\gamma_p} \right)}{\left| \det\left(M_{\gamma_p}^k - 1 \right) \right|^{\frac{1}{2}}} + O(\hbar) \quad (1.8)$$

Where we have used inside (1.8) the representation for the zeroeth order Bessel function $\frac{1}{\pi} \int_0^x \frac{dt \cos(ut)}{\sqrt{x^2 - t^2}} = \frac{J_0(ux)}{2}$. In order to study the limit $x \to \infty$ inside (1.8) we can use the approximation for the Bessel function $J_0(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} \right) + O\left(\frac{1}{x} \right)$

If we took the fractional derivative operator $\sqrt{\frac{m}{2\hbar^2 \pi}} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}$ inside (1.8) we would obtain the trace formulae for the density of states (1.6) , this is deduced from the identity of the Bessel function $\sqrt{\pi} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} J_0\left(a\sqrt{x} \right) = \frac{\cos\left(a\sqrt{x} \right)}{\sqrt{x}}$ [10] , this identity can be easily proved by expanding both functions into a power series around the origin and taking the half-derivative on each term.

Equation (1.8) defined the potential function for the Hamiltonian inside (1.8) which depends on the fractional derivative of the Smooth part of the Eigenvalue staircase $\langle N(E) \rangle = \frac{1}{2\pi\hbar} \iint_{R^2} dx dp H\left(E - H(qx, p) \right)$ plus a correction due to the closed orbits of the dynamical system, this correction will turn to be very important for the case of the potential and the Hamiltonian which yield to the Riemann zeros.

A good example of the Trace formula (1.5) is for the case of the Eigenvalue problem $y(0)=0=y(\pi)$ $H=-\frac{d^2y(x)}{dx^2}=E_n y(x)$, in this case the density of states and the trace (1.5) is just the Poisson summatin formula $\sum_{m=-\infty}^{\infty} e^{2\pi ixm}=\sum_{m=-\infty}^{\infty}\delta\left(x-m\right)$, the smooth part of the Eigenvaue staircase is given by $\langle N(E)\rangle=\sqrt{E}$, since the energies of the problems are $E_n=n^2$, and the correction to the inverse of the potential due to the length of the periodic orbits is (in terms of the Besssel function) $\sum_{m=1}^{\infty} J_0\left(2\pi m\sqrt{x}\right)$

- *Riemann zeros and a potential:*

There is exist an analogue of the Gutzwiller's trace formula for the Riemann zeros, if we consider a dynamical system with the Maslov indices $e^{\frac{\pi}{2}k\mu_{\gamma_p}}=-1$ and length of the periodic orbits $S_\gamma(E)=\sqrt{E}\log p_n$ (prime numbers), see [7]

$$\sum_\gamma h(\gamma)=2h\left(\frac{i}{2}\right)-g(0)\ln\pi-2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}g(\ln n)+\frac{1}{2\pi}\int_{-\infty}^{\infty} ds h(s)\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{is}{2}\right) \quad (1.9)$$

Here, $g(k)=\frac{1}{2\pi}\int_0^{\infty} dx h(x)\cos(kx)=g(-k)$ h(x) and g(x) are test functions which form a Fourier transform pair and $\Lambda(n)=\begin{cases}\ln p & n=p^k\\ 0 & \text{otherwise}\end{cases}$ is the Mangoldt function , formula (1.9) gives the relationship between a sum over the imaginary part of the Riemann zeros and a sum over the primes and prime powers.

By analogy with the Trace formula (1.5) the imaginary part of the zeros are not energies but rather the momenta of a certain Hamiltonian , the energies of the Hamiltonian will be the square of the imaginary part for the Riemann zeros $E_n=\gamma_n^2$, if we do the same reasoning we did for the Gutzwiller trace and set $\hbar=2m=1$, then the potential which yields to the imaginary part of the Riemann Zeros is given by

$$f^{-1}(x)=2\sum_{n}\frac{H(x-\gamma_n^{2})}{\sqrt{x-\gamma_n^{2}}}=\frac{4H\left(x+\frac{1}{4}\right)}{\sqrt{4x+1}}-\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}J_0\left(\sqrt{x}\ln n\right)$$ x >0 (1.10)

$$\frac{1}{2\pi}\int_{-\sqrt{x}}^{\sqrt{x}}\frac{dr}{\sqrt{x-r^2}}\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{ir}{2}\right)-\ln\pi\right)$$

We can see inmediatly how the expressions (1.6) and (1.10) are connected, they both have a correction due to the length of the periodic orbits which includes the Bessel function term, in the case of the Riemann zeros , from the definition of Von Mangoldt function we have that the lenghts of the orbits are equal to the log of prime numbers (with repetition).

But what would happen for $x<0$?, due to the boundary condition $y(0)=0$ there is a infinite potential well at x=0 so the potential would be

$$f(x)=\begin{cases}\text{defined implicitly by formula (1.10) for x >0}\\ \infty \qquad \text{for x } \le 0\end{cases} \quad (1.11)$$

If we take the fractional derivative $\frac{1}{2\sqrt{\pi}}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}$ inside (1.10) and use the identities

$$\sqrt{\pi}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}J_0\left(a\sqrt{x}\right)=\frac{\cos\left(a\sqrt{x}\right)}{\sqrt{x}} \qquad \delta\left(f(x)\right)=\sum_{n}\frac{\delta\left(x-x_n\right)}{|f'(x_n)|} \quad (1.12)$$

We obtain the distributional Riemann-Weil trace formula, so the density of states of our Hamiltonian , with the potential defind implicitly inside (14) is just the Riemann-Weil trace formula (on the momentum variable) [7]

$$\begin{aligned}&\sum_{n=0}^{\infty}\delta\left(k-\gamma_n\right)+\sum_{n=0}^{\infty}\delta\left(k+\gamma_n\right)=\frac{1}{2\pi}\frac{\zeta}{\zeta}\left(\frac{1}{2}+ik\right)+\frac{1}{2\pi}\frac{\zeta'}{\zeta}\left(\frac{1}{2}-ik\right)-\frac{\ln\pi}{2\pi}\\&+\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+i\frac{k}{2}\right)\frac{1}{4\pi}+\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}-i\frac{k}{2}\right)\frac{1}{4\pi}+\delta\left(k-\frac{i}{2}\right)+\delta\left(k+\frac{i}{2}\right)=Tr\left\{\delta\left(E-H\right)\right\}\end{aligned} \quad (1.13)$$

Where we have used the Shokhotsky's formula representation for the delta function $-\frac{1}{\pi}\Im m\left(\frac{1}{x+i\varepsilon-a}\right)=\delta\left(x-a\right)$ with $a=\pm\frac{i}{2}$.

In case $x>>>1$, the smooth density of states can be well approximated by $\left\langle N(x)\right\rangle\approx\frac{\sqrt{x}}{2\pi}\ln\left(\frac{\sqrt{x}}{2\pi e}\right)$ so in this case the trace formula inside (1.10) becomes

$$f^{-1}(x) \approx \frac{1}{\sqrt{\pi}} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \left(\sqrt{x} \ln\left(\frac{\sqrt{x}}{2\pi e} \right) \right) + \frac{2}{\sqrt{\pi}} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \arg \zeta \left(\frac{1}{2} + i\sqrt{x} \right) + O\left(\frac{1}{\sqrt{x}} \right) \quad (1.14)$$

We have used inside (1.14) the zeta regularization [ ] for the Dirichlet series $\frac{\zeta'}{\zeta}\left(\frac{1}{2}+is\right) =_{reg} -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\frac{1}{2}+is}}$ so in this case the term

$$\frac{2}{\sqrt{\pi}} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \arg \zeta \left(\frac{1}{2} + i\sqrt{x} \right) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} J_0\left(\sqrt{x} \ln n \right) \quad (1.15)$$

Unfortunately the expressions for the inverse of the potential (1.10) and (1.14) can not be analytically invert (we will study the asymptotic behaviour in the nex section), however any function can be numerically inverted so the need only to reflect every point of $f^{-1}(x)$ through the line $y = x$ to get $f(x)$

2. A TOY MODEL OF RIEMANN ZEROS WITH AN EXPONENTIAL POTENTIAL

For big energies the Eigenvalue staircase for a Hamiltonian whose energies are the square of the Riemann zeros is given by

$$N(x) = \frac{\sqrt{x}}{2\pi} \ln\left(\frac{\sqrt{x}}{2\pi e} \right) + \frac{7}{8} + O\left(\frac{1}{\sqrt{x}} \right) + \frac{1}{\pi} \arg \zeta \left(\frac{1}{2} + i\sqrt{x} \right) \quad (2.1)$$

Then the smooth part is given approximately by $N_{smooth}(E) = \frac{\sqrt{E}}{2\pi} \ln\left(\frac{\sqrt{E}}{2\pi e} \right)$

To compute the half-derivative we use the representation for the logarithm $\ln(x) \approx \frac{x^{\varepsilon}-1}{\varepsilon}$ $\varepsilon \to 0$, $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ in this case we get

$$f^{-1}(x) \approx \frac{\left(4\pi^2 e^2\right)^{-\varepsilon/2} A(\varepsilon) x^{\varepsilon/2} - B}{\sqrt{\pi}\varepsilon} \qquad f(x) \approx 4\pi^2 e^2 \left(\frac{\varepsilon\sqrt{\pi} x + B}{A(\varepsilon)} \right)^{\frac{2}{\varepsilon}} \quad (2.2)$$

The constants are $A(\varepsilon)=\dfrac{\Gamma\left(\frac{3+\varepsilon}{2}\right)}{\Gamma\left(1+\frac{\varepsilon}{2}\right)}$ and $B=\Gamma\left(\frac{3}{2}\right)=\frac{\sqrt{\pi}}{2}$, and we have used

the property of the half-derivative of powers of 'x' $\dfrac{d^{\frac{1}{2}}x^n}{dx^{\frac{1}{2}}}=\dfrac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}x^{n-\frac{1}{2}}$,

(Nishimoto [10])

The last expression inside (2.2) is equal to an exponential , so for the case of a Hamiltonian with boundary conditions $y(0)=0=y(\infty)$ and that gives only the 'smooth ' part of the staircase of the zeros via the WKB approximation $2\int_0^{a=a(E)}\sqrt{E_n-\lambda e^{4x}}\,dx \approx N_{smooth}(E)=\frac{\sqrt{E}}{2\pi}\ln\left(\frac{\sqrt{E}}{2\pi e}\right)$ the potential is the following

$$f_0(x)=\begin{cases}4\pi^2\exp\left(2-\dfrac{2}{\sqrt{\pi}}\dfrac{\partial G(s)}{\partial s}\Big|_{s=0}\right)e^{4x} & \text{x >0} \\ \infty & x\le 0\end{cases} \qquad \lambda=4\pi^2e^2\exp\left(-\frac{2}{\sqrt{\pi}}\frac{\partial F(s)}{\partial s}\Big|_{s=0}\right)$$

(2.3)

With $G(s)=\dfrac{\Gamma\left(\frac{3}{2}+s\right)}{\Gamma(1+s)}$. So our toy model or approximate model for the Riemann zeros is given by the Hamiltonian on the half line $[0,\infty)$

$$E_ny(x)=-\frac{d^2y(x)}{dx^2}+\lambda e^{4x}y(x) \quad y(0)=0=y(\infty) \;\; E_n\approx\gamma_n^2 \quad \zeta\left(\frac{1}{2}+i\sqrt{E_n}\right)=0 \quad (2.4)$$

And $\lambda\approx 16\pi^2$ has been previously defined inside (2.3) .

An advantage of this model is that is exactly solvable (Amore,[1]), if we impose boundary conditions on the half line $[0,\infty)$ the quantization conditions for the energies are

$$0=C_1J_\mu\left(\frac{\sqrt{-\lambda}}{2}\right)+C_2J_{-\mu}\left(\frac{\sqrt{-\lambda}}{2}\right) \qquad \mu=\frac{i\sqrt{E_n}}{2} \qquad C_1,C_2\in C \qquad (2.5)$$

For any value of C_1,C_2 , condition (2.5) is fulfilled if $J_{\pm i\sqrt{\frac{E}{4}}}\left(\frac{\sqrt{-\lambda}}{2}\right)=0$

So the energies appear inside the index of a Bessel function, in general this problem may be generalized to arbitrary boundary conditons on the half line

$[u_0,\infty)$ for some real u_0 so $y(u_0)=0=y(\infty)$, if we choose also that $C_1=-C_2$ then we may choose the u_0 (if such u_0 exists) so

$$1=\frac{J_{\frac{ix}{2}}\left(\sqrt{-\frac{\lambda}{4}e^{2u_0}}\right)}{J_{\frac{-ix}{2}}\left(\sqrt{-\frac{\lambda}{4}e^{2u_0}}\right)}\approx\frac{\Gamma\left(\frac{1}{4}-\frac{ix}{2}\right)\zeta\left(\frac{1}{2}-ix\right)}{\Gamma\left(\frac{1}{4}+\frac{ix}{2}\right)\zeta\left(\frac{1}{2}+ix\right)}\pi^{ix} \qquad (2.6)$$

Equation (2.6) is just the functional equation for the Riemann Zeta function on the critical line $s=\frac{1}{2}+ix$, this means that the quantization condition (2.5) may give the Riemann zeros and it is equivalent to the functional equation for the Riemann zeta function

3. ZETA REGULARIZATION FOR FUNCTIONAL DETERMINANTS AND THE RIEMANN XI- FUNCTION $\xi(s)$

Berry [5] has suggested the following Quantization condition for the Energies of a Quantum system

$$\Delta(E)=\det\left(E-H\right)=0 \qquad (3.1)$$

Here $\Delta(E)=\prod_{n=0}^{\infty}\left(E-E_n\right)$ is the functional determinant of the system , for example for the Harmonic oscillator and the infinite potential well we have

$$\prod_{n=0}^{\infty}\left(1-\frac{E}{n^2}\right)=\frac{\sin\left(\pi\sqrt{E}\right)}{\pi\sqrt{E}} \qquad \prod_{n=0}^{\infty}\left(1-\frac{E}{n+\frac{1}{2}}\right)e^{\frac{E}{n+1}}=\frac{\cos\left(\pi E\right)}{\sqrt{\pi}}\frac{\Gamma\left(E+\frac{1}{2}\right)}{e^{-\gamma E}} \qquad (3.2)$$

Here $\gamma=0.57721..$ is the Euler-Mascheroni constant.
In order to define a Functional determinant, one of the best method to use is the Zeta regularization [8] the zeta regularized determinant for an operator T having real eigenvalues $\{\lambda_n\}$ is

$$\det(T)=\prod_{n=0}^{\infty}\lambda_n=\exp\left(-\frac{\partial Z(0)}{\partial s}\right) \qquad Z(s)=\sum_{n=0}^{\infty}\frac{1}{\lambda_n^s} \qquad \partial_s Z(0)=-\sum_{n=0}^{\infty}\frac{\ln\lambda_n}{1} \qquad (3.3)$$

Here Z(s) is the spectral zeta function associated to the operator T , in many cases we do not know this function so we need to use the representation

$$Z(s)=\frac{1}{\Gamma(s)}\int_0^\infty dt\Theta(t)t^{s-1} \qquad \Theta(t)=\sum_{n=0}^{\infty}e^{-tE_n} \quad (3.4)$$

This Theta function is defined only for t >0 , for our case with the potential defined implicitly by the equation $f^{-1}(x)=\sum_{n=0}^{\infty}\frac{H(x-\gamma_n^2)}{\sqrt{x-\gamma_n^2}}=2\sqrt{\pi}\frac{d^{\frac{1}{2}}N(x)}{dx^{\frac{1}{2}}}$ we can use the Semiclassical approximation for the Theta function

$$\Theta_{WKB}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dx\int_0^\infty dpe^{-tp^2-tf(x)}=\frac{1}{\sqrt{\pi t}}\int_0^\infty dxe^{-tf(x)}=\sqrt{\frac{t}{\pi}}\int_0^\infty dre^{-tr}\frac{f^{-1}(r)}{dr} \quad (3.5)$$

From the properties of the Laplace transform $\int_0^\infty dtf(t)t^k=(-1)^k\frac{\partial F}{\partial s}$ $\int_0^\infty dtf(t)=F(s)$ $k=\frac{1}{2}$ and from the identity $\int_{-\infty}^{\infty}dxe^{-ax^2}=\sqrt{\frac{\pi}{a}}$ the last integral inside (3.5) is equal to $\sum_{n=0}^{\infty}e^{-t\gamma_n^2}$ with $\zeta\left(\frac{1}{2}+i\gamma_n\right)=0$. If we take the Mellin transform $\frac{1}{\Gamma(s)}\int_0^\infty dt\Theta_{WKB}(t)t^{s-1}e^{-tE^2}$ inside (3.4) we obtain $\sum_{n=0}^{\infty}\frac{1}{\left(E^2+\gamma_n^2\right)^s}=Z(E,s)$,

and the sum is extended to the positive imaginary part of the Riemann zeros , from ths last expression we can define the Riemann Xi-function on the critical line as the quotien of 2 functional determinants

$$\frac{\det\left(H-E\right)}{\det(H)}=\frac{\prod_{n=0}^{\infty}(\gamma_n^2-E)}{\prod_{n=0}^{\infty}\gamma_n^2}=\prod_{n=0}^{\infty}\left(1-\frac{E}{E_n}\right)=\frac{\xi\left(1/2+i\sqrt{E}\right)}{\xi\left(1/2\right)}=\exp\left(-\frac{d}{ds}Z(s,E)\Big|_{s=0}+\frac{d}{ds}Z(s,0)\Big|_{s=0}\right)$$

(3.6)

So, from the expression (3.6) one observes that the functional determinant of a Hamiltonian $H=p^2+f(x)$ with a potential defined implicitly by

$$f^{-1}(x)=\sum_{n=0}^{\infty}\frac{H(x-\gamma_n^2)}{\sqrt{x-\gamma_n^2}}=2\sqrt{\pi}\frac{d^{\frac{1}{2}}N(x)}{dx^{\frac{1}{2}}}=\frac{2}{\sqrt{\pi}}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\arg\zeta\left(\frac{1}{2}+i\sqrt{x}\right) \quad (3.7)$$

Is exactly to the Riemann Xi-function on the critical line, hence Riemann Hypothesis must be true, since the Riemann xi-function is the Charasteristic Polynomial (Functional determinant) of an Hermitian operator in one dimension

Appendix A: Useful formulae:

For a Polynomial or a power function x^m and for the Heaviside step function, the fractional derivative of any order can be computed as follows

$$D^{\alpha}x^{m} = \frac{\Gamma(m+1)}{\Gamma\left(m-\alpha+1\right)}x^{m-\alpha} \qquad D^{\alpha}H(x) = \frac{H(x)}{\Gamma(1-\alpha)}\frac{1}{x^{\alpha}} \qquad D^{\alpha} = \frac{d^{\alpha}}{dx^{\alpha}} \quad \text{(A.1)}$$

If we expanded the Bessel function and the cosine function into a power series around x =0

$$J_0(\sqrt{x}) = \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^{2n}n!n!} \qquad \frac{\cos(\sqrt{x})}{\sqrt{x}} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n-\frac{1}{2}}}{(2n)!} \quad \text{(A.2)}$$

We could prove inmediatly the identity $\sqrt{\pi D_x}J_0\left(a\sqrt{x}\right) = \dfrac{\cos\left(a\sqrt{x}\right)}{\sqrt{x}}$

For the case of the function $\sqrt{x}\ln(x)$ the evaluation of the fractional derivative is a bit harder , [11] and it is defined

$$\frac{d^{\alpha}}{dx^{\alpha}}\left(\sqrt{x}\ln(x)\right) = z^{\frac{1}{2}-\alpha}.\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-\alpha\right)}.\left(\ln(x)+\Psi\left(\frac{3}{2}\right)-\Psi\left(\frac{3}{2}-\alpha\right)\right) \quad \text{(A.3)}$$

Here , we have used the definition of the Digamma function $\Psi(x) = \dfrac{\Gamma'}{\Gamma}(x)$.

For a Quantum system, the quantization condition in terms of the EXACT Eigenvalue staircase can be formulated

$$N(E) = n+\frac{1}{2} \quad \text{or} \quad \cos\left(\pi N(E)\right) = 0 \qquad N(E) = \sum_{n=0}^{\infty}H(E-E_n) \quad \text{(A.4)}$$

The Bohr-Sommerfeld quantization condition for the system come from approximating the exact sum over Energies by an integral over the Phase Space and then integrating over the momentum variable

$$n+\frac{1}{2} = \sum_{n=0}^{\infty}H(E-E_n) \approx \frac{1}{2\pi\hbar}\iint_{R^2}dxdpH(E-H(x,p)) = \frac{1}{\pi\hbar}\int_0^a dx\sqrt{2m\left(E-f(x)\right)} \quad \text{(A.4)}$$

With the Hamiltonian $H=\frac{p^2}{2m}+f(x)$ and $f(a)=E$ a turning point of the system where the momentum of the particle is 0

From the semi-group property for the fractional derivatives $D_x^a D_x^b = D_x^{a+b}$ for positive a and be , if we take the half derivative of the potential

$$\frac{D_x^{\frac{1}{2}}}{2\sqrt{\pi}} f^{-1}(x)=\frac{1}{2\sqrt{\pi}} D_x^{\frac{1}{2}}\left(2\sqrt{\pi} D_x^{\frac{1}{2}} N(x)\right)=D_x^{\frac{1}{2}+\frac{1}{2}} N(x)=\rho(x)=\sum_{n=0}^{\infty}\delta\left(x-\gamma_n^2\right) \quad \text{(A.5)}$$

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