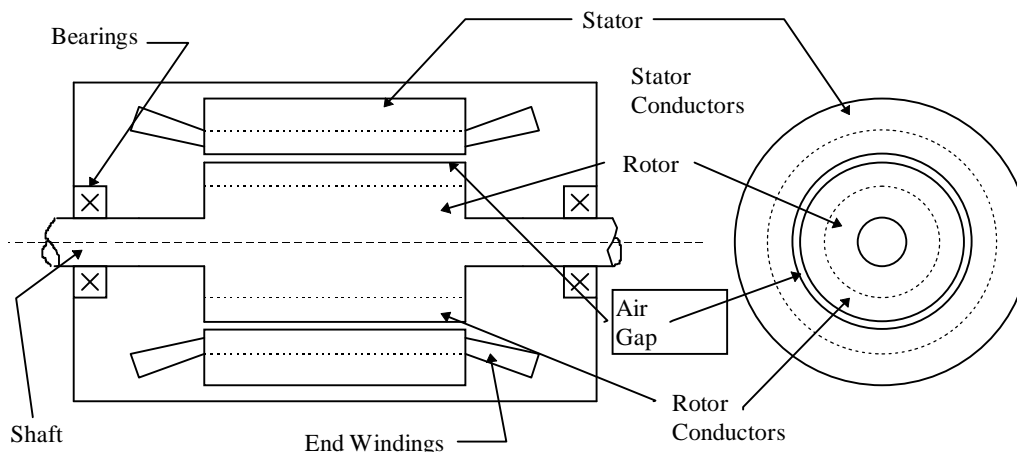# 1   Introduction

Figure 1: Form of Electric Machine

This section of notes discusses some of the fundamental processes involved in electric machinery. In the section on energy conversion processes we examine the two major ways of estimating electromagnetic forces: those involving thermodynamic arguments (conservation of energy) and field methods (Maxwell's Stress Tensor). But first it is appropriate to introduce the topic by describing a notional rotating electric machine.

Electric machinery comes in many different types and a strikingly broad range of sizes, from those little machines that cause cell 'phones and pagers to vibrate (yes, those *are* rotating electric machines) to turbine generators with ratings upwards of a Gigawatt. Most of the machines with which we are familiar are rotating, but linear electric motors are widely used, from shuttle drives in weaving machines to equipment handling and amusement park rides. Currently under development are large linear induction machines to be used to launch aircraft. It is our purpose in this subject to develop an analytical basis for understanding how all of these different machines work. We start, however, with a picture of perhaps the most common of electric machines.

2   Electric Machine Description:

Figure 1 is a cartoon drawing of a conventional induction motor. This is a very common type of electric machine and will serve as a reference point. Most other electric machines operate in

a fashion which is the same as the induction machine or which differ in ways which are easy to reference to the induction machine.

Most (but not all!) machines we will be studying have essentially this morphology. The rotor of the machine is mounted on a shaft which is supported on some sort of bearing(s). Usually, but not always, the rotor is inside. I have drawn a rotor which is round, but this does not need to be the case. I have also indicated rotor conductors, but sometimes the rotor has permanent magnets either fastened to it or inside, and sometimes (as in Variable Reluctance Machines) it is just an oddly shaped piece of steel. The stator is, in this drawing, on the outside and has windings. With most of the machines we will be dealing with, the stator winding is the armature, or electrical power input element. (In DC and Universal motors this is reversed, with the armature contained on the rotor: we will deal with these later).

In most electrical machines the rotor and the stator are made of highly magnetically permeable materials: steel or magnetic iron. In many common machines such as induction motors the rotor and stator are both made up of thin sheets of silicon steel. Punched into those sheets are slots which contain the rotor and stator conductors.

Figure 2 is a picture of part of an induction machine distorted so that the air-gap is straightened out (as if the machine had infinite radius). This is actually a convenient way of drawing the machine and, we will find, leads to useful methods of analysis.
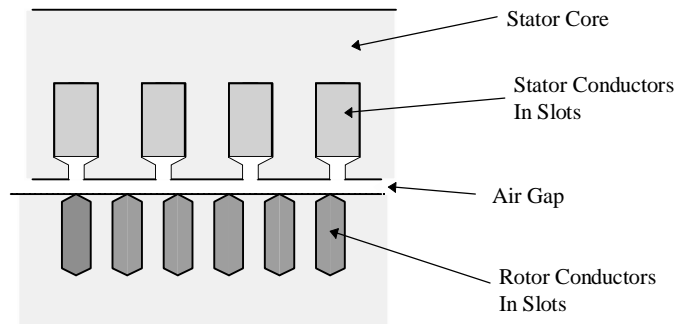

Figure 2: Windings in Slots

What is important to note for now is that the machine has an air gap g which is relatively small (that is, the gap dimension is much less than the machine radius r). The air-gap also has a physical length l. The electric machine works by producing a shear stress in the air-gap (with of course side effects such as production of "back voltage"). It is possible to define the average air-gap shear stress, which we will refer to as τ. Total developed torque is force over the surface area times moment (which is rotor radius):

$$T = 2\pi r^2 \ell < \tau >$$

Power transferred by this device is just torque times speed, which is the same as force times

surface velocity, since surface velocity is $u = r\Omega$:

$$P_m = \Omega T = 2\pi r \ell < \tau > u$$

If we note that active rotor volume is , the ratio of torque to volume is just:

$$\frac{T}{V_r} = 2 < \tau >$$

Now, determining what can be done in a volume of machine involves two things. First, it is clear that the volume we have calculated here is not the whole machine volume, since it does not include the stator. The actual estimate of total machine volume from the rotor volume is actually quite complex and detailed and we will leave that one for later. Second, we need to estimate the value of the useful average shear stress. Suppose both the radial flux density Br and the stator surface current density Kz are sinusoidal flux waves of the form:

$$B_r = \sqrt{2}B_0 \cos(p\theta - \omega t)$$

$$K_z = \sqrt{2}K_0 \cos(p\theta - \omega t)$$

Note that this assumes these two quantities are exactly in phase, or oriented to ideally produce torque, so we are going to get an "optimistic" bound here. Then the average value of surface traction is:

$$< \tau >= \frac{1}{2\pi} \int_0^{2\pi} B_r K_z d\theta = B_0 K_0$$

The magnetic flux density that can be developed is limited by the characteristics of the magnetic materials (iron) used. Current densities are a function of technology and are typically limited by how much effort can be put into cooling and the temperature limits of insulating materials. In practice, the range of shear stress encountered in electric machinery technology is not terribly broad: ranging from a few kPa in smaller machines to about 100 kPa in very large, well cooled machines.

It is usually said that electric machines are *torque* producing devices, meaning tht they are defined by this shear stress mechanism and by physical dimensions. Since power is torque times rotational speed, high power density machines necessarily will have high shaft speeds. Of course there are limits on rotational speed as well, arising from centrifugal forces which limit tip velocity.

Our first step in understanding how electric machinery works is to understand the mechanisms which produce forces of electromagnetic origin.

3 Energy Conversion Process:

In a motor the energy conversion process can be thought of in simple terms. In "steady state", electric power input to the machine is just the sum of electric power inputs to the different phase terminals:

$$P_e = \sum_i v_i i_i$$

Mechanical power is torque times speed:
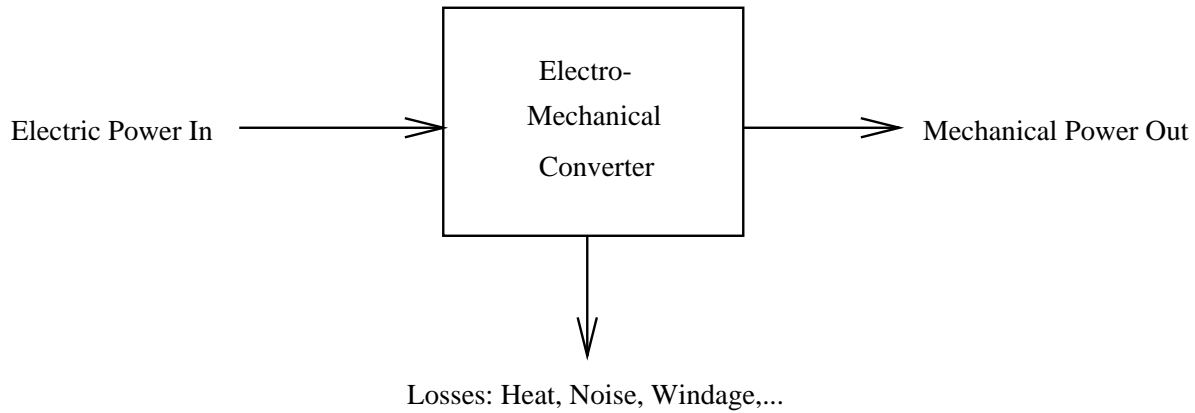
$$P_m = T\Omega$$

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Figure 3: Energy Conversion Process

And the sum of the losses is the difference:

$$P_d = P_e - P_m$$

It will sometimes be convenient to employ the fact that, in most machines, dissipation is small enough to approximate mechanical power with electrical power. In fact, there are many situations in which the loss mechanism is known well enough that it can be idealized away. The "thermodynamic" arguments for force density take advantage of this and employ a "conservative" or lossless energy conversion system.

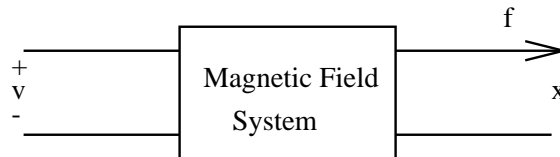## 3.1   Energy Approach to Electromagnetic Forces:

Figure 4: Conservative Magnetic Field System

To start, consider some electromechanical system which has two sets of "terminals", electrical and mechanical, as shown in Figure 4. If the system stores energy in magnetic fields, the energy stored depends on the *state* of the system, defined by (in this case) two of the identifiable variables: flux (λ), current (i) and mechanical position (x). In fact, with only a little reflection, you should be able to convince yourself that this state is a single-valued function of two variables and that the energy stored is independent of how the system was brought to this state.

Now, all electromechanical converters have loss mechanisms and so are not themselves conservative. However, the magnetic field system that produces force is, in principle, conservative in the

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sense that its state and stored energy can be described by only two variables. The "history" of the system is not important.

It is possible to chose the variables in such a way that electrical power *into* this conservative system is:

$$P^e = vi = i\frac{d\lambda}{dt}$$

Similarly, mechanical power *out of* the system is:

$$P^m = f^e\frac{dx}{dt}$$

The difference between these two is the rate of change of energy stored in the system:

$$\frac{dW_m}{dt} = P^e - P^m$$

It is then possible to compute the change in energy required to take the system from one state to another by:

$$W_m(a) - W_m(b) = \int_b^a i d\lambda - f^e dx$$

where the two states of the system are described by $a = (\lambda_a, x_a)$ and $b = (\lambda_b, x_b)$

If the energy stored in the system is described by two state variables, λ and x, the *total differential* of stored energy is:

$$dW_m = \frac{\partial W_m}{\partial \lambda}d\lambda + \frac{\partial W_m}{\partial x}dx$$

and it is also:

$$dW_m = i d\lambda - f^e dx$$

So that we can make a direct equivalence between the derivatives and:

$$f^e = -\frac{\partial W_m}{\partial x}$$

In the case of rotary, as opposed to linear, motion, torque T^e takes the place of force f^e and angular displacement θ takes the place of linear displacement x. Note that the product of torque and angle has the same units as the product of force and distance (both have units of work, which in the International System of units is *Newton-meters* or *Joules*.

In many cases we might consider a system which is electricaly *linear*, in which case inductance is a function only of the mechanical position x.

$$\lambda(x) = L(x)i$$

In this case, assuming that the energy integral is carried out from $\lambda = 0$ (so that the part of the integral carried out over x is zero),

$$W_m = \int_0^\lambda \frac{1}{L(x)}\lambda d\lambda = \frac{1}{2}\frac{\lambda^2}{L(x)}$$

This makes

$$f^e = -\frac{1}{2}\lambda^2\frac{\partial}{\partial x}\frac{1}{L(x)}$$

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Note that this is numerically equivalent to

$$f^e = -\frac{1}{2}i^2\frac{\partial}{\partial x}L(x)$$

This is true *only* in the case of a linear system. Note that substituting $L(x)i = \lambda$ too early in the derivation produces erroneous results: in the case of a linear system it produces a sign error, but in the case of a nonlinear system it is just wrong.

3.1.1 Example: simple solenoid

Consider the magnetic actuator shown in cartoon form in Figure 5. The actuator consists of a circular rod of ferromagnetic material (very highly permeable) that can move axially (the x-direction) inside of a stationary piece, also made of highly permeable material. A coil of N turns carries a current I. The rod has a radius R and spacing from the flat end of the stator is the variable dimension x. At the other end there is a radial clearance between the rod and the stator g. Assume $g \ll R$. If the axial length of the radial gaps is $\ell = R/2$, the area of the radial gaps is the same as the area of the gap between the rod and the stator at the variable gap.
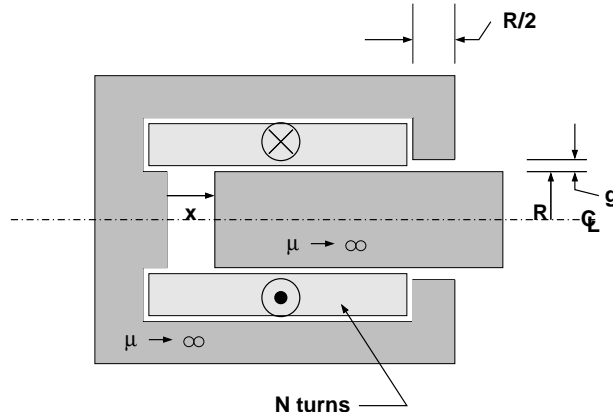

Figure 5: Solenoid Actuator

The permeances of the variable width gap is:

$$\mathcal{P}_1 = \frac{\mu_0\pi R^2}{x}$$

and the permeance of the radial clearance gap is, if the gap dimension is small compared with the radius:

$$\mathcal{P}_2 = \frac{2\mu_0\pi R\ell}{g} = \frac{\mu_0\pi R^2}{g}$$

The inductance of the coil system is:

$$L = \frac{N^2}{\mathcal{R}_1 + \mathcal{R}_2} = N^2\frac{\mathcal{P}_1\mathcal{P}_2}{\mathcal{P}_2 + \mathcal{P}_2} = \frac{\mu_0\pi R^2 N^2}{x + g}$$

Magnetic energy is:

$$W_m = \int_0^{\lambda_0} i \, d\lambda = \frac{1}{2} \frac{\lambda^2}{L(x)} = \frac{\lambda_0^2}{2} \frac{x+g}{\mu_0 \pi R^2 N^2}$$

And then, of course, force of electric origin is:

$$f^e = -\frac{\partial W_m}{\partial x} = -\frac{\lambda_0^2}{2} \frac{d}{dx} \frac{1}{L(x)}$$

Here that is easy to carry out:

$$\frac{d}{dx} \frac{1}{L} = \frac{1}{\mu_0 \pi R^2 N^2}$$

So that the force is:

$$f^e(x) = -\frac{\lambda_0^2}{2} \frac{1}{\mu_0 \pi R^2 N^2}$$

Given that the system is to be excited by a current, we may at this point substitute for flux:

$$\lambda = L(x)i = \frac{\mu_0 \pi R^2 N i}{x+g}$$

and then total force may be seen to be:

$$f^e = -\frac{\mu_0 \pi R^2 N^2}{(x+g)^2} \frac{i^2}{2}$$

The force is 'negative' in the sense that it tends to reduce x, or to close the gap.

3.1.2 Multiply Excited Systems

There may be (and in most electric machine applications there will be) more than one source of electrical excitation (more than one coil). In such systems we may write the conservation of energy expression as:

$$dW_m = \sum_k i_k d\lambda_k - f^e dx$$

which simply suggests that electrical input to the magnetic field energy storage is the sum (in this case over the index k) of inputs from each of the coils. To find the total energy stored in the system it is necessary to integrate over all of the coils (which may and in general will have mutual inductance).

$$W_m = \int \underline{i} \cdot d\underline{\lambda}$$

Of course, if the system is *conservative*, $W_m(\lambda_1, \lambda_2, \ldots, x)$ is uniquely specified and so the actual path taken in carrying out this integral will not affect the value of the resulting energy.

3.1.3   Coenergy

We often will describe systems in terms of inductance rather than its reciprocal, so that current, rather than flux, appears to be the relevant variable. It is convenient to derive a new energy variable, which we will call *co-energy*, by:

$$W'_m = \sum_i \lambda_i i_i - W_m$$

and in this case it is quite easy to show that the energy differential is (for a single mechanical variable) simply:

$$dW'_m = \sum_k \lambda_k di_k + f^e dx$$

so that force produced is:

$$f_e = \frac{\partial W'_m}{\partial x}$$
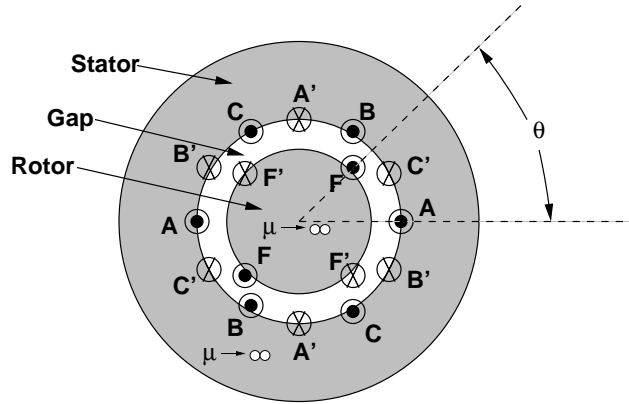
3.2   Example: Synchronous Machine

Figure 6: Cartoon of Synchronous Machine

Consider a simple electric machine as pictured in Figure 6 in which there is a single winding on a rotor (call it the *field* winding and a polyphase armature with three identical coils spaced at uniform locations about the periphery. We can describe the flux linkages as:

$$
\begin{aligned}
\lambda_a &= L_a i_a + L_{ab} i_b + L_{ab} i_c + M\cos(p\theta) i_f \\
\lambda_b &= L_{ab} i_a + L_a i_b + L_{ab} i_c + M\cos(p\theta - \frac{2\pi}{3}) i_f \\
\lambda_c &= L_{ab} i_a + L_{ab} i_b + L_a i_c + M\cos(p\theta + \frac{2\pi}{3}) i_f \\
\lambda_f &= M\cos(p\theta) i_a + M\cos(p\theta - \frac{2\pi}{3}) i_b + M\cos(p\theta + \frac{2\pi}{3}) + L_f i_f
\end{aligned}
$$

It is assumed that the flux linkages are sinusoidal functions of rotor position. As it turns out, many electrical machines work best (by many criteria such as smoothness of torque production)

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if this is the case, so that techniques have been developed to make those flux linkages very nearly sinusoidal. We will see some of these techniques in later chapters of these notes. For the moment, we will simply assume these dependencies. In addition, we assume that the rotor is magnetically 'round', which means the stator self inductances and the stator phase to phase mutual inductances are not functions of rotor position. Note that if the phase windings are identical (except for their angular position), they will have identical self inductances. If there are three uniformly spaced windings the phase-phase mutual inductances will all be the same.

Now, this system can be simply described in terms of coenergy. With multiple excitation it is important to exercise some care in taking the coenergy integral (to ensure that it is taken over a valid path in the multi-dimensional space). In our case there are actually five dimensions, but only four are important since we can position the rotor with all currents at zero so there is no contribution to coenergy from setting rotor position. Suppose the rotor is at some angle θ and that the four currents have values i_{a0}, i_{b0}, i_{c0} and i_{f0}. One of many correct path integrals to take would be:

$$
\begin{aligned}
W'_m \; = \; & \int_0^{i_{a0}} L_a i_a di_a \\
& + \int_0^{i_{b0}} \left(L_{ab} i_{a0} + L_a i_b \right) di_b \\
& + \int_0^{i_{c0}} \left(L_{ab} i_{a0} + L_{ab} i_{b0} + L_a i_c \right) di_c \\
& + \int_0^{i_{f0}} \left(M \cos(p\theta) i_{a0} + M \cos(p\theta - \frac{2\pi}{3}) i_{b0} + M \cos(p\theta + \frac{2\pi}{3}) i_{c0} + L_f i_f \right) di_f
\end{aligned}
$$

The result is:

$$
\begin{aligned}
W'_m \; = \; & \frac{1}{2} L_a \left(i_{a0}^2 + i_{b0}^2 + i_{co}^2 \right) + L_{ab} \left(i_{ao} i_{b0} + i_{ao} i_{c0} + i_{co} i_{b0} \right) \\
& + M i_{f0} \left(i_{a0} \cos(p\theta) + i_{b0} \cos(p\theta - \frac{2\pi}{3}) + i_{c0} \cos(p\theta + \frac{2\pi}{3}) \right) + \frac{1}{2} L_f i_{f0}^2
\end{aligned}
$$

Since there are no variations of the stator inductances with rotor position θ, torque is easily given by:

$$
T_e = \frac{\partial W'_m}{\partial \theta} = -p M i_{f0} \left(i_{a0} \sin(p\theta) + i_{b0} \sin(p\theta - \frac{2\pi}{3}) + i_{co} \sin(p\theta + \frac{2\pi}{3}) \right)
$$

3.2.1 Current Driven Synchronous Machine

Now assume that we can drive this thing with currents:

$$
\begin{aligned}
i_{a0} & = I_a \cos \omega t \\
i_{b0} & = I_a \cos \omega \left(t - \frac{2\pi}{3} \right) \\
i_{c0} & = I_a \cos \omega \left(t + \frac{2\pi}{3} \right) \\
i_{f0} & = I_f
\end{aligned}
$$

and assume the rotor is turning at *synchronous* speed:

$$p\theta = \omega t + \delta_i$$

Noting that $\cos x \sin y = \frac{1}{2} \sin(x - y) + \frac{1}{2} \sin(x + y)$, we find the torque expression above to be:

$$
\begin{aligned}
T_e \quad &= -pMI_aI_f \quad \left(\frac{1}{2} \sin \delta_i + \frac{1}{2} \sin \left(2\omega t + \delta_i \right) \right) \\
&+ \left(\frac{1}{2} \sin \delta_i + \frac{1}{2} \sin \left(2\omega t + \delta_i - \frac{4\pi}{3} \right) \right) \\
&+ \left(\frac{1}{2} \sin \delta_i + \frac{1}{2} \sin \left(2\omega t + \delta_i + \frac{4\pi}{3} \right) \right)
\end{aligned}
$$

The sine functions on the left add and the ones on the right cancel, leaving:

$$T_e = -\frac{3}{2}pMI_aI_f \sin \delta_i$$

And this is indeed one way of looking at a synchronous machine, which produces steady torque if the rotor speed and currents all agree on frequency. Torque is related to the *current torque angle* δ_i. As it turns out such machines are not generally run against current sources, but we will take up actual operation of such machines later.

4   Field Descriptions: Continuous Media

While a basic understanding of electromechanical devices is possible using the lumped parameter approach and the principle of virtual work as described in the previous section, many phenomena in electric machines require a more detailed understanding which is afforded by a continuum approach. In this section we consider a fields-based approach to energy flow using Poynting's Theorem and then a fields based description of forces using the Maxwell Stress Tensor. These techniques will both be useful in further analysis of what happens in electric machines.

4.1   Field Description of Energy Flow: Poyting's Theorem

Start with Faraday's Law:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

and Ampere's Law:

$$\nabla \times \vec{H} = \vec{J}$$

Multiplying the first of these by $\vec{H}$ and the second by $\vec{E}$ and taking the difference:

$$\vec{H} \cdot \nabla \times \vec{E} - \vec{E} \cdot \nabla \times \vec{H} = \nabla \cdot \left(\vec{E} \times \vec{H} \right) = -\vec{H} \cdot \frac{\partial \vec{B}}{dt} - \vec{E} \cdot \vec{J}$$

On the left of this expression is the divergence of electromagnetic energy flow:

$$\vec{S} = \vec{E} \times \vec{H}$$

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Here, $\vec{S}$ is the celebrated Poynting flow which describes power in an electromagnetic field sysstem. (The units of this quantity is watts per square meter in the International System). On the right hand side are two terms: $\vec{H} \cdot \frac{\partial \vec{B}}{dt}$ is rate of change of magnetic stored energy. The second term, $\vec{E} \cdot \vec{J}$ looks a lot like power dissipation. We will discuss each of these in more detail. For the moment, however, note that the divergence theorem of vector calculus yields:

$$\int_{\text{volume}} \nabla \cdot \vec{S} dv = \oiint \vec{S} \cdot \vec{n} da$$

that is, the volume integral of the divergence of the Poynting energy flow is the same as the Poynting energy flow over the surface of the volume in question. This integral becomes:

$$\oiint \vec{S} \cdot \vec{n} da = - \int_{\text{volume}} \left(\vec{E} \cdot \vec{J} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \right) dv$$

which is simply a realization that the total energy flow into a region of space is the same as the volume integral over that region of the rate of change of energy stored plus the term that *looks like* dissipation. Before we close this, note that, if there is motion of any material within the system, we can use the empirical expression for transformation of electric field between observers moving with respect to each other. Here the 'primed' frame is moving with respeect to the 'unprimed' frame with the velocity $\vec{v}$

$$\vec{E}' = \vec{E} + \vec{v} \times \vec{B}$$

This transformation describes, for example, the motion of a charged particle such as an electron under the influence of both electric and magnetic fields. Now, if we assume that there is material motion in the system we are observing and if we assign $\vec{v}$ to be the velocity of that material, so that $\vec{E}'$ is measured in a frame in which thre is no material motion (that is the frame of the material itself), the product of electric field and current density becomes:

$$\vec{E} \cdot \vec{J} = \left(\vec{E}' - \vec{v} \times \vec{B} \right) \cdot \vec{J} = \vec{E}' \cdot \vec{J} - \left(\vec{v} \times \vec{B} \right) \cdot \vec{J} = \vec{E}' \cdot \vec{J} + \vec{v} \cdot \left(\vec{J} \times \vec{B} \right)$$

In the last step we used the fact that in a scalar triple product the order of the scalar (dot) and vector (cross) products can be interchanged and that reversing the order of terms in a vector (cross) product simply changes the sign of that product. Now we have a ready interpretation for what we have calculated:

If the 'primed' coordinate system is actually the frame of material motion,

$$\vec{E}' \cdot \vec{J} = \frac{1}{\sigma} |\vec{J}|^2$$

which is easily seen to be dissipation and is positive definite if material conductivity σ is positive.

The last term is obviously conversion of energy from electromagnetic to mechanical form:

$$\vec{v} \cdot \left(\vec{J} \times \vec{B} \right) = \vec{v} \cdot \vec{F}$$

where we have now identified force density to be:

$$\vec{F} = \vec{J} \times \vec{B}$$

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This is the Lorentz Force Law, which describes the interaction of current with magnetic field to produce force. It is not, however, the complete story of force production in electromechanical systems. As we learned earlier, changes in geometry which affect magnetic stored energy can also produce force. Fortunately, a complete description of electromechanical force is possible using only magnetic fields and that is the topic of our next section.

4.2   Field Description of Forces: Maxwell Stress Tensor

Forces of electromagnetic origin, because they are transferred by electric and magnetic fields, are the result of those fields and may be calculated once the fields are known. In fact, if a surface can be established that fully encases a material body, the force on that body can be shown to be the integral of force density, or *traction* over that surface.

The traction τ derived by taking the cross product of surface current density and flux density on the air-gap surface of a machine (above) actually makes sense in view of the empirically derived Lorentz Force Law: Given a (vector) current density and a (vector) flux density. This is actually enough to describe the forces we see in many machines, but since electric machines have permeable magnetic material and since magnetic fields produce forces on permeable material even in the absence of macroscopic currents it is necessary to observe how force appears on such material. A suitable empirical expression for force density is:

$$\vec{F} = \vec{J} \times \vec{B} - \frac{1}{2} \left(\vec{H} \cdot \vec{H} \right) \nabla \mu$$

where $\vec{H}$ is the magnetic field intensity and μ is the permeability.

Now, note that current density is the curl of magnetic field intensity, so that:

$$
\begin{aligned}
\vec{F} &= \left(\nabla \times \vec{H} \right) \times \mu \vec{H} - \frac{1}{2} \left(\vec{H} \cdot \vec{H} \right) \nabla \mu \\
&= \mu \left(\nabla \times \vec{H} \right) \times \vec{H} - \frac{1}{2} \left(\vec{H} \cdot \vec{H} \right) \nabla \mu
\end{aligned}
$$

And, since:

$$\left(\nabla \times \vec{H} \right) \times \vec{H} = \left(\vec{H} \cdot \nabla \right) \vec{H} - \frac{1}{2} \nabla \left(\vec{H} \cdot \vec{H} \right)$$

force density is:

$$
\begin{aligned}
\vec{F} &= \mu \left(\vec{H} \cdot \nabla \right) \vec{H} - \frac{1}{2} \mu \nabla \left(\vec{H} \cdot \vec{H} \right) - \frac{1}{2} \left(\vec{H} \cdot \vec{H} \right) \nabla \mu \\
&= \mu \left(\vec{H} \cdot \nabla \right) \vec{H} - \nabla \left(\frac{1}{2} \mu \left(\vec{H} \cdot \vec{H} \right) \right)
\end{aligned}
$$

This expression can be written by components: the component of force in the i'th dimension is:

$$F_i = \mu \sum_k \left(H_k \frac{\partial}{\partial x_k} \right) H_i - \frac{\partial}{\partial x_i} \left(\frac{1}{2} \mu \sum_k H_k^2 \right)$$

The first term can be written as:

$$\mu \sum_k \left(H_k \frac{\partial}{\partial x_k} \right) H_i = \sum_k \frac{\partial}{\partial x_k} \mu H_k H_i - H_i \sum_k \frac{\partial}{\partial x_k} \mu H_k$$

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The last term in this expression is easily shown to be divergence of magnetic flux density, which is zero:

$$\nabla \cdot \vec{B} = \sum_k \frac{\partial}{\partial x_k} \mu H_k = 0$$

Using this, we can write force density in a more compact form as:

$$F_k = \frac{\partial}{\partial x_i} \left(\mu H_i H_k - \frac{\mu}{2} \delta_{ik} \sum_n H_n^2 \right)$$

where we have used the Kroneker delta $\delta_{ik} = 1$ if $i = k$, 0 otherwise.

Note that this force density is in the form of the divergence of a tensor:

$$F_k = \frac{\partial}{\partial x_i} T_{ik}$$

or

$$\vec{F} = \nabla \cdot \underline{\underline{T}}$$

In this case, force on some object that can be surrounded by a closed surface can be found by using the divergence theorem:

$$\vec{f} = \int_{\text{vol}} \vec{F} dv = \int_{\text{vol}} \nabla \cdot \underline{\underline{T}} dv = \oiint \underline{\underline{T}} \cdot \vec{n} da$$

or, if we note surface traction to be $\tau_i = \sum_k T_{ik} n_k$, where n is the surface normal vector, then the total force in direction i is just:

$$\vec{f} = \oint_s \tau_i da = \oint \sum_k T_{ik} n_k da$$

The interpretation of all of this is less difficult than the notation suggests. This field description of forces gives us a simple picture of surface traction, the force per unit area on a surface. If we just integrate this traction over the area of some body we get the whole force on the body.

Note one more thing about this notation. Sometimes when subscripts are repeated as they are here the summation symbol is omitted. Thus we would write $\tau_i = \sum_k T_{ik} n_k = T_{ik} n_k$.

4.3   Example: Linear Induction Machine

Figure 7 shows a highly simplified picture of a single sided linear induction motor. This is not how most linear induction machines are actually built, but it is possible to show through symmetry arguments that the analysis we can carry out here is actually valid for other machines of this class.

This machine consists of a stator (the upper surface) which is represented as a surface current on the surface of a highly permeable region. The moving element consists of a thin layer of conducting material on the surface of a highly permeable region. The moving element (or 'shuttle') has a velocity u with respect to the stator and that motion is in the x direction. The stator surface current density is assumed to be:

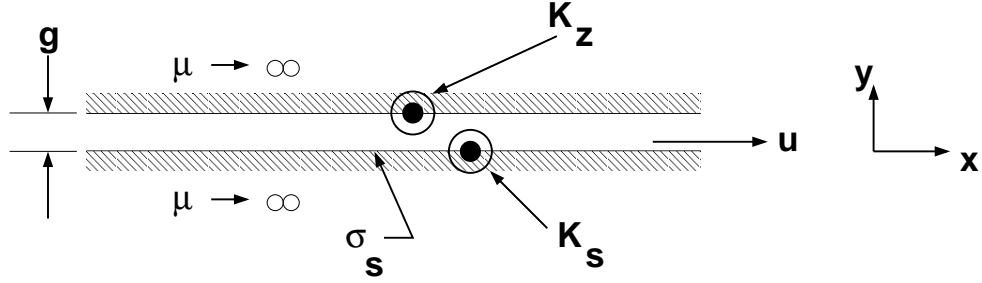$$K_z = \text{Re} \left\{ \underline{K}_z e^{j(\omega t - kx)} \right\}$$

Figure 7: Simple Model of single sided linear induction machine

Note that we are ignoring some important effects, such as those arising from finite length of the stator and of the shuttle. Such effects can be quite important, but we will leave those until later, as they are what make linear motors interesting.

Viewed from the shuttle for which the dimension in the direction of motion is $x' - x - ut'$, the relative frequency is:

$$\omega t - kx = (\omega - ku)\, t - kx' = \omega_s t - kx'$$

Now, since the shuttle surface can support a surface current and is excited by magnetic fields which are in turn excited by the stator currents, it is reasonable to assume that the *form* of rotor current is the same as that of the stator:

$$K_s = \text{Re}\left\{\underline{K}_s e^{j(\omega_s t - kx')}\right\}$$

Ampere's Law is, in this situation:

$$g\frac{\partial H_y}{\partial x} = K_z + K_s$$

which is, in complex amplitudes:

$$\underline{H}_y = \frac{\underline{K}_z + \underline{K}_s}{-jkg}$$

The y- component of Faraday's Law is, assuming the problem is uniform in the z- direction:

$$-j\omega_s \underline{B}_y = jk\underline{E}'_z$$

or

$$\underline{E}'_z = -\frac{\omega_s}{k}\mu_0 \underline{H}_y$$

A bit of algebraic manipulation yields expressions for the complex amplitudes of rotor surface current and gap magnetic field:

$$\underline{K}_s = \frac{-j\frac{\mu_0 \omega_s \sigma_s}{k^2 g}}{1 + j\frac{mu_0 \omega_s \sigma_s}{k^2 g}}\underline{K}_z$$

$$\underline{H}_y = \frac{j}{kg}\frac{\underline{K}_z}{1 + j\frac{mu_0 \omega_s \sigma_s}{k^2 g}}$$

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To find surface traction, the Maxwell Stress Tensor can be evaluated at a surface just below the stator (on this surface the x- directed magnetic field is simply $\underline{H}_x = \underline{K}_z$. Thus the traction is

$$\tau_x = T_{xy} = \mu_0 H_x H_y$$

and the average of this is:

$$< \tau_x >= \frac{\mu_0}{2}\text{Re}\left\{\underline{H}_x \underline{H}_y *\right\}$$

This is:

$$< \tau_x >= \frac{\mu_0}{2}\frac{1}{kg}\frac{|\underline{K}_z|^2 \frac{\mu_0 \omega_s \sigma_s}{k^2 g}}{1 + \left(\frac{\mu_0 \omega_s \sigma_s}{k^2 g}\right)^2}$$

Now, if we consider electromagnetic power flow (Poynting's Theorem): in the y- direction:

$$S_y = E_z H_x$$

And since in the frame of the shuttle $\underline{E}'_z = -\frac{\omega_s}{k}\mu_0 \underline{H}_y$

$$< S'_y >= -\frac{1}{2}\frac{\omega_s}{k}\frac{\mu_0}{kg}\frac{\frac{\mu_0 \omega_s \sigma_s}{k^2 g}}{1 + \left(\frac{\mu_0 \omega_s \sigma_s}{k^2 g}\right)^2}|\underline{K}_z|^2 = -\frac{\omega_s}{k} < \tau_x >$$

Similarly, evaluated in the frame of the stator:

$$< S_y >= -\frac{\omega}{k} < \tau - x >$$

This shows what we already suspected: the electromagnetic power flow from the stator is the force density on the shuttle times the wave velocity. The electromagnetic ower flow into the shuttle is the same force density times the 'slip' velocity. The difference between these two is the power converted to mechanical form and it is the force density times the shuttle velocity.

4.4   Rotating Machines

The use of this formulation in rotating machines is a bit tricky because, at lest formally, directional vectors must have constant identity if an integral of forces is to become a total force. In cylindrical coordinates, of course, the directional vectors are *not* of constant identity. However, with care and understanding of the direction of traction and how it is integrated we can make use of the MST approach in rotating electric machines.

Now, if we go back to the case of a circular cylinder and are interested in torque, it is pretty clear that we can compute the circumferential force by noting that the normal vector to the cylinder is just the radial unit vector, and then the circumferential traction must simply be:

$$\tau_\theta = \mu_0 H_r H_\theta$$

Assuming that there are no fluxes inside the surface of the rotor, simply integrating this over the surface gives azimuthal force. In principal this is the same as surrounding the surface of the rotor by a continuum of infinitely small boxes, one surface just outside the rotor and with a normal facing outward, the other surface just inside with normal facing inward. (Of course the MST is zero on this inner surface). Then multiplying by radius (moment arm) gives torque. The last step is to note that, if the rotor is made of highly permeable material, the azimuthal magnetic field just outside the rotor is equal to surface current density.

5 Generalization to Continuous Media

Now, consider a system with not just a multiplicity of circuits but a continuum of current-carrying paths. In that case we could identify the co-energy as:

$$W'_m = \int_{\text{area}} \int \lambda(\vec{a}) d\vec{J} \cdot d\vec{a}$$

where that area is chosen to cut all of the current carrying conductors. This area can be picked to be perpedicular to each of the current filaments since the divergence of current is zero. The flux λ is calculated over a path that coincides with each current filament (such paths exist since current has zero divergence). Then the flux is:

$$\lambda(\vec{a}) = \int \vec{B} \cdot d\vec{n}$$

Now, if we use the vector potential $\vec{A}$ for which the magnetic flux density is:

$$\vec{B} = \nabla \times \vec{A}$$

the flux linked by any one of the current filaments is:

$$\lambda(\vec{a}) = \oint \vec{A} \cdot d\vec{\ell}$$

where $d\vec{\ell}$ is the path around the current filament. This implies directly that the coenergy is:

$$W'_m = \int_{\text{area}} \int_J \oint \vec{A} \cdot d\vec{\ell} d\vec{J} \cdot d\vec{a}$$

Now: it is possible to make $d\vec{\ell}$ coincide with $d\vec{a}$ and be parallel to the current filaments, so that:

$$W'_m = \int_{\text{vol}} \vec{A} \cdot d\vec{J} dv$$

5.1 Permanent Magnets

Permanent magnets are becoming an even more important element in electric machine systems. Often systems with permanent magnets are approached in a relatively ad-hoc way, made equivalent to a current that produces the same MMF as the magnet itself.

The constitutive relationship for a permanent magnet relates the magnetic flux density $\vec{B}$ to magnetic field $\vec{H}$ and the property of the magnet itself, the *magnetization* $\vec{M}$.

$$\vec{B} = \mu_0 \left(\vec{H} + \vec{M} \right)$$

Now, the effect of the magnetization is to act as if there were a current (called an *amperian current*) with density:

$$\vec{J}^* = \nabla \times \vec{M}$$

Note that this amperian current "acts" just like ordinary current in making magnetic flux density. Magnetic co-energy is:

$$W'_m = \int_{\text{vol}} \vec{A} \cdot \nabla \times d\vec{M} dv$$

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Next, note the vector identity $\nabla \cdot \left(\vec{C} \times \vec{D} \right) = \vec{D} \cdot \left(\nabla \times \vec{C} \right) - \vec{C} \cdot \left(\nabla \times \vec{D} \right)$ Now,

$$W'_m = \int_{\text{vol}} -\nabla \cdot \left(\vec{A} \times d\vec{M} \right) dv + \int_{\text{vol}} \left(\nabla \times \vec{A} \right) \cdot d\vec{M} dv$$

Then, noting that $\vec{B} = \nabla \times \vec{A}$:

$$W'_m = - \oiint \vec{A} \times d\vec{M} d\vec{s} + \int_{\text{vol}} \vec{B} \cdot d\vec{M} dv$$

The first of these integrals (closed surface) vanishes if it is taken over a surface just outside the magnet, where $\vec{M}$ is zero. Thus the magnetic co-energy in a system with only a permanent magnet source is

$$W'_m = \int_{\text{vol}} \vec{B} \cdot d\vec{M} dv$$

Adding current carrying coils to such a system is done in the obvious way.