N. Komanda, A. F. Martin, A. Nijenhuis, R. E. Prather, A. Riese, R. M. Robinson, R. L. Schilling (Germany), A. Stein, Anchorage Math Solutions Group, Western Maryland College Problems group, and the proposer. One solution of part (a) only and three incorrect solutions were received.

The Smallest Factorial That Is a Multiple of n

6674 [1991, 965]. *Proposed by Paul Erdős, Hungarian Academy of Sciences, Budapest, Hungary.*

If n is an integer greater than 1, let $P(n)$ denote the largest prime factor of n. Prove that $x|P(n)!$ for almost all n, i.e., prove that if

$$S(x) = \{n \le x : n \nmid P(n)!\},$$

then

$$\lim_{x\to\infty} |S(x)|/x = 0.$$

Solution by Ilias Kastanas, California State University, Los Angeles, CA. Let $S'(x) = \{n \le x : n|P(n)!\}$; we will show that $d = \lim_{x\to\infty} S'(x)/x$ is equal to 1.

Let Q denote the set of square free natural numbers, and introduce

$$Q(x) = |Q \cap \{1,2,\dots,x\}|$$
$$Q_\alpha = |\{n : n \in Q \text{ and all prime factors of } n < \alpha\}|.$$

For any integer $k > 0$ let $f(k)$ be the least prime greater than $3k$. If $q \in Q$ and q is divisible by some prime $\ge f(k)$, then $k^2q|P(q)!$, since $2k$ and $3k$ are factors of $P(q)!/q$. So, there are at least $Q(x/k^2) - Q_{f(k)}$ integers of the form k^2q in $S'(x)$, and

$$\lim_{x\to\infty} \frac{Q(x/k^2) - Q_{f(k)}}{x} = \lim_{x\to\infty} \frac{Q(x/k^2)}{x} = \frac{1}{\zeta(2)}\frac{1}{k^2}.$$

For the last equation, see G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford, 1980, Theorem 333 in §18.5. Since the sets k^2Q are disjoint for distinct k, we have

$$1 \ge d \ge \frac{1}{\zeta(2)} \sum_{k=1}^{n} \frac{1}{k^2}$$

for any $n > 0$. Therefore,

$$d \ge \frac{1}{\zeta(2)} \cdot \zeta(2) = 1.$$

Editorial comment. In preparing the problem for publication, the editors obtained a solution from Kevin Ford based on a division into cases depending on the number of prime factors of n and the size of $P(n)$. He was able to show that $|S(x)| = O(x/\log x)$ by this method, and indicated that stronger results could be obtained using *sieve theoretic* estimates.

No other solutions were received.

A Characterization of Special Quadratic Irrationals

10218 [1992, 362]. *Proposed by David Dwyer, University of Evansville, Evansville, IN.*