HOROLOGII OSCILLATORII

PART FOUR A. [p. 91]

Concerning the centre of Oscillation.

[Note : the centre of oscillation (or percussion) of a physical or compound pendulum is the point at which the mass may be concentrated to give the same period as a simple pendulum, on being suspended in a similar manner and set in motion about the axis of rotation.]

The investigation of the centre of oscillation or movement was one proposed by the most learned Mersenne, and it was famous amongst the geometers of the day; indeed this problem was attempted by me when I was still a boy, along with many others, and I have kept the letters Mersenne sent me; and neither should we exclude the work of Descartes recently published, which contains answers to these things proposed by Mersenne. Moreover, he wanted to find the centre of oscillation as I shall do here, for sectors of a circle, which was either suspended from the vertex of the angle or from the middle of the arc, and disturbed from the side; likewise for circular segments, and for triangles suspended either from a vertex or from the middle of the base. These problems are reduced to the case of the simple pendulum, that is, a weight hanging from a string that has been found at some distance, so that the oscillations are made in the same time as that of the object suspended in the manner indicated. It would truly have been worth the effort as well, if perhaps I had found some satisfying answers, for certainly I grew to hate the investigation, for the answers that were then desired could not be obtained by anyone. Since I was thus restrained from making any progress, since there was nothing I could find that even made things clear at the start, I withdrew from a making a longer investigation; just as on being repulsed at the threshold, as it were. Several conspicuous men had hoped to make some progress with the problems, Descartes, Honore Fabry, and others, but by no means reached the goal, except in a few of the easier cases; yet nothing suitable was produced from these demonstrations, as it seemed to me. Thus, perhaps, a comparison can be made between these results already established with the demonstrations that I intend to show here; which certainly I consider to be established from more reliable principles, and found to be in complete agreement with experiment. The opportunity for me to make a fresh attack on these problems has come about by the need to combine them with my thinking about pendulum clocks, while the moving weight, as well as being of interest on its own merits, due to its application to these clocks, is set out in the description of the working of these clocks. Hence with better beginnings than from the original start, I have at last overcome all the difficulties, producing not only solutions of the type envisaged by Mersenne, but also the solutions of other problems, more difficult than these, can be found too; hence a way is found for determining, by sound reasoning, the centre of oscillation for [p. 92] lines, surfaces, and solid bodies. Indeed, there was the pleasure associated with finding the solutions to these problems, sought after by others for a long time, and understanding the laws and rules associated with these things; also I have seized upon the usefulness of these results, which with appreciation I have applied to the problem about clocks, which I have found in an easy and expeditious manner. Another thing too can be agreed upon, which I have become aware of by doing more work on the clocks, that surely will endure over the centuries, that I am able to deliver a most accurate definition of the measurement of length; which can be found added to the end of these notes as a final touch.

DEFINITIONS.

1.

A Pendulum may be said to be some given figure with weight, whether it should be a line, a surface, or a solid, thus suspended so that it is able to continue its to and fro motion with a force due to its own weight about some point, or rather axis, which is understood to be parallel to the horizontal plane [e. g. a nail driven into a support perpendicular to the plane of the pendulum, lying in the horizontal plane]**.**

II.

The axis parallel to the horizontal plane, about which the motion of the pendulum is understood to be made, is called the axis of oscillation.

III.

A simple pendulum is understood to be composed from a weightless thread or inflexible line with a weight attached to one end; it is considered that the force of gravity acting on the weight has been gathered together to act at one point.

IV.

A composite pendulum is understood to consist of some number of weights, keeping the same distances apart both between themselves and with the axis of oscillation. Hence any suspended figure with the weights provided can be called a composite pendulum, as far as understanding that it can be divided into some number of parts.

V.

Pendulums are said to be isochronous if the oscillations are performed for similar arcs in the same time. [p. 93; thus, large or small arcs of the same radius]

VI.

The plane of oscillation is said to be that plane understood to be drawn through the centre of gravity of the suspended figure, at right angles to the axis of oscillation.

VII.

The line of the centre is drawn through the centre of gravity of the figure perpendicular to the axis of oscillation.

VIII.

The line of the perpendicular, a line in the plane of oscillation drawn from the axis of oscillation, perpendicular to the horizontal plane.

IX.

The centre of oscillation or of movement of some figure, is said to be some point in the line of the centre, such that the distance from the axis of oscillation of the figure is the same length as that of a simple isochronous pendulum.

X.

The axis of the weight is some line passing through the centre of gravity of the figure.

XI.

A plane figure or a line placed in a plane is said to be disturbed in the plane, when the axis of oscillation is in the same plane as the line or figure .

XII.

Truly the same figure or line is said to be disturbed laterally, when the axis of oscillation is at right angles to the figure or line.

XIII.

When the weights are represented by straight lines, it is understood that the numbers associated with the weights or lines is in the same ratio that the weights express between themselves.

HYPOTHESES.

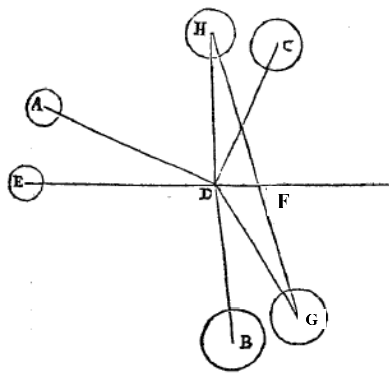
I.

If some weights begin to move under the force of gravity, then it is not possible for the centre of gravity of these weights to ascend to a greater height than that found at the beginning of the motion. [p.94.]

Moreover for these weights, the height is considered to be the distance from a horizontal plane, and the weights are placed relative to that plane, according to the perpendicular distances they are trying to descend. Because all the weights experience the same motion as that arising from the motion of the centre of gravity, either put in place explicitly, or found from the laws of moments, for there is no place in a description of the motion without a consideration of that of the centre of gravity.

Truly, by our stated hypothesis, not even a small stone can be moved in this way, or anything else we might wish to show, for according to everyday experience, it is commonly agreed that a weight cannot raise itself up by the force of gravity acting on it. [In Huygens' day, before Newton *Principia*, there was no distinction between mass and weight, hence with Huygens we use the word 'weight', when in truth we should sometimes be calling it 'mass']
For in the first place, if we propose some heavy body, it is impossible beyond doubt that it is able to raise itself higher by the force of gravity than its original height; moreover, to ascend means that the centre of gravity of the weight rises. But the same thing must be conceded for any number of weights, joined together by inflexible lines, since nothing forbids more than one itself when some greater number is to be considered. And thus, neither will the common centre of gravity of these combined weights be able to ascend higher.

If a number of weights are now put in place but not connected together, then we can establish the common centre of gravity of these too. Indeed the height of the centre of gravity of these [above the reference plane] will be some amount, and of the same size, I say, as that height considered for all of these weights attached to each other just considered : accordingly indeed all these free weights can be deduced to have their common centre of gravity at the same height as before, with no other force indicated than that belonging to the weights themselves, and which may be joined together by as many inflexible lines as you wish, and to be moving around with the same centre of gravity ; for which there is no need for a extra force or power to be determined. Whereby, it is not possible for a certain group of weights, placed in a horizontal plane, all to be raised equally above that plane, by their own force of gravity; thus neither is it possible for any group of weights, however disposed, to arrange themselves so that the centre of gravity is



at a greater height than initially. However, concerning what we have said about any group of weights : without the action of any [external] force, these can be made to lie in a horizontal plane passing through the common centre of gravity, as we now show.

A, B, and C are weights in the given position, the common centre of gravity of which is D, through which a horizontal plane can be drawn, a section of which is the line EF. DA, DB and DC are now inflexible lines, to which the weights are always connected; which can be moved again to E, so that A lies in the plane EF. Truly all the short connecting lines are moved through equal angles, now B is at G and C at H.

Again, B and C are now understood to be connected by the line HG, which cuts the plane EF in F; where by necessity the centre of gravity of these two weights also lies on the connection HG [p. 95] ; and when the three weights are placed at E, G, and H, the centre of gravity is at D, and the centre of gravity of the weight which is at E, also lies in the plane EDF. Therefore the weights H and G are again moved just as with the axis through F, without any force, and at the same time both are led to lie in the plane EF, as thus now all three, where formerly were at A, B, and C, are now at the height of the centre of gravity D itself, with which they are in equilibrium, as is apparent from the translation : which was to be shown. And the same is found for any number of weights.

Moreover this hypothesis of ours also applies to [solid bodies in liquids and] liquid bodies, and not only can all the theorems concerning floating bodies of Archimedes be shown, but also many other theorems. Certainly, if the inventors who try in an ineffective way to construct perpetual motion machines knew about this new principle, then they would easily grasp their mistakes, and understand by mechanical reasoning why some things cannot be made to work.

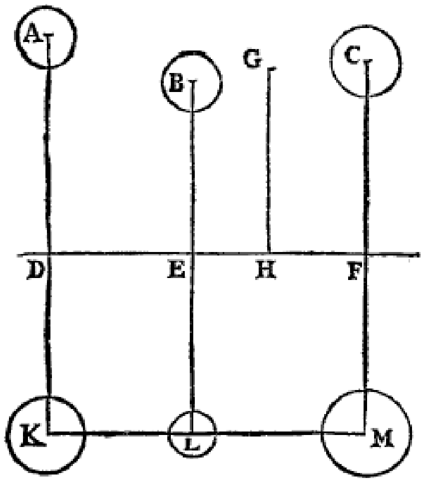
II.

In the absence of air, and with the removal of all other known impediments to motion, as in the following demonstrations that we wish to understand, the centre of gravity of a disturbed pendulum, travels through equal arcs in falling and rising.

For a simple pendulum this demonstration is Prop. 9 of the section concerning the falling of weights. The same indeed is held for composite pendulums, and is found from experiment ; for indeed, a pendulum of some figure is found to perform suitable equal motions [p. 96], except that the motion is impeded more or less by the air present.

PROPOSITION I.

If, for some weights present in one section of a plane, perpendiculars are drawn in that plane from the centres of gravity of the individual weights ; then the sum of the products of these individual lines taken with their respective weights, has the same effect as the perpendicular drawn in the same plane from the common centre of gravity of all the weights, taken with the sum of all the weights.



A, B, and C shall be the weights placed in the same part of the plane, the section of which is DF, and the perpendiculars themselves AD, BE, and CF are drawn. Moreover, the centre of gravity of all the weights A, B, and C is G, from which the perpendicular GH is drawn in the same plane. I say that the sum of the products, which are made by the weights taken with their own perpendiculars, is equal to the product of the line GH by the sum of the weights A, B, and C.
[Note that we have a plan view, with gravity acting into the plane of the page; initially, the length GH must be conjectured, as the proposition finds its length.]

For the perpendiculars, drawn from the individual weights, are understood to be continued to the far side of the section DF in the plane, and the individual lengths DK, EL, and FM are themselves each set equal to HG; and all the lines, referred to as inflexible rods, are parallel to the horizontal; and at the points K, L, and M are placed weights in the same manner, all of which are in equilibrium with the weight opposite to themselves A, B, and C across the line DEF in the plane [which acts as a fulcrum; thus, there are equal and opposite turning effects or moments for each pair of weights K and A, L and B, M and C.] Moreover, the length AD to DK is thus as the weight K to the weight A, and hence DA multiplied by the magnitude of the weight A is equal to DK (or GH) by K. Similarly [p. 97] EB by B is equal to EL (or GH) by L; & FC by C is equal to FM, (or GH), in M. Hence the sum of the products of AD by A, BE by B, and CF by F is equal to the sum of the products of GH by the sum of K, L, and M. Moreover, K, L, and M are also in equilibrium with A, B, and C suspended together from their centre of gravity G. [This latter idea is needed to remove all reference to K, L, and M in the proposition.] Then, since the distance GH is equal to the individual lengths DK, EL, and FM, it is necessary that the sum of the weights A, B, and C in this case is equal to the sum of K, L, and M. Thus, the sum of the products of GH in the sum of A, B, and C, is equal to the products from DA in A, EB in B, & FC in C ; Q.e.d.
[In short : $(A \times AD) + (B \times BE) + (C \times FC) = (A + B + C) \times GH$.]

Thus truly in this demonstration the lines AD, GH, and CF were put in a plane parallel to the horizontal, which was established; it is apparent, if all the weights were changed together into some the situation, the same equality of the products would remain, with all the lines indeed as before. Whereby the proposition is agreed upon.

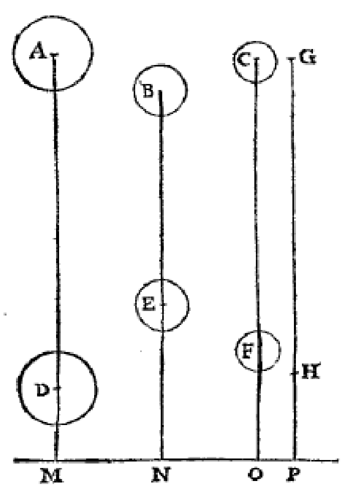
PROPOSITION II.

With the previous positions, if all the weights A, B, and C are equal; I say that the sum of all the perpendiculars* AD, BE, and CF*, is equal to the perpendicular GH drawn from the centre of gravity, multiplied by a number according to the number weights.

Indeed with the sum of the products of the individual weights by their own perpendiculars equal to the product of GH by the sum of the weights ; and here, on account of the weights being equal, this sum of the products is equal to the product of one weight by the sum of all the perpendicular distances; and likewise the product of GH by all the weights, is the same as that product of one of the weights by GH, multiplied by a number according to the number of weights : it is apparent that the sum of the perpendiculars by necessity is now equal to GH itself, multiplied by a number equal to the number of weights present. Q.e.d.
[In this case, GH is equal to the mean length of the lines AD, BE, and CF.]

PROPOSITON III.

If weights of certain magnitudes all descend or ascend, through any unequal intervals; then the distances of descent or ascent of these, multiplied by their respective weights, give a sum equal to that produced by the distance of descent or ascent of the common centre of gravity, multiplied by the sum of all the weights. [p.98]



Let the magnitudes be A, B, and C, which descend from A, B, and C to D, E, and F ; or from D, E, and F ascend to A, B, and C. The common centre of gravity of all these weights, such as A, B, and C, has the same height as the point G; as indeed that of D, E, and F has the same height as the point H. I say that the sum of the products of the height AD by A, BE by B, and CF by C, is equal to the product of GH by the sum of A, B, and C.

Indeed it is understood that the horizontal plane is cut by the line MP, and the lines AD, BE, CF, and GH produced meet this line in M, N, O, and P.

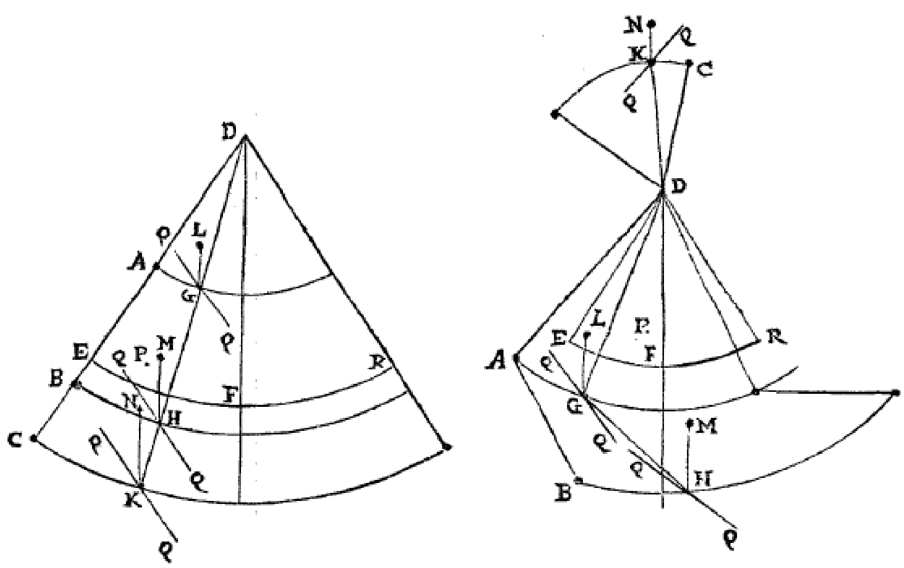
Indeed since the sum of the products of AM by A, BN by B, and CO by C, is equal to the product of GP by the sum of A, B, and C (by Prop. I of this section). Similarly, the sum of the products DM by A, EN by B, and FO by C, is equal to the product of HP by the sum of A, B, and C; and it follows that the difference of the first and second products is equal to the product of GH by the sum of A, B, and C. I say that the difference has been shown to be equal to the products of AD by A, BE by B, and CF by C. Hence, this is also equal to the product of GH by the sum of A, B, and C. Q.e.d.
[Thus, of $(AD \times A) + (BE \times B) + (CF \times C) = GH \times (A + B + C)$. We may note that Huygens does not distinguish between mass and weight; for in the previous proposition, gravity is used to take moments of forces to find the centre of gravity, and then in this

theorem the weights are swung round into the vertical plane, keeping the same relative positions, and the position of the centre of gravity remains in the same place.]

PROPOSITON IV.

If a pendulum is composed from a number weights, and it is released from rest, then any part of the whole oscillation is carried out by the weights together; and thus again it is understood that the individual weights of the pendulum, without a common bond, convert the acquired speed by rising up for as long as they are able to ascend; from this fact, it follows that the common centre of gravity of all the weights returns to the same height that it had at the start of the oscillation. (p. 99)

The pendulum is composed from some number of weights A, B, C, and the rods or surfaces to which they are attached are considered weightless. The pendulum is suspended from the axis drawn through the point D, which is understood to be perpendicular to the plane that is shown here. The centre of gravity E lies in the same plane as the weights A, B, and C ; and the line DE from the centre, is inclined to the perpendicular line DF, by the angle EDF : to which line the pendulum is obviously attracted all the time. From this position the pendulum is released, and completes some part of an oscillation, so that the weights A, B, and C arrive at the points G, H, and K. Henceforth, with the common rod abandoned, [we may now consider each weight A, B, and C attached to D by its own weightless string] it is understood that the speeds acquired can be converted to upwards motion, (since this can be done by the masses striking certain inclined planes such as QQ) and rising for as long as they are able, truly to L, M, and N. When they arrive at these points, the common centre of gravity is the point P. I say that this point P is at the same height as the point E.

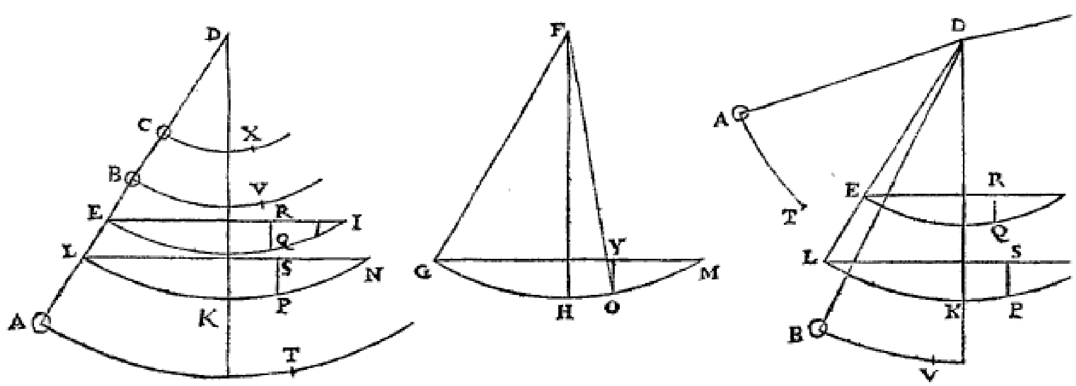


For in the first place it is agreed that the point P cannot be higher than the point E, from the first assumed hypothesis. But we will show that neither can it be lower. For indeed, if it were possible for P to be lower than E, and the weights are understood to fall to lower heights than those to which they can ascend, which are LG, MH, and NK. Hence indeed it is agreed that they have acquired the same speeds needed to ascend to these heights (Prop. 4, part. 2), that is, these speeds acquired from the motion of the pendulum from CBAD to KHGD. Whereby, if with the said speeds the weights are now re-attached to the rod or surface, and they continue the motion began along the arcs ; the question can now be resolved of what happens, if they are considered to continue rather than to rebound from the inclined plane QQ, as before being attached to the rod, (p.100) in the following manner, by which the restored motion of the pendulum continues, or equally, if the motion continues without any interruption. Thus as the centre of gravity of the pendulum E, by descending and ascending, runs through equal arcs EF and FR, and hence it returns to the same height R as E. Moreover, if the centre of gravity E is put higher than the corresponding centre of gravity P for the positions L, M, and N, then R will be lower than P: and hence the centre of gravity of the weights at L, M, and N will be less in height in the descent than the height in the ascent, which is absurd (by Hyp.1 of this section.) Therefore the centre of gravity P is not lower than E. But neither is it higher. Therefore it necessarily must be equal in height. Q.e.d.

PROPOSITION V.

For a given pendulum composed from any number of weights, if the individual weights are multiplied by the square of their distance from the centre of oscillation, and the sum of the products is divided by that product, which is the sum of the weights multiplied by the distance of the common centre of gravity of all the weights from the axis of oscillation; then the quotient arising from this is the length of the simple isochronous pendulum, or the distance between the axis and the centre of oscillation of the composite pendulum.

The component weights of the pendulum are A, B, and C, (only the weights of which need be considered, and neither the geometrical figure nor the magnitude of the swing),



suspended from the axis, which is understood to pass through the point D in a plane, observed at right angles. In which plane the common centre of gravity shall be at E also; for we are not concerned with the different weights. The distance of the point E from the axis, truly the line ED, is called d. Likewise the distance AD of the weight A is e; BD, f; and CD, g. Thus by multiplying the individual weights by the squares of the distances, [p.101], the sum of the products is $aee + bff + cgg$. And again, by multiplying the sum of the weights by the distance d of the centre of gravity, the product is equal to $ad + bd + cd$. (Prop. 1, of this section). Hence, the first product divided by this amount, gives $\frac{aee+bff+cgg}{ad+bd+cd}$. To which, according to the proposition, the length of the simple pendulum FG can be equated, to be called x ; I say that this pendulum is isochronous with the compound pendulum.

[Thus, a compound pendulum can be replaced by an equivalent simple pendulum, supposed to swing in the arc of an inverted cycloid as discussed in the previous sections, like that of Huygens' clock, but this figure is not shown. It may be useful to quote the formula first in the modern setting for small angles : if I is the moment of inertia of the pendulum about some axis, m the mass, and d the distance of the centre of mass from the axis of rotation, then the pendulum satisfies the equation $I\ddot{\vartheta} = -mgd\theta$, for an angular displacement θ from the vertical. This gives rise to a period $T = 2\pi\sqrt{\frac{I}{mgd}}$; in terms of the symbols introduced by Huygens, this becomes $T = 2\pi\sqrt{\frac{ae^2+bf^2+cg^2}{(a+b+c)gd}} \equiv 2\pi\sqrt{\frac{x}{g}}$ for the equivalent simple pendulum.
Thus, $x = \frac{ae^2+bf^2+cg^2}{(a+b+c)d}$ is the length of the equivalent simple pendulum. We see that the numerator is the moment of inertia of the compound or physical pendulum composed of the three masses about the axis of rotation, while the torque of moment exerted on the pendulum by the combined masses is proportional to the denominator.

One may assume that Huygens used some sort of proportionality argument to reach his result, perhaps somewhat as follows :

$\sum_{\text{point masses } i} m_i v_i r_i = (\sum m_i r_i^2)\omega$ for the angular momentum of the point masses m_i at distances r_i from the axis of rotation, and for which $Md\omega$ is the equivalent motion of the centre of mass M at a distance d from the axis ; hence, if everything is equated to an equivalent point mass at some distance x, we can form the ratio

$\frac{\sum m_i r_i^2)\omega}{(\sum m_i)d\omega} = \frac{Mx^2\omega}{Mx\omega}$, which gives the above result on substitution a, b, c, etc.

One can also consider the ratio of the second moment of the masses to the first moment of the masses oscillating, and apply the ratio to an equivalent point mass oscillating at some distance x; however, neither approach suggested here necessarily represents how Huygens came upon his ratio. The point is that the formula gives the correct experimental values, and thus could be regarded as part of some theoretical analysis not available at the time.
The modern reader may be surprised to find something equivalent to the formula for the moment of inertia of several point masses established here; as indeed there are also several results that we would now interpret in terms of an interchange of potential and kinetic energy; these results are however, all deduced from simple kinematic principles, and thus gave investigators such as Newton and Johan. Bernoulli a solid foundation for further developments. The fact that Huygens often relied on the *reductio ad absurdum* type of argument presumably indicates that he did not wish to present the analytical method by which he obtained his result originally. We now return to the text.]

Indeed as the simple pendulum FG and the line of the centre of gravity DE are put in place, with equal angles to the perpendicular, with one line set at a distance from FH, and the other set at a distance from DK, and the weights released, and on the line DE, DL is taken equal to FG.
[Thus, the length of the equivalent simple pendulum FG is taken equal to the length of the compound pendulum from the axis D to the centre of oscillation at some point L.]
And thus the weight G of the pendulum FG, traverses the arc GM in a whole oscillation, [recall that an oscillation in these days was defined as a swing from one side to the other, and not the return swing back to the start as is the case now.]
that the line of the perpendicular FH cuts in the middle; truly the point L for the similar and equal arc LN, is divided by DK in the middle. And likewise the centre of gravity E, runs through the similar arc EI. Because, for whatever points are taken in the arcs GM and NL, with these similarly divided as by O and P, the same speeds are established for the weight G at O, and for the point L at P; hence it is agreed that both run through the arcs in equal times, and hence the pendulum FG is isochronous with the pendulum composed from A, B, and C. Moreover, this proposition can be shown in the following way. [By a *reductio ad absurdum* proof.]

In the first place, assume it is possible for the speed of the point L to be greater when it arrives at P than the speed of the point G at O. However, it is agreed, that the point L traverses the arc LP in the same time that the centre of gravity E traverses the similar arc EQ. Perpendiculars are drawn upwards from the points Q, P, and O which cross the chords to the arcs EI, LN, and GM at R, S, and Y; and SP is called y. Thus, as LD (or x) is to ED (or d), thus SP (or y) is to RQ; and thus RQ is equal to $\frac{dy}{x}$. Since the weight G has such a speed at O, which prevails for the descent as well as the ascent through the same height, indeed through the arc OM, or to ascend the perpendicular distance OY, equal to PS; therefore by assumption the point L, when it arrives at P, has a greater speed there than that required to ascend through the height PS. While truly L passes through P, likewise the weights A, B, and C pass through arcs similar to LP, obviously AT, BV, CX. The speed of the point L at P, to the speed of the point A at T, when they are constrained by the same rod, is thus as the distances DL to DA.
[Since the ratio of the speeds at L and A, or $v_A / v_P = \omega.LD / \omega.AD = LD / AD$, for some common instantaneous angular velocity ω.]
But as the square of the speed of the point L at P is to the square of the speed of the point A at T, thus as the height to which (with that speed) L is able to rise, to the height by which with this speed A is able to rise (Prop. 3 & 4, part 2; which deal with the kinematics of falling bodies). Hence also, as the square of the distance DL, which is xx, to the square of the distance DA, which is ee, thus is the height to which the weight can rise with the speed at the point L, when it is at P, (which height has been supposed to be greater than PS or y) to the height by which the weight can ascend with the speed of the weight A at T;
[*i.e.* $v_P^2 / v_A^2 = LD^2 / AD^2 = x^2 / e^2 = y/z_T$; where z_T is the height that A can rise to at T, and which is not defined by Huygens using a letter.]

if truly upon coming to T, with the rest of the pendulum (p. 102), the motion of A can be changed separately. Thus A will therefore rise to a height greater than $\frac{eey}{xx}$, by supposition. [*i. e.* $z_T > \frac{eey}{xx}$.]

By the same reasoning, the height to which the weight B can ascend, with the speed acquired by passing through the arc BV, is greater than $\frac{ffy}{xx}$. And the height to which the weight C can ascend, with the speed acquired by passing through the arc CX, is greater than $\frac{ggy}{xx}$. Hence, with the individual heights multiplied by their own weights, there is a sum of the products greater than $\frac{aeey+bffy+cggy}{xx}$, which is assumed to be greater too than $\frac{ady+bdy+cdy}{x}$.

[For if we assume that: $z_T > \frac{eey}{xx}$; $z_V > \frac{ffy}{xx}$; $z_X > \frac{ggy}{xx}$,
then: $az_T + bz_V + cz_X > \frac{aeey+bffy+cggy}{xx}$].

For since the length x is equal to $\frac{aee+bff+cgg}{ad+bd+cd}$; then $adx + bdx + cdx$ is equal to $aee + bff + cgg$. And on multiplying everything by y, and dividing by xx, then $\frac{ady+bdy+cdy}{x}$ is equal to $\frac{aeey+bffy+cggy}{xx}$. From which, what has been said above follows.

[*i. e.* $az_T + bz_V + cz_X > \frac{ady+bdy+cdy}{x}$.]

Moreover, the sum of the products is equal to the height that the common centre of gravity of the weights A, B, C rises, to be multiplied by the sum of the weights themselves $a + b + c$; truly if the individual weights, as said, are able separately to be moved in that time. Truly the quantity $\frac{ady+bdy+cdy}{x}$ can be produced from the fall of the centre of gravity of the same weights, (which descent is RQ, or $\frac{dy}{x}$, as found above) multiplies by the same sum of the weights $a + b + c$. Hence when the earlier product was shown to be greater than the other product, it follows that the ascent of the centre of gravity of the weights A, B, and C, when the parts of the pendulum arrive at T, V, and X, that the individual speeds acquired to be converted on rising, are greater than for the descent of the centre of gravity, for A, B, and C moving to T, V, and X; which is absurd, since the said ascent and descent must be equal, according to what has gone before.

In the same manner, if the said speed of the point L, as it comes to P, is less than the speed of the weight G when it arrives at O ; we can show that the possible ascent of the centre of gravity of the weights A, B, and C is less than the descent, which disagrees with the previous proposition. Whereby it remains that the speed of the point L when it is moved to P, is equal to the speed of weight G at O. From which, as said above, it follows that the simple pendulum FG is isochronous with the pendulum composed from A, B, and C.

[Thus, Huygens appeals to an *abductio ad absurdum* argument to validate his reasoning; which really amounts to saying : if it did anything else, then you would get an answer inconsistent with experiment; one must thus respect his simple kinematic arguments in producing the correct results in the first place. There was no conservation of energy principle to appeal to in these days; and as with Newton's *Principia*, it was in the backdrop always, and never centre stage. People like P.G.Tait used to argue that there was such a principle at work, while Leibnitz and Bernoulli talked about a *vis viva,* that we might interpret as some agent present that caused things to happen, in a time independent manner. It took the development of thermodynamics to appreciate the energy conservation principle, and that was 200 years or so away in the future. One could go on and demonstrate Huygens proposition from the conservation of energy principle, but that is not the point of the historical work that we examine, for it must be presented as it was ; if you are left unsatisfied, then all that one can say is that is how it was done at the time. Occasionally people who dabble in the history of mathematics want to improve on it by adding concepts and ideas that came later, in an anachronistic manner, which more or less renders their efforts meaningless.]

PROPOSITION VI.

For a given pendulum composed from some equal weights: if the sum of the squares for the distances, by which each of the weights depart from the axis of oscillation, is multiplied by the distance of the common centre of gravity from the same axis of oscillation, that in turn is multiplied by the number of weights, then there arises the length of the simple pendulum to which the composite pendulum is isochronous. (p. 103).

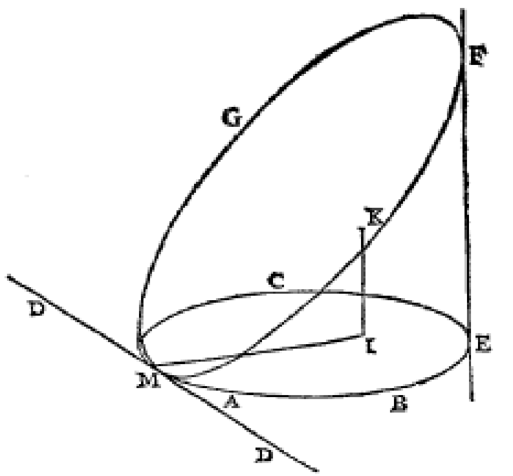
The same weights are put in place as formerly, but all the weights are understood to be equal to each other, and the individual weights are called *a*. Again truly nothing concerning their size need be considered, but what they have as their distance apart.

Thus the length of the isochronous simple pendulum, by the preceding proposition, will be $\frac{aee+aff+agg}{ad+ad+ad}$. Or, since the quantity divided and the quantity dividing are both divided by a, the length now becomes, $\frac{ee+ff+gg}{3d}$. From which the sum of the squares of the distances of the weights from the axis of oscillation is signified, divided by the distance of the common centre of gravity from the same axis of oscillation, and the number of weights, which here is 3. Indeed this number is easily seen, by which it is necessary to respond to the distance d with the number of weights. Whereby the proposition is agreed upon.

Because if the equal weights are joined together on a straight line, and suspended from the upper end; the distance of the centre of gravity from all the weights together, from the axis of oscillation multiplied by the number of weights, is equal to the sum of all the distances of all the weights from the same centre of oscillation axis (by Prop. 2, of this section); and hence, in this case, the length of the isochronous simple pendulum is also found, for the compound pendulum, if the sum of the squares of the individual weights from the axis of oscillation, divided by the sum of all these distances.
[The following definition of a wedge is used to work out sums of squares in plane figures.]

DEFINITION XIV.

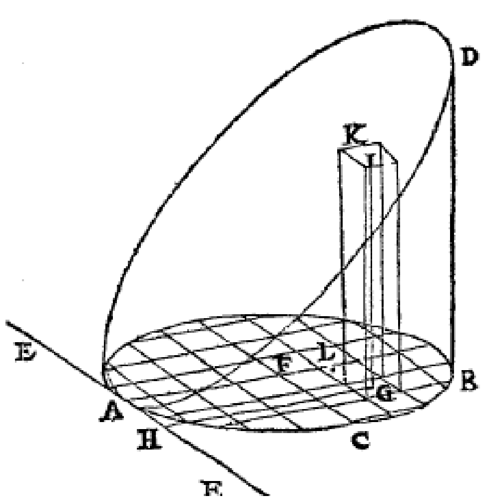
If there is a certain figure and an extended line to which it is a tangent lying in the same plane ; and on the perimeter of the figure there is another straight line in the perpendicular plane, which is carried around in a vertical plane, which describes a certain [cylindrical] surface, which then is cut by a plane drawn through the said tangent, and inclined to the figure of the said surface [at a certain angle]; a volume is understood to be described from these two planes and from the part of the surface, said to be intercepted between both of the planes. A wedge has been cut from the base of that figure.



In the added diagram, ABEC is the given figure ; the line touching that figure MD; [p. 104] to which truly EF is carried around ; moreover the wedge is the solid figure from the planes ABEC and MFG, and from the part of the surface understood to be described by the line EF.

DEFINITION VIII.

The distance between the line, through which the wedge has been cut, and the point on the base, in which the perpendicular falls from the centre of gravity of the wedge, is called the subcentre of the wedge. Truly in the same figure, if K is the centre of gravity of the wedge, the line KI is drawn perpendicular to the base ABEC of this, and again the perpendicular IM to AD ; then IM is what we call the subcentre.



PROPOSITION VII.

The wedge upon some plane figure that has been cut by a plane inclined at 45 degrees, has a volume equal to that obtained by multiplying the area of the same figure by a height equal to the distance of the centre of gravity of the figure, from the line through which the wedge has been cut.

A wedge is formed on the plane figure ACB, by a plane ABD cutting at an angle of 45 degrees, and in contact with the tangent line EE to the figure ACB, placed in the same plane. The centre of gravity of the plane figure is F, from which the perpendicular FA is drawn to the line EE. I say that the wedge ACB has a volume equal to the product of the area of the figure ACB by a height equal to FA.

For it is understood that the figure ACB has been divided into a large number of the smallest elemental equal areas, one of which is G. [p.105] Thus it is agreed, if a single one of these is taken in a product with its distance from the line EE, the sum of the products of all these is equal to the product of the line AF with the sum of all the small areas (Prop. 1 of this section; essentially, the principle of moments applied to the elements of the area)**,** that is, it is equal to that for which the altitude AF is multiplied by the area of the figure ACB. And these particular small areas such as G, multiplied by their distances GH, are equal to parallelepipeds or to the smallest rectilinear prisms erected on these, and which terminate on the surface of the oblique plane AD, such as GK; this is because the heights of these are equal to the distances GH, on account of the angle of inclination of 45 degrees of the planes AD and ACB. It is apparent that the volume of the whole wedge ABD is formed from the sum of these. Hence the wedge volume is equal to the area of the base ACB, multiplies by the height of the line FA. Q.e.d.

[Thus, loosely speaking, the volume of the wedge V is equal to the sum of the elemental prisms ΔV_i, which in turn is equal to the sum of the heights GK or h_i of each prism by their elemental areas ΔA_i , which is turn is equal to the sum of the lengths GH or g_i of each by their elemental areas ΔA_i, which by Prop. 1 in turn is equal to the common length AF times the whole area of the plane figure A :
$V = \sum \Delta V_i = \sum \Delta A_i .h_i = \sum \Delta A_i .g_i = FA \times \sum \Delta A_i = FA \times A.$ This of course takes us back to the area weighing techniques of Archimedes. Gregorius also was interested in such shapes, which he called *ungula* as they resemble hoofs, in the 1650's. The reader can find a part of Gregorius' great *Opus* translated on this website.]

PROPOSITION VIII.

If a straight line is tangent to the figure, and it is understood that the figure is divided up into the smallest equal parts, and from the individual parts the perpendiculars are drawn to that line: the sum of the squares of all these is equal to a certain rectangle, multiplied by the number of these small parts ; indeed the rectangle has sides equal to the distance of the centre of gravity of the figure from the given tangent line and the similar distance from the subcentre of the wedge, which is cut by a plane perpendicular to the figure.

With everything put in place as in the preceding figure, [the angle of inclination of the cutting plane is still set at 45 degrees], LA is the subcentre of the wedge ABD arising from the line EE. Therefore it is necessary to show that the sum of the squares of all the distances of the elemental parts of the figure [p. 106] ACB from the line EE is equal to the rectangle formed by FA and LA, multiplied by the number of elemental parts.

Certainly, it is agreed from the preceding demonstration, that the heights of the individual parallelepipeds such as GK, are equal to the distances of the small parts such as G from the line AE on the base. Whereby, if the parallelepiped GK is now multiplied by the distance GH, then this is the same as if the particular small area G is multiplied by the square of the distance GH; and likewise the same thing can be done for all the other elemental parts. Thus, the sum of all the products of the parallelepipeds by their distances from the line AE, is likewise equal to the product of the wedge volume ABD by the distance LA (from Prop. 1 of this section; [by taking moments]), since the point L lies below the centre of gravity of the wedge. Hence also, the sum of the products of the elemental areas of the individual points G, by the squares of their distances from the line AE, is equal to the product of the wedge volume ABD by the line LA, that is, the product of the figure ACB by the rectangle formed from the sides FA and LA. For the wedge ABD, is equal to the product from the figure ACB in the line FA (Prop. preceding). Again, since the figure ACB is equal to the product from one particular G, by the number of same small parts; it follows, that the said product from the figure ACB by the rectangle from FA and LA, is equal to the product from the particular G by the rectangle from FA and LA, to be multiplied according to the number of small parts G. To which hence also the said sum of the products will be equal, to the particular parts G by the squares of their distances from the line AE, or from one particular G by the sum of all the squares. Whereby, with the multiplication omitted on both sides for a particular G, it is necessary that the same sum of the squares is equal to the rectangle from FA and LA, to be multiplied according to the number of small parts in which the figure ACB is understood to be divided. Q.e.d.

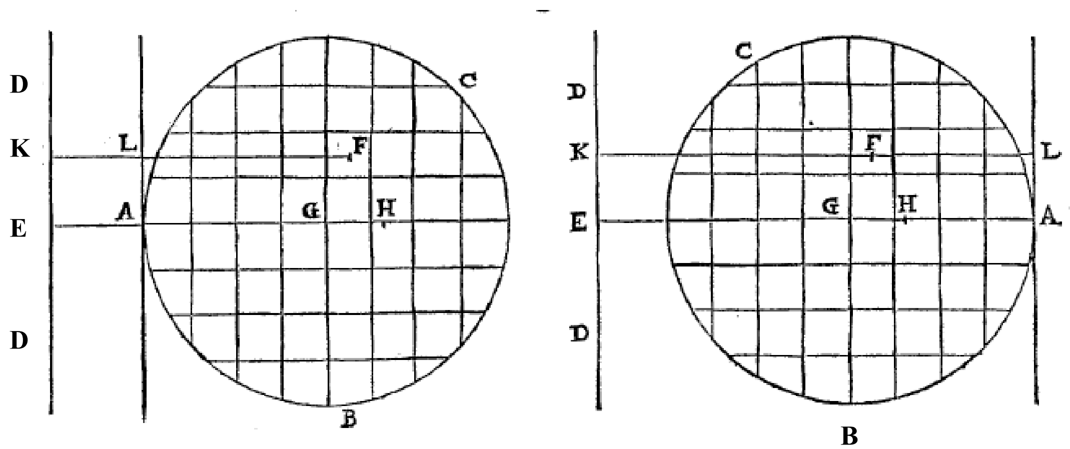
[Thus, in an obvious notation, using the previous result $V = FA \times A = FA \times N\Delta A$: $LA \times V = \sum(\Delta V_i \times d_i) = \sum \Delta A_i . d_i^2 = FA \times LA \times A = FA \times LA \times N\Delta A.$ From which, when all the N elemental areas ΔA_i are equal, it follows that $\sum d_i^2 = FA \times LA \times N.$ This result is needed in evaluating the sum of the squares over an area in conjunction with determining the sum of the squares of the elemental masses in the shape, essentially the moment of inertia of the shape, from the modern standpoint. In an obvious extension of the 3 masses considered in Prop. 5, for the case when the axis of rotation is a tangent to the plane figure, we now have : $\sum r_i^2 = FA \times LA \times N$, where FA is the distance (d) from the point of suspension to the centre of mass and LA is the distance (d') from the point of suspension to the centre of oscillation; for if we assume the area of the plate is A, and m_i = A/N where the plate has unit density, and the distances given are d and d', then $\frac{\sum m_i r_i^2)}{(\sum m_i)d} = \frac{Md'^2}{Md'}$; giving $d' = \frac{\{A/N\}\sum r_i^2}{d.A}$ or $Ndd' = \sum r_i^2$.]

PROPOSITION IX.

For a given plane figure in the same plane as a straight line, which may or may not cut the figure, on which the perpendiculars fall from the individual equal elemental areas into which the figure is understood to be divided; it is required to find the sum of the squares of all the perpendiculars; or the area multiplied by the number of the elemental areas that is equal to the said constant sum of the squares.
[This proposition is the forerunner of the modern parallel-axis theorem relating the moments of inertia about different parallel axes.]

ABC is the given plane figure, and the line ED lies in the same plane; with the figure understood to be divided into a number of equal elemental areas, and it is considered that from each and every one of these elemental areas perpendiculars are drawn to the line ED, thus as from a particular area F, FK has been drawn. It is required to find the sum of the squares formed by all these perpendiculars. [p.107]

For the line AL is drawn parallel to the given line ED, which is a tangent to the figure, and the whole of which is placed outside the figure. Moreover the tangent to the figure is



placed either on the same side of the figure as ED, or on the opposite side. Truly the distance of the centre of gravity of the figure from the line AL is the line GA, cutting ED in E; and the subcentre of the wedge, on the figure cut by the horizontal plane through the line AL is HA. I say that the sum of the squares sought is equal to the rectangle AGH together with the square EG, to be multiplied by the number of elemental areas into which the figure is understood to be divided.

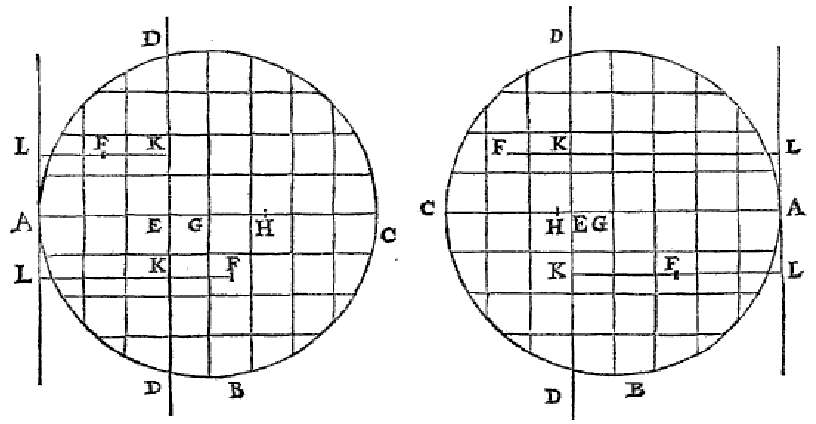
For FK crosses the tangent AL in L, even if that line is required to be produced. Thus in the first place, for the case where the line ED stand at a distance to the left of the figure, and the tangent AL has been drawn on the same side, the proposition can thus be made clear. The sum of all the squares such as FK is equal to the sum of all the squares KL, with twice the sum of all the rectangles KLF, and with the addition of the sum of all the squares LF,
[as $\Sigma FK^2.\Delta A_i = \Sigma(FL + LK)^2.\Delta A_i = \Sigma FL^2.\Delta A_i + 2\Sigma FL.LK.\Delta A_i + \Sigma LK^2.\Delta A_i$].
But the squares LK are equal to the same squares EA.
[$\Sigma LK^2\Delta A_i = \Sigma EA^2\Delta A_i$.]

And it is agreed that the sum of all the rectangles KLF are equal to the rectangle EAG, since the sum of the lines such as FL are equal to GA with the same total number of elements.
[for $\Sigma \mathrm{FL.LK}\Delta A_i = \Sigma \mathrm{EA.LF}\Delta A_i = \mathrm{EA}\Sigma \mathrm{LF}\Delta A_i = EA.AG\Sigma\Delta A_i$]
(by Prop. 2 of this section). Also, it follows from the previous proposition that the sum of the squares LF is equal to the same total number of elemental areas multiplied by the rectangle HAG, that is, the same total number of elemental areas multiplied by the square AG and with the sum of the rectangles AGH.
[i. e. $\Sigma \mathrm{LF}^2\Delta A_i = \mathrm{HA.AG} \times \Sigma\Delta A_i = (\mathrm{AG} + \mathrm{GH}).\mathrm{AG} \times \Sigma\Delta A_i = \mathrm{AG}^2 \times \Sigma\Delta A_i +$ $\mathrm{AG.GH} \times \Sigma\Delta A_i$].
Hence, the sum of the squares FK is equal to the sum of the squares EA, with the sum of twice the rectangles EAG, and in addition the sum of the rectangles AGH with the sum of the squares AG, all by the same number of elemental areas.
[*i. e*. $\Sigma \mathrm{FK}^2.\Delta A_i = \Sigma(\mathrm{EA}^2 + 2.EA.AG + \mathrm{AG}^2 + \mathrm{AG.GH}).\Delta A_i$]
And these three themselves, surely the square EA with twice the rectangle EAG and with the square AG make the square EG. Hence it is apparent that the sum of all the squares FK is equal to the sum of the squares EG, together with the sum of the rectangles AGH, for the same number of elemental areas. Q.e.d.
[*i. e*. $\Sigma \mathrm{FK}^2.\Delta A_i = \Sigma((\mathrm{EA} + \mathrm{AG})^2 + \mathrm{AG.GH}).\Delta A_i = \Sigma(\mathrm{EG}^2 + \mathrm{AG.GH}).\Delta A_i$. Hence, if the distance of the axis of rotation from the tangent is k to the left of the tangent line, then

$$\sum_{ED.} r'^2_i = \sum (r_i + k)^2 = \sum r_i^2 + 2k\sum r_i + Nk^2 = Ndd' + 2Ndk + Nk^2 = N(d+k)^2 + Ndd'',$$

where d = AG and d'' = GH. The other cases that follow can be analysed in a similar manner.]

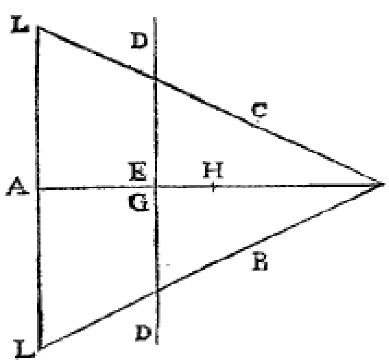
Again in all the remaining cases, [where the line DE lies within the figure, and where the base figures are different] the squares of all the FK are equal to the sum of the squares KL for the same total, less twice the whole of the same rectangles KLF, plus all the same squares LF, all for the same total number of elements; that is, the sum of the squares EA, less twice the sum of the rectangles EAG, plus the sum of the squares AG, plus all the rectangles AGH. [p.108] Thus, for all these cases, the square EA plus the square AG, less twice the rectangle EAG, is equal to the square EG.
[*i. e.* , as $\pm \mathrm{EG} = \mathrm{EA} - \mathrm{GA}$, $\Sigma \mathrm{FK}^2.\Delta A_i = \Sigma(\mathrm{EA}^2 - 2.EA.AG + \mathrm{AG}^2 + \mathrm{AG.GH}).\Delta A_i$
$= \Sigma(\mathrm{EG}^2 + \mathrm{AG.GH}).\Delta A_i$.]
Hence again, the squares of all the FK are equal to the same sum of the squares EG, together with all the rectangles AGH. Whereby the proposition is agreed upon.



Hence it follows, that the rectangle AGH has the same magnitude for a given subcentre of the wedge AH, for a tangent line AL on either side; that is, either to be cut by the one or the other tangent line AL. Thus the AG of one case is to AG of the other case, as HG of the one is to HG of the other. Therefore the lines AG are thus in the same ratio between themselves in both situations for the wedge cut by AL, as can be gathered from Prop. 7 of this section; and in the same manner for the ratios of GH to GH for the two cases.

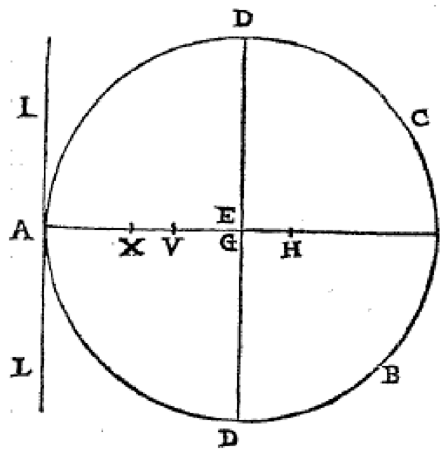
It is also apparent, that for the given centre of gravity of a plane figure G and subcentre of the wedge H, if the other wedge is cut by the other parallel tangents AL, then the subcentre of that wedge is given too.

PROPOSITION X.



With the same settings as in the preceding proposition ; if the given line ED passes through G, the centre of gravity of the figure ABC; then the sum of the squares of the distances of the individual elemental areas, into which the figure is understood to be divided, [p. 109] ***from the line ED, is equal to the rectangle AGH alone, multiplied by the number of elemental areas.***

For this has been shown, when the square EG is then zero. [We can relate this result to that in the previous theorem as follows: $Ndd' = \sum r_i^2$; then if the axis of rotation passes through the centre of gravity, or of mass, rather than along a tangent, then the new sum of squares is
$\sum_{C.of.G.} r'^2_i = \sum (r_i - d)^2 = \sum r_i^2 - 2d\sum r_i + Nd^2 = Ndd' - Nd^2 = Nd(d' - d)$, as required.]



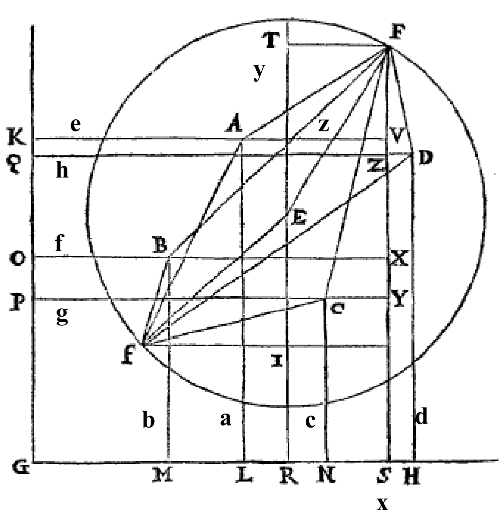
PROPOSITION XI.

Again with everything in position as in the penultimate proposition; if DE is the axis of the plane figure ABC, dividing that figure into two equal and similar parts, and in addition, VG is the distance of the centre of gravity of the left-half figure DAD from the line ED; and also GX is the subcentre of the wedge on the same figure cut by ED ; then which case the rectangle XGV **[multiplied by the number of elemental areas in the whole area ABC]** ***is equivalent to the rectangle AGH*** **[multiplied by the number of elemental areas in ABC] .**

For the rectangle XGV, multiplied by the number of elemental areas in the figure DAD, is equal to the sum of the squares of the perpendiculars from the individual elemental areas of the same half figure falling on the line ED [by Prop.8 of this section]. And hence likewise the same rectangle XGV, multiplying the sum of the individual elemental areas of the whole figure ABC, is equal to the squares of the perpendiculars sent from all the individual elemental areas of this figure to the line ED [for a rectangle congruent to XGV can be constructed on the other half-disc to which the distances on the other half-disc are summed] ; that is, to the rectangle AGH multiplied by the number of elemental areas, as agreed upon from the preceding proposition. Hence it follows that the rectangle XGV and AGH are equivalent to each other. Q.e.d.

PROPOSITION XII.

For some number of points in a plane, if some circle is drawn from the centre of gravity of these; and moreover, if lines are drawn from all the given points to some point on the circumference of the circle, [p.110] ***then the sum of the squares of all these lengths is always equal to the same amount.***



ABCD are the given points : and the centre of gravity of these, or the point at which the whole weight can be considered to act, is E ; and some circle F*f* is described with centre E, and by taking some point such as F in the circumference of this circle, the lines AF, BF, CF, and DF are drawn to this point F, from the given points A, B, C, and D. I say that the sum of all the squares taken together are equal to a certain given amount, which is always the same for whatever point F is taken on the circumference.

[The single letters a, b, c, etc, have been added by the translator for clarity; the original diagram can be found in the Latin text following the translation.]

For the lines GH and GK are drawn at right angles, and all the given points are placed in the same quadrant formed by these lines. From each point perpendiculars are sent to both lines : AL, AK; BM, BO; CN, CP; and DH, DQ. Moreover, from the centre of gravity E, and from the point F, to either of the two lines, GH or GK, perpendiculars ER and FS are drawn. And likewise, from the given points, in the same line FS, the perpendiculars AV, BX, CY, and DZ are drawn. And FT is drawn perpendicular to ER. Now again define the following :

AL = a	AK = e	radius EF = z
BM = b	BO = f	GS = x
CN = c	CP = g	
DH = d	DQ = h	

Since E is the centre of gravity of the points A, B, C, D ; if the perpendiculars AL, BM, CN, and DH are added together, and the sum is divided by the number of parts, which are the number of points given, then ER is the equivalent single line for these (by Prop. 2, of this section). Similarly, the sum of the perpendiculars AK, BO, CP, and DQ divided by the same number of parts [p. 111], gives the single perpendicular equal to all of these, drawn from E to the line GK, namely RG (by Prop. 2 of this section). Thus, if the sum of AL, BM, CN, and DH, or $a+b+c+d$ is called l : and the sum of AK, BO, CP, and DQ, or $e+f+g+h$, is called m : and the number of given points is called θ; then ER $=\frac{l}{\theta}$; & RG $=\frac{m}{\theta}$. Since GS is x, then RS or $\text{FT}=x-\frac{m}{\theta}$; or $\frac{m}{\theta}-x$, if GR is greater than GS; and the square is always $FT^2=x^2-2\frac{xm}{\theta}+\frac{mm}{\theta\theta}$, where by taking from the square FE $= zz$, the square $TE^2=zz-xx+2\frac{xm}{\theta}-\frac{mm}{\theta\theta}$ is left. Hence $TE=\sqrt{zz-xx+2\frac{xm}{\theta}-\frac{mm}{\theta\theta}}$. Moreover ER $=\frac{l}{\theta}$. Thus $TR=\frac{l}{\theta}\pm\sqrt{zz-xx+2\frac{xm}{\theta}-\frac{mm}{\theta\theta}}$. [Note: the symbol ± is not used by Huygens.] Where TR, for the sake of brevity, is called y. Now again we collect the sum of the squares of all the terms FA, FB, FC, and FD. The square of AF is equal to the sum of the squares of AV and VF. But AV is equal to the difference of the two lengths VK and AK, or of the two lengths SG and AK; and hence AV = x - e or e - x; and the square $\text{AV}^2=xx-2ex+ee$. VF is equal to the difference of the two lengths FS and VS or of the two FS and AL; and hence VF = y - a or a - y; and the square $\text{VF}^2=yy-2ay+aa$. With the squares of AV and VF added :

$\text{FA}^2=xx-2ex+ee+yy-2ay+aa$. In the same way these squares are found :

$$\text{FB}^2=xx-2fx+ff+yy-2by+bb.$$

$$\text{FC}^2=xx-2gx+gg+yy-2cy+cc.$$

$$\text{FD}^2=xx-2hx+hh+yy-2dy+dd.$$

The sum of these, if we put the squares $ee+ff+gg+hh=nn$; $aa+bb+cc+dd=kk$; is thus : $\theta\, xx-2mx+nn+\theta yy-2ly+kk.$ If a certain θ is the number of given points and likewise of the squares, and the positions are : $e+f+g+h=m,\ \&\ a+b+c+d=l.$

In that sum, if in the terms for θyy & $2ly$, $\frac{l}{\theta} \pm \sqrt{zz - xx + 2\frac{xm}{\theta} - \frac{mm}{\theta\theta}}$ is substituted in place of y, then these terms become :

$+\theta\, yy = \frac{ll}{\theta} + 2l\sqrt{zz - xx + 2\frac{xm}{\theta} - \frac{mm}{\theta\theta}} + \theta\, zz - \theta\, xx + 2zm - \frac{mm}{\theta}$. &

$-2ly = -2\frac{ll}{\theta} - 2l\sqrt{zz - xx + 2\frac{xm}{\theta} - \frac{mm}{\theta\theta}}$, or

$+\theta\, yy = \frac{ll}{\theta} - 2l\sqrt{zz - xx + 2\frac{xm}{\theta} - \frac{mm}{\theta\theta}} + \theta\, zz - \theta\, xx + 2zm - \frac{mm}{\theta}$.

& $-2ly = -2\frac{ll}{\theta} + 2l\sqrt{zz - xx + 2\frac{xm}{\theta} - \frac{mm}{\theta\theta}}$.

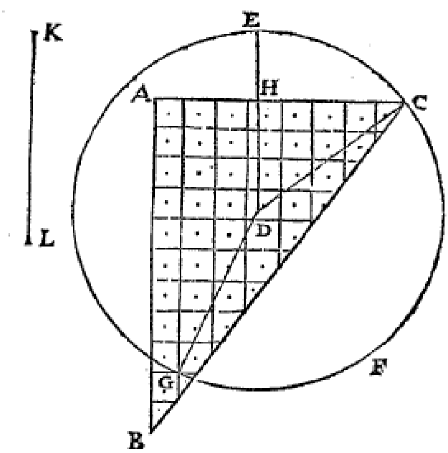
Hence, in both cases, for θyy - $2ly$ we have $-\frac{ll}{\theta} + \theta\, zz - \theta\, xx + 2xm - \frac{mm}{\theta}$. For which, with the adjacent quantities left, the preceding sum with the contents $\theta\, xx - 2xm + nn + kk$, gives the whole sum, truly the sum of the squares of FA, FB, FC, and FD $= \theta\, zz + nn + kk - \frac{mm+ll}{\theta}$. Which appears to be the constant given, since all the quantities are given; and the same is always found to be the case, wherever the point F is taken on the surface of the circle. Q.e.d.

For if some points are put in place having different weights, put in some ratios to each other, such as if point A weighs as 2, B as 3, C as 4, and D as 7, and by finding the centre of gravity of these, the circle is again described, and to a point of this circumference from the given points lines are drawn, and the squares of the individual points are taken multiplied according to the weight of each point; as the square AF is doubled, BF by three, CF by four, and DG by seven; I say that again the sum of all the [modified] squares to the given distance, and it is always the same, wherever on the circumference the point is taken. This is indeed apparent from the preceding demonstration, if we imagine the points themselves to be multiplied according to the number of the weight attributed to each weight; as it were at A there are two points joined together, three at B, four at C, and seven at D, and all of these equally heavy.

[Thus, the moment of inertia of the point masses taken on an axis at any point on the circumference of the circle with centre at the centre of mass or gravity perpendicular to the plane of the page I_Z, is equal to the sum of the moments of inertia of the same point masses about the *x* and *y* axes, I_X and I_Y; this is an extension of the modern perpendicular axis theorem, which is usually stated from the centre of mass.]

PROPOSITION XIII.

If a plane figure, or a line present in a plane, is suspended in different ways from points which are taken in the same plane, and equally distant from the centre of gravity of the points; the motion disturbed laterally is isochronous for the different points of suspension.



ABC is the plane figure, or the line present in a plane, the centre of gravity of which is D; where the circumference of the circle ECF is described in the same plane. I say, that if from some point on the circumference, such as E or G, the suspended figure is disturbed from the side; then these points of suspension are isochronous to each other and to the same simple pendulum.

Initially, the suspension is from the point E, where moreover it lies outside the figure, as here, and the line EH, by which the figure hangs, is itself considered to be fixed to the figure in a rigid manner.

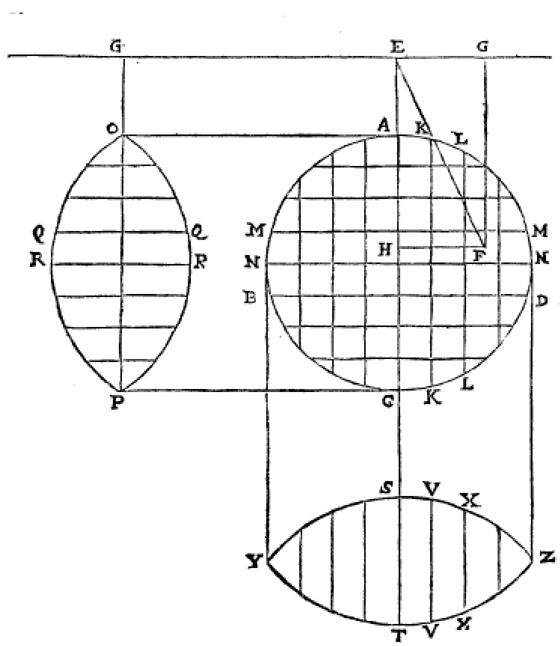
The figure ABC is understood to be divided up into [a large number of] equal elemental parts, and lines to be drawn from the centres of gravity of all these elemental areas to the point E; indeed it is observed [in the diagram], when the figure is set in motion from the side, that the motion is perpendicular to the axis about which it oscillates. Therefore the squares of all these perpendiculars, divided by the line ED, and multiplied by the number of elemental areas into which the figure has been divided, brings about the length of the equivalent simple pendulum KL that is isochronous to the figure, (by Prop. 6 of this section), [p. 113] sit KL. Moreover, with the figure suspended from the point G, the length of the simple isochronous pendulum is again found, by dividing the squares of all the lines which are drawn from all the elemental areas to the point G, by the line GD, and multiplying according to the number of elemental areas (by Prop. 6 of this section). Indeed when the points G and E lie on the circumference described with centre D, which is the centre of gravity of the figure ABC, or the centre of gravity of all the points, which are the same distance from the centre of the elemental areas of the figure; hence the sum of the squares from the lines, which are drawn from the elemental areas to the point G, is equal to the sum of the squares from the lines which are drawn from the same elemental areas to the point E (by the preceding Proposition). These sums of squares, from either point of suspension, are applied to equal magnitudes : as can be seen, by suspension from the point E, to the line ED, and multiplied by the number of elemental areas; otherwise by suspension from the point G, to the line DG, and multiplied by the same number of elemental areas. Hence it is apparent, from this more recent application, when the suspension is from the point G, to make a length of isochronous pendulum equal to that of the first application, that is, the same as to KL itself.

In the same way, if the figure is suspended from C, or from some other point on the circumference ECF, then there is agreement with the isochronous pendulum KL. Thus the proposition is agreed upon.

[This result follows immediately from the previous proposition and the earlier proposition that the appropriate moment of inertia is equal to N × ED × KL, where ED is the distance from the point of suspension to the centre of gravity or mass, and KL K is the centre of oscillation.]

PROPOSITION XIV.

For a given solid figure, and a line of indeterminate length, which either lies outside the figure, or passes through it; and with the figure understood to be divided [p.114] ***into*** [a large number of] ***equal minimal volume elements, perpendiculars are understood to be drawn from all of these volumes to the given line; and it is required to find the sum of all the squares which result from these, or to find a constant multiplying the number of elements, to which the said sum of squares shall be equal.***



ABCD is the given solid figure, and the line of indeterminate length which passes through the point E, is understood to be at right angles to the plane of the page : and which either cuts the figure, or lies completely outside the figure. It is required to find the sum of all the squares such as FE, with it being understood that the solid figure ABCD is composed from a large number of equal minimal volume elements, such as F, and the lines are drawn perpendicular to the given line through E, as is the case for FE shown here.

The figure is cut by the plane EAC, drawn through the said given line and passing through the centre of gravity of the figure. [This is a plane perpendicular to the page, and also vertical to the eye.] Likewise another plane is understood to be drawn passing through the same given line and through EG, which is itself at right angles [This is another plane perpendicular to the page, but horizontal to the eye.]

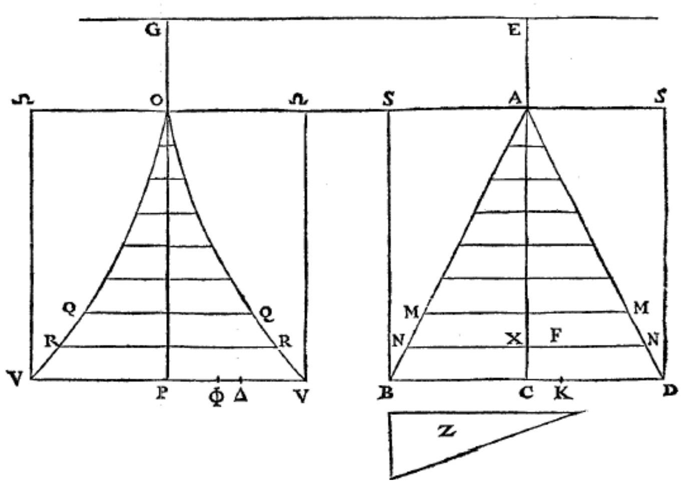
Now it is agreed, the square of this line, which is drawn from any of the said elemental volumes [p. 115], to the given perpendicular line through E, such as FE, is equal to the sum of the squares of the lines FG and FH, which are constructed perpendicular, from the same elemental volume, to the planes through EG and EC as said before. (Prop. 47, Book 1, Euclid). Whereby, if we can find the sum of the squares, which is constructed from the sums of all the perpendiculars from the elemental volumes, which lie in the said plane through EG and EC, then we will also have this sum equal to the sum of the squares from the perpendiculars, which fall on the given line through the point E from all the elemental volumes. [This theorem corresponds to the well-known result : $I_Z = I_X + I_Y$, for moments of inertia of a shape about perpendicular axis, but not through the centre of mass.]

The first sum of these squares is found as follows. The first plane figure to be given OQP is put in place, to the side of the solid figure ABCD, of the same height as that figure, which shall be constructed as follows: the sections of the plane figure OQP by the lines QQ and RR correspond to the plane sections MM and NN of the solid figure ABCD, and are parallel to these ; and the same shall be the case for all these lines between themselves and the ratio of these planes, which are taken on both sides and which correspond to each other in order; so that the line RR is to the line QQ as the plane NN to MM. Since therefore, if the plane figure OQP is understood to be divided into the same total number of equal elemental parts, as there are that make up the volume ABCD, then also in any segment of the plane figure, such as QQRR, there is the same total number of particles as there are in the segment of the solid figure MMNN.
Hence, the sum of the squares of the perpendiculars of all the elemental areas of the figure OQP in the plane EG, is equal to the sum of the squares of the elemental volumes of the perpendiculars of the solid figure, produced in the same plane EG. Moreover that sum of squares will be produced, if they are given in the figure OQP, in the wedge constructed for that plane figure, which was required by proposition 9 of this section. Hence with these given, the sum of the squares is given too, from the perpendiculars which are drawn in the plane EG from all the elemental volumes of the solid ABCD.

Now, likewise, the plane figure SYTZ is put in place; which is the same length as the solid figure ABCD, *i. e.*, bounded by the tangent planes BY and DZ, and parallel to the plane EAC. This figure is of such a kind that a section by the lines VV and XX, and by lines parallel to these, gives the same ratio of these lines between themselves as of the planes of the solid figure, if they are taken to correspond mutually to each other. Thus again the sum of the squares of the perpendiculars, from the individual elements of the figure SYTX and falling on the line ST, is equal to the sum of the squares of the perpendiculars which are drawn in the plane AC, from the elemental volumes of the solid figure ABCD. Moreover, the sum of these squares will be given, if the distance of the centre of gravity of the figure SYTZ from the lines BY or DZ are known; and in the same manner the distance of the centre of gravity of the wedge [p.116] from the plane is found, itself being cut by the same line (by Prop. 9. of this section). Or, for the figure SYTZ with some ordinates present, as ST shall be the axis of this figure, the same sum of the squares is given, if the distance of the centre of gravity of the half figure SZT from the axis ST is given, and likewise the centre of gravity of the wedge, drawn upon the same half figure, with the plane cut by the axis (by Prop. 11. of this section). Hence, with these given, the sum of the squares from the perpendiculars is also given which, coming from all the elemental volumes of the solid figure ABCD, are understood to be drawn in the plane EAC. Moreover we have already found the sum of the squares from all the perpendiculars in the plane drawn through EG. Hence the aggregate of both these sums can be obtained, *i. e*., as shown above, the sum of the squares of the perpendiculars which, from all the elemental areas of the solid figure ABCD, fall on a given line passing through E, and which has been set up perpendicular to the plane of the page. Q.e.d.
[Thus, the method is set out in a general way; the nature of the solid dictates the particular form of these projected volumes into equivalent plane areas.]

PROPOSITION XV.

With the same figure in place, it may happen that the solid figure ABCD is of such a kind that the centre of gravity of the plane figure SYTZ in proportion to the solid figure, is not given as a known distance from the tangents BY or DZ [see diagram in previous prop.] ***; or equivalently, the subcentre of the wedge to be cut by the plane through the same BY or DZ may not known. Nevertheless, in such circumstances for the proportional plane figure OQP viewed on the left of ABCD*** [in the above diagram]***, if the distance Φ P is given, by which the centre of gravity of the half figure OPV departs from the line OP, then it is possible to find the sum of the squares of the distances of the elemental volumes of the solid figure ABCD relative to the plane EC. Moreover, it is required that all sections such as NN and MM are similar to the plane sections, and the centre of gravity of the sections passes through the plane EC, as occurs for prisms, pyramids, cones, and for conoids, and many other figures. And it is necessary to be given the distances of the centres of gravity of these plane sections from the axis of oscillation above the parallel tangents out of the plane of the page ; and the subcentre of the wedge to be used can be put in place on the plane figure, and with the planes of the wedge drawn passing through the same tangent lines.***



For example, if BD is the greatest of these sections, and it is understood that a line is drawn through B parallel to the axis through E out of the page, *i. e.*, at right angles to the plane which is seen here, and it is necessary to be given the distance of the centre of gravity of the section BD from the given line through B, which is BC; and likewise the subcentre of the wedge, to be cut upon the section BD, by the plane drawn through the line passing through B normal to the plane of the page, which is the subcentre BK.

[Note that the base of the solid conical shape ABCD is shown here from an end elevation, and may be taken as some arbitrary oval or rectilinear shape not shown in detail, and this shape is the plane figure for the base of the wedge formed on the section ; the subcentre K hence lies beyond C as before; points that correspond to these points C and K lie on vertical lines that eventually meet at A as the sections are decreased in size; also, OVV is the plane figure that corresponds to the weight of the solid figure ABD (one might even imagine the solid figure squashed flat with only lateral distortion); if the areas of the sections of the solid figure increase as the squares of the distances from the central axis AC, then the bounding curves are parabolic in the left-hand plane figure.]

And indeed with these things given, and with PV bisected in Δ, if one makes the ratio ΔP to PΦ, thus as the rectangle BCK is to some area Z [to be determined] ;

[thus Z takes the place of the constant in the proportionality which has to be found for individual cases; the plane ratio is determined for the sum of the squares for the half-figure, while the solid figure has the sum of the squares evaluated from the central axis.] I say that this rectangle Z itself, multiplied by the number of elemental volumes of the solid figure ABCD, is equal to the sought sum of the squares of the distances of the same elemental volumes from the plane EC. [The Proposition to be proven; it is fairly obvious what the rectangle Z must be equal to.]

For the squares of the distances of the individual elements lying in the plane of the section BD from the plane EC passing through its centre of gravity ; or the squares of the distances of the individual elements of the solid segment BNND from the same plane, is agreed to be equal to the rectangle BCK, multiplied by the number of given elemental volumes (by Prop. 8 of this section). [i. e., the sum of the squares of the elemental areas from EC is first established for the plane area BD.]

[Also, recall proposition 9 for the case in which EG, or the axis about which the squares are taken, passes through the centre of gravity of the shape, that the sum of the squares of the elements in the plane figure, to which BNND approximates, is equal to BC.CK × number of elemental elements in the segment, which we can consider to be a cylinder of infinitesimal height and of some particular cross-section.]

Similarly, if the distance of the centre of gravity for the plane of the section NN is NX, from a tangent line passing through N parallel to the axis through E out of the page; then the subcentre of the wedge cut on that section is the line NF; and the sum of the squares from the distances of the elemental areas of the section NN from the plane EC, (or the squares from the distances of the individual elemental volumes of the segment of the solid figure NMMN, from the same plane), is equal to the rectangle NXF, to be multiplied by the number of elements in the section NN, or of the segment NMMN, [on repeating the wedge analysis].

Moreover BD and NN have been divided similarly in C and K, and in X and F. Hence the rectangle BCK is to the rectangle NXF as the square BD is to the square NN. [That is: BC/BD = NX/NN from symmetry, and CK/BD = XF/NN from similar triangles associated with the wedges; hence, $\frac{BC.CK}{NX.XF} = \frac{BD^2}{NN^2}$, and

$$\frac{\sum_{BD} x_i^2 . \Delta A_i}{\sum_{NN} x_i^2 . \Delta A_i} = \frac{BC.CK \times N_{BD}}{NX.XF \times N_{NN}} .]$$

Now, the ratio of the number of individual elements in the section BD, to the number of individual elements in the section NN, is as the areas of the sections themselves, *i. e.*, as the square BD to the square NN.

[*i. e.* $\frac{N_{BD}}{N_{NN}} = \frac{BD^2}{NN^2} = \frac{VV}{RR}$.]

Thus the rectangle BCK, multiplied by the number of individual elements of the section BD, to the rectangle NXF, multiplied by the number of individual elements of the section NN, is as the square of the ratio of the square BD to the square NN; *i. e.*, that which the square of VV has to the square RR, in the proportional plane figure.

[$\frac{BC.CK.N_{BD}}{NX.XF.N_{NN}} = (\frac{BD^2}{NN^2})^2 = (\frac{VV}{RR})^2$.]

Hence the first mentioned sum of the squares, for the distances of the elements BNND from the plane EC is to the other sum of squares for the distances of the elements of the segment NMMN, are as the square VV to the square RR.

[$\frac{\sum_{BD} x_i^2 .\Delta A_i}{\sum_{NN} x_i^2 .\Delta A_i} = \frac{BC.CK \times N_{BD}}{NX.XF \times N_{NN}} = \frac{VV^2}{RR^2}$.]

Likewise by the same reasoning, it can be shown that the sums of the squares for the distances of the elements in the remaining segments of the solid figure ABCD are between themselves in the ratio of the squares of the lines in the figure OVV, and which correspond to the base of this segment. Whereby the sum of the squares, from the distances of all the individual elements of all the segments of the solid ABCD to the plane EC, to the sum of the squares of just as many individual elements of the segments, equal to the maximum segment, *i. e*., of the cylinder or prism BDSS, having the same base and height as the solid figure ABCD, is in the same ratio as the sum of the squares of the lines VV, RR, QQ, etc., to the square of the same amount equal to VV, *i.e*., as the solid of revolution OVV about the axis OP, to the cylinder VVΩΩ, which has the same base and altitude; hence indeed, to the ratio of the solid figure OVV to the cylinder VVΩΩ, which has the same base and height.

[Thus, if the moment of inertia of the solid figure is I_{ABC}, with the other moments of inertia labeled in a like manner, then $I_{ABC}/I_{SSBD} = I_{OVV}/I_{\Omega\Omega VV}$, or : $\frac{\sum_{ABCD} x_i^2 .\Delta A_i}{\sum_{BDSS} x_i^2 .\Delta A_i} = \frac{\sum_{OVV} VV_i^2}{\sum_{\Omega\Omega VV} VV^2}$.]

Hence, the ratio of the solid figure of rotation OVV to the cylinder VVΩΩ, is agreed to be composed from the ratio of the planes from which they are generated by rotation about OP, *i.e*., from the ratio of the plane figure OPV, to the rectangle PΩ, and from the ratio of the distances by which the centres of gravity of these are distant from the axis OP; *i. e*., by the ratio PΦ to PΔ.

[*i. e.* $\frac{I_{OVV}}{I_{VV\Omega\Omega}} = \frac{\sum_{OVV} x^2 .\Delta A_i}{\sum_{VV\Omega\Omega} x^2 .\Delta A_i} = \frac{\text{area } OPV}{\text{rect.} P\Omega} \times \frac{P\Phi}{P\Delta}$]

And indeed the first ratio of these, surely that of the plane OPV to the rectangle PΩ, is the same as that which the solid figure ABCD has to the cylinder or prism BDSS, *i.e*., the same as of the number of individual elements of the solid figure ABCD, to the number of individual elements of the cylinder or prism BDSS. Truly the other ratio, surely PΦ to PΔ, is the same by construction as that which the area Z has to the rectangle BCK.

[*i. e.* $\frac{\text{area } OPV}{\text{rect.} P\Omega} = \frac{\text{area } ABCD}{\text{rect.} BDSS}$; and $\frac{P\Phi}{P\Delta} = \frac{\text{area } Z}{\text{rect.} BCK}$.]

Thus again, the said sum of the squares, from the distances of all the individual elements of the solid figure ABCD to the plane EC, to the sum of the squares, from the distances of all the individual elements of the cylinder or prism BDSS from the same plane, has that ratio which is composed first from the ratio of the number of individual elements of the solid figure ABCD to the number of individual elements of the cylinder or prism BDSS, and second from the ratio of the area Z to the rectangle BCK: *i. e*., the ratio which the rectangle Z has, multiplied by the number of elemental prisms or cylinders BDSS.

$$[\frac{I_{ABCD}}{I_{BDSS}} = \frac{\sum\limits_{ABCD} x^2.\Delta A_i}{\sum\limits_{BDSS} x^2.\Delta A_i} = \frac{N_{ABCD}}{N_{BDSS}} \times \frac{\text{area } Z}{\text{rect.} BCK}.]$$

But the fourth of the magnitudes is equal to the second sum of squares : surely the rectangle BCK, multiplied by the number of individual elements of the cylinder or prism BDSS is equal to the sum of the squares from the distances of the individual elements of the same prism or cylinder BDSS from the plane EC; and if indeed the same rectangle BCK is multiplied by (p.119) the number of individual elements of the segment BNND, then it is equal to the sum of the squares of the distances of the individual elements of the same segment from the plane EC (by Prop. 8, of this section). Thus the third is equal to the first part of the ratio: surely the plane Z multiplied by the number of individual elements of the solid figure ABCD, is equal to the sum of the squares from the distance of the individual elements of the same ABCD to the plane EC. (by Prop. 14, book 5, Euclid). Q.e.d.

Indeed it is to be noted, when the solid ABD is a solid of revolution about the axis AC, that the same rectangle BCK is made equal to the fourth part of the square BC; since the subcentre of the wedge, from the cut upon the circle BD by the plane drawn through the tangent at B, is the line CK which is equal to $\frac{5}{4}$ of the radius BC. Hence, if PV is placed equal to BC, it follows, by making the ratio PΔ to P Φ thus as the rectangle BCK to the area Z, or $\frac{1}{4}$ of the square BC to the area Z; that is, equal to PΔ squared to Z; then the area Z is equal to the rectangle ΔPΦ.

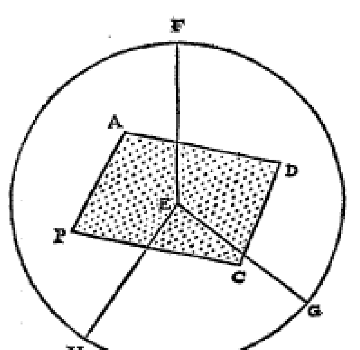
[*i. e.*, $\frac{P\Delta}{P\Phi} = \frac{\text{rect.} BCK}{\text{area } Z} = \frac{\frac{1}{4}BC^2}{\text{area } Z} = \frac{P\Delta^2}{\text{area } Z}$; from which the area $Z = P\Delta.P\Phi$.]

And then hence the rectangle ΔPΦ , multiplied by the number of elemental volumes of the solid figure ABD, is equal to the sought sum of the squares from all the perpendiculars, which fall on the plane EC from the same elements.

PROPOSITION XVI.

Some figure is to be considered, whether it is a line, surface or volume; if it is suspended in different ways, and set in motion upon axes between parallel lines which are equidistant from the centre of gravity of the figure, then these oscillations are isochronous to each other.

[This proposition is concerned with finding a simple plane figure onto which the mass of an arbitrary pendulum can be projected, to give a pendulum with the same period, however it is suspended from the circumference of a given circle.]



A magnitude of some description is put in place, and the centre of gravity of this shape is the point E, and it is understood to be suspended from an axis passing through F at right angles to the plane of the page. Thus the plane of oscillation is in the same plane; in which, with centre E and radius EF, the circumference FHG is described; and from some other point on the circumference such as H, the object is again suspended and set in motion, with the oscillations in the same plane. I say that these oscillations about H are isochronous with these set in motion about the axis through F. [p.120]

Indeed it is understood that the proposed magnitude is divided up into a large number of equal elemental shapes, and thus from both points of suspension the plane of oscillation remains the same, with regard to the parts of the magnitude under consideration. It can be seen, if from all the elemental parts into which the magnitude is divided, perpendiculars fall that can be gathered together in the said plane of oscillation [of the page], and that these will occur for the same points from both points of suspension. Moreover, these are the points which are apparent in the area [as the feet of the perpendiculars] shown in the area ABCD.

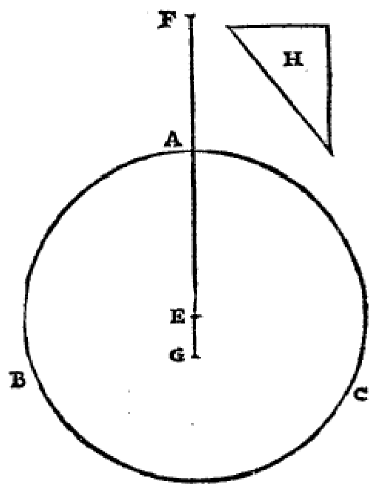
[Thus, a projection of the mass of the solid figure onto the plane of oscillation ABCD is considered, thus the object is a little more general than in the previous proposition, which was restricted to conoidal or pyramidal shapes.]

Therefore, since E is the centre of gravity of the proposed magnitude, equilibrium is maintained for any point about this axis, which passes through the point E at right angles to the plane ABCD ; and it is easily seen, since if all the aforementioned points are signified in the area ABCD, for which the weight is equally distributed, then the centre of gravity for these points also will be E always. Since truly, in order that this can be done, if in any point several perpendiculars are to coincide, then these points are understood to give rise to doubles in the total, and a multiple of these for the whole weight is to be taken. Thus from these considerations, it is apparent that the centre of gravity is again the point E.

Again it is apparent that the sum of the squares from the lines, which are drawn from all the given points to the point F, is the same as the sum of the squares from these lines [in the plane ABCD], which can be drawn from the individual elements of the proposed magnitude perpendicular to the axis of oscillation passing through F; obviously for the lines themselves, for which the squares are understood, the lines are understood to have the same length in both places. Similarly also, when the suspension is from an axis through H, it is apparent that the sum of the squares from the lines, which are drawn to the point H from all the points in the designated area ABCD, is the same as the sum of the squares for these lines drawn from all the elements of the proposed magnitude, perpendicular to the axis of oscillation passing through H. Hence in either case, if the sum of the squares from the lines which, from all the points mentioned, are drawn to the point F or H, is divided by the lines EF or EH, multiplied by the number of elements into which the proposed magnitude is understood to be divided, there will arise from this application the length of the simple pendulum which is isochronous with the magnitude suspended from F or H.

Moreover the sum of the squares in both cases is the same (Prop. 11, of this section) ; and the lines EF and EH are equal to each other also; and have the same number of elemental parts. Hence, when the amounts shown in the working are equal, and from which these are applied, both are equal to each other, also the lengths arising from the divisions are equal to each other, that is, the lengths of the isochronous pendulums of the proposed magnitude suspended from F or H. By which the proposition is agreed upon.

[Thus, the sum of the squares of the equivalent masses in the plane area is constant, as is the moment of inertia about any point on the circumference, and hence the radius of oscillation is also constant, leading to isochronous oscillations.]



PROPOSITION XVII. [p.121]

If a given rectangular area, on multiplication by the number of elemental parts into which the suspended figure is understood to be divided, gives a product equal to the squares of all the distances from the axis of oscillation; then if that result is divided by the length of the line equal to the distance between the axis of oscillation and the centre of gravity of the suspended magnitude, then the length of the simple pendulum to which it is isochronous arises.

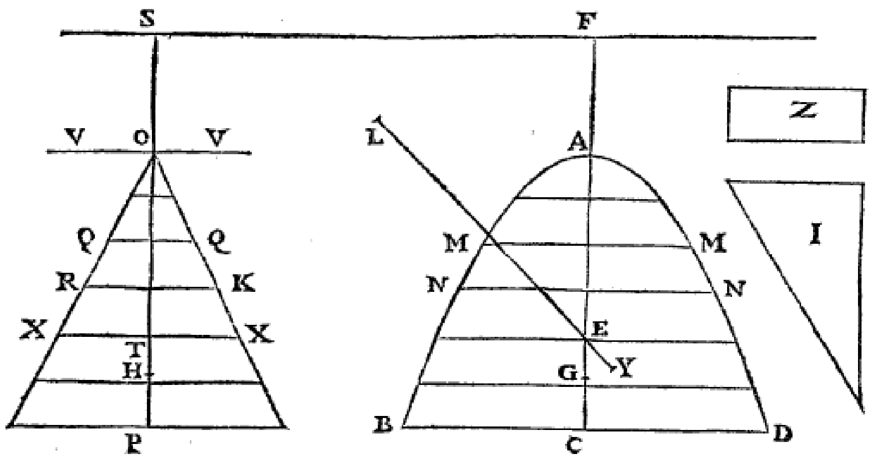
ABC is the figure, the centre of gravity of which is E, suspended from the axis which is set up through the point F to the observed plane. With the figure divided up into the equal small elements, from which all the perpendiculars are understood to fall on the axis of oscillation : let this figure to be found equal H, which multiplied by the number of said elemental parts, as shown above in the previous propositions, to be equal to the sum of the squares of all the said perpendiculars. With the division of the plane H by the line FE, the length FG is found. I say that this length is equal to the length of the equivalent simple pendulum, having isochronous oscillations to the magnitude ABC, if it is disturbed about the axis through F.
[By a generalisation of Proposition 6, the length of the equivalent simple pendulum is :
$FG = \frac{\sum_F x_i^2 .\Delta A_i}{\sum_F x_i .\Delta A_i} = \frac{I_F}{FE.A}$, where A is the area of the swinging object, assuming the densities have cancelled out.]

For indeed the sum of the squares for the distances from the axis F, divided by the distance FE multiplied by the number of elemental parts, makes the length of the simple isochronous pendulum (by Prop. 6, of this section). Truly the plane H is equal to this sum of the squares multiplied by the same number of elemental particles. Hence the plane H, multiplied by the same number of elemental parts, divided by the distance FE, multiplied [p.122] by the number of elemental parts; or, with the common multiplication omitted, if the plane H is divided by the distance FE, then there arises also the length of the simple isochronous pendulum. Thus the length EG is agreed upon. Q.e.d.

PROPOSITION XVIII.

If a given rectangular area, multiplied by the number of elemental particles of the suspended magnitude, is equal to the squares of the distances from the axis of gravity, parallel to the axis of oscillation; that area, I say, if divided by the line equal to the distance between both the said axes, gives a line equal to the interval by which the centre of oscillation is below the centre of gravity of the same magnitude.

Let ABCD be the given magnitude, the centre of gravity of which is E; and which is suspended from the axis, which is understood to pass through the point F on a line at right angles to the plane of the page, and which has the centre of oscillation G. Again another



axis is understood to pass through the centre of gravity G, parallel to the axis through F. With the known magnitude divided into the smallest elemental parts, the plane I multiplied by the number of elemental particles is equal to the squares of the distances from the said axis through E. By dividing the plane I by the distance FE, a certain line is produced. I say that line is equal to the interval EG, by which the centre of oscillation lies below the centre of gravity of the magnitude ABCD.

Indeed a general demonstration is given in order that we can understand this proposition : the plane figure OQP is understood to be analogous to the magnitude ABCD, adopted at the side ; indeed the plane figure is cut by the same horizontals as the magnitude ABCD, and which has segments intercepted between two planes, in the same ratio between themselves with the segments of the said magnitude, to which they themselves correspond ; and the individual segments of the figure OQP, divided into all the equal elemental parts, which are contained by the corresponding segments themselves in figure ABCD. Moreover these ratios can be formed for any kind of magnitude ABCD, either a line, or a surface, or a volume. Truly the centre of gravity of the figure OQP, shall always be T, with the same height to be shown with the centre of gravity of the magnitude ABCD; and likewise, if the plane to the horizontal is drawn through F, it cuts the line through the centre of the figure OQP, as you wish here in S, with equal distances ST and FE.

Moreover again it is agreed that the squares of the distances from the axis of oscillation F, divided by the distance FE multiplied by the number of elemental parts, gives the length of the isochronous pendulum (by Prop. 6 of this section); and which length can be put as FG. Truly the sum of these squares has been seen to be equal to the squares of the distances from the horizontal plane through F, together with the squares of the distances from a plane FE drawn to the vertical, passing through the axis F and the centre of gravity (Prop. 47, book1, Euclid). And the sum of the squares of the distances of the magnitude ABCD from the plane to the horizontal through F, is equal to the squares of the distances of the of the figure OQP from the line SF. Which squares (if O is the highest point in the figure OQP, and H the centre of gravity of the wedge drawn on that section by the plane through the line OV, parallel to SF) are equal to the rectangle OTH and the square ST, multiplied by the number of elemental parts of the said figure (by Prop. 9 of this section), or of the magnitude ABCD. Truly the sum of the squares of the distances of the magnitude ABCD from the plane FE, and the distance of the axis of oscillation F from the centre of gravity E, are always the same : that hence we can put equal to the area Z, to be multiplied by the number of elemental parts of the magnitude ABCD.

Thus since the squares of the distances of the magnitude ABCD, from the axis of oscillation F, are equal to those, surely the sum of the square ST with the rectangle OTH and the plane Z, multiplied by the number of elemental parts of the same magnitude; if all of these are divided by the length FE or ST, the length of the pendulum isochronous to the magnitude ABCD is produced (by Prop. 6 of this section). But from the division of the square ST by its side ST, ST or FE is produced. Hence the rest EG is that which arises from the division of the rectangle OTH and the plane Z, by the same ST or FE.

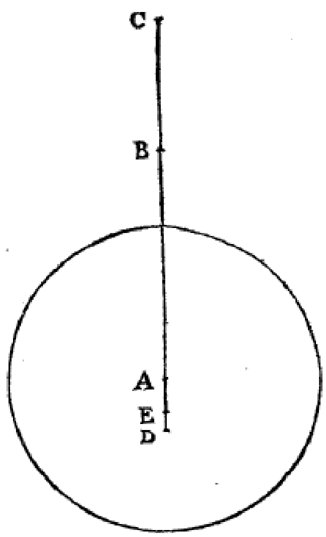
Wherefore it remains for us to show that the rectangle OTH with the plane Z is equal to the plane I. For then it will be agreed, that the plane I, divided by the distance FE, to give rise to a length equal to EG itself. Moreover, that can be shown as follows. The rectangle OTH, multiplied by the number of elemental parts of the figure OQP, or of the magnitude ABCD, [p. 124] is equal to the sum of the squares of the distances of the figure from the line XT (by Prop. 10 of this section), which is drawn through the centre of gravity T parallel to SF ; and hence also to the squares of the distances of the magnitude ABCD, from the plane KK to the horizontal, drawn through the centre of gravity E; when both distances are the same. But the plane Z, similarly multiplied, has been put equal to the squares of the distances of the magnitude ABCD from the plane FE to the vertical. And it is indeed apparent that squares of the distances from the plane FE, together with the said squares of the distances from the plane to the horizontal through E, are equal to the squares of the distances from the centre of gravity through E, which are parallel to the axis F parallels (Prop. 47, book 1. Euclid). Thus the rectangle OTH together with the plane Z, multiplied by the number of elemental parts of the magnitude ABCD, is equal to the squares of the distances of the same magnitudes from the axis through E. But also the plane I, multiplied by the same number of elemental parts is equal to the same squares of the distance. Hence the plane I is equal to the rectangle OTH taken with the plane Z. Which remained to be shown.

Hence again it can be shown, as was shown in Proposition 16, that for any magnitude whatever, if it is suspended from one place or another and set in motion, from parallel axes which are equidistant from its centre of gravity, then the motions are isochronous to each other.

Indeed, with the magnitude ABCD suspended either from the axis F or from the axis L parallel to this; it is apparent that both have the same sum of the squares from the axis through E, from the parallel axes F or L. Hence also the plane I, multiplied by the number of elemental parts of this shape, in either case is equal to the same sum of the squares. Truly this plane, divided by the distance of the centre of gravity from the axis of oscillation, which in either case is put as the same, gives the distance which the centre of oscillation lies below the centre of gravity. Hence also this distance in either case will be the same. Just as if, with the suspension made from I, the said distance will be EY, which is itself equal to EG; and the total YL is equal to GF; and thus in each suspension the same simple pendulum will be isochronous to the magnitude ABCD.

PROPOSITION XIX.

If the same magnitude, now with a shorter or longer suspension, is set in motion; thus the distances of the axis of oscillation from the centre of gravity are to each other, thus in the inverse ratio of the distances of the centres of oscillation from the same centre of gravity.



The magnitude, the centre of gravity of which is A, is suspended first and set in motion from the axis B, hence indeed from the axis C ; the centre of oscillation in the first case is D, and in the second case it is E. [p.125]. I say that as BA is to CA thus EA is to DA.

As indeed, when suspended from B, the distance AD is found, which truly is the distance that the centre of oscillation is below the centre of gravity, by dividing a certain area by the distance BA, multiplied by the number of equal elemental parts into which the magnitude is understood to have been divided, equal to the sum of the squares of the distances from the axis through A, parallel to the axis in B(by the preceding Proposition); hence the rectangle BAD is equal to the said area. Likewise, for the suspension from C, when the distance is made equal to AE, by dividing the said area by the distance CA; and the rectangle CAE has an equal area. Thus the rectangles BAD and CAE are equal to each other; and hence the ratio BA to CA is the same as AE to AD. Q.e.d.

Hence it is apparent, for a given simple pendulum, what the length of the isochronous suspension shall be for one suspension, with the centre of gravity of this given; also for any other suspension, shorter or longer, provided it remains in the same plane of oscillation, the length of the isochronous pendulum will be given.

PROPOSITION XX.

The centre of oscillation and the point of suspension are interchangeable between each other.

In the above figure , for which, with the position of the suspension from B, the centre of oscillation is D; also by inverting everything, and with the suspension put from D, [p.126] then B will be the centre of oscillation. For this can be shown from the preceding proposition.

END OF PART IVA.

HOROLOGII OSCILLATORII

PARS QUARTA.

De centro Oscillationis.

Centrum Oscillationis, seu Agitationis, investigationem olim mihi, fere adhuc puero, aliisque multis, doctissimus Mersennus proposuit, celebre admodum inter illius temporis Geometras problema, prout ex litteris ejus ad me datis colligo, nec non ex Cartesii haud pridem editis, quibus ad Mersennianas super his rebus responsum continetur. Postulabat autem centra illa ut invenirem in circuli sectoribus, tam ab angulo quam a medio arcu suspensus, atque in latus agitatis, item in circuli segmentis, & in triangulis, nunc ex vertice, nunc ex media basis pendentibus. Quod eo redit, ut pendulum simplex, hoc est, pondus filo appendum reperiatur ea longitudine, ut oscillationes faciat temporum eorundem ac illae, uti dictum est, suspensae. Simul vero pretium operae, si forte quaesitis satisfecissem, magnum sane & invidiosum pollicebatur. Sed a nemine id quod desiderabat tunc obtinuit. Nam me quod attinet, cum nihil reperirem quo vel primus aditus ad contemplationem eam patesceret; velut a limine repulsus, longiori investigatione tunc quidem abstinui. Qui vero rem sese confecisse sperabant viri insignes, Cartesius, Honoratus Fabrius, aliique, nequaquam scopum attigerunt, nisi in paucis quibusdam facilioribus, sed quorum tamen demonstrationem nullam idoneam, ut mihi videtur, attulerunt. Idque comparatione eorum quae hic trademus manifestum fore spero, si quis forte quae ab illis tradita sunt, cum nostris hisce contulerit; quae quidem & certioribus principiis demonstrata arbitror, & experimentis prorsus convenientia reperi. Occasio vero ad haec denuo tendanda, ex pendulorum automati nostri temperandorum ratione oblata est, dum pondus mobile, praeter id quod in imo est, illis applico, ut in descriptione horologii fuit explicatum. Hinc melioribus auspiciis atque a prima origine rem exorsus, tandem difficultates omnes superavi, nec tantum problematum Mersennianorum solutionem, sed alia quoque illis difficiliora reperi, [p. 92] & viam denique, qua in lineis, superficiebus, solidisque corporibus certa ratione centrum illlud investigare liceret. Unde quidem, praeter voluptatem inveniendi quae multum ab aliis quaesita fuerant, cognoscendique in his rebus naturae leges decretaque, utilitatem quoque eam cepi, cujus gratia primo animum ad haec applicueram, reperta illa horologii temperandi ratione facili & expedita. Accessit autem hoc quoque, quod pluris faciendum arbitror, ut certae, saeculisque omnibus duraturae, mensurae definitionem absolutissimam per haec tradere possem ; qualis est ea quae ad finem horum adjecta reperietur.

DEFINITIONES.

1.

Pendulum dicatur figura quaelibet gravitate praedita, sive linea fuerit, sive superficies, sive solidum, ita suspensa ut circa punctum aliquod, vel axem potius, qui plano horizontalis parallelus intelligitur, motum reciprocum vi gravitatis suae continuare possit.

II.

Axis ille horizontalis plano parallelus, circa quem penduli motus fieri intelligitur, dicatur axis Oscillationis.

III.

Pendulum simplex dicatur quod filo vel linea inflexili, gravitatis experte, constare intelligitur, ima sui parte pondus affixum gerente; cujus ponderis gravitas, velut in unum punctum collecta, censenda est.

IV.

Pendulum vero compositum, quod pluribus ponderibus constat, immutabiles distantias servantibus, tum inter se, tum ab axe Oscillationis. Hinc figura quaelibet suspensa, ac gravitate praedita, pendulum compositum dici potest, quatenus cogitatu in partes quotlibet est divisibilis.

V.

Pendula isochrona vocentur, quorum Oscillationes, per arcus similes, aequalibus temporibus peraguntur. [p. 93]

VI.

Planum Oscillationis dicatur illud, quod per centrum gravitatis figurae suspensae duci intelligitur, ad axem oscillationis rectum.

VII.

Linea centri, recta quae per centrum gravitatis figurae ducitur, ad axem oscillationis perpendicularis.

VIII.

Linea perpendiculi, recta in plano oscillationis, ducta ab axe oscillationis, ad horizontis planum perpendicularis.

IX.

Centrum oscillationis vel agitationis figurae cujuslibet, dicatur punctum in linea centri, tantum ab axe oscillationis distans, quanta est longitudo penduli simplicis quod figurae isochronum sit.

X.

Axis gravitatis, linea quaevis recta, per centrum gravitatis figurae transiens.

XI.

Figura plana, vel linea in plano sita, in planum agitari dicatur, cum axis oscillationis in eodem cum figura lineave est plano.

XII.

Eadem vero latus agitari dicantur, cum axis oscillationis ad figurae lineaeve planum rectus est.

XIII.

Quando pondera in rectas lineas duci dicentur, id ita est intelligendum, ac si numeri lineaeve, quantitates ponderum rationemque inter se mutuam exprementes, ita ducantur.

HYPOTHESES.

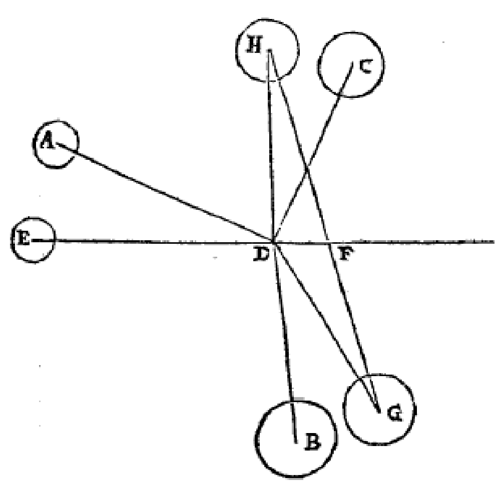
I.

Si pondera quotlibet, vi gravitatis suae, moveri incipiant; non posse centrum gravitatis ex ipsis compositae altius, quam ubi incipiente motu reperiebatur, ascendere. [p.94.]

Altitudo autem in his secundum distantiam a plane horizontali consideratur, graviaque ponuntur ad hoc planum, secundum rectas ipsi perpendiculares, descendere conari. Quod idem ab omnibus, qui de centro gravitatis egerunt, vel ponitur expresse, vel a legentibus supplendum est, cum absque eo centri gravitatis consideratio locum non habeat.

Ipsa vero hypothesis nostra quominus scrupulum moveat, nihil aliud sibi velle eam ostendemus, quam quod nemo unquam negavit, gravia nempe sursum non ferri. Nam primo, si unum quodpiam corpus grave proponamus, illum vi gravitatis suae altius ascendere non posse extra dubium est. ascendere autem tunc intelligitur scilicet, cum ejus centrum gravitatis ascendit. Sed & idem de quotlibet ponderibus, inter se per lineas inflexiles conjunctis, concedi necesse est, quoniam nihil vetat ipsa tanquam unum aliquod considerari. Itaque neque horum commune gravitatis centrum ultro ascendere poterit.

Quod si jam pondera quotlibet non interse connexa ponantur, illorum quoque aliquod commune centrum gravitatis esse scimus. Cujus quidem centri quanta erit altitudo, tantam ajo & gravitatis ex omnibus compositae altitudinem censeri debere; siquidem omnia ad eandem illam centri gravitatis altitudinem deduci possunt, nulla alia accersita potentia quam quae ipsis ponderibus inest, sed tantum lineis inflexibus ea pro lubitu conjungendo, ac circa gravitatis centrum movendo; ad quod nulla vi neque potentia determinata opus est. Quare, sicut fieri non potest ut pondera quaedam, in plano eodem horizontali posita, supra illud planum, vi gravitatis suae, omnis aequaliter attollantur; ita nec quorumlibet ponderum, quomodocunque dispositorum, centrum gravitatis ad majodrem quam habet altitudinem pervenire poterit. Quod autem diximus pondera quaelibet, nulla adhibita vi, ad planum horizontale, per centrum commune gravitatis eorum transiens, perduci posse, sic ostendetur.



Sint pondera A, B, C, positione data, quorum commune gravitatis centrum sit D, per quod planum horizontale ductum ponatur, cujus sectio recta EF. Sint jam lineae inflexiles DA, DB, DC, quae pondera sibi invariabiliter connectant; quae porro moveantur, donec A sit in plano EF ad E. Virgis vero omnibus per aequales angulos delatis, erunt jam B in G, & C in H.

Rursus jam B & C connecti intelligantur virga HG, quae secet planum EF in F; ubi necessario quoque erit centrum gravitatis binorum [p. 95] istorum ponderum connexorum, cum trium, in E, G, H, positorum, centrum gravitatis sit D, & ejus quod est in E, centrum gravitatis sit quoque in plano EDF. Moventur igitur rursus pondera H, G, super puncto F, velut axe, absque vi ulla, ac simul utraque ad planum EF adducuntur, adeo ut jam tria, quae prius erant in A, B, C, ad ipsam sui centri gravitatis D altitudinem, suo ipsorum aequilibrio, translata appareat. quod erat ostendendum. Eademque de quotcunque aliis est demonstratio.

Haec autem hypothesis nostra ad liquida etiam corpora valet, ac per eam non solum omnia illa, quae de innantibus habet Archimedes, demonstrari possunt, sed & alia pleraque Mechanicae theoremata. Et sane, si hac eadem uti scirent novorum operum machinatores, qui motum perpetuum irrito conatu moliuntur, facile suos ipsi errores deprehenderent, intelligerentque rem eam mechanica ratione haud quaquam possibilem esse.

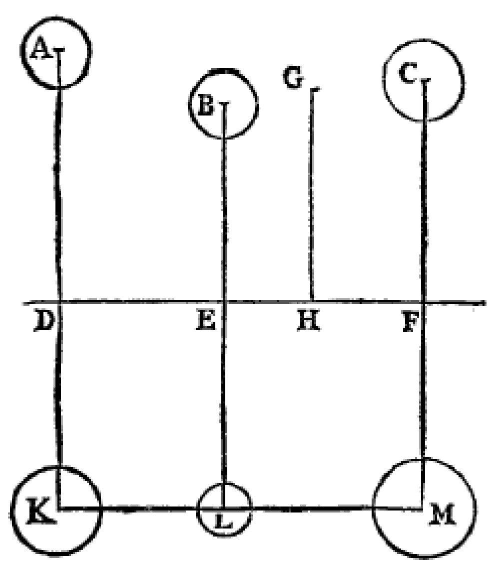
II.

Remoto aeris, alioque omni impedimento manifesto, quemadmodum in sequentibus demonstrationibus id intelligi voluimus, centrum gravitatis penduli agitati, aequales arcus descendendo ac ascendendo percurrere.

De pendulo simplici hoc demonstratum est propositione 9 de Descensu gravium. Idem vero & de composito tenendum esse declarat experientia; si quidem, quaecunque fuerit penduli figura, [p. 96] aeque apta continuando motui reperitur, nisi in quantum plus minusve aeris objectu impeditur.

PROPOSITIO I.

Ponderibus quotlibet ad eandem partem plani existentibus, si a singulorum centris gravitatis agantur in planum illud perpendiculares ; hae singulae in sua pondera ducte, tantumdem simul efficient, ac perpendicularis, a centro gravitatis omnium in planum idem cadens, ducta in pondera omnia.



Sint pondera A, B, C, sita ad eandem partem plani, cujus sectio recta DF, inque ipsum a singulis ponderibus ducantur perpendiculares AD, BE,CF. Sit autem G punctum centrum gravitatis ponderum omnium A, B, C, a quo ducatur perpendicularis in idem planum GH. Dico summam productorum, quae fiunt a singulis ponderibus in suas perpendiculares, aequari producto ab recta GH in omnia pondera A, B, C.

Intelligantur enim perpendiculares, a singulis ponderibus eductae, continuari in lateram partem plani DF, sintque singulae DK, ED, FM, ipsi HG aequales; omnesque lineae, inflexiles virgas referant, ad horizontem parallelas; & ponantur in K, L, M, gravitates ejusmodi, quae singulae cum sibi oppositis A, B, C, aequilibrium faciant ad intersectionem plani DEF. Omnes igitur K, L, M, aequiponderabunt omnibus A, B, C. Erit autem, sicut longitudo AD ad DK, ita pondus K ad pondus A, ac proinde DA ducta in magnitudinem A, aequabitur DK, sive GH, ductae in K. Similiter [p. 97] EB in B aequabitur EL, sive GH, in L; & FC in C aequabitur FM, sive GH, in M. Ergo summa productorum ex AD in A, BE in B, CF in F, aequabitur summae productorum ex GH in omnes K, L, M. Quum autem K, L, M, aequiponderent ipsis A, B, C, etiam iisdem A, B, C, ex centro ipsorum gravitatis G suspensis, aequiponderabunt. Unde, cum distantia GH aequalis sit singulis DK, EL, FM, necesse est magnitudines A, B, C, simul sumptas, aequari ipsis K, L, M. Itaque & summa productorum ex GH in omnes A, B, C, aequabitur productis ex DA in A, EB in B, & FC in C ; quod erat demonstratum.

Etsi vero in demonstratione positae fuerint rectae AD, GH, CF, horizonti parallelae, & planum ad horizontem erectum; patet, si omnia simul in alium quemlibet situm transponantur, eandem manere productorum aequalitatem, cum rectae omnes sint eadem quaedem quae prius. Quare constat propositum.

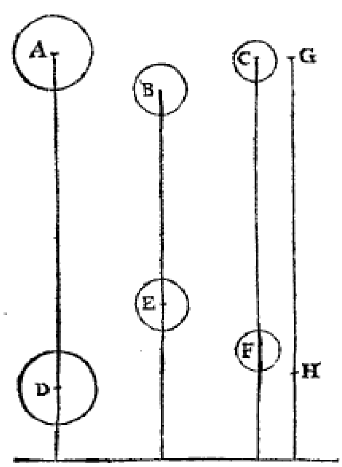
PROPOSITIO II.

Positis quae prius, si pondera omnia* A, B, C*, sint aequalia; dico summam omnium perpendicularium* AD, BE, CF*, aequari perpendiculari, a centro gravitatis ducta, GH, multiplici secundum ponderum numerum.

Quum enim summa productorum, a ponderibus singulis in suas perpendiculares, aequatur producto ex GH in pondera omnia; sitque hic, propter ponderum aequalitatem, summa illa productorum aequalis producto ex uno pondere in summam omnium perpendicularium; itemque productum ex GH in pondera omnia, idem quod productum ex pondere uno in GH, multiplicem secundum ponderum numerum : patet summam perpendicularium necessario iam aequari ipse GH, multiplici secundum ponderum necessario jam aequari ipse GH, multiplici secundum ponderum numerum. quod erat demonstratum.

PROPOSITO III.

Si magintudines quaedam descendant omnes, vel ascendant, licet inaequalibus intervallis; altitudines descensus vel ascensu cujusque, in ipsam magintudinem ductae, efficient summam productorum aequalem ei, quae sit ex altitudine descensus vel ascensus centri gravitatis omnium magnitudinum, ducta in omnes magnitudines.
[p.98]



Sunto magnitudines A, B, C, quae ex A, B, C, descendant in D, E, F ; vel ex D, E, F, ascendant in A, B, C. Sitque earum centrum gravitatis omnium, dum sunt A, B, C, eadem altitudine cum puncto G; cum vero sunt in D, E, F, eadem altitudine cum puncto H. Dico summam productorum ex altitudine AD in A, BE in B, CF in C, aequari producto ex GH in omnes A, B, C.

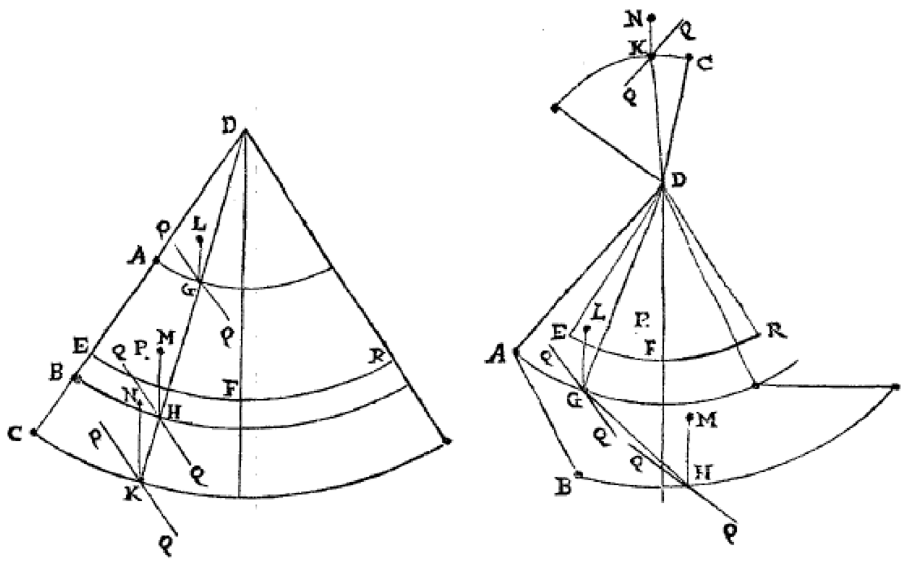
Intelligatur enim planum horizontale cujus sectio recta MP, atque in ipsum incidant productae AD, BE, CF, & GH, in M, N,O, P.

Quia igitur summa productorum ex AM in A, BN in B, CO in C, aequalis est facto ex GP in omnes A, B, C (Prop. I huius). Similiterque summa productorum ex DM in A, EN in B, FO in C, aequalis facto ex HP in omnes A, B, C; sequitur & excessum priorum productorum supra posteriora, aequari facto in GH in omnes magnitudines A, B, C. Dictum vero excessum aequari manifestum est productis ex AD in A, BE in B, CF in C. Ergo haec simul etiam aequalia erunt producto ex GH in omnes A, B, C. quod erat demonstrandum.

PROPOSITO IV.

Si pendulum e pluribus ponderibus compositum, atque e quiete dimissum, partem quamcunque oscillationis integrae confecerit, atque inde porro intelligantur pondera ejus singula, relicto communi vinculo, celerates acquisitas sursum convertere, ac quousque possunt ascendere; hoc facto, centrum gravitatis ex omnibus compositae, ad eandem altitudinem reversum erit, quam ante inceptam oscillationem obtinebat. (p. 99)

Sit pendulum compositum ex ponderibus quotlibet A, B, C, virgae, vel superficiei pondere carenti, inhaerentibus. Sitque suspensum ab axe per D punctum ducto, qui ad planum, quod hic conspicitur, perpendicularis intelligatur. In quo eodem plano etiam centrum gravitatis E, ponderum A, B, C, positum sit; lineaque centri DE, inclinetur ad lineam perpendiculi DF, angulo EDF : attracto, nimirum, eo usque pendulo. Hinc vero dimitti jam ponatur, ac partem quamlibet oscilationis conficere, ita ut pondera A, B, C, perveniant in G, H, K, Unde, relicto deinceps communi vinculo, singula intelligantur acquisitas celeritates sursum convertere, (quod impingendo in plana quaedam inclinata, velut QQ, fieri poterit,) & quousque possunt ascendere, nempe in L, M, N. Quo ibi pervenerint, sit centrum gravitatis omnium punctum P. Dico hoc pari altitudine esse cum puncto E.

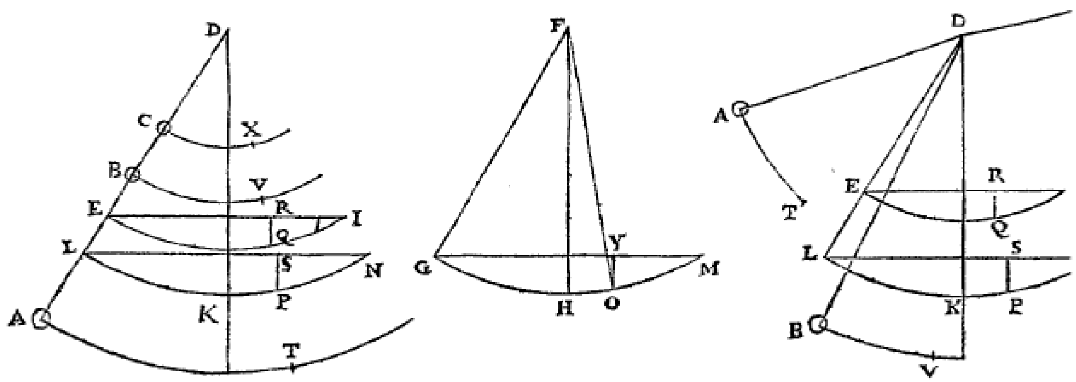


Nam primum quidem, constat P non altius esse quam E, ex prima sumptarum hypothesium. Sed nec humilius fore sic ostendemus. Sit enim, si potest. P humilius quam E, & intelligantur pondera ex iisdem, ad quas ascenderunt, altitudinibus recidere, quae sunt LG, MH, NK. Unde quidem easdem celeritates ipsius acquiri constat, quas habebant ad ascendendum ad istas altitudines(Prop. 4, part. 2), hoc est, eas ipsas quas acquisierant motu penduli ex CBAD in KHGD. Quare, si cum dictis celeritatibus ad virgam superficiemve, cui innexa fuere, nunc referantur, eique simul adhaerescant, motumque secundum inceptos arcus continuent; quod fiet, si priusquam virgam attingant, a planis inclinatis QQ repercussa intelligantur; (p.100) absolvet, hoc modo restitutum pendulum, oscillationis partem reliquam, aeque ac si absque ulla interruptione motum continuasset. Ita ut centrum gravitatis penduli, E, arcus aequales EF, FR, descendendo ac ascendendo percurrat, ac proinde in R eadem ac in E altitudine reperiatur. Ponaebatur autem E esse altius quam P centrum gravitatis ponderum in L, M, N, positorum. Ergo & R altius erit quam P: adeoque ponderum ex L, M, N, delapsorum centrum gravitatis, altius , quam unde descenderat, ascendisset. quod est absurdum. (Hypoth. 1. huj.) Non igitur centrum gravitatis P humilius est quam E. Sed nec altius erat. Ergo aeque altum sit necesse est. quod erat demonstrandum.

PROPOSITIO V.

Dato pendulo ex ponderibus quotlibet composito, si singula ducantur in quadrata distantiarum suarum ab axe oscillationis, & summa productorum dividatur per id quod sit ducendo ponderum summam, in distantiam centrigravitatis communis omnium ab eodem axe oscillationis; orietur longitudo penduli simplicis composito isochroni, sive distantia inter axem & centrum oscillationis ipsius penduli compositi.

Sint pondera pendulum componentia, (quorum nec figura nec magnitudo, sed gravitas tantum consideretur), A, B, C, suspensa ab axe, qui per punctum D, ad planum quod



conspicitur , rectus intelligitur. In quo plano sit quoque eorum centrum commune gravitatis E; nam pondera in diversis esse nihil refert. Distantia puncti E ab axe, nempe recta ED, vocetur d. Item ponderis A distantia AD, sit e; BD, f; CD, g. Ducendo itaque singula pondera in quarata [p.101] suarum distantiarum, erit productorum summa $aee + bff + cgg$. Et rursus, ducendo summam ponderum in distantiam centri gravitatis omnium, productum aequale erit $ad + bd + cd$. (Prop. 1, hujus). Unde, productum prius per hoc divedendo, habebitur $\frac{aee+bff+cgg}{ad+bd+cd}$. Cui longitudini si aequalis statuatur longitudo penduli simplicis FG, quae etiam x vocabitur; dico hoc illi composito isochronum esse.

Ponantur enim tum pendulum FG, tum linea centii DE, aequalibus angulis a linea perpendiculi remota, illud ab FH, haec ab DK, atque inde dimissa librari, & in recta DE sumatur DL aequalis FG. Itaque pondus G penduli FG, integra oscillatione arcum GM percurret, quem linea perpendiculi FH medium secabit. punctum vero L arcum illi similem & aequalem LN, quem medium dividet DK. Itemque centrum gravitatis I, percurret similem arcum EI. Quod si in arcubus GM, NL, sumptis punctis quibuslibet, similiter ipsos dividentibus, ut O & P, eadem celeritas esse ostendatur ponderis G in O, & puncti L in P; constabit inde aequalibus temporibus utroque arcus percurri, ac proinde pendulum FG, pendulo composito ex A, B, C, isochronum esse. Ostendetur autem hoc modo.

Sit primo, si potest, major celeritas puncti L, ubi in P pervenit, quam ponderis G in O. Constat autem, dum punctum L percurrit arcum LP,simul centrum gravitatis E percurrire arcum similem EQ. Ducantur a punctis Q, P, O, perpendiculares sursum, quae occurant subtensis arcuum EI, LN, GM, in R, S, Y. & SP vocetur y. Unde, cum sit ut LD, x, ad ED, d, ita SP, y, ad RQ; erit RQ aequalis $\frac{dy}{x}$. Iam quia pondus G eam celeritatem habet in O, qua valet ad eandem unde descendit altitudinem ascendere, nempe per arcum OM, vel perpendicularem OY ipsi PS aequalem; punctum igitur L, ubi in P pervenerit, majorem ibi celeratatem habebit, quam qua ascenditur ad altitudinem PS. Dum vero L transit in P, simul pondera A, B, C, similes arcus percurrunt ipsi LP, nimirum AT, BV, CX. Estque puncti L celeritas in P, ad celeritatem ponderis A in T, quum vinculo eodem contineantur, sicut distantia DL ad DA. Sed ut quadratum celeritatis puncti L, quam habet in P, ad quadratum celeritatis puncti A in T, ita est altitudo ad quam illa celeritate ascendi potest, ad altitudinem quo hac celeritate ascendi potest (Prop. 3 & 4, part. 2). Ergo etiam, ut quadratum distantiae DL, quod est xx, ad quadratum distantiae DA, quod est ee, ita est altitudo quo ascenditur celeritate puncti L, quum est in P, (quae altitudo major dicta est quam PS sive y) ad altitudinem quo ascenditur celeritate ponderis A in T; si nempe postquam in T pervenit, relicto pendulo, (p. 102) seorsim motum suum sursum converteret. Quae proinde altitudo major erit quam $\frac{eey}{xx}$.

Eadem ratione, erit altitudo ad quam ascenderet pondus B, celeritate acquisita per arcum BV, major quam $\frac{ffy}{xx}$. Et altitudo ad quam ascenderit pondus C, celeritate acquisita per acrum CX, major quam $\frac{ggy}{xx}$. Unde, ductis singulis altitudinibus istis in sua pondera, erit summa productorum major quam $\frac{aeey+bffy+cggy}{xx}$, quae proinde major quoque probatur quam $\frac{ady+bdfy+cdy}{x}$. Nam quia posita est longitudo x aequalis $\frac{aee+bff+cgg}{ad+bd+cd}$; erit $adx+bdx+cdx$ aequale $aee+bff+cgg$. Et ductis omnibus in y, & dividendo per xx, erit $\frac{adx+bdx+cdx}{\mathrm{x}}$ aequale $\frac{aeey+bffy+cggy}{\mathrm{xx}}$. Unde quod dictum est consequitur. Est autem summa ista productorum aequalis ei, quod sit ducendo altitudinem, ad quam ascendit centrum gravitatis commune ponderum A, B, C, in summam ipsorum ponderum, $a+b+c$; si nempe singul, uti dictum, seorum quosque possunt moveantur. Quantitas vero $\frac{ady+bdy+cdy}{\mathrm{x}}$ producitur ex descensu centri gravitatis eorundem ponderum, (qui descensus est, sive $\frac{dy}{x}$, ut supra inventum fuit,) in eadem quoque ponderum summam $a+b+c$. Ergo quum prius productum altero hoc majus ostensum fuerit,sequitur ascensum centri gravitatis ponderis A, B, C, si relicto pendul ubi pervenere in T, V, X, singula celeratates acquisitas sursum convertant, majorem fore ejusdem centri gravitatis descensu,dum ex A, B, C, moventur in T, V, X. quod est absurdum, cum dictus ascensus descensui aequalis esse debeat, per antecedentem.

Eodem modo, si dicatur celeritatem puncti L, ubi pervenerit in P, minorem esse celeritate ponderis G quum in O pervenerit; ostendemus ascensum possibilem centri gravitatis ponderum A, B, C, minorem esse quam descensum, quod eidem propositione antecedenti repugnat. Quare relinquitur ut eadem, sit celeritas puncti L, ad P translati, quae ponderis G in O. Unde, ut superius dictum, sequitur pendulum simplex FG composito ex A, B,C, isochronum esse.

PROPOSITIO VI.

Dato pendulo ex quotcunque ponderibus aequalibus composito; si summa quadratorum a distantiis, quibus unumquodque pondus abest ab axe oscillationis, plicetur ad distantiam centri gravitatis communis ab eodem oscillationis axe, multiplicem secundum ipsorum ponderum numerum, orietur longitudo penduli simplicis composito isochroni. (p. 103).

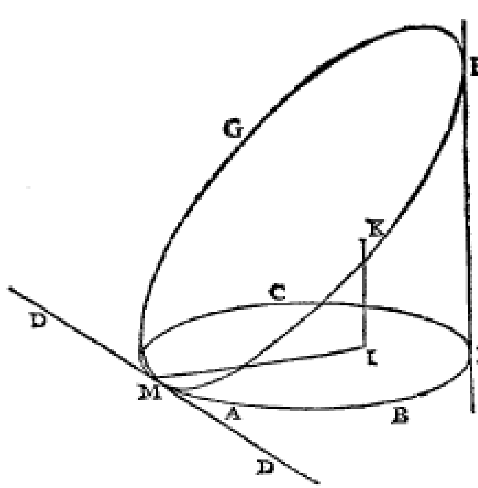
Sint posita eadem quae prius, set pondera omnia inter se aequalia intelligantur, & singula dicantur a. Rursus vero nulla eorum magnitudino consideretur, sed pro minimus habeantur, quantum ad extensionem.

Itaque penduli simplicis isochroni longitudo, per propositionem antecedentem, erit $\frac{aee+aff+agg}{ad+ad+ad}$. Vel, quia quantitas divisa ac dividens utraque per *a* dividitur, fiet nunc eadem longitudo, $\frac{ee+ff+gg}{3d}$. Quo significatur summa quadratorum a distantiis ponderum ab axe oscillationis, applicata ad distantiam centri gravitatis omnium ab eodem oscillationis axe, multiplicem secundum numerum ipsorum ponderum, qui hic est 3. faciel enim perspicitur numerum hunc, in quem ducitur distantia *d*, respondere neccessario ipsi ponderum numero. Quare constat propositum.

Quod si pondera aequalia in unam lineam rectam conjuncta sint, atque ex termino ejus superiore suspensa; constat distantiam centri gravitatis, ex omnibus compositae, ab axe oscillationis, multiplicem secundum ponderum numerum, aequari summae distantiarum omnium ponderum ab eodem oscillationis axe (Prop. 2, huius); ac proinde, hoc casu, habebitur quoque longitudo penduli simplicis,composito isochroni, si summa quadratorum a distantiis ponderum singulorum ab axe oscillationis, dividatur per summam earundem omnium distantiarum.

DEFINITIO XIV.

Si fuerint in eodem plano, figura quaedam, & linea recta quae ipsam extrinsecus tangat; & per ambitum figura alia recta, plano ejus perpendicularis, circumferatur, superficiemque quandam describat, quae deinde secetur plano per dictam tangentem ducto & ad dictae figurae planum inclinato; solidum comprehensum a duobus planis istis, & parte superficiei descriptae, inter utrumque planum intercepta, vocatur. Cuneus super figura illa, tanquam basi, abscissus.



In schemate adjecto, est ABEC figura data; recta eam tangens MD; [p. 104] quae vero per ambitum ejus circumfertur, EF; cuneus autem figura solida planis ABEC, MFG, & parte superficiei, a recta EF descriptae, comprehensa.

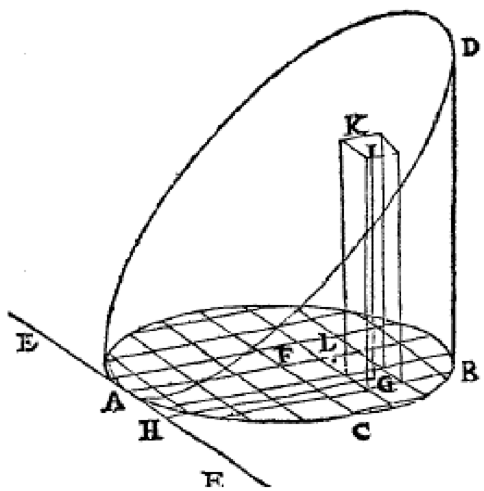
DEFINITIO VIII.

Distantia inter rectam, per quam cuneus abscissus est, & punctum baseos, in quod perpendicularis cadit a cunei centro gravitatis, dicatur cunei

Subcentrica. Nempe in figura eadem, si K sit centrum gravitatis cunei, recta vero KI ad badin ejus ABEC perpendicularis ducta sit, & rursus IM perpendicularis ad AD ; erit IM, quam subcentricam dicimus.

PROPOSITIO VII.

Cuneus super plana figura qualibet abscissus, plano inclinato ad angulum semirectum, aequalis est solido, quod sit ducendo figuram eandem, in altitudinem aequalem distantiae centri gravitatis figurae, ab recta per quam abscissus est cuneus.



Sit, super figura plana ACB, cuneus ABD abscissus plano ad angulum semirectum inclinato, ac transeunte per EE, rectam tangentem figuram ACB, inque ejus plano sitam. Centrum vero gravitatis figurae sit F, unde in rectam EE ducta sit perpendicularis FA. Dico cuneum ACB aequalem esse solido, quod sit ducendo figuram ACB in altitudinem ipsi FA aequalem.

Intelligatur enim figura ACB divisa in particulas minimas aequales [p.105] quarum una G. Itaque constat, si harum singulae ducantur in distantiam suam ab recta EE, summam productorum fore aequalem ei quod sit ducendo rectam AF in particulas omnes (Prop. 1 hujus), hoc est, ei quod sit ducendo figuram ipsam ACB, in altitudinem aequalem AF. Atqui particulae singulae ut G, in distantias suas GH ductae, aequales sunt parallepipedis, vel prismatibus minimis, super ipsas erectis, atque ad superficiem obliquam AD terminatis, quale est GK; quia horum altitudines ipsis distantiis GH aequantur, propter angulum semirectum inclinationis planorum AD & ACB. Patetque ex his parallelipipedis totum cuneum ABD componi. Ergo & cuneus ipse aequabitur solido super base ACB, altitudinem habenti rectae FA aequalem. quod erat demonstrandum.

PROPOSITIO VIII.

Si figuram planam linea recta tangat, divisaque intelligatur figura in particulas minimas aequales, atque a singulis ad rectam illam perpendiculares ducta : erunt omnium harum quadrata, simul sumpta, aequalia rectangulo cuidam, multiplici secundum ipsarum particularum numerum; quod nempe rectangulum sit a distantia centri gravitatis figurae ab eadem recta, & a subcentrica cunei, qui per illam super figura abscinditur.

Positis enim caeteris omnibus quae in constructione praecenenti, sit LA cunei ABD subcentrica in rectam EE. Oportet igitur ostendere, summam quadratorum omnium a distantiis particularum [p. 106] figurae ACB aequari rectangulo ab FA, LA, multiplici secundum particularum numerum.

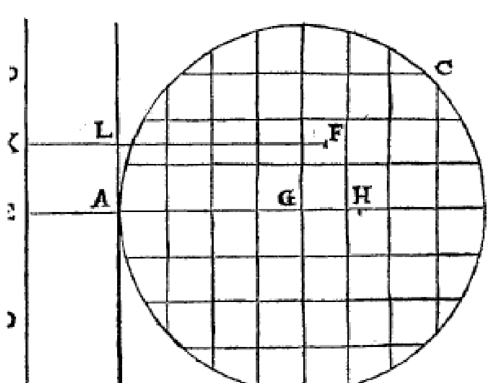
Et constat quidem ex demonstratione praecedenti, altitudines parallelipipedorum singulorum, ut GK, aequales esse distantiis particularum, quae ipsorum bases sunt, ut G, ab recta AE. Quare, si jam parallelepipedum GK ducamus in distantiam GH, perinde est ac si particula G ducatur in quadratum distantiae GH. Eodemque modo se res habet in reliquis omnibus. Atqui producta omnia parallelepipedorum in distantias suas ab recta AE, aequantur simul producto ex cuneo ABD in distantiam LA (Prop. 1 hujus), quia cuneus gravitat super puncto L. Ergo etiam summa productorum a particulis singulis G, in quadrata suarum distantiarum ab recta AE, aequabitur producto ex cuneo ABD in rectam LA, hoc est, producto ex figura ACB in rectangulum ab FA, LA. Nam cuneus ABD, aequalis est producto ex figura ACB in rectam FA (Prop. praeced.). Rursus quia figura ACB aequalis est producto ex particula una G, in numerum ipsarum particularum; sequitur, dictum productum ex figura ACB in rectangulum ab FA, LA, aequari producto ex particula G in rectangulum ab FA, LA, multiplici secundum numerum particularum G. Cui proinde etiam aequalis erit dicta summa productorum, a particulis G in quadrata suarum distantiarum ab recta AE, sive a particula una G in summam omnium horum quadratorum. Quare, omissa utrinque multiplicatione in particulam G, necesse est summam eandem quadratorum aequari rectangulo ab FA, LA, multiplici secundum numerum particularum in quas figura ACB divisa intelligitur. quod erat demonstrandum.

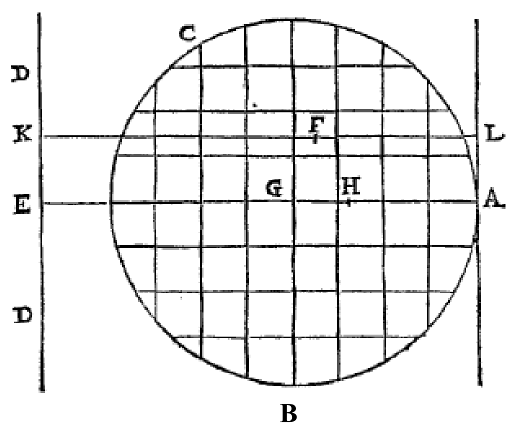
PROPOSITIO IX.

Data figura plana & in eodem plano linea recta, quae vel secet figuram ven non, ad quam perpenduculares cadant a particulis singulis minimis & aequalibus, in quas figura divisa intelligitur; invenire summam quadratorum ab omnibus istis perpendicularibus; sive planum, cujus multiplex, secundum particularum numerum, dictae quadratorum summae aequale sit.

Sit data figura plana ABC, & in eodem plano recta ED; divisaque figura cogitatu in particulas minimas aequales, intelligantur ab unaquaque earum perpendiculares ductae in rectam ED, sicut a particula F ducta est FK. Oporteatque invenire [p.107] summam quadratorum ab omnibus istis perpendicularibus.

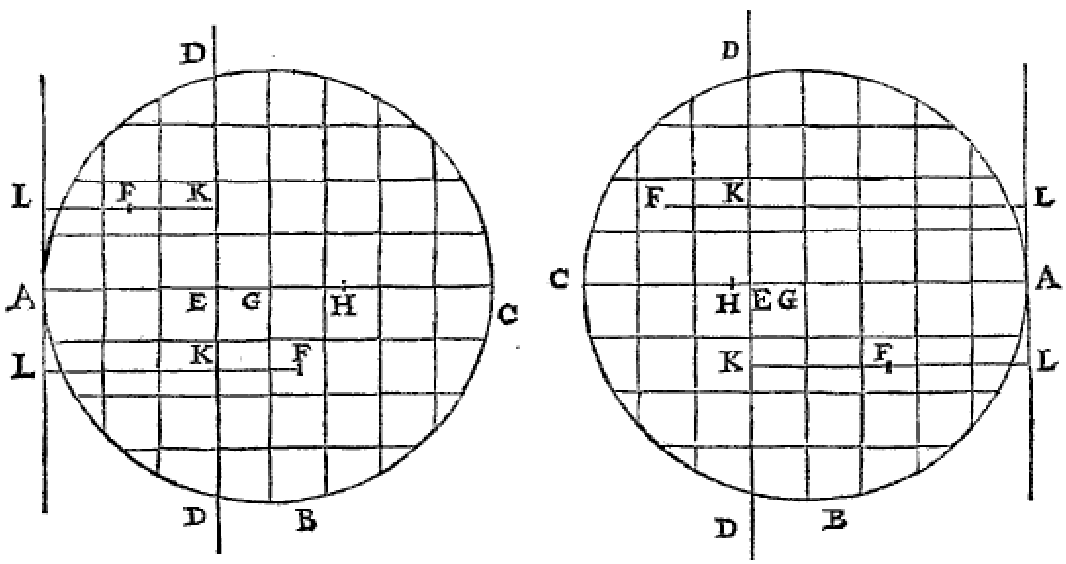
Sit datae ED parallela recta AL, quae figuram tangat, ac tota extra eam posita sit. Potest autem figuram vel ab eadem parte ex qua est ED, vel a parte opposita contingere. Distantia vero centri gravitatis figurae ab rectae AL sit recta GA, secans ED in E; & subcentrica cunei, super figura abscissi plano per rectam AL, sit HA. Dico summam quadratorum quaesitam aequari rectangulo AGH una cum quadrato EG, multiplicibus secundum particularum numerum, in quas figura divisa intelligitur.





Occurrat enim FK, si opus est producta, tangenti AL in L puncto. Itaque primum, eo casu quo recta ED a figura distat, & tangens AL ad eandem figurae partem ducta est, sic propositum ostendetur. Summa omnium quadratorum FK aequatur totidem quadratis KL, una cum bis totidem rectangulis KLF, & totidem insuper quadratis LF. Sed quadrata KL aequantur totidem quadratis EA. Et rectangula KLF aequatur esse constat totidem rectangulis EAG, quia omnes FL aequales totidem GA (Prop. 2, hui.) Et denique quadrata LF aequantur totidem rectangulis HAG (Prop. praeced.), hoc est, totidem quadratis AG cum totidem rectangulis AGH. Ergo quadrata omnia FK aequalia erunt totidem quadratis EA, cum totidem duplis rectangulis EAG, atque insuper totidem quadratis AG cum totidem rectangulis AGH. Atque tria ista; nempe quadratum EA cum duplo rectangulo EAG & quadrato AG; faciunt : quadratum EG. Ergo apparet quadrata omnia FK aequari totidem quadratis EG, una cum totidem rectangulis AGH. Quod erat ostendendum.

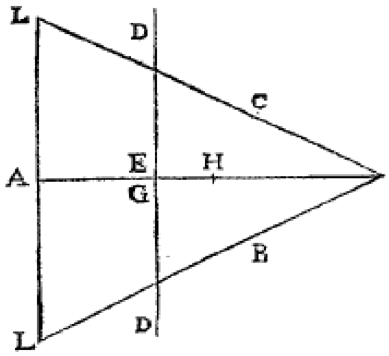
Porro in reliquis omnibus casibus, quadrato omnia FK aequantur totidem quadratis KL, minus bis totidem rectangulis KLF, plus totidem quadratis LF; hoc est, totidem quadratis EA, minus totidem duplis rectangulis EAG, plus totidem quadratis AG, cum totidem rectangulis AGH. [p.108] Atqui, omnibus hisce casibus, sit quadratum EA, plus quadrato AG, minus duplo rectangulo EAG, aequale quadrato EG. Ergo rursus quadrata omnia FK aequalia erunt totidem quadratis EG, una cum totidem rectangulis AGH. Quare constat propositum.



Hinc sequitur, rectangulum AGH eadem magnitudine esse, utriusvis cunei subcentrica fuerit AH; hoc est, sive per hanc, sive per illam tangentium parallelarum AL abscissi. Itaque AG unius casus ad AG alterius, ut HG hujus ad HG illius. Sicut autem rectae AG inter se, ita in utroque case cunei per AL abscissi, ut colligitur ex prop. 7 huj. Ergo ita quoque reciproce GH ad GH.

Apparet etiam, dato figurae planae centro gravitatis G, & subcentrica cunei, per alterutram tangentium parallelarum AL abscissi, dari quoque cunei, per tangentem alteram AL abscissi, subsentricam.

PROPOSITIO X.

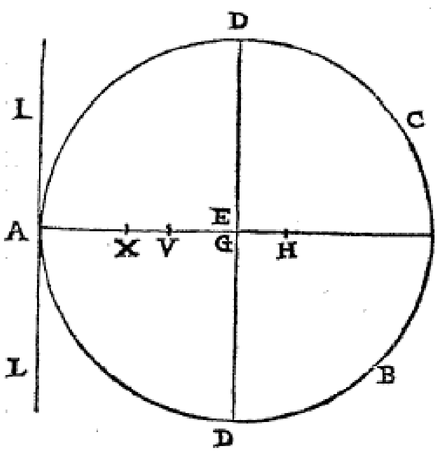


Positis quae in propositione praecedenti; si data recta ED transeat per G, centrum gravitatis figurae ABC; erit summa quadratorum a distantiis particularum, in quas figura divisa intelligitur,[p. 109] ***ab recta ED, aequalis rectangulo soll AGH, multiplici secundum ipsarum particalarum numerum.***

Hoc enim manifestum est, quum nullum tunc sit quadratum EG.

PROPOSITIO XI.

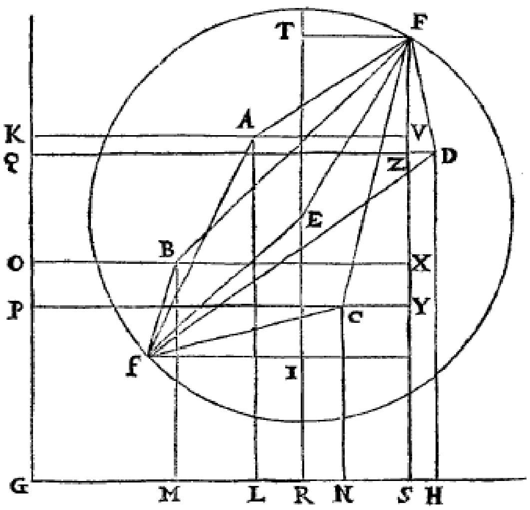
Positis rursus caeteris ut in praecedentium punultima; si DE sit axid figurae planae ABC, in duas aequales similesque portiones eam dividens, sitque insuper VG distantia centri gravitatis dimidiae figurae DAD ab recta ED, cunei vero, super ipsam abscissi per ipsam ED, subcentrica GX; erit rectangulum XGV aequale rectangulo AGH.



Est enim rectangulum XGV, multiplex secundum numerum particularum figurae DAD, aequale quadratis omnibus perpendicularium a particulis ejusdem figurae dimieae in rectam ED cadentium [Prop.8, huj.]. Ac proinde idem rectangulum XGV, multiplex secundum numerum particularum totius figurae ABC, aequale erit quadratis perpendicularium, ab omnibus particulis figurae hujus in rectam ED demissarum; hoc est, rectangulo AGH multiplici secundum eundem particularum numerum, ut constat ex propos. preacedenti. Unde sequitur rectanguli rectangula XGV, AGH inter se aequalia esse. quod erat demonstrandum.

PROPOSITIO XII.

Datis in plano punctis quotlibet; si ex centro gravitatis eorum circulus quilibet describatur; ducantur autem ab omnibus datis punctis, ad punctum aliquod in circuli illius circumferentia lineae rectae; erit summa quadratorum ab omnibus semper eidem plano aequalis.



Sint data puncta ABCD : centrumque gravitatis eorum, sive magnitudinum aequalium ab ipsis suspensarum, sit E; & centro E describatur circulus quilibet F*f*, in cuius circumferentia sumpto puncto aliquo, ut F, ducantur ad id, a datis punctis, rectae AF, BF, CF, DF. Dico earum omnium quadrata, simul sumpta, aequalia esse plano cuidam dato, semperque eidem, ubicunque in circumferentia punctum F sumptum fuerit.

Ducantur enim rectae GH,GK, angulum rectum constituentes, & quarum unicuique omnia data puncta sint posita ad eandem partem. Et a singuli in utramque harum perpendiculares agantur AL, AK; BM, BO; CN, CP; DH, DQ. A centro autem gravitatis E, & a puncto F, in alterutram duarum, GH vel GK, perpendiculares ER, FS. Et item, a datis punctis, in ipsam FS perpendicularis AV, BX, CY,DZ. Et FT perpendicularis in ipsam ER. Porro sit jam

AL = d	AK = e	radius EF = z
BM = b	BO = f	GS = x
CN = c	CP = g	
DH = d	DQ = h	

Quia autem E est centrum gravitatis punctorum A, B, C, D; si addantur in unum perpendiculares AL, BM, CN, DH, compositaque ex omnibus dividatur in tot partes, quot sunt data puncta; earum partium uni aequalis erit ER (Prop. 2, huj.). Similiterque, divisa in totidem partes summa [p. 111] perpendicularium AK, BO, CP, DQ, earum uni aequalis erit perpendicularis, ducta ex E in rectam GK, sive ipse RG (Prop. 2 huj.). Itaque, si summa omnium AL, BM, CN, DH, sive $a+b+c+d$ vocetur l : summa vero omnium, AK, BO, CP, DQ, sive $e+f+g+h$, vocetur m : & numerus, datorum punctorum multitudinem exprimens, dicatur θ; erit $\mathrm{ER} = \frac{l}{\theta}$; & $\mathrm{RG} = \frac{m}{\theta}$. Cumque GS sit x, erit RS sive $\mathrm{FT} = x - \frac{m}{\theta}$; vel $\frac{m}{\theta} - x$, si GR major quam GS; & semper quadratum $FT = x - 2\frac{xm}{\theta} + \frac{mm}{\theta\theta}$, quo ablato quadrato FE = *ZZ,* relinquetur quadratum $TE = zz - xx + 2\frac{xm}{\theta} - \frac{mm}{\theta\theta}$. Et proinde $TE = \sqrt{zz - xx + 2\frac{xm}{\theta} - \frac{mm}{\theta\theta}}$. Erat autem $\mathrm{ER} = \frac{l}{\theta}$. Itaque $TR = \frac{l}{\theta} +$ vel - $\sqrt{zz - xx + 2\frac{xm}{\theta} - \frac{mm}{\theta\theta}}$. Quae TR, brevitatis gratia, dicatur y.

Colligamus jam porro summam quadratorum omniun FA, FB, FC, FD. Quadratum AF aequatur quadratis AV, VF. Est autem AV aequalis differentiae duarum VK,AK, sive duarum SG, AK; ac proinde AV = x - e ver e - x; & qu. $\mathrm{AV} = xx - 2ex + ee$. VF vero aequalis est differentiae duarum FS, VS sive duarum FS, AL; ac proinde VF = y - a vel a - y; & qu. $\mathrm{VF} = yy - 2ay + aa$. Additisque quadratis AV, VF, sit quadratum

FA $= xx - 2ax + ee + yy - 2ay + aa$. Eodemque modo invenientur quadrata haec;

$\text{FA} = xx - 2ex + ee + yy - 2ay + aa$.
$\text{FB} = xx - 2fx + ff + yy - 2by + bb.$
$\text{FC} = xx - 2gx + gg + yy - 2cy + cc.$
$\text{FD} = xx - 2hx + hh + yy - 2dy + dd.$

Horum vero summa; si ponamus quadrata $ee + ff + gg + hh = nn$; $aa + bb + cc + dd = kk$; erit ista, $\theta\, xx - 2mx + nn + \theta yy - 2ly + kk.$ Si quidem θ erat numerus datorum punctorum ideoque & quadratorum, positumque fuerat $e + f + g + h = m$, & $a + b + c + d = l.$

In ista vero summa, si in terminis θyy & $2ly$, pro y, ponatur id cujus loco positum erat, nempe $\frac{l}{\theta}$ + vel - $\sqrt{zz - xx + 2\frac{xm}{\theta} - \frac{mm}{\theta\theta}}$, fiet

$+\theta\, yy = \frac{ll}{\theta} + 2l\sqrt{zz - xx + 2\frac{xm}{\theta} - \frac{mm}{\theta\theta}} + \theta\, zz - \theta\, xx + 2zm - \frac{mm}{\theta}.$ &

$-2ly = -2\frac{ll}{\theta} - 2l\sqrt{zz - xx + 2\frac{xm}{\theta} - \frac{mm}{\theta\theta}}$, vel

$+\theta\, yy = \frac{ll}{\theta} - 2l\sqrt{zz - xx + 2\frac{xm}{\theta} - \frac{mm}{\theta\theta}} + \theta\, zz - \theta\, xx + 2zm - \frac{mm}{\theta}.$

& $-2ly = -2\frac{ll}{\theta} + 2l\sqrt{zz - xx + 2\frac{xm}{\theta} - \frac{mm}{\theta\theta}}$.

Ac proinde, utroque casu, pro θyy - $2ly$ habebitur $-\frac{ll}{\theta} + \theta\, zz - \theta\, xx + 2xm - \frac{mm}{\theta}.$ Quo appositis reliqui quantitatibus, summa praedicta contentis $\theta\, xx - 2xm + nn + kk$, fiet tota summa, nempe quadratorum FA, FB, FC, FD $= \theta\, zz + nn + kk - \frac{mm+ll}{\theta}.$ Quod apparet esse planum datum, cum hae quantitates omnes datae sint; semperque idem reperiri, ubicunque in circumferentia sumptum fuerit punctum F. quod erat demonstratum.

Quod si puncta data diversas gravitates habere ponantur, invicem commensurabiles, ut si punctum A ponderet ut 2, B ut 3, C ut 4, D ut 7, eorumque reperto gravitatis centro, circulus rursus describatur, ad cujus circumferentiae punctum, a datis punctis rectae ducuntur, ac singularum quadrata multiplicia sumantur secundum numerum ponderis puncti sui; ut quadratum AF duplum, BF trium, CF quadruplum, DG septulum; dico rursus summam omnium aequalem fore spatio dato, semperque eidem, ubicunque in circumferentia punctum sumptum fuerit. Patet enim hoc ex praecedenti demonstratione, si imaginemur puncta ipsa multiplicia secundum numeros attributae cuique gravitatis; quasi nempe in A duo puncto conjuncta sint, in B tria, in C quatuor, in D septem, atque illa omnia aequaliter gravia.

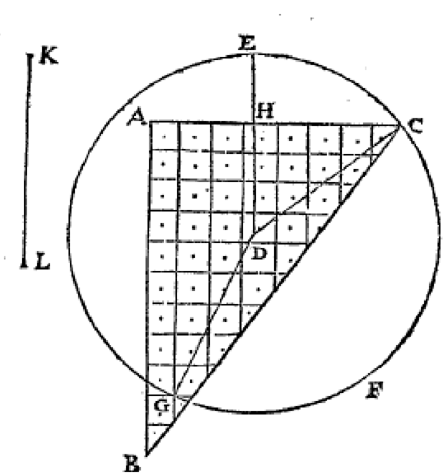
PROPOSITIO XIII.

Si figura plana, vel linea in plano existens, aliter atque aliter suspendatur a punctis, quae, in eodem plano accepta, aqualiter a centro gravitatis suae distent; agitata motu in latus ipse isochrona est.

Sit figura plana, vel linea plano existens ABC, cujus centrum gravitatis D. quo eodem centro, circumferentia circuli in eodem plano describatur, ECF. Dico, si a quovis in illa puncto, ut E, C, vel G, suspensa figura agitetur in latus; sibi ipsi, sive eidem pendulo simplici, isochronam esse.

Sit prima suspensio ex E puncto, quando autem est extra figuram, ut hic, putandum est lineam EH, ex qua figura pendet, rigidam esse, atque immobiliter ipse affixam.

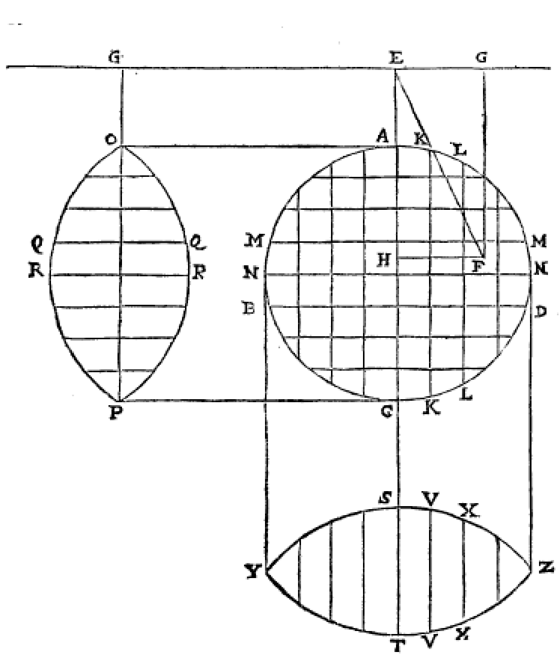
Intelligatur figura ABC divisa in particulas minimas aequales, a quarum omnium centris gravitatis, ad punctum E, rectae ductae sint; quas quidem manifestum est, quum moveatur figura motu in latus, esse ad axem agitationis perpendicules. Harum igitur omnium perpendicularium quadrata, divisa per rectam ED, multiplicatem secundum numerum particularum in quas figura divisa est, efficiunt longitudinem penduli simplicis, figurae isochroni (Prop. 6 huj.), quae [p. 113] sit KL. Suspensa autem figura ex puncto G, rursus longitudo penduli simplicis isochroni invenitur, dividendo quadrata omnia linearum, quae a particulis figurae ducuntur ad punctum G, per rectam GD, multiplicem secundum earundem particularum numerum (Prop. 6 huj.). Quum igitur puncta G & E sint in circumferentia descripta centro D, quod est centrum gravitatis figurae ABC, sive centrum gravitatis punctorum omnium, quae centra sunt particularum figurae aequalium; erit proinde summa quadratorum a lineis, quae a dictis particulis ad punctum G ducuntur, aequalis summae quadratorum a lineis quae ab iisdem particulis ducantur ad punctum E (Prop. praeced.). Hae vero quadratorum summae, utraque suspensione, applicantur ad magnitudines aequales : quippe, in suspensione ex E, ad rectam ED, multiplicatem secundum numerum omnium particularum; in suspensione autem ex G, ad rectam DG, multiplicem secundum earundem particularum numerum. Ergo patet, ex applicatione hac posteriori, quum nempe suspensio est ex G, fieri longitudinem penduli isochroni eadem atque ex applicatione priori, hoc est, eandem ipsi KL.



Eodem modo, si ex C, vel alio quovis puncto circumferentiae ECF, figura suspendatur, eidem KL isochrona esse probabitur. Itaque constat propositum.

PROPOSITIO XIV.

Data figura solida, & linea recta interminata, qua vel extra figuram cadat, vel per eam transeat; divisque [p.114] ***figura cogitatu in particulas minimas aquales, a quibus omnibus ad datam rectam perpendiculares ductae intelligantur; invenire summam omnium quae ab ipsis fiunt quadratorum, sive planum, cujus muliplex secundum particularum numerum, dicta quadratorum summae aequale sit.***



Sit data figura solida ABCD, & linea recta quae, per punctum E transiens, ad planum hujus paginae erecta intelligatur: quaeque vel secet figuram, vel tota extra cadat. Intellectoque, a singulis particulis minimis aequalibus, solidum ABCD constituentibus, velut F, rectas duci perpendiculares in datam rectam per E, quemadmodum hic FE, oporteat omnium quadratorum FE summam invenire.

Secetur figura plano EAC, per dictam datam lineam & per centrum gravitatis figurae ducto. Item aliud planum intelligatur pe eandem lineam datam, perque EG, quae ipsi est ad angulos rectos.

Constat jam, quadratum rectae cujusque, quae a particula dictarum [p. 115] aliqua, ad lineam datam per E perpendicularis ducitur, sicut FE, aequari quadratis duarum FG, FH, quae, ab eadem particula, in plana per EG & EC ante dicta, perpendiculares aguntur (47.lib. 1. Eucl.). Quare, si cognoscere possimus summam quadratorum, quae fiunt ab omnibus perpendicularibus, quae a particulis universis cadunt in plana dicta per EG & per EC : habebimus etiam huic aequalem summam quadratorum a per pendicularibus, quae ab universis iisdem particulas cadunt in rectam datam per E punctum.

Illa vero prior quadratorum summa colligetur hoc modo. Ponatur primo figuram planam dari OQP, ad latus figurae solidae ABCD, ejusdem cum ipsa altitudinis, quae que sit ejusmodi, ut secta lineis rectis QQ, RR, quae respondeant planis figuram solidam ABCD secantibus MM, NN, & his parallelis; eadem sit dictarum linearum inter se, quae & planorum horum ratio, si nempe sumantur utrinque quae in oreine sibi respondent. Ut si linea RR sit ad QQ quaemadmodum planum NN ad MM. Quod si igitur figura plana OQP, in totidem particulas minimas aequales divisa intelligatur, quot intelliguntur in solido ABCD, erunt etiam in unoquoque segmento figurae planae, velut QQRR, tot numero particulae, quod sunt in figurae solidae segmento MMNN, isti segmento respondente; ac proinde & summa quadratorum, a perpendicularibus omnium particularum figurae OQP in planum EG, aequabitur summae quadratorum, a perpendicularibus omnium particularum figurae solidae, in idem planum EG productis. Illa autem quadratorum summa data erit, si dentur in figura OQP, cuneoque illius, quae propos. 9 huj. requiri diximus. Ergo his datis, dabitur quoque summa quadratorum, a perpendicularibus quae, a particulis omnibus solidi ABCD, ducuntur in planum EG.

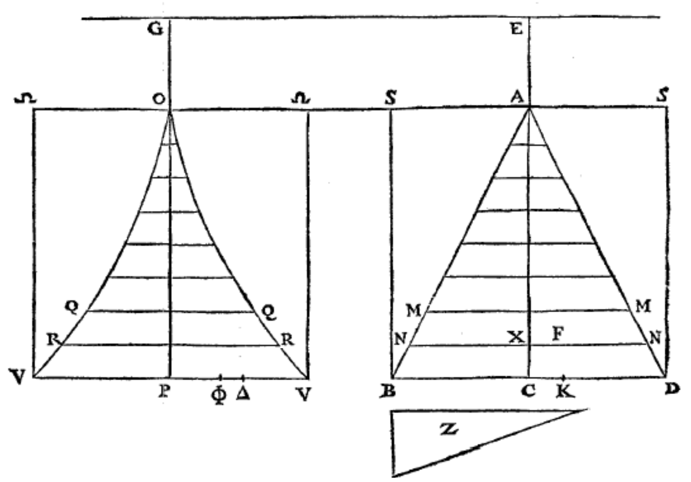
Ponatur nunc alia item figura plana SYTZ, ejuseem cum solido ABCD latitudinis, hoc est, quam includant plana BY, DZ solidum contingentia, ac parallela plano EAC, quaeque sit ejusmodi, ut, secta lineis rectis VV, XX, & his parallelis, faciat eandem inter se rationem linearum harum atque illorum planorum, si sumantur quae sibi mutuo respondent. Itaque rursus quadrata simul omnia perpendicularium, a particulis figurae SYTX in rectam ST cadentium, aequalia erunt quadratis omnibus perpendicularium quae, a particulis solidi ABCD, ducuntur in planum AC. Illorum autem summa quadratorum data erit, si detur distantia centri gravitatis figurae SYTZ ab recta BY vel DZ; nec non distantia indidem centri gravitatis [p.116] cunei sui abscissi plano per eandem rectam (Prop. 9. huj.) Vel, figura SYTZ ordinata existente, ut ST sit axis ejus, eadem quadratorum summa dabitur, si detur distantia centri gravitatis figurae dimidiae AZT ab axe ST, item centri gravitatis cunei, super eadem dimidia figura, abscissi plano per axem ducto (Prop. 11. huj.). Ergo, his datis, dabitur quoque summa quadratorum a perpendicularibus quae, a particulis omnibus solidi ABCD, ductae intelliguntur in planum EAC. Invenimus autem & summam quadratorum, a perpendicularibus omnibus in planum per EG ductis. Ergo & aggregatum utriusque summae habebitur, hoc est, per superius ostensa, summa quadratorum perpendicularium quae, a particulis omnibus solidi ABCD, cadunt in rectam datam per E transeuntem, & ad paginae hujus planum erectam. quod erat demonstratum.

PROPOSITIO XV.

Iisdem positis, si solidum ABCD sit ejusmodi, ut figurae plana SYTZ, ipsi proportionalis, non habeat notam distantiam centri gravitatis a tangentibus BY vel DZ, vel, ut subcentrica cunei super ipsa abscissi, plan pe easdem BY vel DZ, ignoretur; in figura tamen proportionali, quaea latere est, OQP, detur distantia Φ P, qua centrum gravitatis figurae dimidiae OPV abest ab axe OP; licebit hinc invenire summam quadratorum a distantiis particularum solidi ABCD a plano EC. Oportet autem ut sectiones omnes, NN, MM, sint plana similia; utque per omnium centra gravitatis transeat planum EC; quemadmodum in prismate , pyramide, cono, conoidibus, multisque aliis figuris contingit. Atque eorum planorum distantias centri gravitatis, super tangentibus axi oscillationis parallelis, datas esse necesse est ; uti & subcentricas cuneorum, qui super ipsis abscindantur, ductis planis per easdem tangentes.

Veluti, si maxima dictarum sectionum sit BD, & in B intelligatur recta parallela axi E, hoc est, erecta ad planum quod hic conspicitur, oportet datam esse distamtiam centri gr. sectionis BD a dicta linea in B, quae sit BC; itemque subcentricam cunei, super sectione BD a dicta linea in B, quae sit BC; itemque subdcentricam cunei. super sectione BD abscissi, plano ducto per eandem lineam in B, quae subcentrica sit BK.

Etenim his datis, divisaque PV bifariam in Δ, si fiat sicut ΔP ad PΦ, ita rectangulum BCK ad spatium quoddam Z; dico hoc ipsum, multiplex per numerum particularum solidi ABCE , aequari summae quaesitae quadratorum, a distantiis earundem particularum a plano EC.



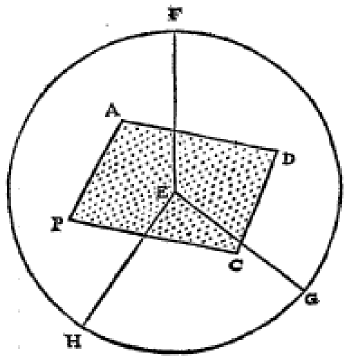
Quadrata enim a distantiis particularum planae sectionis BD,a plano EC, quod per centrum gravitatis suae transit; sive quadrata a distantiis particularum solidarum segmenti BNND a plano eodem, aequari constat rectangulo BCK, multiplici per numerum dictarum particularum (Prop. 8.huj.). Similiter, si planae sectonis NN distantia centri gravitatis, ab recta quae in N intelligitur axi E parallela, sit NX; subcentrica vero cunei super ipsa abscissi, plano per eandem rectam, sit NF; erunt quadrata a distantiis particularum planarum sectionis NN a plano EC; sive quadra a distantiis particularum solidarum segmenti NMMN, a plano eodem, aequalia rectangulo NXF, multiplici per numerum particularum ipsarum sectionis NN, ve segmenti NMMN. Est autem BD divisa similiter in C & K, atque NN in X & F. Ergo rectangulum BCK ad rectangulum NXF, sicut quadratum BD ad quadratum NN.

Est autem & numerus particularum sectionis BD, ad numerum particularum sectonis NN, sicut sectiones ipsae; hoc est, sicut quadratum BD ad quadratum NN. Itaque rectangulum BCK, multiplex per numerum particularum sectionis BD, ad rectangulum NXF, multiplex per numerum particularum sectionis NN, duplicatam habebit rationem quadrati BD ad quadratum NN; hoc est, eam quam quadratum VV ad quadratum RR, in figura proportionali. Erit igitur & dictae prior summa quadratorum, a distantiis particularum segmenti BNND a plano EC, ad summam alteram quadratorum, a distantiis particularum segmentis NMMN, ut qu. VV ad qu. RR. Eademque ratione ostendetur, summas quadratorum a distantiis particularum in reliquis segmentis solidi ABCD, esse inter se in ratione quadratorum quae fiunt a rectis in figura OVV, quae base cuiusque segmenti respondent. Quare summa quadratorum, a distantiis particularum omnium segmentorum solidi ABCE a plano EC, erit ad summam quadratorum, a distantiis particularum segmentorum totidem, maximo segmento aequalium, hoc est, cylindri vel prismatis BDSS, eandem cum solido ABCD basim altitudinemque habentis, sicut quadrata omnia rectarum VV, RR, QQ, & c. ad quadrata totidem maximo VV aequalia, hoc est, sicut solidum rotundum OVV circa axem OP, ad cylindrum VVΩΩ, qui basim & altitudinem habeat eandem. Hanc vero rationem solidi OVV ad cylindrum VVΩΩ, qui basin & altitudinem habeat eandem. Hanc vero rationem solidi OVV ad cylindrum VVΩΩ, componi constat ex ratione planorum quorum conversione generantur, hoc est, ex ratione plani OPV, ad rectangulum PΩ, & ex ratione distantiarum quibus horum planorum centra gravitatis absunt ab axe OP; hoc est, & ex ratione PΦ ad PΔ. Et prior quidem harum rationum, nempe plani OPV ad rectangulum PΦ, eadem est quae solidi ABCE ad cylindrum vel prisma BDSS, hoc est, eadem quae numeri particularum solidi ABCD, ad numerum particularum cylindri vel prismatis BDSS. Alter vero ratio, nempe PΦ ad PΔ, est eadem, ex constructione quae spatii Z ad rectangulum BCK. Habebit itaque dicta summa quadratorum, a distantiis omnium particularum solidi ABCE a plano EC, ad summam quadratorum, a distantiis omnium particularum cylindri vel prismatis BDSS ab eodem plano, rationem eam quae componitur ex ratione numeri particularum solidi ABCD. ad numerum particularum cylindri vel prismatis BDSS, & ex ratione spatii Z ad rectangulum BCK : hoc est, rationem quam habet rectangulum Z, multiplex per numerum particularum cylindri prismatis BDSS. Atqui quarta harum magnitudinum aequalis est secundae; nempe rectangulum BCK, multiplex per numerum particularum cylindri vel prismatis BDSS, aequale summae quadratorum, a distantiis particularum ejusdem prismatis vel cylindri BDSS a plano EC; siquidem rectangulum idem BCK, multiplex (p.119) per numerum particularum segmenti BNND, aequatur quadratis distantiarum particularum ejusdem segmenti a plano EC (Prop. 8, huj.). Ergo & tertia primae aequabitur; nempe planum Z, multiplex per numerum particularum solidi ABCD, summae quadratorum, a distantiis particularum solidi ejusdem ABCD a plano EC. (Prop. 14, lib 5, Eucl.). quod erat demonstratum.

Notatandum vero, quando solidum ABD rotundum est circa axem AC, fieri semper rectangulum BCK aequale quartae parti quadrati BC; quoniam subcentrica cunei, abscissi super circulo BCD, plano per tangentem in B, nempe recta BK, aequatur $\frac{5}{4}$ radii BC. Unde, si PV aequalis posita sit BC, sequitur, faciendo ut PΔ ad P Φ ita rectangularum BCK, hoc est, $\frac{1}{4}$ quadrati BC, hoc est,qu. PΔ ad planum aliud Z, fore hoc rectangulo ΔPΦ aequale. Ac proinde tunc ipsum rectangulum ΔPΦ , multiplex secundum numerum particularum solidi ABD, aequari summae quaesiti quadratorum a perpendicularibus omnibus, quae a particulis iisdem cadunt in planum EC.

PROPOSITIO XVI.

Figura quavis, sive linea fuerit, sive superficies, sive solidum; si aliter atque aliter suspendatur, agiteturque super axibus inter se parallelis, quique a centro gravitatis figurae aequaliter distent, sibi ipse isochrona est.



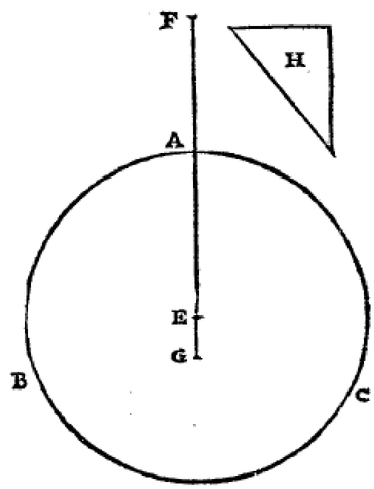
Proponatur magnitudo quaevis,cujus centrum gravitatis E punctum, sitque primo suspensa ab axe, qui per F intelligatur hujus paginae plano ad angulos rectos. Itaque idem planum erit & planum oscillationis. In quo si centro E, radio EF, describatur circumferentia FHG, sumptoque in illa puncto quovis, ut H, magnitudo secundo suspendi intelligatur ab axe in hoc puncto infixo, atque agatari, manente eodem oscillationis plano. Dico isochronam fore sibi ipsi agitatae circa axem in F.

[p.120] Intelligatur enim dividi magnitudo proposita in particulas minimas aequales. Itaque, quia in utraque illa suspensione idem manet oscillationis planum, respectu partium magnitudinis; manifestum est, si ab omnibus particulis, in quas divisa est magnitudo, perpendiculares cadere concipiantur in dictum oscillationis planum, illas utraque suspensione occurrere ipsi in punctis iisdem. Sint autem haec puncta ea quae apparent in spatio ABCD.

Quam igitur E sit centrum gravitatis magnitudinis propositae, ipsaque proinde circa axem, qui per E punctum erectus est ad planum ABCD, quovis situ aequilibrium servet; facile perspicitur, quod si punctis omnibus ante dictis, quae in spatio ABCD signantur, aequalis gravitas tribuatur, eorum quoque omnium centrum gravitatis futurum est punctum E. Quod si vero, ut fieri potest, in puncta aliqua plures perpendiculares coincidant, illa puncta quasitoties geminata intelligenda sunt, gravitatesque toties multiplces accipiendae. Atque ita consideratorum, patet rursus centrum gravitatis essse E punctum.

Porro summam quadratorum ab rectis, quae ducuntur a dictis punctis omnibus ad punctum F, eandem esse paret cum summa quadratorum ab iis rectis, quae a singulis particulis magnitudinis propositae ducuntur perpendiculares in axem oscillationis per F transeuntem ; quippe cum lineae ipsae, quarum quadrata intelliguntur, utrobique eandem habeant longitudinem. Similiter etiam, cum suspensio est ex axe per H, patet summam quadratorum ab rectis, quae ab omnibus punctis, in spatio ABCD signatis, ducuntur ad punctum H, eandem esse cum summa quadratorum, ab iis quae, a particulis omnibus magnitudinis propositae, ducuntur perpendiculares in axem oscillationis per H transeuntem. Ergo utroque casu, si summa quadratorum ab rectis quae, a punctis omnibus praedictis, ducuntur ad puncta F vel H, dividatur per rectas EF vel EH, multiplices secundum numerum particularum in quas magnitudo proposita divisa intelligitur, orietur ex applicatione hac longitudo penduli simplicis, quod magnitudini suspensae ex F vel H isochronum sit. Est autem summa quadratorum utroque casu aequalis (Prop. 11, huj.) ; & rectae quoque EF, EH, inter se aequales; & particularum idem numerus. Ergo, quum & applicatae quantitates, & quibus illae applicantur, utrobique aequales sint, etiam longitudines ex applicatione ortae aequales erunt, hoc est, longitudines pendulorum isochronorum magnitudini propositae suspensae ex F vel H. Quare constat propositum.

PROPOSITIO XVII. [p.121]



Dato plano, cujus multiplex per numerum particularum, in quas suspensa figura divisa intelligitur, aequetur quadratis omnium distantiarum ab axe oscillationis; si illud applicetur ad rectam, aequalem distantiae inter axem oscillationis & centrum gravitatis suspensae magnitudinis, orietur longitudo penduli simplicis ipsi isochroni.

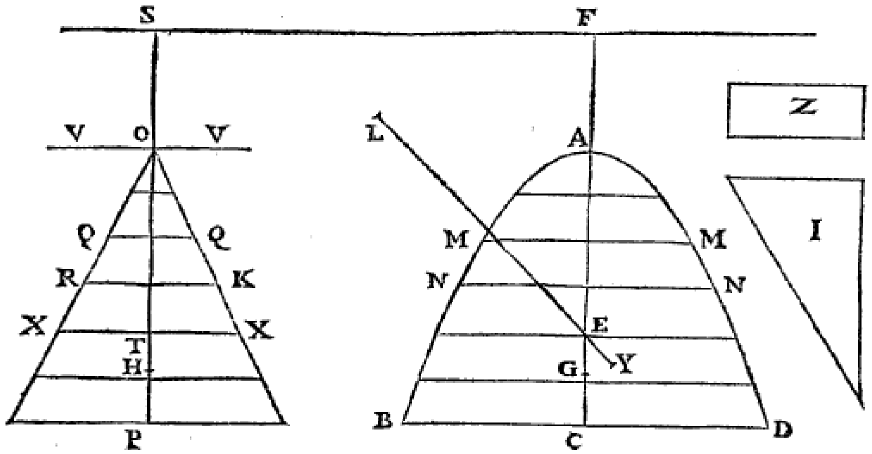
Sit figura ABC, cujus centrum gravitatis E, suspensa ab axe qui, per F punctu ad planum quod conspicitur, erectus sit. Ponendoque divisam figuram in particulas minimas aequales, a quibus omnibus, perpendiculares cadere intelligantur : esto, per superius ostensa, inventum planum H, cujus multiplex per numerum dictarum particularum, aequetur quadratis omnibus dictarum perpendicularium. Applicatoque plano H ad rectam FE, fiat longitudo FG. Dico hanc esse longitudinem penduli simplicis, isochronas oscillationes habentis magnitudini ABC, agitatae circa axem per F.

Quia enim summa quadratorum, a distantiis ab axe F, applicata ad distantiam FE, multiplicem secundum partium numerum, facit longitudinem penduli simplicis isochroni (Prop. 6, huj.). Isti vero quadratorum summae aequale ponitur planum H, multiplex per eundem particularum numerum. Ergo & planum H, multiplex per eundem particularum numerum;. Ergo & planum H, multiplex per eundem particularum numerum, si applicetur ad distantiam FE, multiplicem [p.122] secundum particularum numerum; sive, omissa communi multiplicitae, si planum H applicetru ad distantiam FE; orietur quoque longitudo penduli simplicis isochroni. Quam proinde ipsam longitudinem EG esse constat. quod erat demonstrandum.

PROPOSITIO XVIII.

Dato spatium planum, cujus multiplex secundum numerum particularum suspensae magnitudinis, aequetur quadratis distantiarum ab axe gravitatis, axi oscillationis parallelo; id, inquam, spatium si applicetur ad rectam, aequalem distantiae inter utrumque dictorum axium, orietur recta aequalis intervallo, quo centrum oscillationis inferius est centro gravitatis ejusdem magnitudinis.

Esto magnitudo ABCD, cujus centrum gravitatis E; quaeque suspensa ab axe , qui per punctum F ad planum hujus paginae erectus intelligitur, habeat centrum oscillationis G. Porro axi per F intelligatur axis alius, per centrum gravitatis E transiens, parallelus.



Divisaque magnitudine cogitatu in particulas minimas aequales,sit quadratis distiarum, ab axe dicto per E, aequale planum I, multiplex nempe secundum numerum dictorum particularum; applicatoque plano I ad distantiarum FE, fiat recta quaedam. Dico eam aequalem esse intervallo EG, quo centrum oscillationis inferius est centro gravitatis magnitudinis ABCD.

Ut enim universali demonstratione quod propositum est comprehendamus : intelligatur plana figura, magnitudini ABCD analoga, ad latus adopta, OQP; quae nempe, secta planis horizontalibus iisdem cum magnitudine ABCD, habeat segmenta intercepta inter bina quaeque plana, in eadem inter se ratione cum segmentis dictae magnitudinis, quae ipsis respondent; sintque segmenta singula figurae OQP, divisa in tot particulas aequales, quod continentur segmentis ipsis respondentibus in figura ABCD. Haec autem intelligi possunt fieri, qualiscunque fuerit magnitudo ABCE, sive linea, sive superficies, sive solidum. Semper vero centrum gravitatis figurae OQP, quod sit T, eadem altitudine esse manifestum est cum centro gravitatis magnitudinis ABCD; ideoque, si planum horizontale, per F ductum, secet lineam centri figurae OQP, velut hic in S, aequales esse distantias ST, FE.

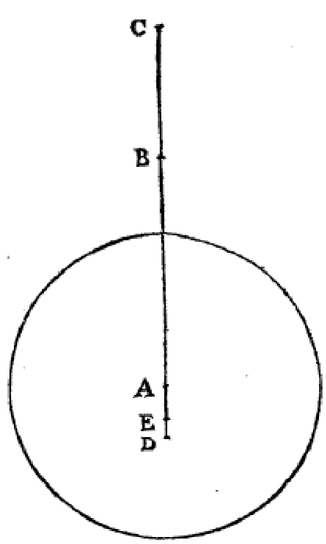
Porro autem constat quadrata distantiarum, ab axe oscillationis F, applicata ad distantiam FE, multiplicem secundum numerum particularum, efficere longitudinem penduli isochroni (Prop. 4, huj.); quae longitudo posita fuit FG. Illorum vero quadratorum summam, aequalem esse perspicuum est, quadratis distantiarum a plano horizontali per F, una cum quadratis distantiarum a plano verticali FE, per axem F & centrum gravitatis E ducto (Prop. 47, lib. 1. Eucl.). Atque quadrata distantiarum magnitudinis ABCE a plano horizontali per F, aequantur quadratis distantiarum figurae OQP ab rectas SF. Quae quadrata (si O sit punctum supremum figurae OQP, & H centrum gravitatis cunei super ipsa abscissi, plano per rectam OV, parallelam SF) aequalia sunt rectangulo OTH & quadrato ST, multiplicibus secundum numerum particularum dictae figurae (Prop. 9, huj.), sive magnitudinis ABCD. Quadrata vero distantiarum magnitudinis ABCD a plano FE, quantumcumque axis oscillationis F distet a centro gravitatis E, semper eadem sunt : quae proinde putemus aequari spatio Z, multiplici secundum numerum particularum magnitudinis ABCD.

Itaque quoniam quadrata distantiarum magnitudinis ABCD, ab axe oscillationis F, aequantur istis, quadrato nimirum ST, rectangulo OTH, & plano Z, multiplicibus per numerum particularum ejusdem magnitudinis; si applicentur haec omnia ad distantiam FE sive ST, orietur longitudo FG penduli isochroni magnitudini ABCD (Prop. 6, huj.). Sed ex applicatione quadrati ST ad latus suum ST, orietur ipsa ST, sive FE. Ergo reliqua EG est ea quae oritur ax applicatione rectanguli OTH, & plani Z, ad eandem ST vel FE.

Quare superest ut demonstremus rectangulum OTH, cum plano Z, aequari plano I. Tunc enim constabit, etiam planum I, applicatum ad distantiam FE, efficere longitudinem ipsi EG aequalem. Illud autem sic ostendetur. Rectangulum OTH, multiplex secundum numerum particularum figurae OQP, sive magnitudinis ABCD, [p. 124] aequatur quadratis distantiarum figurae ab recta XT (Prop. 10, huj.), quae per centrum gravitatis T ducitur ipsi SF parallela; ac proinde etiam quadratis distantiarum magnitudinis ABCD, a plano horizontali KK, ducto per centrum gravitatis E; cum distantiae utrobique sint eaedem. At vero planum Z, similiter multiplex, aequale positum fuit quadratis distantiarum magnitudinis ABCD a plano verticali FE. Ac patet quidem quadrata haec distantiarum a plano FE, una cum dictis quadratis distantiarum a plano horizontali per E, aequalia esse quadratis distantiarum ab axe gravitatis per E, qui sit axi F parallelus (Prop. 47, lib. 1. Eucl.). Itaque rectangulum OTH una cum plano Z, multiplicia secundum numerum particularum magnitudinis ABCD, aequalia erunt quadratis distantiarum ejusdem magnitudinis a dicto axe per E. Sed & planum I, multiplex secundum eundem particularum numerum, aequale positum fuit iisdem distantiarum quadratis. Ergo planum I aequale est rectangulo OTH & plano Z simul sumptis. quod ostendendum superat.

Hinc rursus manifestum sit, quod propositione 16 demonstratum fuit; nempe magnitudinem quamlibet, si aliter atque aliter suspendatur atque agitur, ab axibus parallelis, qui a centro gravitatis suae aequaliter distent, sibi ipse isochronam esse.

Sive enim magnitudo ABCD suspendatur ab axe F, sive ab axe L illi parallelo; patet eadem utrobique esse quadrata distantiarum ab axe per E, qui sit axibus F vel L parallelus. Unde & planum I, cujus multiplex, secundum numerum particularum, aequatur quadratorum summae, utroque casu idem erit. Hoc vero planum, applicatum ad distantiam centri gravitatis ab axe oscillationis, quae utroque case eadem ponitur, efficit distantiam qua centrum oscillationis inferius est centro gravitatis; Ergo etiam haec distantia utroque casu eadem erit. Velut si, factasuspensione ex I, fuerit dicta distantia EY, erit ipsa aequalis EG; & tota YL aequalis GF; adeoque in suspensione utraquae idem pendulum simplex isochronum sit magnitudini ABCD.



PROPOSITIO XIX.

Si magnitudo eadem, nunc brevius nunc longius suspensa, agitur ; erunt, sicut distantae axium oscillationis a centro gravitatis inter se, ita contraria ratione distantiae centrorum oscillationis ab eodem gravitatis centro.

Sit magnitudo, cujus centrum gravitatis A, suspensa prium atque agitata ab axe in B, deinde vero ab axe in C; sitque in prima [p.125] suspensione centrum oscillationis D, in posteriori vero centrum oscillationis E. Dico esse ut BA ad CA ita EA ad DA.

Quam enim, in suspensione ex B, efficiatur distantia AD, qua nempe centrum oscillationis inferius est centro gravitatis, applicando ad distantiam BA spatium quoddam, cujus multiplex secundum numerum particularum minimarum aequalium, in quas magnitudo divisa intelligitur, aequatur quadratis distantiarum ab axe per A, parallelo axi in B (Prop. prec.) ; erit proinde rectangulum BAD dicto spatio aequal. Item, suspensione ex C, quum fiat distantia AE, applicando idem dictum spatium ad distantia CA; erit & rectangulum CAE eidem spatio aequale. Itaque aequalia inte se rectangula BAD, CAE; ac proinde ratio BA ad CA eadem quae AE ad AD. quod erat demonstrandum.

Hinc patet, dato pendul simplici, quod magnitudini suspensae isochronum sit un una suspensione, datoque ejus centro gravitatis; etiam in alia omni suspendione, longiori vel breviori, dummodo idem maneat planum oscillationis, longitudinem penduli isochroni datum esse.

PROPOSITIO XX.

Centrum oscillationis & punctum suspensionis inter se convertuntur.

In figura superioi, quia, posita suspensione ex B, centrum oscillationis est D; etiam invertendo omnia, ponendoque suspensionem ex D, [p.126] erit tunc centrum oscillationis B. Hoc enim ex ipsa propositione praecedenti manifestum est.