especially useful in pointing out the nature of the error involved in the frequent definition of the partial correlation coefficient as that between two variables "when the other variables are held constant." It can not too strongly be emphasized that *no* variable is held *constant* in the partial correlation measurement. This process of measurement seeks merely to eliminate from the indicated correlation between two variables the part which arises from their mutual linear dependence upon certain other specified variables. The development in the book is especially happy in that it shows the only way in which constant or "assigned" values enter the concept of partial correlation.*

The remaining chapters, on *Random sampling fluctuations*, *The Lexis theory*, and *The development of the Gram-Charlier series*, are of somewhat less general interest than the earlier portions of the book. They are, however, important to the general reader; and the exposition of the probable error (standard error, rather than probable, is considered in the text) derivations and their significance is especially worthwhile.

The reviewer has not sought to verify in detail the symbolic portions of the text; but, if there be any mistakes, they are not such as to impede careful reading and full understanding of the discussion.

A difficult task has been handled by the author with admirable skill; and statisticians generally will be grateful for a volume which is available for courses and reference, for mathematicians and "other workers."

W. L. CRUM

PRINCIPIA: VOLUMES II AND III

Principia Mathematica. By Alfred North Whitehead and Bertrand Russell. Volumes II and III. Second Edition. Cambridge, University Press, 1927. xxxi+742 pp., and viii+491 pp.

The second and third volumes of the second edition of Whitehead and Russell's *Principia Mathematica* do not differ from the corresponding volumes of the first edition. An account of the changes which the authors think desirable is contained in the introduction and appendices to the first volume of the second edition, of which a review has previously appeared,† but the text of all three volumes has been left unchanged.

The second and third volumes of the *Principia Mathematica* are devoted to building up, on the basis of the system of logic developed in the first volume, the theories of cardinal numbers, relations and relation-numbers, series, well-ordered series and ordinal numbers, and finally of the continuum and of real numbers. The task of developing these theories on the basis of the theorems and processes of logic only, as well as that, undertaken in the first volume, of investigating logic itself by mathematical methods,

* In this connection, the reviewer regrets the parenthetical use (p. 101) of the words "held constant," although he is sure that no careful reader will be misled by them after going through the preceding discussion in the text.

† B. A. Bernstein, this Bulletin, vol. 32 (1926), pp. 711–713.

are both of the highest importance to our understanding of the foundations of mathematics. The magnitude of the undertaking thoroughly justifies the formidable appearance of the three volumes, which contain what is, so far, the most nearly successful attempt to accomplish these tasks. In spite of serious difficulties which remain unsolved, the work has established its claim to be ranked as a notable achievement.

Volume II begins with an account of cardinal numbers, based on the definition, given originally by Frege, that the cardinal number of a class α is the class of all classes similar to α. This definition has a pragmatic justification, in that it leads to cardinal numbers which have the properties we require. But it seems worth while to notice the possibility of another definition* more in accord with our intuitive idea of number, namely that the cardinal number of a class α is the abstraction from α with respect to the propositional function *x is similar to y*. In order to make this definition, we require the following postulate†: If ϕ is any propositional function of two variables which is transitive and symmetric, and if A is any term such that $\phi(A,x)$ is true for some x, then A_ϕ, the abstraction from A with respect to ϕ, exists and has the property that $A_\phi = B_\phi$ if and only if $\phi(A,B)$ holds. The obvious objection is that this is an unnecessary additional postulate which ought to be avoided, but this objection, we believe, can be removed by a consideration of the treatment of classes in the first volume of the *Principia Mathematica.*

According to this treatment the existence of classes is not assumed and the word class itself is not defined, but a determinate meaning is given to propositions about classes. On this basis, classes appear as shadowy things without actual existence. In a certain sense this is the correct view, for classes are, essentially, not a part of reality but fictions devised for their usefulness as instruments in understanding reality. But the same statement is true of propositions and propositional functions and, in fact, all the terms of logic and of mathematics. A postulate asserting the existence of a logical or mathematical term ought, indeed, to be understood merely as making this term, by fiat, a part of a certain abstract structure which is being built. The treatment of classes given by Whitehead and Russell seems not well balanced in that it ascribes a greater degree of reality to propositions and propositional functions than to classes. A postulate asserting the existence of classes probably ought to be included in their treatment.

If, however, we accept the postulate about abstractions proposed above, it is not necessary to postulate the existence of classes, because they can be introduced as follows. A class is determined by a propositional function, but classes differ from propositional functions in that two equivalent but distinct propositional functions determine the same class. By analogy with Whitehead and Russell's definition of cardinal number, we would, except for the obvious circle, define the class determined by a propositional

* G. Peano, Formulaire de Mathématiques, vol. 3 (1901), p. 70.

† See B. Russell, *The Principles of Mathematics*, 1903, p. 166, where a similar postulate is proposed.

function ϕ as the class of propositional functions equivalent to ϕ. On the basis of the postulate which asserts the existence of abstractions, we could define the class determined by a propositional function ϕ as the abstraction from ϕ with respect to equivalence, and so avoid the necessity of another postulate for the introduction of classes.

These considerations would be changed, of course, if we accepted the postulate proposed in the introduction to the second edition of the *Principia Mathematica*, that functions of propositions are always truth-functions, and that a function can occur in a proposition only through its values, for this implies that equivalent propositional functions are identical, and hence that classes and propositional functions may be regarded as the same. But the authors themselves are doubtful about this proposal and their arguments in support of it are not fully convincing. In fact "A asserts p," which they take as a crucial instance, and which is certainly not a truth-function, seems to be a function of the proposition p. The statement that A asserts p is correctly analyzed not as meaning that A utters certain sounds (the analysis proposed by Whitehead and Russell) but that A utters sounds which have a certain content of meaning, and it is this content of meaning which constitutes the proposition p, in the usual sense of the word proposition. The proposal under discussion can mean only that the word proposition is to be used in some quite different sense, in which case the *Principia Mathematica* would fail to give any account of propositions in the usual sense of the word.

Continuing our consideration of the contents of volumes II and III, we find of especial interest the construction of the system of inductive cardinals (positive integers) on the basis of logical postulates only, because it is well known how the real number system, the space of euclidean or projective geometry, and other important mathematical systems, can be constructed on the basis of the system of positive integers, apart from difficulties which arise in connection with the theory of logical types.

The success of this construction of the system of inductive cardinals depends largely on the adoption of the following definition: Consider all those classes of cardinal numbers which contain 0 and which have the property that if they contain ν then they contain $\nu+1$; the common part of all these classes is the class of inductive cardinals. The numbers 0 and 1 and the operation of addition have, of course, been previously defined. Apart from difficulties connected with the theory of types (which theory we hope to see supplanted or greatly modified) this definition gives us the complete system of positive integers with all its usual properties, something certainly not present in the original assumptions (except in the sense that these assumptions enable us to construct the system of positive integers in the way just described).

This definition seems to be the only one that will lead to the desired result. It might suggest itself to consider the cardinal numbers of those classes which are similar to no proper part of themselves, but, without the aid of the multiplicative axiom or a weakened form of it, it is not known how to prove that this class of cardinals is not more extensive than that of the inductive cardinals defined as above.

The axiom of infinity, which it is found necessary to assume in proving important familiar properties of the finite cardinals, is of interest because, as one of the authors has pointed out elsewhere,* the existence of infinite classes can be proved if we disregard restrictions imposed by the theory of types.

Of the remaining subjects treated, that of real numbers and the treatment of Dedekindian and continuous series are of greatest interest.

It is well known how to construct the system of real numbers once the series of positive integers is given, but the usual method takes no account of difficulties raised by the theory of types. According to their original scheme, the authors proposed to overcome these difficulties by means of the axiom of reducibility, but in the introduction to the second edition they abandon this axiom in favor of the postulate mentioned above, that functions of propositions are always truth-functions, and that a function can occur in a proposition only through its values. As a result of this substitution the system of real numbers can no longer be adequately dealt with by any method known to the authors.

If for no other reason than that it leads to important results to which the other does not, the axiom of reducibility seems to be distinctly preferable to the postulate which Whitehead and Russell now propose as a substitute for it.

It is to be hoped, however, that the question will ultimately be settled by the complete abandonment of the theory of logical types or by an alteration in it more radical than any yet proposed. For the theory of types, although adequate to obtain the results we require, provided that the axiom of infinity and the axiom of reducibility are admitted, introduces many complications and creates some awkward situations, one of them the following, referred to at the beginning of volume II of the *Principia Mathematica*. Having proved the theorems that we require about functions of the first $n-1$ types, then in order to obtain the same theorems about functions of the nth type we must make a new assumption of all our postulates, applying them to functions of the nth type instead of functions of some lower type, and must then prove all our theorems anew. We "see," by symbolic analogy, that this can always be done. But the statement that this is so is impossible under the theory of types.

The well known contradictions, of which a list is given in the first volume of the *Principia Mathematica*, must, of course, be dealt with, and the theory of types at present affords the best known method of doing this. Our hope is that another method can be found which will entail fewer complications.

Alonzo Church†

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* B. Russell, *The Principles of Mathematics*, 1903, p. 357.

† National Research Fellow in Mathematics.