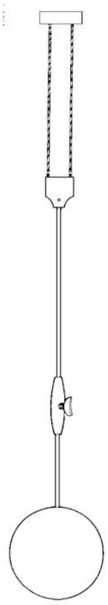
THINGS ARE SELDOM WHAT THEY SEEM - CHRISTIAAN HUYGENS, THE PENDULUM AND THE CYCLOID



by Alan Emmerson

In December 1656, Dutch mathematician and scientist Christiaan Huygens [1] invented what is regarded as the first pendulum regulated clock [2] and he had Salomon Hendrikszoon Coster build an example early in 1657, or so we are told. Huygens was 28 years old.

At about the same time, we are also told, Huygens became aware that if the amplitude of the pendulum's swing changed, the time of swing would also change - the pendulum was not isochronous.

The clocks built by Coster and several drawings attributed to Huygens, clearly show that they attempted to incorporate features in their clocks to alter the path of the pendulum bob to overcome this problem. Specifically, there were clocks which had curved metal plates, now known as "chops", on either side of the pendulum suspension. These were placed so that the suspension thread could wrap around them over an arc of about 50° either side of the rest position. The path followed by the pendulum bob was therefore by definition an involute of the shape of the chops.

It is said that Huygens deduced that, if the chops were cycloidal, the bob of a pendulum would swing along a cycloidal path, rather than the circular arc of the simple pendulum, and the pendulum would then be isochronous.

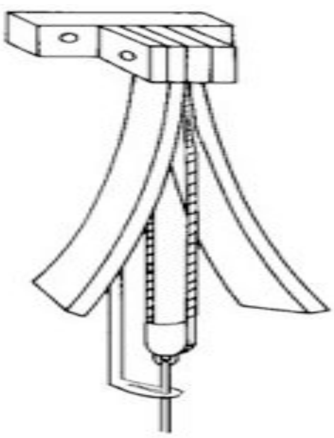


Figure 1 Huygens' Chops

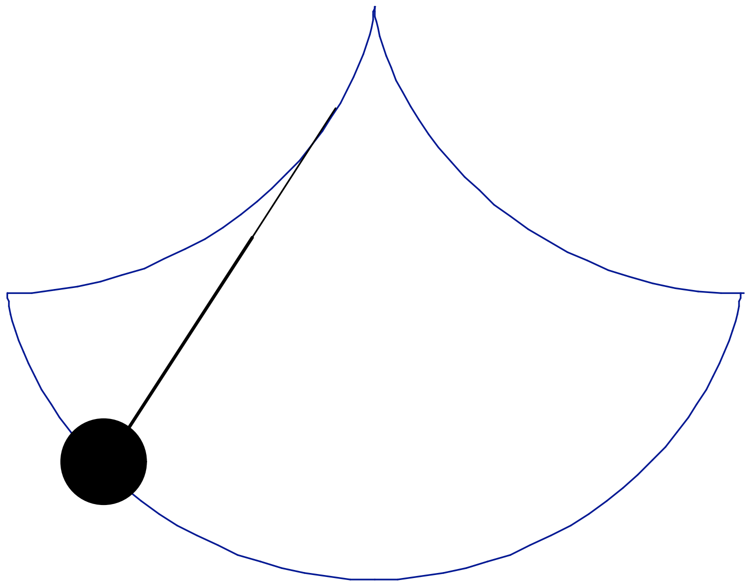


Figure 2. Cycloid and Involute

Since being unexpectedly credited [3] with "discovering" something which I thought was self evident, that Huygens' isochronous cycloidal principle would not work with a pendulum having rigid components, I have become intrigued by the story of Huygens and the cycloid. Unfortunately, the popular twentieth century secondary sources in clocks and the history of science I have read are unconvincingly perfunctory, vague or ambiguous and do not seem to withstand scrutiny. It is common to find, for example, casual assertions that the chops on a particular clock were fitted to make the pendulum bob swing in a cycloidal arc. The Rijksmuseum voor de Geschiedenis der Natuurwetenschapen in Leiden has a clock attributed to Huygens/Coster with allegedly cycloidal chops and claims that it was made in 1657 and that it is the oldest pendulum clock. [4] Yet Richard Good FBHI[5] says that Huygens did not discover the anisochronism until December 1659. Good says that in the same year Huygens proved that the pendulum should swing along a cycloid, and that cycloidal chops were fitted to all later clocks. Haswell [6] says that all happened in 1665. Coster died at the end of 1659. Like a similar clock at The Time Museum, Rockford, Illinois, the Rijksmuseum clock is spring driven whereas the clock Huygens described in 1658 as his invention was weight driven. Landes [7] writes that Huygens arrived at the cycloid by experimenting with the shape of the chops and by subsequent analysis. But as we will see that begs one or two questions. The often accepted authority Britten's [8] is also apt to be confusing. Referring to Huygens it says, "it was not until then [1658] that he was able to discover the formula which determines its performance. That is, that the time occupied by the swing of a pendulum varies as the square root of the length of its arc and inversely as the force of gravity. This irregularity is known as circular error". Even the most promising of texts appears to contain unfortunate phrases. Thus Plomp [9] says, "…we know the exact date on which Huygens constructed his first pendulum clock : December 25th 1656." But, anyone who has ever separated the plates of a clock knows that a clock is not built in a day. Should Plomp's apparent lack of precision cast doubt on the rest of his text? There are many published biographical essays on the life and work of Christiaan Huygens. [10] They differ frequently over significant details and are weak on technicalities. The UK Science Museum world wide web site is disappointingly imprecise. [11]

The confusion is assisted by Huygens' having published two works on the subject with similar titles. *Horologium* of 1658 and *Horologium Oscillatorium* [12] of 1673. In *Horologium* Huygens describes a pendulum controlled clock, not his first clock, to their Lordships the Governors of Holland with a view to establishing his priority of invention.[13] In contrast, *Horologium Oscillatorium* is a significant work in applied mathematics. It is subtitled Geometrical Demonstration Concerning the Motion of Pendula as Applied to Clocks. Huygens' biographer C.D. Andriesse tells us that large proportion of *Horologium Oscillatorium* was actually written during 1660. [14] The work was published thirteen years later,

So, although there are already very much more scholarly writings than mine on the works of Huygens, none I know of give satisfactory answers to these questions:

> When, if ever, were cycloidal chops first fitted for the purpose of making the pendulum isochronous?
> If chops were fitted before that, what was their purpose?

In this paper I have tried to lay out the sequence of Huygens' endeavours to show what was chicken and what was egg.

SOME PRELIMINARIES

The C16th Pendulum

The pendulum was established in science well before Huygens arrived. Around 1602, Galileo [15] had made the experimentally based hypothesis that the time of swing was constant according to the standards of measurement that were then applied for astronomy. [16] Within wide limits, the time for a complete swing was not affected by the size or material of the bob, provided that the rod or cord was the same length. Galileo had determined experimentally that the time of swing was inversely proportional to the square root of the length of the pendulum. [17]

Although the pendulums of Galileo's experiments would have been anisochronous, Galileo clearly believed the pendulum was isochronous for he put a great deal of effort into trying to explain from his theory of motion why this should be so.

Because Galileo was working with pendulums four or five metres long, space requirements probably forced him to use small amplitudes of swing around 15° for which , as shown in Appendix A, the departure from isochronism is small. Nevertheless, had Galileo been able to compare the times of swing of pendulums freely swinging simultaneously, at constant 15° and 10° amplitude, but being otherwise identical, he would have observed something like Figure 4. Ten minutes of observation would have been adequate to detect the difference between the times of swing and to reveal the general anisochronism of the pendulum

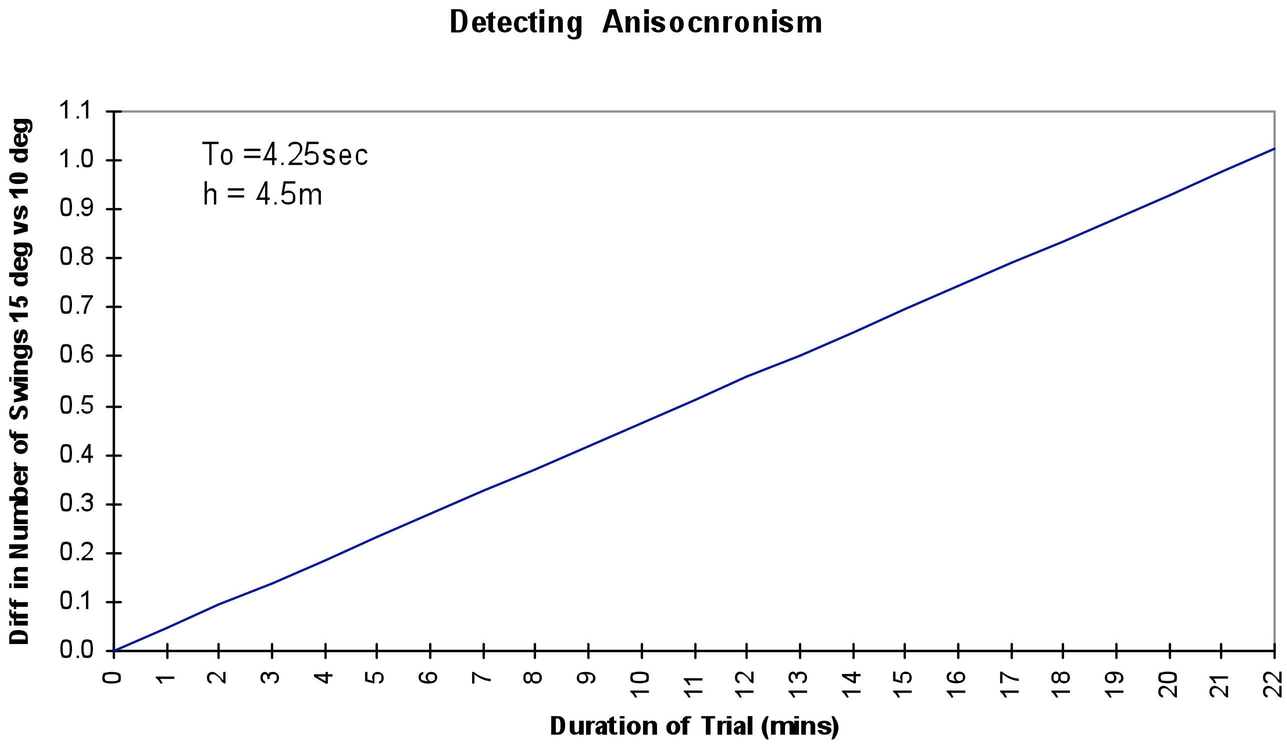


Figure 4 Comparison of Freely Swinging Pendulums - Two Different but Constant Amplitudes

However, the amplitude of a real pendulum decays because of friction. Galileo could have maintained the swing of a pendulum by impulsing it with his hand, but would have been loathe to do so because of the implication for the deductions from the experiment. Thus Galileo was dealing with pendulums in which the amplitude continuously decayed. With long pendulums used by Galileo, having wooden bobs, the amplitude of swing becomes very small quite quickly. The amplitude decayed to negligible, it seems, in about seven minutes.[18] Correspondingly, the anisochronism becomes negligible for most of the duration of the experiment and is undetectable. The mathematical background to this behaviour is set out in Appendix A and is summarised in equation A10. Figure 5 illustrates for two long pendulums swinging together, with initial amplitudes of 15° and 20°.

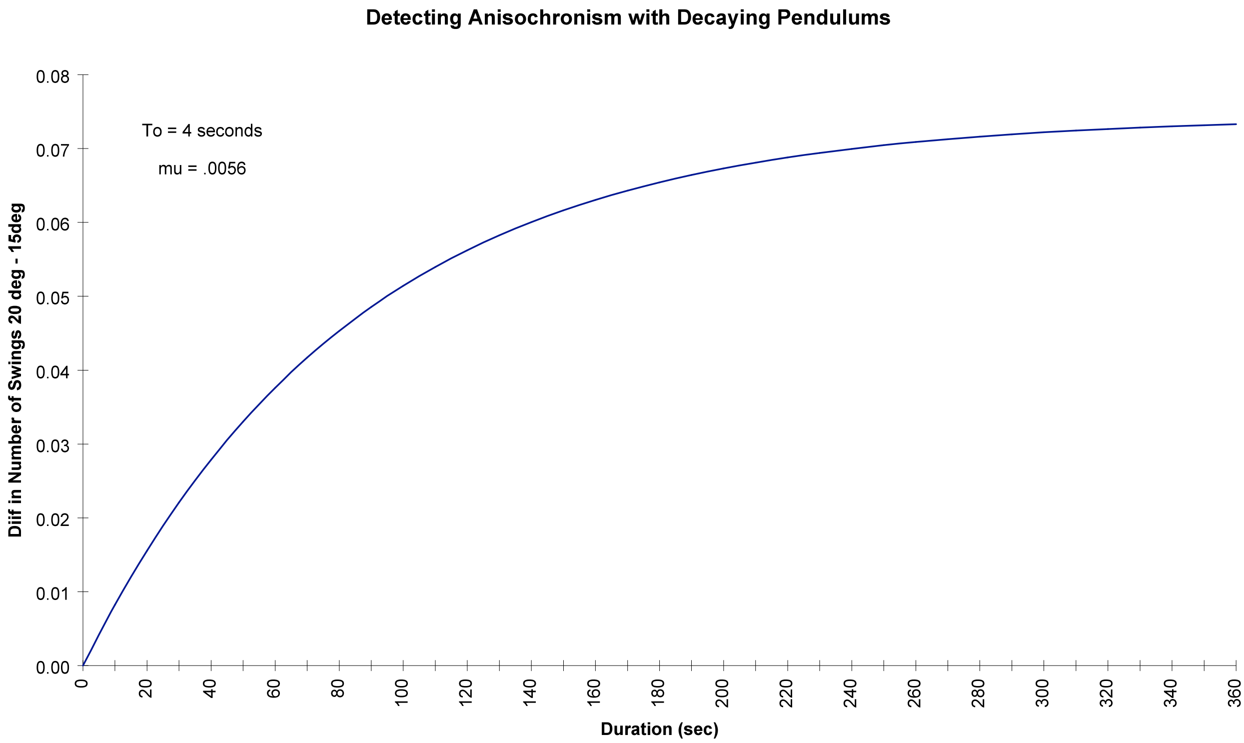


Figure 5 Comparison of Decaying Pendulums - Two Different Initial Amplitudes

That is a plausible explanations of Galileo's not noticing the fundamental anisochronism of the pendulum.

Galileo's experiments did not require a time standard to measure the performance of the trial pendulum. Four out of five of Galileo's propositions about pendulums were based on observing two pendulums of different sorts running simultaneously. The other related to conservation of energy. The need to calibrate the pendulum arose when the pendulum itself began to be used as a time standard. Galileo proposed that accurate measurement of time intervals for the purpose of astronomy could be achieved by counting the swings of a calibrated pendulum. By extension, the principle could potentially be extended to finding longitude and Galileo proposed such a scheme in 1636.

There was a sizeable reward for anyone demonstrating a method of "finding the longitude". As early as 1598, King Philip III of Spain had offered a life pension of 2000 ducats, a perpetual pension of 6000 ducats and an immediate grant of 1000 ducats. Huygens wrote very definitely about a considerable sum offered by the Government of Holland. [19]

Even as late as 1639, Galileo did not have a satisfactory explanation connecting the behaviour of the pendulum to his law of falling due to gravity. In 1638, Baliani derived Galileo's law of fall from the behaviour of the pendulum . He told Galileo, who was then able to invert the derivation, but was unable to publish the result. Baliani revised and substantially extended this work in 1646. [20]

By the 1640s, the notion of the rigid or "compound" pendulum was well established, so much so that in 1647 the relationship of its time of swing to that of the simple pendulum was the subject of a war of words between mathematicians Roberval and Descartes.

It had become the common practice of astronomers to estimate the elapse of time by counting the swings of a weight suspended from a light chain and impelled from time to time by the hand of an assistant.

A new unit of time measurement had evolved, the second sexagesimal division of the hour, or as we know it the "second". In April 1642, the astronomer Riccioli and nine associates kept a "seconds pendulum going for 24 hours, counting 87,998 oscillations." [21] What is more, Riccioli repeated this experiment twice in the following months.

Huygens' invention of 1656 was intended to automate this procedure by adapting the existing common clockwork mechanism so as to count the swings of a pendulum and also sustain the motion of the pendulum in the presence of dissipative forces.[22] He too hoped that his invention would permit the "finding of the longitude" [23]

Shortly before his death in January 1641, Galileo designed in his head, for he was by then blind, a method of harnessing the pendulum to a clock. Galileo's son Vincenzio and associate Vincenzo Viviani were unable to bring the design to fruition before Vincenzio died in May 1649. [24] So in both the question of keeping time and the science of mechanics, Huygens took up the reins from Galileo.

The Role of Marin Mersenne

Marin Mersenne (1588-1648) was a French theologian, priest, mathematician, scientist and philosopher. From 1620 onwards he corresponded or met with some eighty, perhaps all, of the eminent mathematicians scientists and philosophers of the time. He acted as a clearing house for their work. In 1633, 1634 and 1639 he translated Galileo's work on mechanics from Italian into French and it is largely through Mersenne that Galileo's mechanics became known outside Italy. In 1646, at the age of 17 and while still at university, Huygens wrote to Mersenne about Galileo. [25]

Mersenne did some experiments of his own using a pendulum to keep time and he confirmed that the time of swing was inversely proportional to the square root of the length of the pendulum. In 1644, he may have experimentally confirmed the length of the pendulum beating seconds. Mersenne used the pendulum for measuring time intervals and he recommended this method to Huygens [26].

The Cycloid

The cycloid, the path followed by a point on the circumference of a circle as that circle rolls along a straight line.

The cycloid was perhaps first investigated by Nicolaus de Cusa (Cardinal Cusanus) in 1451. Subsequently its properties attracted the interest of many of the world's great mathematicians and physicist - Bouvelles, Roberval, Galileo, Toricelli, Descarte, the Bernoullis, Fermat, Leibnitz, Wren and Pascal.

Through the offices of Marin Mersenne, Huygens would have been aware of most of the work done on the cycloid. He himself had worked on the curve, and in 1658 and 1659 Pascal [27] acknowledged Huygens' achievement. By 1665, the cycloid was probably the most studied curve in history.

Dating Events in C17th

There is a potential ambiguity of twelve months in interpreting the dates given for any event in the story of Huygens and his contemporaries. This arises from changes made at various times to the choice of the month that began the year. In England and many European countries the year began on 25th March, so that January, February and March in, say, 1580 came after December 1580. This practice changed, and from

1582, the change was often, but not always, concurrent with the adoption of the Gregorian calendar, a process which continued sporadically for the next two hundred and more years. England changed in January 1752 (NewStyle).

Various parts of the Netherlands changed to the Gregorian calendar in 1583 or 1700/01 according to religious influences. Huygens' town of Zuilichem is in Gelderland which adopted the Gregorian calendar in January 1700. Huygen lived in Paris from 1665 to 1673. In France, as in Italy and Germany, the various localities also changed to the Gregorian calendar at different times. The change to the choice of the month to start the year seems to have happened rather more uniformly. Most of the part of continental Europe we are interested in seems to have adopted 1 January as the start of the new year before 1600.

Nevertheless, a letter or other paper dated between 1 January and 25 March may have been written twelve months earlier or later than a first glance would suggest.

ESTABLISHING THE PRECURSORS

Precursors to the Cycloidal Chops

There are four essential precursors to clocks being designed with cycloidal chops to make their pendulums isochronous. A problem caused by anisochronism must have been recognised. The cycloid must have been identified as a tautochrone [28]. There must have been a method of determining exactly which cycloid was required for a particular pendulum. It must have been shown that the evolute of the cycloid is itself a cycloid.

To establish the chronology we should turn to primary documents. The closest we have are translations of *Horologium Oscillatorium* and Huygens' letters and papers that document most of his scientific work, now collected in twenty two volumes and known as the *Oeuvres Complète*s *de Christiaan Huygens*. [29]

A Problem Caused By Anisochronism ?

There is no doubt that Huygens knew before *Horologium* was written, which was before the publication date of September 1658, that the pendulum was anisochronous. Huygens and others had been using the pendulum to measure time intervals in astronomy for several years. In *Horologium,* Huygens writes "It is asserted with truth that wide and narrow oscillations of the same pendulum are not traversed in absolutely equal time, but that the larger arcs take a little longer, which it is possible to demonstrate by a simple experiment. For if two pendulums, equal in weight and length, are released at the same time, one far from the perpendicular, the other only a little deflected, it will be perceived that they are not long in unison, but that of which the swings are smaller outstrips the other." [30]

Figure 6 illustrates such an experiment with two freely swinging pendulums nominally beating seconds. After six minutes, one pendulum is 0.4 of a swing ahead of the other - nearly in opposite phase.

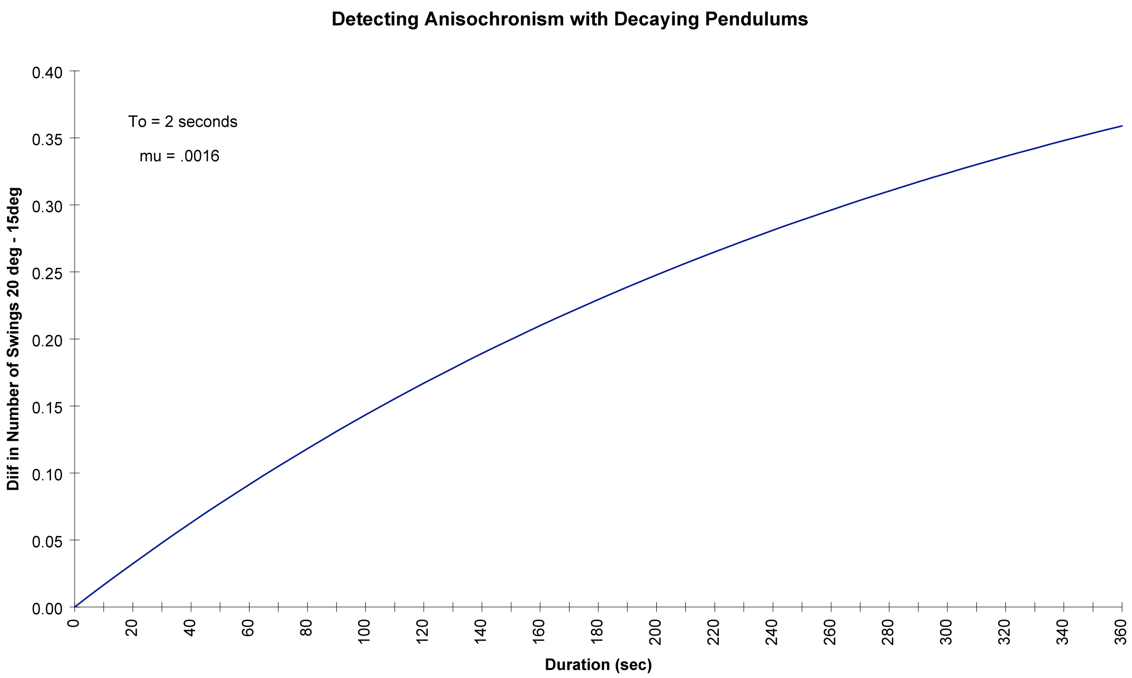


Figure 6 Comparison of Decaying Pendulums - Two Different Initial Amplitudes

Was this a problem? Huygens did not really think so but intended to follow it up. "Yet as I have said, my time piece is less likely to an inequality of this kind, because all the vibrations are of equal amplitude. Nevertheless, it remains not entirely free from inequalities, although these are very tiny, and as is needful, I intend to pursue the matter." [31] Indeed he had already been doing so for quite some time There is a record of his experimenting with chops in May and June of 1657.[32]

So this precursor, observing the anisochronism, was established by mid 1657, if not earlier.

The Cycloid Identified As A Tautochrone

Some months after the death of Coster, and three years after constructing his first clock, Huygens set out to actually analyse the motion of the simple pendulum. The description of this work is not part of *Horologium Oscillatorium* . It is in a self contained paper entitled On Determination of the Period of a Simple Pendulum [33] dated December 1659 [34]. By geometrically based argument, and calling on his earlier work [35] on centrifugal force, Huygens showed that the period of the pendulum was a function of only the length of the pendulum. This of course implied that the pendulum as analysed was isochronous. In the analysis, however, he had used an approximation which was not true for all angles of swing when the bob swung in a circle.

Huygens knew that in practice the period of the pendulum also depended on the amplitude of swing. He then asked himself what path the pendulum would have to follow for the approximation to be true for all angles of swing. That is, what path would make the pendulum isochronous. He found that the requisite curve was one in which the tangent would be drawn by exactly the same method that was then used for drawing the tangent to a cycloid. [36] This cycloid would have a vertical axis equal to half the length of the pendulum. This result of course applied only to the simple pendulum. A derivation using modern methods is set out in Appendix B.

Thus the second necessary precursor was established early in 1660.

Huygens then set about the inverse problem - proving that the cycloid was a tautochrone. He achieved this by a lengthy argument that became Part II of *Horologium Oscillatorium* with the conclusion as Proposition XXV.

The Evolute Of The Cycloid Is A Cycloid

After discovering that the isochronous path was a cycloid, the obvious next step for Huygens was to determine what shape his chops should be so that, as the pendulum suspension thread wrapped and unwrapped around them, the centre of the bob would describe a cycloid. In other words, he set out to find the evolute of the cycloid.

He had previously dealt with strings carrying weights and unwrapping around cylinders in his investigation of centrifugal force. Huygens work notes for 20 December 1659 show him numerically correlating the coordinates of a cycloid with the angular displacement of the pendulum string[37]. *Oeuvres Complètes* shows us that by Summer 1660 [38] Huygens had determined that the chops should be semi-cycloids, congruent with the intended path of the pendulum bob. Huygens conveyed his conclusion to the senior scientists of the day, so that by December 1661 Huygens' cycloidal pendulum was actually being investigated as a means of establishing a standard of length. [39] The conclusion was actually published much later as Proposition VI of Part III of *Horologium Oscillatorium*.,

Thus it was not until the European Summer of 1660 that all the prerequisites for designing and fitting cycloidal chops were satisfied.

HUYGENS' EARLY CLOCKS

The First Clock

The first announcement that Huygens had invented a new more accurate clock was in a letter [40] he wrote to Professor van Schooten on 12 January 1657. Huygen is not specific in this letter about the date he finished the clock..

However, we do know that on 26 December 1657 Huygens wrote to to Ismael Boulliau. [41] He was responding to a letter from Boulliau telling Huygens that The Grand Duke of Tuscany was reputed to have a clock just like that which Huygens had shown Boulliau in April 1657. Huygens prefaced his reply by noting that it was just a year and a day since he had made the first model of his clock.[42] From this we can infer that Huygens had a pendulum clock of some sort, working or not, on 25 December 1656. We can also infer that by April 1657, Huygen, perhaps with Coster's help, had a clock that worked well enough to be worth copying.

In the same letter to Boulliau, Huygens seemed to foreshadow the imminent conversion of a tower clock to pendulum control in a nearby town. This was to have a pendulum about 21 *pied* long weighing 40 or fifty *livres*. Perhaps this was the tower clock at Scheverlingh which Coster and Huygens modified and which Huygens later described to Jean Chapelain in a letter dated 28 March 1658 [43] just before they received a contract for a similar job at Utrech [44]. That clock had a rigid pendulum 24 *pied* long and weighing 50 *livre*. The suspension cords for the pendulum were six *pouce* long.

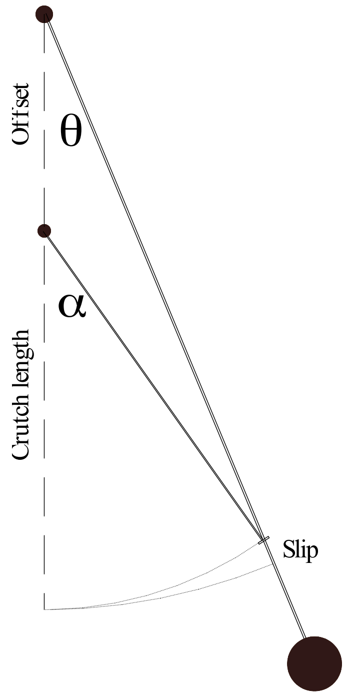
The nature of Huygens' modification was shown by a sketch in the letter to Chapelain which I have copied here as Fig 7. We note that there are no chops in the sketch. The particular relevance of this sketch is that Huygens said it demonstrated the principle of his pendulum clocks.

Fig 7 Huygens' Sketch of 28 March 1558

This leads to the inference that Huygens produced his first pendulum controlled clock by modifying an existing clock - possibly a spring driven table clock of the sort which was common in early C17th. Now this is something that might have been accomplished in one day as Plomp says . This would have involved standing the table clock on its edge with the corresponding transformation shown in Figure 8. A table clock verge escapement of this vintage would probably have had a fifteen tooth crown wheel. As Table 1 of Appendix C shows, this would require a pendulum swinging about 30° either side of centre. (The bob of a 6 inch pendulum would have swung 3 inches either side of centre.) Such a pendulum would be particularly susceptible to anisochronism - as demonstrated in Appendix A.

Fig 8 Huygens' Modification of 25th December 1656 - Conjectural Schematic

Huygens has left us no explanation of why he introduced the crutch. We can speculate firstly that he was concerned that, if he hung the pendulum directly from the verge arbour, the pivot and balance-cock would be taking lateral loads for which they were not designed.. In the course of the modification, Huygens would have quickly realised that if the pendulum string was to remain straight while being impulsed by the escapement, either the pendulum should be impulsed near the bob or the bob needed to be fairly heavy - perhaps heavier than could be sustained swinging through 30°. Once he realised that a rigid pendulum rod was needed to accept impulse torque or lateral impulse force, the crutch could be shortened. [45] With the pendulum suspended from well above the verge arbor, this would have given the opportunity for the crutch to act to reduce the pendulum arc necessary for the escapement to work as shown in Figure 9. On the other hand, Huygens may have used the crutch to implement a lever pair hat could be used to match the torque supplied by the spring and escapement to the torque needed by the pendulum. Huygens would perhaps reluctantly have accepted the increased slip caused by the offset.



Offset/Crutch Length	Theta at Alpha=30 degrees	Slip/Crutch Length percent
1.0	15.0	7
0.9	15.8	6
0.8	16.7	6
0.7	17.7	6
0.6	18.8	5
0.5	20.1	5
0.4	21.6	4
0.3	23.2	3
0.2	25.1	2
0.1	27.4	1
0.0	30.0	0

Fig 9 Angular and Slip Effects of Crutch

I remain curious about why we find no contemporary record of Huygens' modification and testing or calibration of this first clock nor any announcement to their Lordships the Governors of Holland? Perhaps the clock did not work at all well. That would not be surprising.

I find it a compelling notion that somewhere in the chronology there must have been a working pendulum clock with only one hand and no chops. Until such a clock had been built and run, Huygens would not have appreciated that he had sufficient precision to justify two hands and also a problem with anisochronism.

When then did the chops and the minute hand appear?

Coster's Clock

The next thing we know is that Huygens took his ideas and presumably his first effort to an established clockmaker Salomon Hendrikszoon Coster and surveyor and clockmaker Johan van Kal. [46] We do not know exactly when or why.

There are two versions of what happenedthen. Britten's Old Clocks and Watches and their Makers[47] has it that "He [Huygens] assigned his rights in the invention to Coster who submitted it to the States-General and was granted a patent (Octroo) for twenty one years from 16 June 1657." Eminent Dutch clock collector and author J.L. Sellink writes[48] "It is known however that Huygens obtained a patent for his discovery and that Salomon Coster from The Hague obtained the licence to manufacture and sell clocks built on this principle for 21 years. After Coster's death in 1659 the licence passed [49] to Claude Pascal and Severin Oosterwijck. It is now known … that Huygens request to patent his method in Paris was refused."

So who was granted the patent? We ask this question in seeking to discover the inventor because, under today's laws, the industrial right, the monopoly, is granted by the state to the person who first had the idea and made a prototype. The patentee is the inventor. The patentee then licences one or more manufacturers.

However, we have little knowledge of the state of industrial rights law in the United Netherlands in 1657 - if indeed there was any. For comparison, the Monopoly Act in England became law in 1642, but I understand that the Patent Law in France was not enacted until 1791.

Plomp[50] says " …the States General of the United Netherlands granted Salomon Coster the exclusive right ('octroy' or 'privilege') for a period of 21 years to make and sell clocks in the Netherlands constructed according to the invention of Christiaan Huygens" Plomp's authority for this is an editorial note in *Oeuvres Complètes* [51].

Documents[52] associated with the granting of the *privilege* to Coster show that the document was actually issued to Coster and that it covered "[new inventions in horology which had been developed by Christiaan Huygen and shaped in the hands of Salomon Coster and Johan van Kal, clockmakers]".

If there were ever a technical description of the invention which Coster was allowed to duplicate it has been lost.

Now the Rijksmuseum voor de Geschiedenis der Natuurwetenschapen has, in Leiden, a clock in the style known as a Hague clock which has chops and which is labelled "*Salomon Coster Haghe met privilege 1657*" and we might therefore infer that it was built either in 1657 or to the design standard of 1657. However, *met privilege 1657* may indicate the date of the licence not the date of manufacture and not the design standard. Plomp's survey of the Dutch pendulum clocks of the period [53] records no surviving clock labelled *met privilege 1658* or *met privilege 1659.* [54] which would reveal the labelling practices of the day.

Oeuvres Complètes Vol XVII, published in 1932, [55] describes a Coster pendulum clock, owned by the Rijksmuseum and located at the Natural Science museum in Leiden. Comparison of recent photographs with those in *Oeuvres Complètes* confirm this is the clock in the previous paragraph.. It has chops, a spherical screw-adjusted pendulum bob, motion work for two hands and photographs allow one easily to believe that it has stop-work on the mainspring barrel - although stop work is not mentioned in the textual description which is otherwise quite detailed.

At the conclusion to that article, the author remarks " We have found in the Hague a second example of the same clock. [It has the same train count and is other wise identical except that it has no chops, the Coster cartouche is not dated, the pendulum bob is a copper disc without screw adjustment.] ".

The Science Museum in London and the Time Museum also each have a similar clock and say that seven such clocks are known to have survived into the twentieth century. At least one of them, already mentioned, does not have chops. The chops on the clock at Leyden are not cycloidal and are clearly an after thought.. The clock has a remnant of the pendulum suspension that would have been used before chops were introduced. The clock in the Science Museum has no such remnant. One might well think that the basis of the Leiden clock predates the first use of chops. These clocks throw light on the configuration of the clock of the *privilege*, but the true origin of the chops remains in the dark.

Incidentally, buried in this somewhere is the invention of coaxial motion work for hours and minutes.

So we are left with the simple fact that some time before 16 June 1657 Huygens, Coster and van Call produced a working pendulum controlled clock. We do not really know what it looked like. Distribution of credit between Huygens and Coster remains unresolved.

These events, though, suggest that Coster may have contributed substantially to the design of the working pendulum clock.

While this first Coster clock just may have had chops, they would not have been cycloidal, other than by coincidence.

The Clock of Horologium.

In *Horologium* of September 1658 Huygens documented the design of a pendulum controlled clock with a view to establishing his priority of invention. The design is illustrated in Figure 10.

The text and the design show that by this date Huygen was certain that the propensity for errors due to the anisochronous pendulum was greater for pendulums with larger amplitudes. Huygens wrote "With large arcs the swings take longer, in the way I have explained, therefore some inequalities in the motion of a timepiece exist from this cause …" As can be seen from the drawing, Huygens went to some trouble to reduce the amplitude of the pendulum while maintaining sufficient amplitude of the verge to release the escapement crown wheel. Huygens had introduced a pirouette , a contrate and pinion pair between the crutch and the verge. The crutch is short and the pendulum pivot is some distance above the crutch pivot.

This design had an unusual dial layout - a minute dial with concentric seconds hand, showing hours on a subsidiary dial.

There is more to be learned from the text. Completing the quotation above, Huygen wrote " With large arcs the swings take longer, in the way I have explained, therefore some inequalities in the motion of a timepiece exist from this cause, and, although it may seem to be negligible, when the clocks were so constructed that the movement of the pendulum was somewhat greater [than at present] I have used an appliance as a remedy for this also."

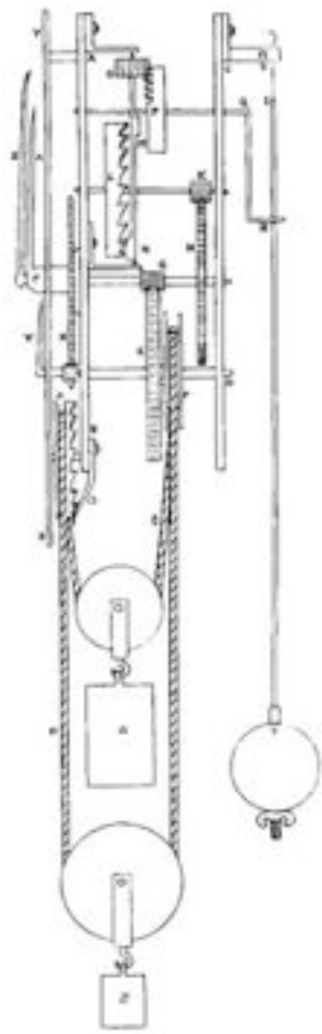


Figure 10 Clock of *Horologium*

As we know Huygens had already experimented with chops, I think we might fairly infer that this appliance was in fact a pair of chops intended to increase the speed of the pendulum in the longer arcs, but, there is no direct evidence on this matter in *Horologium* .

We may similarly infer that Huygens was satisfied that the chops were not working and that that he and Coster had discontinued their use by September 1658. For Huygens goes on to say, " At the present time, certainly, this method is not the cure."

The drawing in Figure 10 seems to be a a sketch of the concept rather than a practicable clock design. For example, there is nothing to prevent the pirouette contrate wheel from moving out of mesh with its pinion; and the contrate wheel would have to be fitted to its arbor after the arbor was installed in the backplate. It seems quite possible that this clock never progressed beyond the design stage. However, Horologium may be read to imply that Huygens experimented with this style of clock. "Therefore, by rendering all the swings short, … individual times are distinguished by no remarkable difference. … doubling the driving weight does not thereby accelerate the movement of the pendulum or alter the working of the time piece, which was not so in all others hitherto in use." Huygens was probably not aware of the extent to which the recoil of the verge escapement masked the effect of changing the driving torque.

Horologium also shows that before 1658 Huygens was well aware of the causes of varying efficiency in the clock train and the consequences for the rate of the clock. He would have been aware of the effect of lost motion (back lash) and additional friction in the contrate and pinion set. Perhaps it was for that reason that Huygens and Coster either did not take up or did not continue use of the pirouette in domestic clocks.

In any case, browsing through Huygens' work notes, we find evidence of a change or contemplated change back to chops in November 1658 [56] and then in June and October 1659, [57] and that in December 1659 he was attempting to calibrate a pair of chops experimentally,[58] and describing his discovery to his old mathematics tutor Prof van Schooten of Leiden. [59]

The Clock of *Horolgium Oscillatorium*

The clock design in *Horologium Oscillatorium* of 1673 does not show the improvements over the design of 1658 that one might expect in thirteen years, except that we see Huygens has returned to using chops. The explanation is that much of Horologium Oscillatorium was written well before the date of publication and the design probably originates from late in 1660. The crutch pivot coincides approximately with the centre of swing of the pendulum. The crown wheel has 15 teeth and the amplitude of the pendulum would have been about 30°. The design has an unusual dial layout. There are concentric hour and minute hands, but seconds are displayed by a rotating subsidiary dial. It has a rigid pendulum hanging from a bifilar suspension operating between chops. The chops, as drawn, do not conform to the correct cycloidal shape , though that could be no more than a publishing convenience as there are other draughtsman's errors in the drawing.

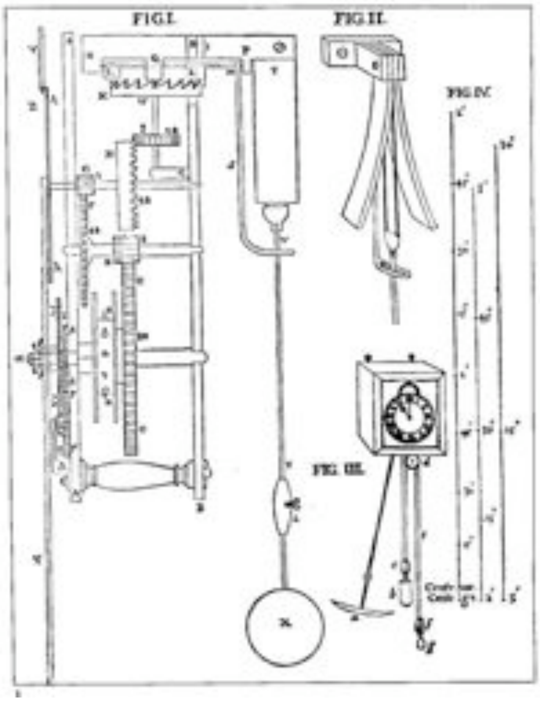


Figure 11 The Clock in *Horologium Oscillatorium*

Features of this drawing suggest that it too is a sketch of a concept rather than a working clock.

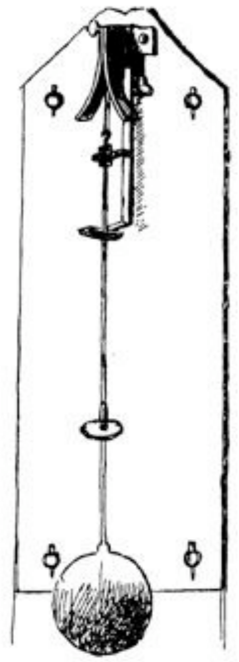
<u>THE LAST WORD</u>

Late in 1661 Huygens, for a second time, sought to determine theoretically the period of a rigid pendulum. (The relevant work notes are dated August to November 1661 [60]) This time he was successful.

But he deduced something else as well. When the work on the rigid body pendulum was published in Part IV of *Horologium Oscillatorium,* Huygens put it unequivocally, at Proposition XXIV, "It is not possible to determine the centre of oscillation for pendula suspended between cycloids." The very reason this is true means that cycloidal chops do not provide an isochronous path for a rigid body pendulum. This is explained in Appendix B.

It is difficult to imagine that an intellect such as Huygens'. did not realize this immediately. In December 1661 he had actually found that the isochronism was not quite perfect. [61] At the time, he attributed this to the elasticity of the suspension threads. One would expect that after December 1661 Huygens ought to have been dissatisfied with cycloidal chops for real pendulums. [62] Nevertheless, he persisted with chops in his subsequent designs for sea clocks, sometimes using either a double fork crutch suggested by the Scotsman Bruce, or a pivoted disc suspended from a light wire frame, which might avoid rigid body rotation..

By the date the manuscript for *Horologium Oscillatorium* was written, Huygens had realised the implication of Proposition XXIV. He suggested the difficulty could be overcome by constructing the pendulum so that the rigid parts did not rotate. In reading Proposition XXIV one can almost sense Huygens' discomfiture as, having advocated cycloidal chops, he tries to make light of this difficulty. I have not discovered when Huygens came to this realisation.



Pendulum controlled by Crutch with Double Fork
OC VolXVII p166

The logic is this:

Either Huygens was very slow to realize that the imposssibility of determining the centre of oscillation for pendula suspended between cycloids also meant that cycloidal chops do not provide an isochronous path for a rigid body pendulum.

Or, the clocks designed by Huygens after, say, 1662 did not have *cycloidal* chops but were intended to have experimentally shaped chops.

Or Huygens had so much mental capital invested in the cycloid that he would not abandon it.

There was no real need for the chops, cycloidal or otherwise, after the Clement/Knibb/Hooke invention of the anchor escapement in 1666. By permiting the escape wheel to release with a much smaller movement of the pallets and pendulum, the anchor escapement greatly relieved the susceptibility of pendulum clocks to errors caused by the anisochronous pendulum. It might be thought curious that Huygens did not mention the anchor escapement in Horo;ogium Oscillatorium. But in the end run the work is a dissertation in theoretical mechanics dating from 1660/61 rather than a treatise on clockmaking in 1673.

CONCLUSION

The chronology of Christiaan Huygens, the pendulum and the cycloid may be summarised as follows.

1656	25 Dec	Huygens modifies table clock, replacing balance by pendulum and crutch
1657		(Coster or van Call builds ?) pendulum controlled clock for Huygens
		Huygens aware of anisochronism and that pendulum is more susceptible to anisochronism at large amplitudes of swing
	May	Huygens experiments with chops
	June	Coster granted licence to manufacture style of clock built for Huygen
1658	Jan Apr	Tower clock at Scheverling and Utrech converted - no chops. Huygens and Coster devise pirrouette mechanism
		Huygens experiments with chops, is unable to make them work accurately
1658	Sept	*Horologium* published
	Oct Nov	Huygens experiments with chops, is still unable to make them work accurately
1659		Coster dies
1659	Dec	Huygens again attemps to calibrate chops experimentally.
		Huygens analyses period of simple pendulum, shows that an isochronous path would be a cycloid, proves cycloid is a tautochrone
1660	summer	Huygens shows involute of cycloid is itself a cycloid
1661	Aug to Nov	Huygens analyses period of rigid pendulum notes that this cannot be done for rigid body swing from a thread
	Dec	Observes cycloidal pendulum not perfectly isochronous
1662 ?		Realizes cycloidal chops do not make rigid pendulum isochronous

So we see that no clock made by Coster would have been deliberately made with cycloidal chops.

There may only have been, say, an 18 month period between when Huygens deduced that a cycloid was the correct shape for the chops of a simple pendulum and when he realised that the chops would not work for a pendulum with rigid elements. It seems quite possible that *cycloidal* chops were never actually fitted to clocks under Huygens' directions outside that period.

> "Things are seldom what they seem.
> Skim milk masquerades as cream
> Turkeys strut in peacocks' feathers.
> Very true. So they do."

© A.J.Emmerson,
Gaythorne, July 2004
Revised Jan 2005

APPENDIX A

DECAY AND ANISOCHRONISM

Definitions

A "simple pendulum" consists of a particle, with mass but no other dimensions, attached to a point fixed in space, by a taut string having fixed length, no stiffness, and no mass, so that the particle is constrained to revolve about this point, along a path which has a constant radius of curvature and is in a single plane parallel to the local gravitational field.

In the jargon, a "compound pendulum" is a rigid body in which a point, other than the centre of mass, is fixed in space by a pivot, so that the body may rotate about that pivot in a single plane parallel to the local gravitational field. The simple pendulum is in fact a special case of the compound pendulum and may be analysed by the same equations. [63]

In deriving the equations of motion of these idealised pendulums, it is usually assumed that the pendulum swings in a stationary frame of reference, that the gravitational field is uniform, that there are no other force fields, and that there are in particular no dissipative forces.

Huygens' pendulum consisted of a rigid bob and rod having mass and suspended from the top of the rod by thread attached to a fixed point.

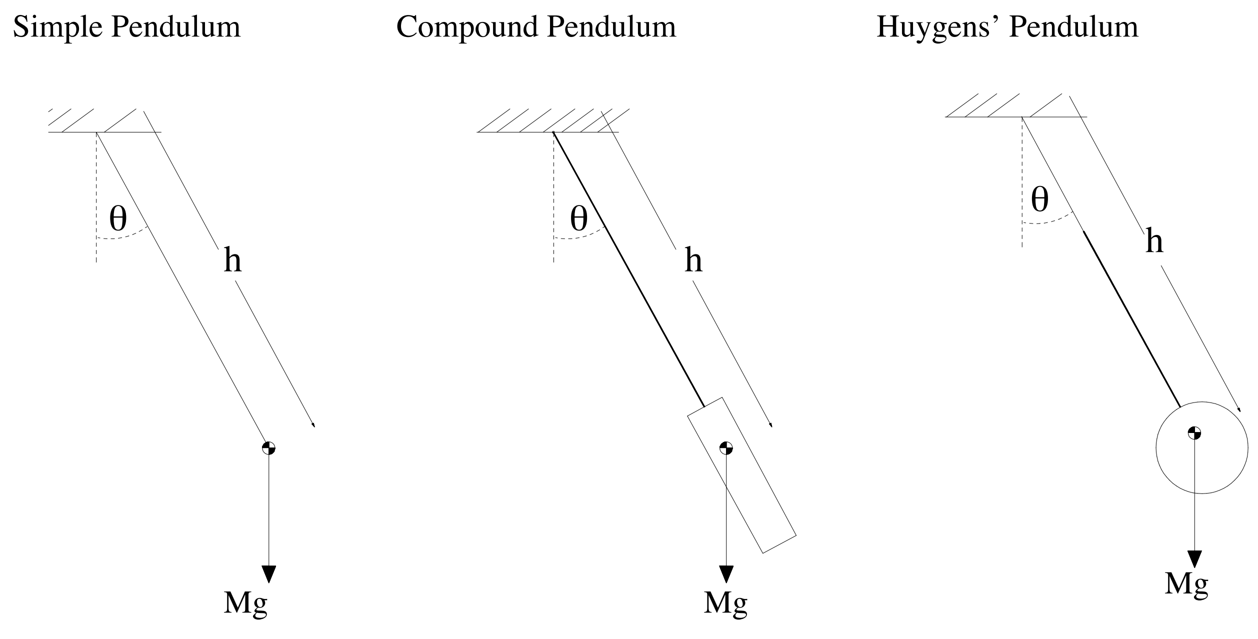


Figure A1. Pendulum Idealisations

The usual notation is:

h is the distance from the pivot axis to the centre of mass of the pendulum.
M is the mass of the pendulum
g is the acceleration due to gravity
θ is the angular displacement of the centre of mass from the rest position
A is the amplitude of swing, the maximum value of θ.
I is the mass moment inertia of the pendulum about the pivot axis, and
k_o is the radius of gyration of the pendulum about its centre of mass

Time of Swing

Using the method of rotational dynamics originated in the late C18th we observe that the applied torque is equal to the rate of change of angular momentum. We can then write:

$$\frac{d^2\theta}{dt^2} = -\frac{Mgh}{I}\sin\theta \qquad \text{--- A1.}$$

An acceptably accurate solution to that differential equation leads to a time for a complete swing cycle of:

$$T = 2\pi\sqrt{\frac{I}{Mgh}}\left(1+\frac{1}{4}\sin^2\frac{A}{2}+\frac{9}{64}\sin^4\frac{A}{2}+...\right) \qquad \text{---------- A2.}$$

For algebraic simplicity we adopt the usual approximation

$$T = 2\pi\sqrt{\frac{I}{Mgh}}\left(1+\frac{A^2}{16}\right)$$

(This approximation incurs an error in the estimation of the time of swing when compared with the complete infinite series. The error is 0.13% at an amplitude of swing near 45°)

We can rewrite this time of swing as $$T = T_0\left(1+\frac{A^2}{16}\right) \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ A3.}$$

And $T_0 = 2\pi\sqrt{\dfrac{I}{Mgh}}$ may be regarded as the time of swing of an hypothetical isochronous pendulum .

We note that $\dfrac{T}{T_0} =$

1.0386 @ 45°
1.0043 @ 15°
1.0005 @ 5°
1.00002 @ 1°

The mechanism of a clock records the number of swings made by the pendulum. In a measurement interval D seconds long the number of swings made by the pendulum is given by

$$N_D = D/T_0\left(1+\frac{A^2}{16}\right) \qquad \text{-------------------------------------- A4.}$$

If two identical pendulums were set running at the same time with constant but different amplitudes, the difference in the number of swings in the trial duration would grow as already shown for example in Figure 4. Every time the difference in number of swings increased by one half, the pendulums would be seen to be out of phase with each other. In the case shown, the phase relationship would reverse about once every 10minutes..

The implicit presumption is that it is possible to maintain the pendulums swinging at such large amplitudes for the duration of the experiment in the face of the dissipation of the pendulums' energy. That is of course the job of the going train of the clock.

Sensitivity to Anisochronism - in a Clock

The amplitude dependent component of the time of swing, $T_0\left(\frac{A^2}{16}\right)$ seconds per swing. is known as "circular error".[64] In itself it is of no consequence. The length of the pendulum can be adjusted to give the right number of swings per day.

The real problem is that changing the amplitude of swing also changes the time of swing.

Differentiating equation A3. gives

$$\frac{dT}{dA} = T_0\frac{A}{8} \qquad \text{--------------------------------A 5.}$$

or,

$$\frac{dT}{T_0} = \frac{A}{8}dA$$

Thus an increase in the amplitude of swing increases the time of swing. The absolute effect is bigger when the time of swing, or the amplitude of swing, is already large.

The reason that this type of error is so troublesome is that there are uncontrollable effects in a clock which act so as to vary the amplitude of swing. The consequence of "circular error" or, more correctly, of the anisochronism, is that, by changing the amplitude, those effects change the time of swing of the pendulum.

The influences at work are those which determine the energy dissipated by the pendulum per unit time and those which replace that energy.

It is possible to get a working idea of the effect on time of swing caused by variations in factors influencing the power absorbed from the pendulum.

From considerations of the potential energy at the top of the swing, we can show that, for a simple pendulum with an angular amplitude of A each side, the energy of the pendulum is:

$$E = Mgh(1-\cos A)$$

The change in that energy as the angular amplitude changes can be obtained by differentiating that expression and gives:

$$\frac{dE}{dA} = Mgh\sin A \quad \text{-------------------------------------} \quad \text{A6.}$$

The actual total power loss has been measured rather precisely for many pendulums. This is usually done by measuring the time taken for the arc of a swinging pendulum to diminish by a given proportion, say one half.. Mathematics, and the observation of real pendulums swinging freely, show that the amplitude changes by decaying with time very nearly as described by the following equation:

$$A = A_0 e^{-\mu t}$$

Where:

t is the time the pendulum has been swinging

A_o is the amplitude at the beginning

μ depends upon the resistance factors of the pendulum and its surroundings and is near enough to constant for small swings. [65]

The rate at which the angular amplitude changes can be found by differentiating that expression to give:

$$\frac{dA}{dt} = -\mu A_0 e^{-\mu t} \quad \text{ie} \quad \frac{dA}{dt} = -\mu A \quad \text{-------------------------} \quad \text{A7.}$$

Now the rate of energy change is $\frac{dE}{dt}$ and $\frac{dE}{dt} = \frac{dE}{dA}\frac{dA}{dt}$ so that from A6 and A7 the power loss for the freely swinging pendulum with a decaying amplitude is:

$$\frac{dE}{dt} = -\mu MghA\sin A \quad \text{--------------------------------} \quad \text{A8.}$$

Thus the power being absorbed depends non linearly on the amplitude of the swing, and on a combination of factors relating to the dynamics of the pendulum. Incidentally, it is not correct to infer from this equation that the energy loss rate is proportional to M. That is because μ also depends on M.

A pendulum will not continue to swing for ever if it keeps losing energy. The energy transferred to the universe outside the pendulum must be replaced. It must be replaced at the same average rate as it is absorbed. For constant amplitude, the power supplied must be equal to the power being absorbed as given in equation A8 .

Thus power supplied is given by $P = \mu MghA\sin A$

If the power supplied changes the amplitude will change. Assuming that there is no asymmetry in the impulse which would itself change the time of swing:

$$\frac{dP}{dA} = \mu Mgh(A\cos A + \sin A)$$

from equation A 5 $\quad \dfrac{dT}{dA} = \dfrac{AT_0}{8}$

So that $\quad \dfrac{dT}{dP} = \dfrac{AT_0}{8}\dfrac{1}{\mu Mgh(A\cos A + \sin A)}$

ie $\quad \dfrac{dT}{dP} = \dfrac{A}{8\mu Mgh(A\cos A + \sin A)}T_0$

Now, $\quad \mu Mgh = \dfrac{P}{A\sin A}$

So $\quad \dfrac{dT}{dP} = \dfrac{A^2 \sin A}{8P(A\cos A + \sin A)}T_0$

That is, the change in the time of swing consequent on a small change in input power is given by

$$dT = \frac{T_0 A^2 \sin A}{8(A\cos A + \sin A)}\frac{dP}{P} \qquad \text{--------------- A9}$$

The trigonometric component of this expression increases with A. The period of a pendulum with 45° amplitude is about ten times more sensitive to power change than one with 15° amplitude.

In a measurement interval D seconds long the number of swings made by the pendulum is given by equation 4

$$N_D = D/T_0\left(1 + \frac{A^2}{16}\right)$$

This number of course will change if the nett power input to the pendulum changes.

$$\frac{dN}{dP} = \frac{dN}{dA}\frac{dA}{dP} = D\frac{d}{dA}\left[T_0\left(1 + \frac{A^2}{16}\right)\right]^{-1}\frac{dA}{dP}$$

but $\quad \dfrac{dP}{dA} = \mu Mgh(A\cos A + \sin A) \quad$ and $P = \mu MghA\sin A$

so $\quad \dfrac{dP}{dA} = \dfrac{P(A\cos A + \sin A)}{A\sin A}$

$$\text{and} \quad \frac{dA}{dP} = \frac{A \sin A}{P(A\cos A + \sin A)}$$

$$\text{Then} \qquad \frac{dN}{dP} = D\frac{d}{dA}\left[T_0\left(1+\frac{A^2}{16}\right)\right]^{-1} \frac{A\sin A}{A\cos A + \sin A}\frac{1}{P}$$

So that the change in the number of swings made in a given duration, caused by a change in the power input (or loss) is given by:

$$dN = -\frac{D}{8T_0}\left(1+\frac{A^2}{16}\right)^{-2} \frac{A^2 \sin A}{A\cos A + \sin A}\frac{dP}{P}$$

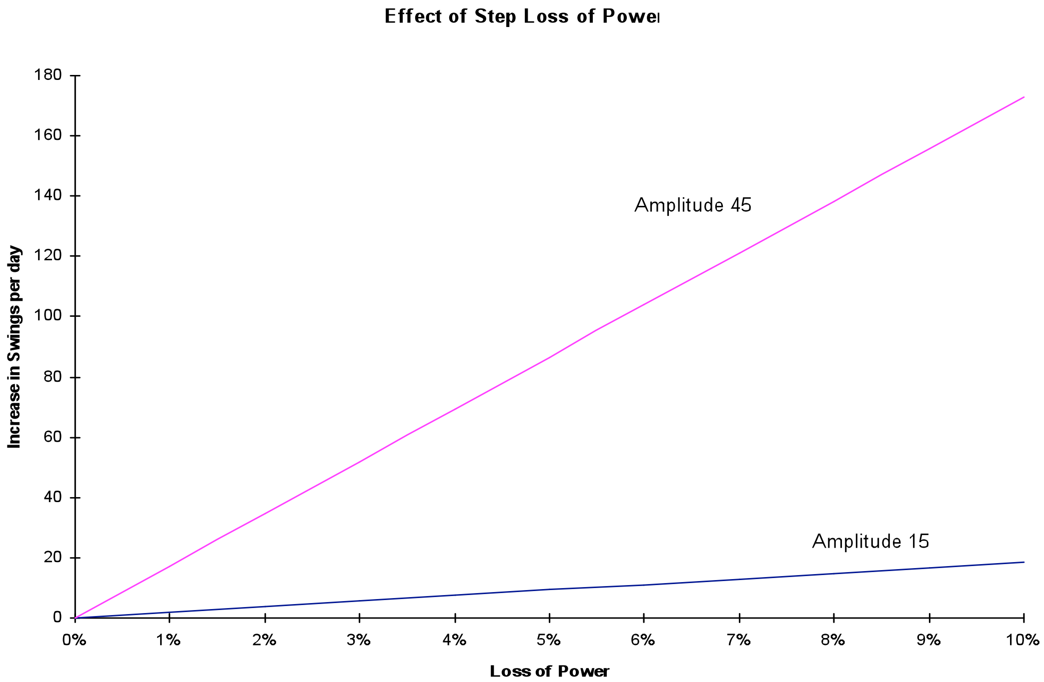


Figure A2

As Figure A2 shows, if the initial amplitude were 15°, a 10% loss of power to the pendulum would barely be noticeable over the course of the day, whereas, with 45° amplitude, 10% power loss would cause a gain of three minutes per day.

Sensitivity to Anisochronism - Freely Swinging Pendulum

Equation A3 gives the time of swing of a free pendulum as $T = T_0\left(1+\frac{A^2}{16}\right)$

If the pendulum amplitude is continuously decaying, then

$$A = A_0 e^{-\mu t}$$

So that at time t $\quad T_t = T_0(1 + \frac{A_0^{\,2}}{16} e^{-2\mu t})$ subject to the approximations above

In the interval of time from t to t+dt the number of swings dN made by the pendulum will be $\frac{dt}{T_t}$

thence $\quad N = \int \left[T_0(1 + \frac{A_0^{\,2}}{16} e^{-2\mu t}) \right]^{-1} dt \quad$ and,

$$N = \frac{1}{-2\mu T_0}\left[-2\mu t - \ln\left(1 + \frac{A_0^{\,2}}{16} e^{-2\mu t}\right)\right] + C \qquad = \frac{1}{T_0}\left[t + \frac{1}{2\mu}\ln\left(1 + \frac{A_0^{\,2}}{16} e^{-2\mu t}\right)\right] + C$$

When t=0, N=0, so that $\quad C = -\frac{1}{2\mu T_0}\ln\left(1 + \frac{A_0^{\,2}}{16}\right)$ and thence

$$N = \frac{1}{T_0}\left\{ t + \frac{1}{2\mu}\left[\ln\left(1 + \frac{A_0^{\,2}}{16} e^{-2\mu t}\right) - \ln\left(1 + \frac{A_0^{\,2}}{16}\right)\right]\right\} \qquad \text{--------------------} \quad \text{A10}$$

MAKING THE PENDULUM ISOCHRONOUS

Huygens' analytical approach to making the pendulum isochronous was to consider the motion of a particle moving along a plane curve in space under the influence of gravity, as in Figure B1. We can look at the problem in the same way using our now standard mathematical techniques which we have inherited from the late C17th.

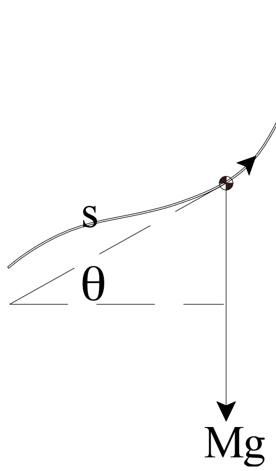


Fig B1

Referring to Figure B1, in which s is the distance along the curve and θ is the angle between the tangent to the curve and the horizontal. The motion will be a simple harmonic oscillation [66], and therefore isochronous if

$$\frac{d^2s}{dt^2} \propto -s \qquad \text{B1}$$

Considering the change in displacement of the particle and resolving the acceleration due to gravity along the path, the equation of motion is

$$\frac{d^2s}{dt^2} = -g\sin\theta$$

Comparing with equation B1, the motion will be simply harmonic if

$$s \propto g\sin\theta$$

that is if

$$s = Cg\sin\theta$$

Thus if the motion were simply harmonic we would have $\dfrac{ds}{d\theta} = Cg\cos\theta$ B2

Note that while this is a sufficient condition for isochronism it is not a necessary condition. There may be some other path that is isochronous but not a simple harmonic motion.

To examine what path the particle must follow for equation B2 to apply, establish a Cartesian coordinate system as shown in Figure B2. We wish to know how the x and y coordinates of the particle should vary as θ changes.

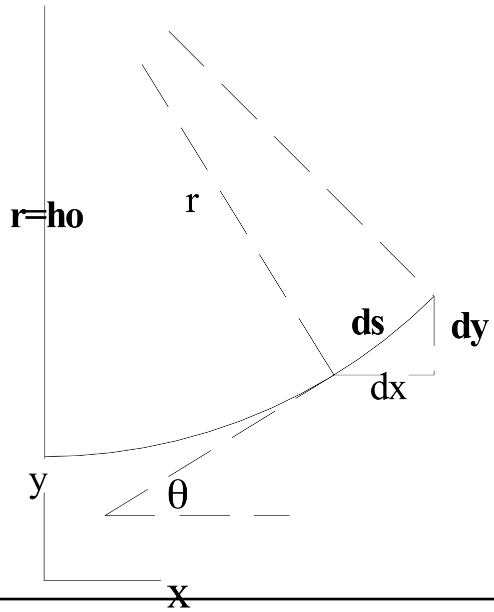


Figure B2

From an initial displacement s, infinitesimally displace the particle through ds, and denote the corresponding changes in x and y as dx and dy as again shown in Figure B2.

Then: $\dfrac{dy}{ds} = \sin\theta$, $\dfrac{dx}{ds} = \cos\theta$, and, for the motion to be simple harmonic, $\dfrac{ds}{d\theta} = Cg\cos\theta$

Thus $\dfrac{dx}{d\theta} = \dfrac{dx}{ds}\dfrac{ds}{d\theta} = Cg\cos^2\theta = \dfrac{Cg}{2}(1+\cos 2\theta)$ and

$$\frac{dy}{d\theta} = \frac{dy}{ds}\frac{ds}{d\theta} = Cg\sin\theta\cos\theta = \frac{Cg}{2}\sin 2\theta$$

Integrating with respect to θ, and setting constants of integration so that x = y = 0 at θ = 0 gives:

$$x = \frac{Cg}{4}(2\theta + \sin 2\theta) \quad \text{and} \qquad y = \frac{Cg}{4}(1-\cos 2\theta) \qquad \text{B3}$$

These are the equations of a cycloid with the cusps pointing upwards in which the rolling circle has radius $\dfrac{Cg}{4}$, and has turned through 2θ. The radius of the rolling circle is not arbitrary.

This derivation is applicable not only to a particle but also to the centre of mass of a rigid body of any shape. Under the action of a constant vertical force, the motion of the centre of mass will be simply harmonic and therefore isochronous if it follows a particular cycloidal path:

because, on the particular cycloidal path $\dfrac{ds}{d\theta} = Cg\cos\theta$ so that $s \propto -g\sin\theta$ and therefore under a constant vertical force $\dfrac{d^2s}{dt^2} \propto -s$, which is a sufficient condition for isochronism

(where θ is the instantaneous inclination of the path to the horizontal axis of the cycloid.)

Huygens' arrangement depended on two geometric facts:

The involute of a cycloid is itself a cycloid, and, accordingly, the end of a taut string unwrapping from a cycloidal chop will follow a cycloidal path.

The tangent to any involute is normal to the evolute at their intersection and thus the taut string of Huygens' pendulum was normal to the path of the centre of mass and the tension in it had no component along the path. The acceleration along the path was then that due to the vertical gravity force alone.

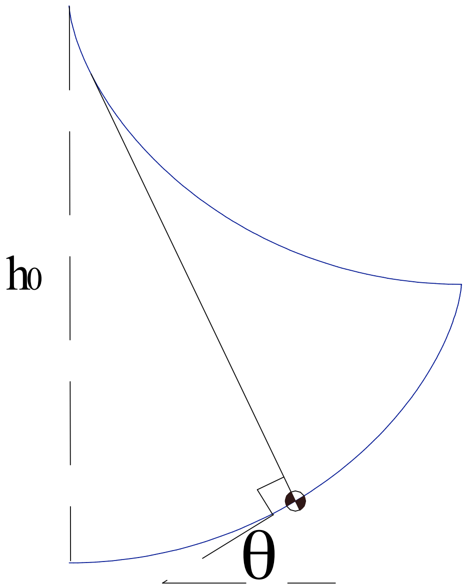


Figure B3 Huygens Arrangement

There is a third consequence of those facts. Because the tangent to any involute is normal to the evolute at their intersection, Huygens' taut string lies instantaneously along the radius of curvature of the path of the bob. If the path is to be that followed by the bob of a simple pendulum initially suspended a distance h_o above the origin of the Cartesian coordinates then, when $\theta = 0$, the radius of curvature $r = h_0$

By definition, curvature = $\dfrac{d\theta}{ds}$ and consequently the radius of curvature $r = \dfrac{ds}{d\theta}$

For simple harmonic motion , from equation B2, $\dfrac{ds}{d\theta} = Cg\cos\theta$

so that for simple harmonic motion $r = Cg\cos\theta$ B4

Substituting $r = h_0$ when $\theta = 0$ in B4 leads to $Cg = h_0$ and from B3 the parametric equations of the cycloidal path for Huygens' pendulum are then

$$x = \frac{h_0}{4}(2\theta + \sin 2\theta) \qquad \text{and} \qquad y = \frac{h_0}{4}(1 - \cos 2\theta) \qquad \text{B5}$$

and we note that the radius of the generating circle is one quarter of the length of the pendulum string.

This method of considering motion along the path using infinitesimal calculus was not available to Huygens who was working well before Newton devised and published his methods. Huygens had available the notions of mechanics that became Newton's first two laws of motion and the methods of analytical geometry. Generally speaking, geometrical methods do not permit the shape of a curve to be inferred from its other characteristics.

Our keen eyed editor Bob Holstrom sent me a bookseller's review of Traite de la Pendule a Cycloide which says that in about 1684 the author of the book P. Baert independently reached the same conclusions as Huygens using different methods.

In discovering that the isochronous pendulum needed the cycloid Huygens was perhaps lucky. To emphasize this point, consider that the complementary problem, [67] that of finding the curve which was a brachistochrone, required infinitesimal calculus and was not solved until 1697. But, it was Huygens' perspicacity which earned his luck.

The Rigid Body Pendulum

Huygens had briefly investigated the rigid body pendulum in response to a request from Mersenne in 1646. He was not able to achieve a result and dropped the inquiry[68]. When Huygens began to regulate the clocks using a supplementary weight sliding on the pendulum rod, the question of the rigid pendulum became more relevant.

Huygens investigated the behaviour of the rigid pendulum, under the heading "centre of oscillation" an historical term of confusing meaning which persisted through the C20th. Without the assistance of the integral calculus, he identified, and showed how to calculate, what we now know as the moment of inertia and centre of gravity and thereby determined the period of what we have come to call the compound pendulum. The relevant work notes are dated August to November 1661. [69] These notes became Part IV of *Horologium Oscillatorium*.

Thus, although Huygens and his correspondents subsequently initiated the methods of rotational and rigid body dynamics, those methods were not available to Huygens in 1659 when he discovered the relevance of the cycloid. The style of analysis above of course precludes consideration of a rigid pendulum.

However, consider Huygens' real pendulum as generalised in Figure B4. A rigid body, centre of mass G is pin jointed at another point A to a string AO fixed at O and unwrapping around a curved cheek OP. The string is instantaneously tangential to the cheek at O'.

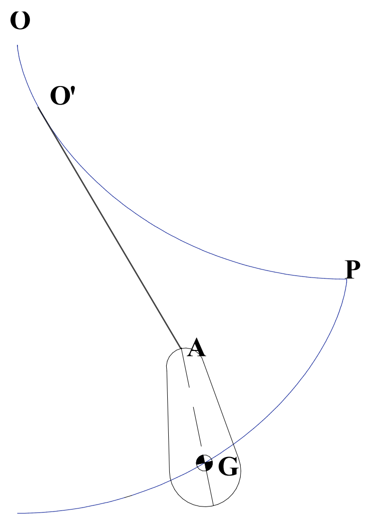


Figure B4

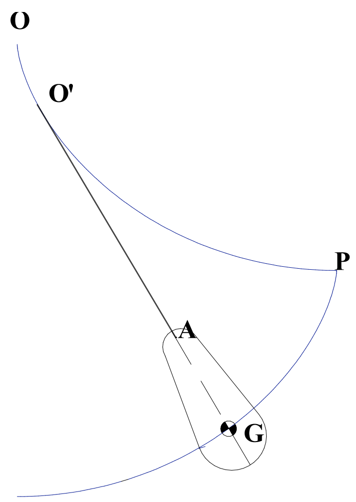


Figure B5

According to the definition of an involute, if G is to move along the involute to OP , then O' A and G must always be colinear as shown in Figure B5. This means that the rigid body must continuously rotate to remain aligned with the string as it wraps and unwraps.. The rate of rotation falls to zero and reverses at the end of each swing and is elsewhere generally not constant. To provide this acceleration the resultant of the forces acting on the body must have a moment about G.

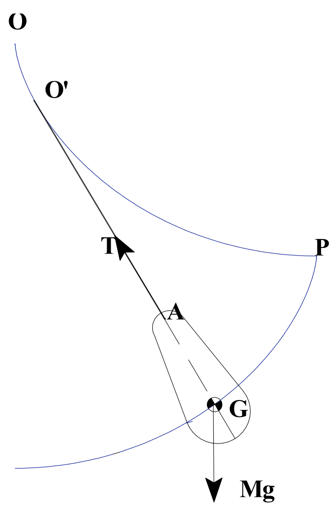


Figure B6

The forces acting on the body are Mg due to gravity and T the tension in the string as shown in Figure B6. If O' A and G are colinear , as shown, these forces have no moment about G, the angular acceleration of the body is zero and the colinearity of O' A G cannot be maintained.

Thus no simple arrangement of a string wrapping around a cheek will cause the centre of mass of the rigid body to follow the involute of the cheek. [1]

Huygens was well aware[70] of this by the end of 1661. It seems likely that Huygens' demurrer went unnoticed, for the unqualified acceptance of the cycloidal path as isochronous went on in horological circles for some considerable time.

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1 Unless A and G are coincident. That is, unless the bob is suspended freely from a pin joint at its centre of mass.

If by some arrangement the centre of mass of a rigid body suspended from a string were persuaded to traverse a cycloid, we would have $\frac{ds}{d\theta} = k\cos\theta$ as a property of any cycloid and thence $s = -k\sin\theta + j$ (where k and j are constants).

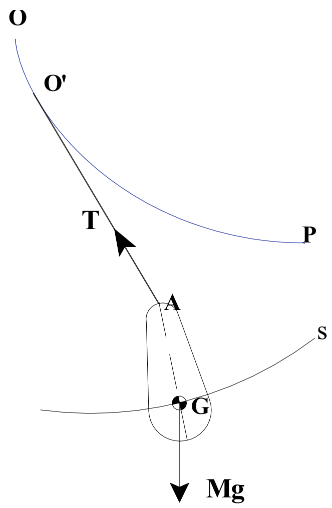

Figure B7

Since, as we have shown, O' A and G cannot remain colinear, the forces on the rigid body will be as shown in Fig B7 The acceleration along the path will be given by:

$$\frac{d^2s}{dt^2} = -g\sin\theta + f(T)$$

but $$s = -k\sin\theta + j$$

so that $$\frac{d^2s}{dt^2} \neq -s \times \text{ constant}$$

and the motion is therefore not simple harmonic.

Thus, if a rigid body is suspended by a string and its centre of mass traverses a cycloid, the motion of the centre of mass will not be isochronous.

There remains the possibility that, despite the centre of mass not following the involute of the cheek, there might be some shape of cheek which causes the motion to be isochronous.

Other clockmakers have been seen to use curved chops to alter the swing of the pendulum notably Arnold and Harrison, suggesting that the principle might work after all. Closer examination of this case shows that in the long run Harrison used chops not to impart isochronism by their shape, but to provide an adjustment mechanism to minimize the variation in rate caused by extraneous influences.

In 1818 Benjamin Gompertz showed that the cycloid was not isochronous for a rigid pendulum and attempted to derive the correct isochronous path. He concluded that the objective could not be achieved by a pendulum's suspension cord wrapping around cheeks.[71]. His work appears to have gone unnoticed by horologists.

Concerns Obscured

Huygens demurrer about the cycloidal pendulum appears in *Horologium Oscillatorium* Part IV Proposition XXIV.[72] As translated he says." It is not possible to determine the centre of oscillation for pendula suspended between cycloids."

The centre of oscillation is an awkward concept at best. I find it confusing and unhelpful. If a rigid body is swinging as a pendulum about an axis of suspension in the body, the centre of oscillation of the body lies on a straight line extending from the the axis of suspension through the centre of mass. The distance of the centre of oscillation along this line from the axis of suspension is equal to the length of a simple pendulum that has the same period as the swinging rigid body. Calculating the position of the centre of oscillation entails knowing the centre of mass and the mass moment of inertia of the body. Most importantly from Huygens' perspective, it permits one to find a two-mass system that is dynamically equivalent to the rigid body.

The point relevant to Proposition XXIV is that if the rigid body is not swinging about a fixed axis in the body the centre of oscillation is not defined. When the pendulum is a rigid body suspended by a string, the rigid body is not swinging about a fixed axis in the body. Thus, " It is not possible to determine the centre of oscillation for pendula suspended between cycloids."

Proposition XXIV tells me that Huygens was aware that the string, bob, and rod did not swing as a rigid body. Not only did the point of suspension move along the chop, but the rod and string could not remain colinear. This meant that the bob would not follow the involute of the chops and the cycloids would not deliver isochronism. The significance of Prop XXIV is obscured until one makes this connection.

The most likely time for Huygens to have made these observations was when he was first working on the period of the compound pendulum created by the addition of a rating weight to the rod. That is 1661. On the other hand, Huygen continued to design clocks apparently with cycloidal chops. Did he miss the connection?

Even the inveterate critic English scientist Robert Hooke missed the point of the Proposition (but not the physical facts.). After receiving a copy of *Horologium Oscillatorium* in 1673, he observed that Huygens' cycloidal pendulum was imperfect saying " … supported partly by threads, ribbons, or other pliable material in order to be bending about the cycloidal cheeks, partly also by a stiff rod or plate, is subject to another great inequality namely to a bending at the place where the stiff and pliable parts are joined together. And this is not notional but very visible to the eye especially if the check be great that is given it by the watch parts [escapement] so that all the pains for the adjustment after M. Zulichem we come short of the idea of perfection in the measure of time which his geometrical demonstrations would insinuate." Hooke, in short did not realise, or did not acknowledge, that Huygens had identified and recorded this problem years earlier.[73]

Multiple Systems

A pendulum with a flexible joint in it, such as a rigid body suspended from a string (or from a leaf spring) does not fall within the definition of a compound pendulum. It is a multiple system, a system having two degrees of freedom. Huygens' real pendulum was a multiple system.

In considering the behaviour of a multiple system we should first reconsider what constitutes isochronism. What aspect of isochronism is important in a clock. We are concerned that the periods between successive occurrences of a specific geometric configuration remain constant. In the conventional arrangement we consider successive passages of the crutch through the vertical. We are concerned that this period should remain constant, principally in the face of changes of energy of the pendulum. In the rigid body pendulum suspended from a string, constant energy is not the same as constant amplitude. To demonstrate this and to show the obstacles in the way of isochronism for real pendulums, consider the elementary constant energy system depicted in Figure B8.

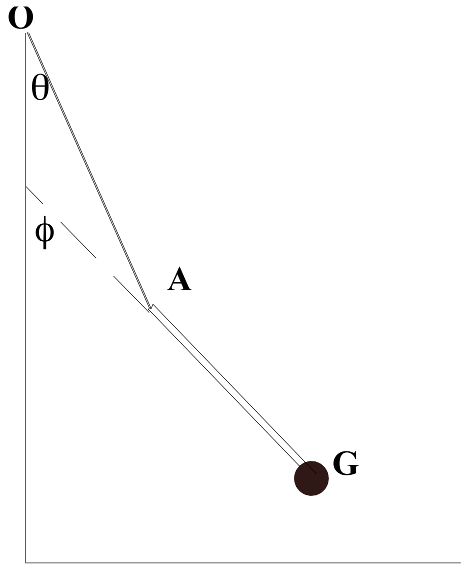

Figure B8

A rigid body AG consisting of a light rod and a concentrated mass is suspended from A by a taut string fixed at O. For simplicity OA = AG = 12inches.

This is a standard problem for which the equations of motion for small amplitudes are:

$$\theta = A_1 \cos(\omega_1 t + \varepsilon_1) + A_2 \cos(\omega_2 t + \varepsilon_2)$$

$$\phi = B_1 \cos(\omega_1 t + \varepsilon_1) + B_2 \cos(\omega_2 t + \varepsilon_2)$$

where $\omega^4 - \frac{k^2 + a^2 + al}{k^2 l} g\varpi^2 + \frac{g^2 a}{k^2 l} = 0$

l being the length of the string, *a* the distance from A to the centre of mass, and *k* the radius of gyration of the rigid elements about G

The equations of motion for this particular system, for small amplitudes are:

$$\theta = A_1 \cos(9.15t) + A_2 \cos(3.5t)$$

$$\phi = B_1 \cos(9.15t) + B_2 \cos(3.5t)$$

The constants B are proportional to A. The constants A are set by the initial amplitudes of θ and ϕ.

If the system is set running with $\theta = \phi$, the string oscillates as shown in Figure B9

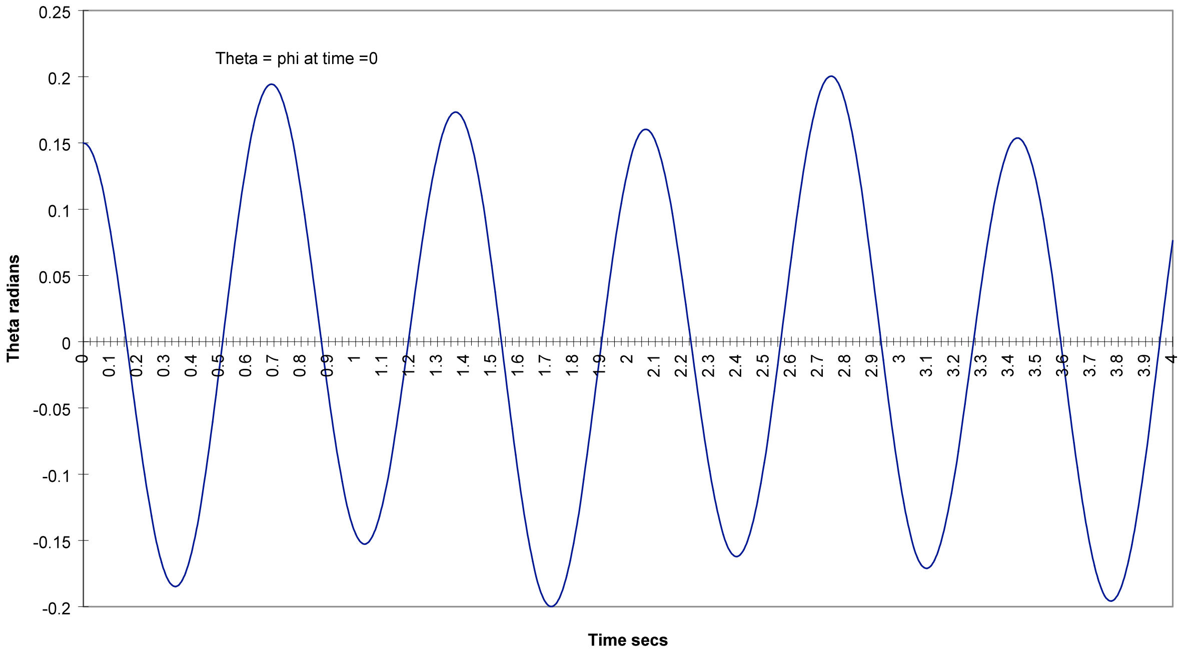

Figure B9 Oscillation of String

Notice the cyclic variation in amplitude. While it may not be so clear, the period of the oscillation is also varying. There is a 12% difference between the maximum and minimum times between successive passages through $\theta = 0$.

If the system is set running with the body of the pendulum 8½° from vertical and the string vertical ie $\phi = 0.15$ and $\theta = 0$, the string oscillates as shown in Figure B10 where the variation of amplitude and period is much more obvious.

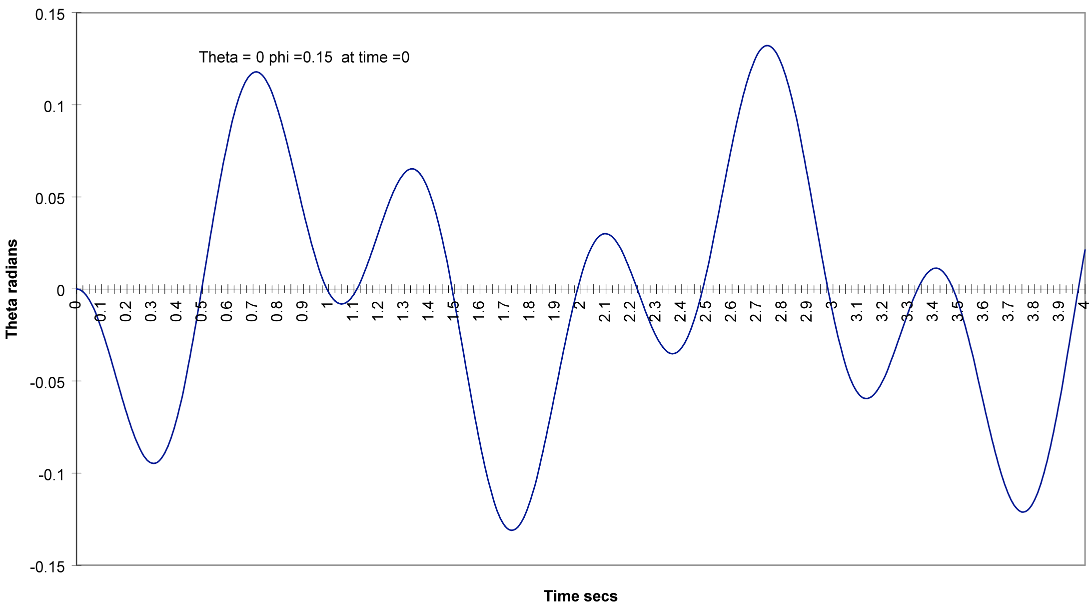

Figure B10 Oscillation of String

Figure B11 compares the variation of the rotation of the string θ with the rotation of the rigid pendulum body ϕ. This illustrates the conclusion already reached that the string and the body do not remain aligned.

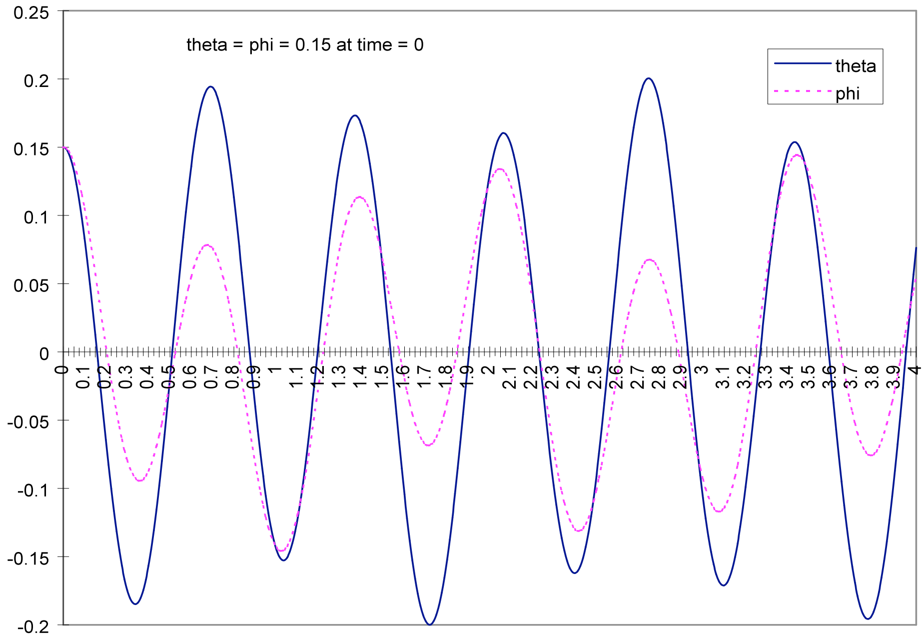

Figure B11

The motion of the centre of mass involves similar variation in amplitude and period as shown in FigB12. A crutch of the usual form would behave similarly.

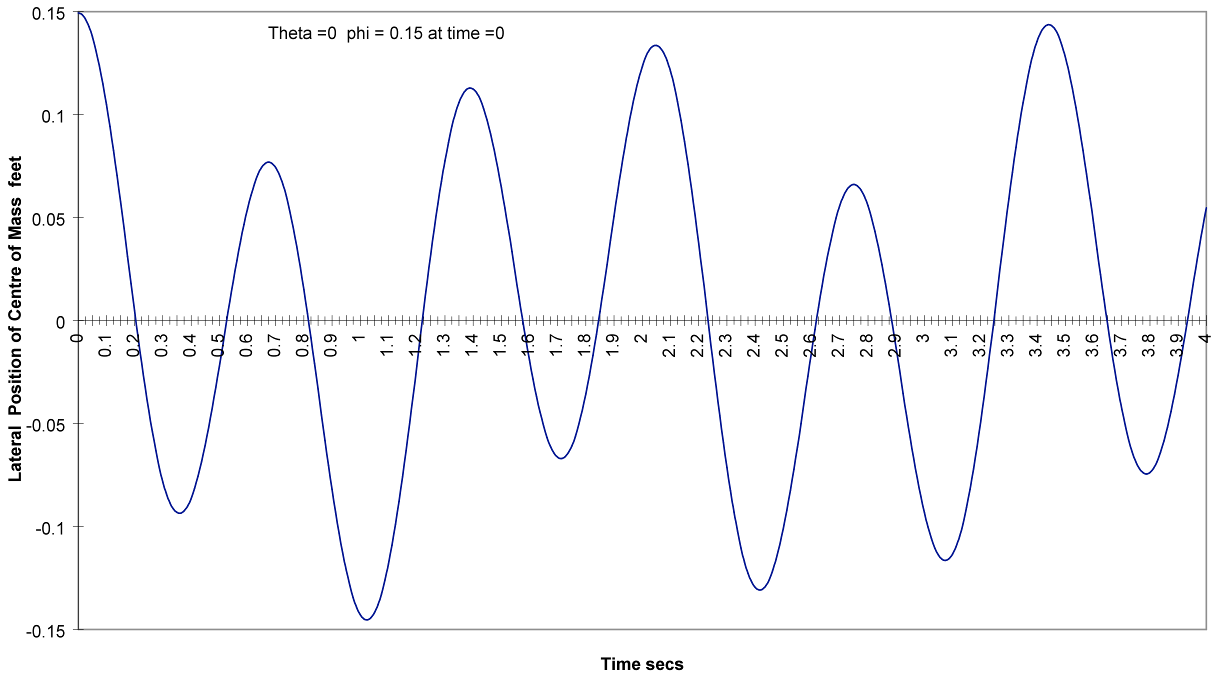

Figure B12 Oscillation of Centre of Mass

Under the same conditions, the path followed by the centre of mass is as shown in Figure B13. Note that the vertical scale has been enlarged.

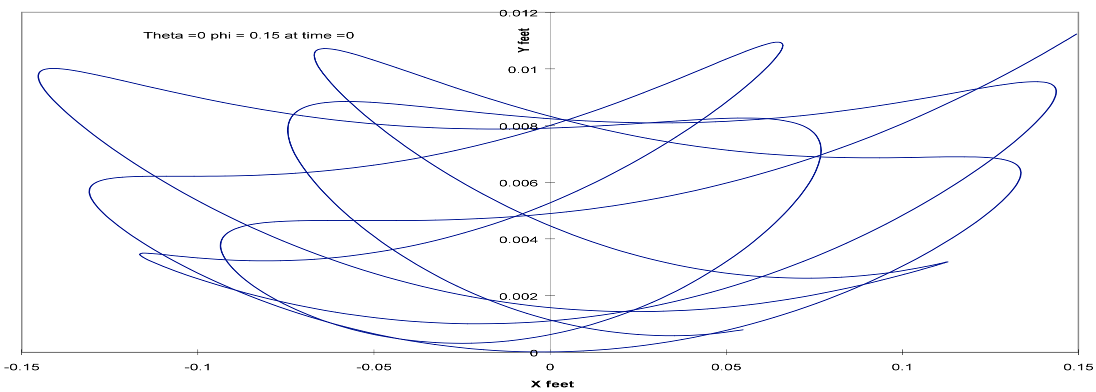

Figure B13

This behaviour is of the same kind that I referred to in Horological Science Newsletter a few years ago[74].

It is just conceivable that a crutch, or similar appliance, might restore the energy lost by the pendulum, apply a moment that maintains the rigid pendulum body colinear with the suspension string and do this with a force having a component along the path of the centre of mass proportional to the displacement. This would be a very clever or very fortunate design.

GEOMETRY OF THE VERGE AND CROWN WHEEL ESCAPEMENT

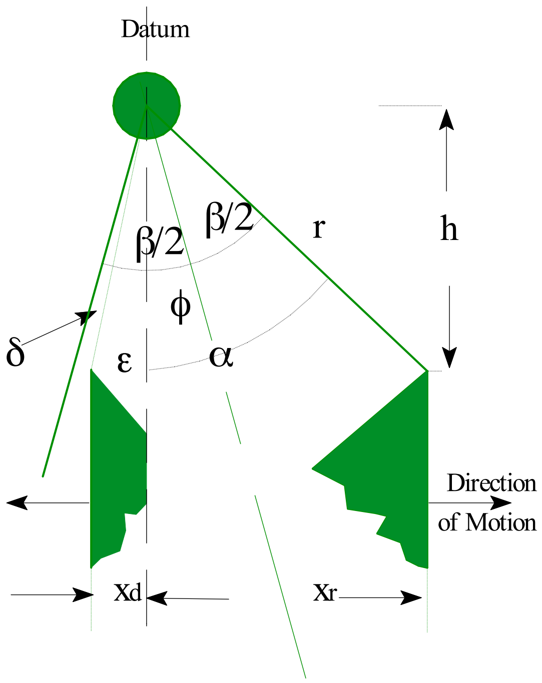

Figure C1

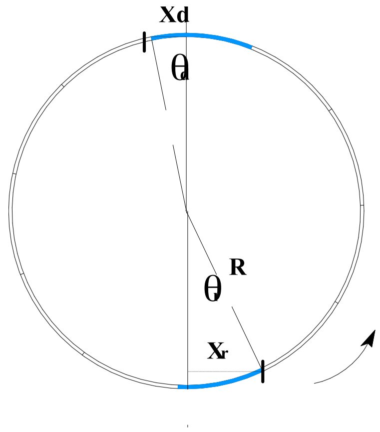

Figure C2

Escapement at the Instant of Release.

The diagrams show, in elevation and plan, an advancing tooth on one side of the crown wheel (nearer to the reader) about to lift the verge pallet clear of the tooth tip and the corresponding tooth on the other side of the wheel about to drop onto the other pallet.

The plane containing the axis of the crown wheel and the axis of the verge is taken as a datum plane.

The angle between the pallets is β
The angular position of the pallet at release is α
The travel or displacement of the axis of symmetry of the pallets from the datum is ϕ.
The clearance angle corresponding to the drop is δ
The lateral displacement of the tip of the tooth at release is $\mathbf{x_r}$
The lateral displacement of the tip of the tooth at drop is $\mathbf{x_d}$
The crown wheel has N teeth

We wish to know the angle of swing at release in terms of the design parameters $\boldsymbol{\beta}$, **h**, **r** and $\boldsymbol{\delta}$

From Figure C1 : $\alpha = \phi + \dfrac{\beta}{2}$, $\alpha = \cos^{-1}\dfrac{h}{r}$, and $\alpha = \sin^{-1}\dfrac{x_r}{r}$

$$\therefore \phi = \alpha - \frac{\beta}{2} = \cos^{-1}\frac{h}{r} - \frac{\beta}{2} \qquad \text{C1}$$

There is however a constraint on the free choice of $\boldsymbol{\beta}, \boldsymbol{h},$ and $\boldsymbol{r}$. There is a requirement that the angle of drop $\boldsymbol{\delta}$ must be positive as shown.

From Figure C1 : $\dfrac{\beta}{2} = \delta + \varepsilon + \phi$, (that is $\varepsilon = \dfrac{\beta}{2} - \phi - \delta$,)

and $\varepsilon = \tan^{-1}\dfrac{x_d}{h}$

Whence $\dfrac{\beta}{2} - \phi - \delta = \tan^{-1}\dfrac{x_d}{h}$ and $x_d = h\tan\left(\dfrac{\beta}{2} - \phi - \delta\right)$ C2

In the symmetrical layout usually adopted, the crown wheel must have an odd number of teeth. Thus the angular distance between the tip of a tooth and the tip of the tooth closest to diametrically opposite is π+½tooth pitch. Consequently, in Figure C2,

$$\theta_r = \frac{1}{2}pitch + \theta_d$$

$$ie\theta_r - \theta_d = \frac{\pi}{N}$$

$$\theta_r = \sin^{-1}\frac{x_r}{R}$$

Also from Figure C2,

$$\theta_d = \sin^{-1}\frac{x_d}{R}$$

$$\therefore \sin^{-1}\frac{x_r}{R} - \sin^{-1}\frac{x_d}{R} = \frac{\pi}{N}$$

Thus for N≥15 the angles θ are small so that :

$$\frac{x_r}{R} - \frac{x_d}{R} = \frac{\pi}{N}$$

$$\text{ie: } x_r = R(\frac{\pi}{N} + \frac{x_d}{R})$$

Substituting from C2 gives

$$x_r = R\left(\frac{\pi}{N} + \frac{h}{R}\tan\left(\frac{\beta}{2} - \phi - \delta\right)\right)$$

We may reasonably anticipate that $\frac{\beta}{2} - \phi - \delta$ is small so that:

$$x_r = R\left(\frac{\pi}{N} + \frac{h}{R}\left(\frac{\beta}{2} - \phi - \delta\right)\right)$$

But $\frac{x_r}{r} = \sin\alpha$ and thus $$\sin\alpha = \frac{R}{r}\left(\frac{\pi}{N} + \frac{h}{R}\left(\frac{\beta}{2} - \phi - \delta\right)\right)$$

or $$\sin\alpha = \frac{R}{r}\frac{\pi}{N} + \frac{h}{r}\left(\frac{\beta}{2} - \phi - \delta\right)$$

that is $$\frac{h}{r}\phi = \frac{R}{r}\frac{\pi}{N} + \frac{h}{r}\frac{\beta}{2} - \frac{h}{r}\delta - \sin\alpha$$

Whence $$\phi = \frac{R}{h}\frac{\pi}{N} + \frac{\beta}{2} - \delta - \frac{r}{h}\sin\alpha \qquad \text{C3}$$

But $\alpha = \cos^{-1}\frac{h}{r}$ so that $$\phi = \frac{\beta}{2} - \frac{r}{h}\sin\cos^{-1}\frac{h}{r} - \delta + \frac{R}{h}\frac{\pi}{N} \qquad \text{C4}$$

Thus the requirement is: $$\frac{\beta}{2} - \frac{r}{h}\sin\cos^{-1}\frac{h}{r} + \frac{R}{h}\frac{\pi}{N} - \phi = \delta \text{ and } \delta > 0 \qquad \text{C5}$$

The requirement $\delta > 0$ can be satisfied by pre setting a value for drop. The practicalities of construction suggest that drop should be expressed as a fraction of the crown wheel tooth pitch. If D is the angular drop of the crown wheel, $r\delta = RD$, for small δ, so that $\delta = \frac{RD}{r}$

So that $$D = k \times \frac{2\pi}{N}$$

And thus $$\delta = \frac{RD}{r} = k \times \frac{R}{r}\frac{2\pi}{N}$$

Substituting in C5 gives $$\frac{\beta}{2} - \frac{r}{h}\sin\cos^{-1}\frac{h}{r} + \frac{R}{h}\frac{\pi}{N} - \phi = \frac{R}{r}\frac{2k\pi}{N} \qquad \text{C6}$$

Transforming and collecting terms in C6 gives

$$\phi = \frac{\beta}{2} - \frac{r}{h}\sin\cos^{-1}\frac{h}{r} + \frac{\pi}{N}\left(\frac{R}{h} - 2k\frac{R}{r}\right) \qquad \text{C7}$$

But $$\phi == \cos^{-1}\frac{h}{r} - \frac{\beta}{2} \qquad \text{C1}$$

There are two approaches to the design of the escapement.

Adding C1 and C7 yields $$2\phi = \cos^{-1}\frac{h}{r} - \frac{r}{h}\sin\cos^{-1}\frac{h}{r} + \frac{\pi}{N}\left(\frac{R}{h} - 2k\frac{R}{r}\right)$$

and this prescribes $$\frac{\beta}{2} == \cos^{-1}\frac{h}{r} - \phi$$

Alternatively:

Subtracting C1 from C7 yields $$0 = \beta - \cos^{-1}\frac{h}{r} - \frac{r}{h}\sin\cos^{-1}\frac{h}{r} + \frac{\pi}{N}\left(\frac{R}{h} - 2k\frac{R}{r}\right)$$

ie $$\beta = \cos^{-1}\frac{h}{r} + \frac{r}{h}\sin\cos^{-1}\frac{h}{r} - \frac{\pi}{N}\left(\frac{R}{h} - 2k\frac{R}{r}\right)$$

resulting in $$\phi = \cos^{-1}\frac{h}{r} - \frac{\beta}{2}$$

These are the characteristic equations for the escapement. They are accurate only when N≥ 15 and $\frac{\beta}{2} - \phi - \delta$ is small . The design criteria for the escapement include $\frac{h}{r} \le 1$ or the crown wheel will run free.

Table C1 shows the values of the angle of swing to release for various pallet angles using practicable values for the other escapement parameters.

Pallets Included Angle b degrees	Angle of Swing at Release f degrees			
	N=15		N=29	
	r	**f**	**r**	**f**
70	3.0	40	1.7	29
80	3.1	36	1.8	26
90	3.3	32	1.9	22
100	3.4	27	2.0	18
110	3.5	23	2.2	15

For: R=15mm, h=0.75mm and k=0.1

Table C1 Verge Escapement , Angle of Swing at Release

NOTES

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1 Chritiaan Huygens van Zuilichem, he was referred to also as Zuilichem or van Zuilichem , often spelled Zulichem.

2 There are earlier claims for Leonardo, Bodeker, Galileo and son, and Hevel/Treffler. vide Clutton C. Baillie G. Ilbert C. ed, Britten's Old Clocks and Watches and their Makers 9th ed, Bloomsbury Books, London, 1989 pp71-73. There is also a claim for Fromanteel vide Edwardes E.L. The Suspended Foliot and New Light on Early Pendulum Clocks, in Antiquarian Horology June 1981

3 Horological Journal, News - Huygens not Isochronous , British Horologcal Institute, Upton UK, September 1999
Emmerson A.J, The Pendulum Revisited, in Horological Science Newsletter, NAWCC Chapter 161 , March 1999

4 Michel H. Scientific Instruments in Art and History, Barrie and Rockliff, London, 1967, p165 and plate 82.

This work comes with the imprimatur of the Royal Astronomical Society Librarian, the curator of the Museum of the History of Science at Oxford, and Academie Internationale d'history des Science .
This clock is also described by Clutton opcit. The clock is engraved "Salomon Coster met privilege 1657" which perhaps means no more than that it was built after the grant of license and patent in June 1657 and no later than end of 1659 when Coster died.

Plomp 1979 p14 describes a similar clock seen in that museum in 1923, but says it has been missing from the museum since the 1930s..

5 Good R. Britten's Watch and Clockmakers Handbook Dictionary and Guide 16th ed , Bloomsbury Books, London, 1987, p242

6 Haswell J.E, Horology, Chapman and Hall Ltd, London, 1937

7 Landes D.S. Revolution in Time revd, Penguin Books Ltd, London, 2000 p128 et seq

8 Clutton C opcit p74

9 Plomp R, Spring-driven Dutch Clocks 1657 -1710, Interbook International BV, Schiedam, 1979, p11

10 For example Christian Huygens (1629-1695) in Rouse Ball , A Short Account of the History of Mathematics 4th ed 1908; and Andriesse C.D, Christian Huygens, Albin Michel 1998. The most reliable works seem to be those of H.J.M. Bos R.J. Blackwell and M.S. Mahoney. And Joelle G. Yoder, Unrolling Time, Christiaan Huygens and the Mathematization of Nature, Cambridge, 1988, ISBN 0521524814 has been commended to me by Paul Middents. I am grateful to Paul Middents for drawing my attention to more than a few important references and for his detailed review of aspects of this paper.

11 www.sciencemuseum.org.uk/on-line/huygens

12 *Horologium Oscillatorium Sive de Motu Pendulorum ad Horlogia Aptato Demonstrationes*

Geometrica, Paris 1673

13 Edwards E.L. trans, *Horologium* by Christiaan Huygens 1658, in Antiquarian Horology December 1970 p35 et seq

14 Andriesse C.D, Christian Huygens, Albin Michel, 1998

15 A facsimile/hypertext hybrid of Galileo's notes on motion are available through the Max Plank Institute for the History of Science at <http:www.mpiwg-berlin.mpg.de>

16 Galileo's original legendary observations were probably of "pendulums" swinging with an amplitude of perhaps 20° at most. His first observation would have been that no matter what the amplitude, the times of swing of the cathedral lamps were the same. (He had no need of his pulse to reach that conclusion; he just had to see there was no phase change)

17 These propositions appear in Galileo's Discourses and Mathematical Demonstration Concerning Two New Sciences, dated 1638 . However, the experiments were performed some thirty years beforehand.. Galileo was virtually under house arrest when he wrote this during the last years of his life. *Discorsi* was smuggled out of Italy and published in Leiden. The pendulum results were without theoretical support and appeared in the dialogue rather than the propositions of *Discorsi*

18 In *Discorsi,* Galileo describes his pendulums as "repeating their goings and comings a good hundred times by themselves" Blackwell opcit p19

19 Gould LtCdr R, The Marine Chronometer, The Holland Press, London 1923, reprinted 1960 p12

20 Büttner, Damerow and Renn, Traces of an Invisible Giant: Shared Knowledge in Galileo's Unpublished Treatises, Max Plank Institute, 2002

21 Ariotti P.E, Aspects of the Conception and Development of the Pendulum in the 17th Century, Arch, History of the Exact Sciences 8, 1972 pp329-410 Cited by Plomp R.opcit p17

Clearly there is a discrepancy of terminology in this report. A seconds pendulum "beats seconds" but has a period of two seconds. If 87,998 events were counted in 24 hours either the events were beats or the pendulum was a half second pendulum and the counted events were swings..

22 In *Horologium,* Huygens specifically states the invention was to improve on contemporary tiresome counting of the excursions of a pendulum. Edwardes L.E op cit p43

23 There are several references in primary source documents to finding the longitude:

Oeuvres Complètes Vol II p 5 12 January 1657
Oeuvres Complètes Vol II pp 7-8 Letter 370, 1 February 1657
Oeuvres Complètes Vol XII p8 undated

Huygens' father had been involved in the matter of Galileo's project for finding longitude proposed to the Dutch States General in 1636. Vide Leopold J The Longitude Timekeepers of Christiaan Huygens in Andrews J ed , The Quest for Longitude,, Harvard University Press, 1996, reprinted in NAWCC Bulletin Vol 43/5 No 334 October 2001, and citing *Horologium* and similar papers by Huygens

[24] These facts are disclosed in a letter from Viviani to Leopoldo de Medici dated 20th August 1659.

[25] Andriesse op it

[26] Encyclopedia Britannica This information is found repeated throughout the literature but without reference to a primary source.

[27] Blackwell R.J, trans Christiaan Huygens' The Pendulum Clock or Geometrical Demonstration Concerning the Motion of Pendula as Applied to Clocks *(Horologium Oscillatorium)* Iowa State University Press 1986

footnote to p82 citing Pascal *Historia trochoidis sive cycloidis,* Paris, 1658 and letter Pascal to Huygens 6 January 1659 appearing in *Ouvres Complètes* Vol II p309 Letter 562, 6 January 1659

[28] Given two points, such that the straight line joining them is neither vertical nor horizontal, a curve drawn between them is a tautochrone if a particle falling along the curve under gravity reaches the bottom of the curve in the same time no matter from what point on the curve it starts.

[29] *Societe Hollandaise des Sciences*, *Oeuvres Complète*s *de Christiaan Huygens*. Vols I to XXII, Martinus Nijhoff, The Hague 1888 to 1950

[30] Edwardes L.E op cit p51

[31] ibid

[32] *Oeuvres Complètes* Vol XVII *L Horologe Pendule de 1657,* p20

[33] *Oeuvres Complètes* Vol XVI p392 1 December 1659

[34] This date is due to Professor Michael S. Mahoney of Princeton University, perhaps the most recent of Huygens' translators. Bos in Blackwall opcit places this work and the work on the cycloid earlier in 1659

[35] *De vi Centriguga , Oeuvres Complètes* Vol XVI pp255-301 is a posthumously published version of this work. At Note 1 the editors assign the start of the work to 21 October 1659. The centrifugal force theorems also appear as Part V of *Horologium Oscilatorium*

[36] *Oeuvres Complètes* Vol XVIII pp18-19

[37] *Oeuvres Complètes* Vol XVII p 98

[38] *Oeuvres Complètes* Vol XVII pp 142-148

[39] *Oeuvres Complètes* Vol III p437 Letter 940

[40] *Oeuvres Complètes* Vol II p5 Letter363 12 Jan57.

[41] *Oeuvres Complètes* Vol II p109 Letter 443 26 December 1657

[42] "*Il y:eust hier un an justement que je fis le premier modelle de cette forte d'horologes.*"

[43] *Oeuvres Complètes* Vol II p 160 Letter 477 28 March 1658

44 *Oeuvres Complètes* Vol XVII p77

45 An important principle is that while ever there is a need to apply torque to a pendulum to sustain its motion the pendulum must have some rigid elements.

46 Also written as Jan (van) Call He was known for making bells

47 Clutton C. Baillie G. Ilbert C. ed, Britten's Old Clocks and Watches and their Makers 9th ed, Bloomsbury Books, London, 1989 p73.

48 Sellnick J.L. Clockmaking in the Netherlands in Smith A. ed The International Dictionary of Clocks, Chancellor Press 1996, page 287

49 On the contrary, *Oeuvres Complètes* Vol 3 Letter 704 of 1 Jan 1660 Note 2 has it that after Coster's death, his widow Jannetje Harmans Hartloop continued the manufacture of the clocks at least until 20Sept 1676. Presumably she held the *privilege* and sub licensed other makers.

50 Plomp Dr R. Spring Driven Dutch Pendulum Clocks 1657 - 1710, Interbook International BV Schiedam 1979, p12

51 *Oeuvres Complètes* Vol 2 pp237-238

52 *Oeuvres Complètes* Vol XVII p79, *Oeuvres Complètes* Vol II p237 Letter 525 16 Jun 57, *Oeuvres Complètes* Vol II p248 Note 3

53 Plomp Dr R opcit

54 It is also possible, of course, that the chops were added to this clock some years after the clock was manufactured. The engraving "*met privilege 1657"* looks like a modification to the cartouche

55 *Oeuvres Complètes* Vol XVII The Pendulum Clock of 1657, p15.

For this translation and similar assistance I am grateful to Professor N. Heckenberg of the University of Queensland

56 Huygens described chops to Petit in a letter at *Oeuvres Complètes* Vol II p270 Letter 546

57 *Oeuvres Complètes* Vol XVII p84 notes 11 October 1659 and letter to Boulliau of 24 Jun 1659.

58 vide *Oeuvres Complètes* Vol XVII pp97-99 December 1659

59 *Oeuvres Complètes* Vol II letter 691 pp521-522 6 December 1659

60 *Oeuvres Complètes* Vol XVI pp 414 Note 2

61 *Oeuvres Complètes* Vol III p437 Letter 940

62 See also *Oeuvres Complètes* Vol XVII pp 101 Note 2

63 The effects of rigidity are not trivial. The difference between the period of a compound pendulum beating seconds and that of a simple pendulum of the same "length" is about 15 minutes per day.

64 Circular error is not an inherent error in a pendulum. Nor is it caused by the pendulum swinging in a circle rather than a cycloid. The "error" lies in estimating the period of a circular arc pendulum on the assumption that $\theta = \sin\theta$.

65 This equation is very convenient and will suffice for the present purpose even though it is an approximation. In practice, the damping factor μ increases with the amplitude of swing. The pendulum in consequence decays rather more quickly from large arcs. But a value of μ determined by experiment will be valid as an average for the range of amplitudes of the experiment.. For further discussion see Rawlings A.L, The Science of Clocks and Watches, British Horological Society, Upton, 1993 3rd ed pp79-102

66 Indeed this principle itself is due to Huygens. He enunciated the principle in 1673 calling it. *incitation parfaite decroisante* (perfectly reducing impulsion ?)

67 Given two points, such that the straight line joining them is neither vertical nor horizontal, find how a curve between them must be drawn if a particle falling along the curve under gravity is to reach the bottom of the curve in the least possible time.

68 Blackwell R.J op cit p107

69 *Oeuvres Complètes* Vol XVI pp 414 Note 2

70 *Horologium Oscillatorium* Part IV Proposition XXIV

71 Gompertz B. On Pendulums Vibrating Between Cheeks, in The Journal of Science and the Arts No V Vol III, pp13- 33, The Royal Institution of Great Britain , James Eastburn & Co, New York, 1818

72 *Horologium Oscillatorium* Vol XVII p347

73 For this quotation I am grateful to Paul Middens

74 A.J. Emmerson, Pendulums Revisited, Horological Science Newsletter 1999-2 pp8-13, NAWCC Chapter 166 March 1999