Estimating Security Price Derivatives Using Simulation

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S imulation has proved to be a valuable tool for estimating security prices for which simple closed form solutions do not exist. In this paper we present two direct methods, a *pathwise method* and a *likelihood ratio method*, for estimating derivatives of security prices using simulation. With the direct methods, the information from a single simulation can be used to estimate multiple derivatives along with a security's price. The main advantage of the direct methods over resimulation is increased computational speed. Another advantage is that the direct methods give unbiased estimates of derivatives, whereas the estimates obtained by resimulation are biased. Computational results are given for both direct methods, and comparisons are made to the standard method of resimulation to estimate derivatives. The methods are illustrated for a path independent model (European options), a path dependent model (Asian options), and a model with multiple state variables (options with stochastic volatility).
(*Simulation; Derivative Estimation; Security Pricing; Option Pricing*)

1. Introduction

In this paper two direct methods for estimating security price derivatives using Monte Carlo simulation are presented, a *pathwise method* and a *likelihood ratio* method. The standard indirect method of estimating a security price derivative is resimulation. In this approach, an initial simulation is run to determine a base price, then the parameter of interest is perturbed and another simulation is run to determine a perturbed price. The estimate of the derivative is the difference in the simulated prices divided by the parameter perturbation. The direct methods proposed in this paper offer increased *computational speed*, in that they avoid the need to run additional simulations. The output from the initial simulation contains considerable information that can be used to estimate security price derivatives directly. This advantage becomes more pronounced as the number of derivatives to be estimated increases. While the resimulation approach requires one additional simulation run for each derivative to be estimated, the direct methods require information only from the initial simulation run. A second advantage of the direct methods is that they give

unbiased estimates of derivatives, whereas the resimulation estimates are generally biased.

Boyle (1977) proposed the use of Monte Carlo simulation for estimating security prices. Since that time, simulation has proved to be a valuable tool in many situations where other tools are not available or are not practical to implement. For example, simulation is useful for estimating security prices when there is no analytical expression for the terminal distribution of the security price, when there are multiple state variables, and when there are path dependent payoffs. Simulation has been used to estimate prices of contingent claims, prices of mortgage backed securities, and to value swaps. For an overview of Monte Carlo methods for security pricing, including references to pricing American-style contingent claims using simulation, see Boyle, Broadie, and Glasserman (1995). For a brief sampling of the literature the reader is referred to Boyle (1977), Hull and White (1987), Johnson and Shanno (1987), Jones and Jacobs (1986), Kemna and Vorst (1990), Schwartz and Torous (1989), and Smith et al. (1990).

The focus of this paper is using simulation to estimate security price derivatives. Derivative information is of

practical and theoretical importance. For practitioners, derivative information is important for hedging, i.e., for reducing the risk of a security or portfolio of securities, when closing the position is not practical or desirable. For example, the derivative of an option price with respect to the price of the underlying security (i.e., the *delta*) indicates the number of units of the security to hold in the hedge portfolio. The corresponding second derivative (i.e., the *gamma*) is related to the optimal time interval required for rebalancing a hedge under transactions costs. Other derivative information is useful for protecting against changes in the associated parameters, e.g., changes in volatility, time to maturity, or interest rates. On the theoretical side, Breeden and Litzenberger (1978) show that the second derivative with respect to the strike price can be interpreted as a state price density. Carr (1993) shows how first and higher order derivatives of an option's price with respect to the initial price of the underlying security can be viewed as an expectation, under an appropriate change of measure, of the corresponding derivative at the terminal date.

To estimate derivatives via simulation, two direct methods are investigated. The *pathwise method* is based on the relationship between the security payoff and the parameter of interest. Differentiating this relationship leads, under appropriate conditions, to an unbiased estimator for the derivative of the security price. In contrast, the *likelihood ratio method* is based on the relationship between the probability density of the price of the underlying security and the parameter of interest. These methods have been studied in the discrete-event simulation literature, but have not received much attention in financial applications. An exception is Fu and Hu (1995). For an overview of the pathwise method (also called perturbation analysis), see Glasserman (1991), and for the likelihood ratio method (also called the score function method) see Rubinstein and Shapiro (1993).

The direct methods are described in detail through an example in §2. For ease of exposition, the example chosen in §2 is a European option on a dividend paying asset, an example for which simulation is not required. While both direct methods lead to unbiased estimators, they differ in their effectiveness and scope of applicability. Second derivative estimators are developed in §3. Computational comparisons of the two direct methods and the indirect resimulation method are given in §4.

To illustrate a range of applications, the examples in §4 include Asian options (path dependent payoffs) and options in a stochastic volatility setting (multiple state variables). Concluding remarks are given in §5. Detailed formulas and technical results are given in the appendices.

2. Derivative Estimates for European Call Options

In this section, the pathwise and likelihood ratio methods are developed for estimating sensitivities of security prices through simulation. For expository purposes, the two methods are introduced through a simple example—one for which simulation is not required. Consider the price p of a European call option on a dividend paying asset that follows a lognormal diffusion. In particular, assume that the risk neutralized price of the underlying asset, S_t, satisfies the stochastic differential equation

$$dS_t = S_t[(r - \delta)dt + \sigma dB_t], \quad (1)$$

where B_t is a standard Brownian motion process (see Hull 1993 for a general discussion of this model). In equation (1), r is the riskless interest rate, δ is the dividend rate, and $\sigma > 0$ is the volatility parameter. Under the risk neutral measure, $\ln(S_T/S_0)$ is normally distributed with mean $(r - \delta - \sigma^2/2)T$ and variance $\sigma^2 T$. The option has a strike price of K and matures at time $T > 0$, with the current time taken to be $t = 0$. In this "Black-Scholes world," the option price is given by

$$p = E[e^{-rT} \max(S_T - K, 0)]. \quad (2)$$

Throughout the paper, E denotes the expectation operator under the risk neutral measure. See Harrison and Kreps (1979) for a justification of the risk neutral pricing approach.

2.1. Pathwise Derivatives
To illustrate the application of the first method, we consider the problem of estimating *vega*, which is $dp/d\sigma$. We do this by defining the discounted payoff

$$P \doteq e^{-rT} \max(S_T - K, 0), \quad (3)$$

(so that $p = E[P]$) and examining how changes in σ determine changes in P. Since σ affects P only through S_T, we begin by examining the dependence of S_T on σ.

The lognormal random variable S_T can be represented as

$$S_T = S_0 e^{(r-\delta-\sigma^2/2)T + \sigma\sqrt{T}Z}, \qquad (4)$$

where Z is a standard normal random variable. Consequently,

$$\frac{dS_T}{d\sigma} = S_T(-\sigma T + \sqrt{T}Z) \qquad (5)$$

$$= \frac{S_T}{\sigma}[\ln(S_T/S_0) - (r - \delta + \tfrac{1}{2}\sigma^2)T].$$

This tells us how a small change in σ affects S_T. Now consider the effect on P of a small change in S_T. If $S_T \geq K$, then the option is in the money and any increase Δ in S_T translates into an increase $e^{-rT}\Delta$ in P. If, however, $S_T < K$, then $P = 0$, and P remains 0 for all sufficiently small changes in S_T. Thus, we arrive at the formal expression

$$\frac{dP}{dS_T} = e^{-rT}1_{\{S_T \geq K\}}, \qquad (6)$$

in which $1_{\{\cdot\}}$ denotes the indicator of the event in braces.[1] Combining (5) and (6) gives

$$\frac{dP}{d\sigma} = \frac{dP}{dS_T}\frac{dS_T}{d\sigma} = e^{-rT}1_{\{S_T \geq K\}}\frac{S_T}{\sigma}$$

$$\times [\ln(S_T/S_0) - (r - \delta + \tfrac{1}{2}\sigma^2)T]. \qquad (7)$$

Each of the terms in this expression is easily evaluated in a simulation, making the estimator $dP/d\sigma$ easy to use. Moreover, it follows from Proposition 1 in Appendix A that this estimator is unbiased, i.e.,

$$E\left[\frac{dP}{d\sigma}\right] = \frac{dp}{d\sigma}.$$

A similar argument leads to an estimator of *delta*, the derivative of the option price with respect to the initial price of the underlying asset. Much as before, we have

$$\frac{dP}{dS_0} = \frac{dP}{dS_T}\frac{dS_T}{dS_0} = e^{-rT}1_{\{S_T \geq K\}}\frac{dS_T}{dS_0}. \qquad (8)$$

[1] Equation (6) is valid if interpreted as a right-hand derivative or as the almost everywhere defined derivative of a Lipschitz function. Technical issues of this type are treated in Appendix A.

Furthermore, from (4) we find that

$$\frac{dS_T}{dS_0} = e^{(r-\delta-\sigma^2/2)T + \sigma\sqrt{T}Z} = \frac{S_T}{S_0}.$$

Substituting this into (8), we arrive at the estimator

$$\frac{dP}{dS_0} = e^{-rT}1_{\{S_T \geq K\}}\frac{S_T}{S_0}. \qquad (9)$$

This estimator is also unbiased, i.e.,

$$E\left[\frac{dP}{dS_0}\right] = \frac{dp}{dS_0}.$$

Similar arguments can be used to develop derivative estimates for options with path dependencies, for which simulation is often the only available computational approach (see §4).

2.2. Derivatives Based on Likelihood Ratios

The second method of estimating derivatives puts the dependence on the parameter of interest in an underlying probability density, rather than in a random variable. We continue with the European option example. It follows from (4) that under the risk neutral measure, the density of S_T is

$$g(x) = \frac{1}{x\sigma\sqrt{T}}n(d(x)), \quad x \geq 0, \qquad (10)$$

where

$$n(z) = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}$$

is the standard normal density and

$$d(x) = \frac{\ln(x/S_0) - (r - \delta - \sigma^2/2)T}{\sigma\sqrt{T}}. \qquad (11)$$

Thus, we can write (2) as

$$p = \int_0^\infty e^{-rT}\max(x - K, 0)g(x)dx. \qquad (12)$$

We now use this representation of p to derive derivative estimates. We begin by considering $dp/d\sigma$. Notice that in (12), σ appears only as an argument of g. Assuming we can interchange the derivative and integral, (12) implies that

$$\frac{dp}{d\sigma} = \int_0^\infty e^{-rT} \max(x - K, 0)\, \frac{\partial g(x)}{\partial \sigma}\, dx. \qquad (13)$$

Multiplying and dividing the integrand in (13) by $g(x)$ and using the identity $(\partial g/\partial \sigma)/g = \partial \ln g/\partial \sigma$, gives

$$\frac{dp}{d\sigma} = \int_0^\infty e^{-rT} \max(x - K, 0)\, \frac{\partial \ln(g(x))}{\partial \sigma}\, g(x)dx$$

$$= E\left[e^{-rT} \max(S_T - K, 0)\, \frac{\partial \ln(g(S_T))}{\partial \sigma} \right].$$

This indicates that the likelihood ratio estimator

$$e^{-rT} \max(S_T - K, 0)\, \frac{\partial \ln(g(S_T))}{\partial \sigma}, \qquad (14)$$

is an unbiased estimator of $dp/d\sigma$ when S_T is simulated under the risk neutral measure. The estimator in (14) is easily implemented using the formula

$$\frac{\partial \ln(g(x))}{\partial \sigma} = -d\, \frac{\partial d}{\partial \sigma} - \frac{1}{\sigma}, \qquad (15)$$

where d is given in (11) and

$$\frac{\partial d}{\partial \sigma} = \frac{\ln(S_0/x) + (r - \delta + \sigma^2)T}{\sigma^2 \sqrt{T}}.$$

A similar argument provides an estimator of the derivative with respect to the initial asset price. Proceeding just as before, we arrive at the equation

$$\frac{dp}{dS_0} = E\left[e^{-rT} \max(S_T - K, 0)\, \frac{\partial \ln(g(S_T))}{\partial S_0} \right],$$

and hence obtain the unbiased likelihood ratio estimator

$$e^{-rT} \max(S_T - K, 0)\, \frac{\partial \ln(g(S_T))}{\partial S_0}, \quad \text{where} \quad (16)$$

$$\frac{\partial \ln(g(x))}{\partial S_0} = \frac{d(x)}{S_0 \sigma \sqrt{T}} = \frac{\ln(x/S_0) - (r - \delta - \sigma^2/2)T}{S_0 \sigma^2 T}.$$

The estimator in (16) is easily used in a simulation of S_T.

2.3. Discussion

We have seen through the European call option example that it is sometimes possible to obtain estimates of derivatives of security prices without resimulation. The same methods apply when closed form expressions are not available and simulation is necessary. In general, the estimators obtained through the pathwise method and

the likelihood ratio method are not the same. Numerical comparisons are presented in §4. At this point, we make some general observations about direct methods compared with resimulation and the scope of the methods discussed above.

Both pathwise derivative estimates and estimates based on likelihood ratios require an interchange of a derivative and an integral (expectation) for unbiasedness. It is largely this requirement that limits their scope, though the limitation is rarely an issue with standard pricing models. Classical conditions for this interchange require fairly strong smoothness conditions on the integrand; see, e.g., Franklin (1944, p. 150–151). These conditions typically are satisfied by the probability densities arising in applications of the likelihood ratio method to pricing models. Indeed, the density in (10) is continuously differentiable in each of its parameters on its domain. In contrast, the pathwise dependence of the payoff of a derivative security may not be smooth. For example, the expression in (3) is continuous in S_T but fails to be differentiable at the point $S_T = K$. As a consequence, somewhat greater care is required with this method in justifying the interchange of derivative and integral. As a rough rule of thumb, if the payoff is continuous, the pathwise method is typically applicable; see Appendix A for a more precise discussion.

Since smoothness is rarely a problem for densities, the main limitation in the application of the likelihood ratio method is that the parameter of interest may not be a parameter of the density at all. This is the case with the strike price in (2); the likelihood ratio method does not apply to this parameter (except possibly through a change of variables). The pathwise method, however, easily covers this case.

It is important to note that the derivation of the likelihood ratio estimator (14) did not make use of any properties of the dependence of the option payoff on the underlying asset price. That is, the particular form of (3) was not important, except for the fact that it displays no explicit dependence on σ. As a consequence, essentially the same estimator applies to *any* derivative security. If the discounted payoff associated with some security is f, meaning that its price is given by $p = E[f(S_T)]$, then its derivative with respect to σ is given by

$$\frac{dp}{d\sigma} = E\left[f(S_T) \frac{\partial \ln(g(S_T))}{\partial \sigma} \right],$$

subject only to the validity of the interchange of derivative and integral. This contrasts markedly with the pathwise method, which depends in an essential way on the form of the payoff. Using the pathwise method, one must derive a different estimator for each payoff function. On one hand, this distinction represents an implementation advantage for the likelihood ratio method; on the other hand, it suggests that the pathwise method is better able to exploit the structure of individual problems.

3. Second Derivatives

The *gamma* of an option, i.e., the second derivative with respect to the initial price of the underlying security, is related to the optimal time interval required for rebalancing a hedge under transactions costs. In this section the direct methods are extended to the estimation of second derivatives.

3.1. Pathwise Second Derivative Estimators

We begin our discussion by considering the simple (if artificial) case of an exponential payoff. Suppose that the payoff of a contingent claim is e^{-S_T} when the final price of the underlying security is S_T. Then the value of this claim today is

$$p = E[e^{-rT}e^{-S_T}].$$

Consider the second derivative of p with respect to the initial price S_0. Differentiating twice inside the expectation gives

$$\frac{d^2p}{dS_0^2} = E\left[e^{-rT}e^{-S_T}\left\{ \left(\frac{dS_T}{dS_0}\right)^2 - \frac{d^2S_T}{dS_0^2} \right\} \right].$$

From (4), we find that $dS_T/dS_0 = S_T/S_0$ and $d^2S_T/dS_0^2 = 0$. Making these substitutions, we get

$$\frac{d^2p}{dS_0^2} = E\left[e^{-rT}e^{-S_T}\left(\frac{S_T}{S_0}\right)^2 \right].$$

Now let p once again be the European option price in (2). From (9) we have

$$\frac{dp(S_0)}{dS_0} = E\left[e^{-rT}\left(\frac{S_T(S_0)}{S_0}\right) 1_{\{S_T(S_0) \geq K\}} \right], \qquad (17)$$

where the dependence of S_T on S_0 is made explicit. Consider a small increase h in S_0. Since the ratio S_T/S_0 does not depend on S_0, we get

$$\frac{dp(S_0 + h)}{dS_0} - \frac{dp(S_0)}{dS_0}$$

$$= E\left[e^{-rT}\left(\frac{S_T}{S_0}\right)(1_{\{S_T(S_0+h) \geq K\}} - 1_{\{S_T(S_0) \geq K\}}) \right]$$

$$= E\left[e^{-rT}\left(\frac{S_T}{S_0}\right) 1_{\{S_T(S_0+h) \geq K > S_T(S_0)\}} \right]$$

$$= E\left[e^{-rT}\left(\frac{S_T}{S_0}\right) 1_{\{S_T(S_0) + dS_T(S_0)/dS_0 h \geq K > S_T(S_0)\}} \right].$$

Dividing by h and letting h decrease to zero, the expectation becomes concentrated at $S_T = K$. Using $dS_T(S_0)/dS_0 = S_T/S_0$, this gives[2]

$$\frac{d^2p}{dS_0^2} = e^{-rT}\left(\frac{K}{S_0}\right)^2 g(K) = e^{-\delta T}\frac{n(d_1(K))}{S_0\sigma\sqrt{T}}, \quad \text{where} \quad (18)$$

$$d_1(x) = [\ln(S_0/x) + (r - \delta + \tfrac{1}{2}\sigma^2)T]$$

$$/(\sigma\sqrt{T}) = -d(x) + \sigma\sqrt{T}.[3]$$

Expression (18) involves no random quantities and thus requires no simulation. Indeed, the result is the well known formula for the gamma of an option, which is usually derived without reference to simulation (see, e.g., Hull 1993, p. 312). The effect of the expectation in (17) is to "smooth" the indicator function. We will see in §4 that similar smoothing arguments result in nontrivial second derivative estimators in settings where no closed form expression exists.

3.2. Likelihood Ratio Second Derivative Estimators

Consider again the problem of estimating the second derivative of p in (2) with respect to the initial asset price

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[2] The same result can be derived in another way. Taking the derivative of (17) again with respect to S_0 gives

$$\frac{d^2p}{dS_0^2} = E\left[e^{-rT}\left(\frac{S_T}{S_0}\right)\frac{\partial}{\partial S_0} 1_{\{S_T(S_0) \geq K\}} \right] = e^{-rT}\left(\frac{K}{S_0}\right)^2 g(K).$$

The second equality uses $d1_{\{S_T(S_0) \geq K\}}/dS_0 = \delta(K)dS_T/dS_0 = \delta(K)S_T/S_0$, where $\delta(\cdot)$ represents the Dirac delta function. This type of argument is used in Carr (1993) in a different context.

[3] The last equality in Equation (18) follows from the identity $e^{-\delta T}n(d_1(K)) = e^{-rT}(K/S_0)n(d(K))$.

S_0. Starting from (12) and differentiating twice under the integral gives

$$\frac{d^2p}{dS_0^2} = \int_0^\infty e^{-rT} \max(x - K, 0) \frac{\partial^2 g(x)}{\partial S_0^2} dx.$$

Multiplying and dividing the integrand by $g(x)$ turns the integral into an expectation and yields

$$\frac{d^2p}{dS_0^2} = E\left[e^{-rT} \max(S_T - K, 0) \frac{\partial^2 g(S_T)}{\partial S_0^2} \frac{1}{g(S_T)} \right].$$

The expression

$$e^{-rT} \max(S_T - K, 0) \frac{\partial^2 g(S_T)}{\partial S_0^2} \frac{1}{g(S_T)},$$

is thus an unbiased likelihood ratio estimator of the second derivative. The estimator can be written more explicitly using

$$\frac{\partial^2 g(S_T)}{\partial S_0^2} \frac{1}{g(S_T)} = \frac{d^2 - d\sigma\sqrt{T} - 1}{S_0^2 \sigma^2 T},$$

where $d = d(S_T)$ is given in (11).

4. Computational Results

This section presents computational comparisons of the two direct methods and the indirect resimulation method through three examples. The first example involves path independent claims, in particular, European options on dividend paying assets. To illustrate the methods on path dependent claims, derivatives of Asian option prices (i.e., options based on an arithmetic average price) are computed in the second example. To illustrate the methods on a model with multiple state variables, derivatives for options with stochastic volatility are computed in the third example.

The resimulation method is described next. Suppose that the security price p depends on a parameter θ and the goal is to estimate $dp/d\theta$ at $\theta = \theta_0$. Denote the simulation estimator of the price at $\theta = \theta_0$ by $P(\theta_0)$. The simulation estimate of the price is the sample average over independent outcomes of $P(\theta_0)$. In the resimulation method, the parameter is perturbed to $\theta_1 = \theta_0 + h$, and the new simulation price estimator $P(\theta_1)$ is computed. The resimulation estimator of the derivative is the forward finite difference $(P(\theta_1) - P(\theta_0))/h$. The resimulation estimate is the average over all trials of this

estimator. The choice of h is discussed in Appendix B. The importance of using common random numbers for both estimators is also discussed in Appendix B. To estimate a second derivative, the parameter is perturbed to $\theta_{-1} = \theta_0 - h$, and the new simulation price estimator $P(\theta_{-1})$ is computed. The resimulation estimator of the second derivative $d^2p/d\theta^2$ at $\theta = \theta_0$ is the central finite difference

$$(P(\theta_{-1}) - 2P(\theta_0) + P(\theta_1))/h^2.$$

An advantage of resimulation compared with the direct methods is that it involves no programming effort beyond what is required for the pricing simulation itself. But this justification seems weak compared with the advantages of the direct estimators. The direct methods provide unbiased estimators, whereas resimulation inherits the bias that results from finite difference approximation to the derivative. Even more important is the fact that the computational savings with direct methods increases with the number of derivatives estimated. Estimating finite differences with respect to n parameters requires $n + 1$ simulations. All n derivatives can be estimated from a single simulation using the direct methods. Thus, they offer a *potential $(n + 1)$-to-one computational advantage*. Many simulation runs are needed to solve for the implied value of a parameter given a security price. In this case, the use of direct estimators of derivatives can lead to significant computational savings. The actual magnitude of the savings depends on the additional computational effort to use a derivative estimate compared to the cost of an additional simulation. Ordinarily, the cost of the former is small relative to the latter. In the first example, however, each trial of the simulation requires only one random number, so the computational savings are not as great.

Variance reduction techniques that apply to the original simulation estimator of a security price can often be used with the three simulation methods for estimating derivatives. In our examples, the control variate method was used to reduce the variance of the estimates. Further discussion of this technique is given at the end of this section.

4.1. Example 1: European Options on Dividend Paying Assets

For European options on dividend paying assets, explicit expressions for all derivatives are available. For

completeness these expressions are given in Proposition 2 in Appendix C. The pathwise estimators and the likelihood ratio estimators are summarized in Propositions 3 and 4, respectively, in Appendix C. These are derived using the arguments in §§2 and 3.

Table 1 contains simulation results for this example. Several points are noteworthy from Table 1. First, the simulation estimates are within two standard errors of the exact values. Second, the resimulation method gives point estimates and standard errors that are almost identical to the pathwise method. One exception is the estimate of gamma, where the pathwise estimate gives an exact result in this case. The use of a small perturbation parameter h leads to biases in the resimulation method that are too small to detect in the results. Third, the standard errors with the likelihood ratio method are typically 1.5 to 4 times greater than the pathwise and resimulation standard errors. The larger standard errors are likely due to the likelihood ratio estimators not depending on the form of the security payoff.

The effectiveness of control variates seems quite sensitive to the estimator with which they are used. With pathwise estimates, the reduction in the estimated standard error is roughly 30–50%, and is very close to the corresponding reduction for the resimulation estimates. In most cases, the impact on the likelihood ratio estimates is somewhat less. However, it is possible that a different control variate would yield different results.

Why do the resimulation and pathwise methods give nearly identical results in this example? Consider, for instance, estimating $dp/d\sigma$. The resimulation estimator is $(P(\sigma_1) - P(\sigma_0))/h$, which can be written as

$$e^{-rT} \frac{\max(S_T(\sigma_1) - K, 0) - \max(S_T(\sigma_0) - K, 0)}{h}.$$

The pathwise estimator is $e^{-rT} 1_{\{S_T(\sigma_0) \ge K\}} dS_T(\sigma_0)/d\sigma$. If common random numbers are used, these estimators differ in two respects. First, they differ when $S_T(\sigma_0) < K$ but $S_T(\sigma_1) \ge K$. In this case the pathwise estimator is exactly zero, but the resimulation estimator can differ significantly from zero. However, for small h, the probability of this situation is also small. In all other cases, the estimators differ in the term

$$(\max(S_T(\sigma_1) - K, 0) - \max(S_T(\sigma_0) - K, 0))/h$$

versus $dS_T(\sigma_0)/d\sigma$. However, for small h, these terms

are nearly equal. In other words, as h decreases to zero, the resimulation estimator converges to the pathwise estimator. This, in fact, *defines* the pathwise estimator. Hence it is not surprising that for small h the results are nearly identical.

4.2. Example 2: Asian Options

In this example derivatives are computed for Asian options, i.e., options on an arithmetic average price. The payoff of these options is path dependent, that is, the payoff depends not only on the terminal security price but on all the previous prices that enter into the average. Closed form expressions for the option price and derivatives are not available for this model. However, analytical approximations have been developed in Turnbull and Wakeman (1991) and Ritchken et al. (1993). Additional analytical results are given in Geman and Yor (1993). We use this example merely to illustrate simulation results for a path dependent example. While analytical approaches are available for some Asian option models, if the stochastic process of the underlying asset is modified slightly, it is straightforward to modify the simulation estimators but the analytical approaches may not carry through.

We assume that the underlying asset satisfies the stochastic differential equation (1). Let T be the maturity of the option written on the average of the last m daily closing prices. Thus, the average price can be written as $\bar{S} = \sum_{i=1}^{m} S_i/m$, where (by a slight abuse of notation) S_i is the price at time $t_i = T - (m - i)/365.25$. For convenience we assume that $T > m/365.25$, i.e., the maturity is greater than the averaging period. The derivative estimators do not change significantly if this is not the case. When Asian options are initiated, the time until the averaging period begins, t_1, is typically much larger than the increment between averaged prices (which is one day in this example).

The estimators for this example are summarized in Propositions 5 and 6 in Appendix C. Here *theta* is defined to be the negative of the derivative of the option price with respect to maturity for a *fixed* averaging increment. In other words, a change in T means a change in the time t_1 until averaging begins. All estimators in Propositions 5 and 6 follow from the same reasoning as the previous ones, though the resulting expressions are more complicated. In particular, the pathwise estimator

Table 1   **European Call Options on Dividend Paying Assets**

	Initial Asset Price (S_0)					
	90	(Std. Err.)	100	(Std. Err.)	110	(Std. Err.)
Call Price						
Exact	1.220		5.126		12.327	
Simulation estimate	1.182	0.034	4.993	0.073	12.171	0.107
Estimate with control	1.211	0.025	5.073	0.033	12.298	0.023
Delta (dp/dS_0)						
Exact	0.222		0.568		0.844	
Resimulation estimate	0.217	0.005	0.561	0.005	0.844	0.004
Resimulation with control	0.221	0.003	0.566	0.003	0.848	0.002
Pathwise estimate	0.217	0.005	0.561	0.005	0.844	0.004
Pathwise with control	0.221	0.003	0.566	0.003	0.848	0.002
Likelihood ratio estimate	0.215	0.008	0.551	0.013	0.817	0.017
LR estimate with control	0.220	0.006	0.562	0.008	0.834	0.010
Vega ($dp/d\sigma$)						
Exact	11.946		17.446		11.435	
Resimulation estimate	11.640	0.268	16.932	0.295	10.719	0.390
Resimulation with control	11.887	0.175	17.236	0.155	11.110	0.220
Pathwise estimate	11.640	0.268	16.932	0.294	10.720	0.390
Pathwise with control	11.887	0.175	17.236	0.155	11.111	0.220
Likelihood ratio estimate	11.490	0.672	16.672	1.086	10.074	1.543
LR estimate with control	11.857	0.600	17.300	0.955	10.969	1.356
Gamma (d^2p/dS_0^2)						
Exact	0.029		0.035		0.019	
Resimulation estimate	0.027	0.006	0.048	0.008	0.021	0.005
Resimulation with control	0.027	0.006	0.048	0.008	0.021	0.005
Pathwise estimate	0.029		0.035		0.019	
Likelihood ratio estimate	0.028	0.002	0.033	0.002	0.017	0.003
LR estimate with control	0.029	0.001	0.035	0.002	0.018	0.002
Rho (dp/dr)						
Exact	3.751		10.344		16.108	
Resimulation estimate	3.672	0.076	10.212	0.098	16.130	0.075
Resimulation with control	3.739	0.053	10.306	0.060	16.188	0.057
Pathwise estimate	3.672	0.076	10.212	0.098	16.130	0.075
Pathwise with control	3.739	0.053	10.305	0.060	16.189	0.057
Likelihood ratio estimate	3.625	0.134	10.012	0.241	15.544	0.361
LR estimate with control	3.723	0.107	10.229	0.162	15.889	0.222
Theta ($-dp/dT$)						
Exact	−8.742		−14.370		−12.415	
Resimulation estimate	−8.524	0.191	−14.005	0.203	−11.979	0.245
Resimulation with control	−8.701	0.124	−14.226	0.090	−12.240	0.119
Pathwise estimate	−8.525	0.191	−14.007	0.203	−11.980	0.245
Pathwise with control	−8.702	0.124	−14.227	0.090	−12.241	0.119
Likelihood ratio estimate	−8.414	0.463	−13.774	0.754	−11.371	1.074
LR estimate with control	−8.677	0.410	−14.240	0.649	−12.048	0.918

Parameters: $r = 0.1$, $K = 100$, $\delta = 0.03$, $\sigma = 0.25$, and $T = 0.2$.
All simulation results based on 10,000 trials.
Resimulation estimates use $h = 0.0001$ except for gamma where $h = 0.05$ is used.

for gamma is no longer a constant. Results for this model are given in Table 2. The results are consistent with those in Example 1, e.g., the point estimates and standard errors are very close for the pathwise and re-simulation methods. An exception is gamma, where the standard errors for the pathwise method are smaller than the resimulation method. This is due to using a larger value for h, which is necessary because of the nondifferentiability of the payoff function. For estimating gamma, a hybrid method was also tested, i.e., the pathwise estimate of delta was resimulated.

4.3. Example 3: Options with Stochastic Volatility

To illustrate the methods on a model with multiple state variables, derivatives for options with stochastic volatility are computed in this example. Following Johnson and Shanno (1987) and Hull and White (1987) we assume that S and σ follow the risk neutralized stochastic processes:

$$dS_t = S_t[(r - \delta)dt + \sigma_t dZ_t], \tag{19}$$

$$d\sigma_t = \sigma_t[\mu dt + \xi dW_t], \tag{20}$$

where Z and W are correlated Brownian motion processes with constant correlation ρ. Johnson and Shanno (1987) present simulation results for this model, and Hull and White (1987) give analytical results for certain special cases and simulation results for other cases. Additional analytical results for a similar model are given in Heston (1993) and Stein and Stein (1991). Our aim is to illustrate the simulation methods for derivative estimation on a model with multiple state variables. Closed form solutions, when available, are generally preferable to simulation methods because of their computational speed advantage. However, changes to the stochastic processes (19) and (20) are easily incorporated in the simulation methods, but the analytical solutions may not be so easily modified.

Our simulation results are based on the following discrete time version of (19)–(20):

$$S_{i+1} = S_i(1 + (r - \delta)\Delta t + \sigma_i\sqrt{\Delta t}Z_i), \tag{21}$$

$$\sigma_{i+1} = \sigma_i(1 + \mu\Delta t + \xi\sqrt{\Delta t}W_i). \tag{22}$$

In (21)–(22), m is the number of time steps in the discretization, $\Delta t = T/m$, $t_i = (i/m)T$, and S_i and σ_i are

the simulated asset prices and volatilities at time t_i, respectively. Also, Z_i and W_i are correlated standard normal random variables. This is a first order Euler approximation to (19)–(20). See Duffie (1996) for a discussion of discrete approximations to continuous time models. See Duffie (1996) and Duffie and Glynn (1993) for related convergence issues.

Pathwise and resimulation results for this example are given in Table 3. Likelihood ratio estimators are not used because the estimators are substantially more complicated in this example and because their performance in the earlier examples was not as promising. Pathwise derivative estimators for this model are given in Proposition 7 in Appendix C. In accordance with (21)–(22), theta is the negative of the derivative with respect to the maturity T with m held fixed; thus, $d(\Delta t)/dT = 1/m$. In addition to the usual derivatives, sensitivities with respect to ξ and μ are also computed. Although the estimators are somewhat more complicated than in the previous example, the results given in Table 3 are similar.

As seen in all three examples, the bias in the resimulation method is small enough that it is not an essential concern. Since the computational effort required by the pathwise and likelihood ratio methods are nearly identical, the difference in standard errors is a strong argument in favor of the pathwise method. Since the resimulation method typically requires much more computational effort than the pathwise method, the nearly identical results for the two methods also favor the pathwise method.

4.4. Control Variates

Variance reduction techniques that apply to the original simulation estimator of a security price can often be applied to derivative estimators. Among the most powerful tools is the control variate technique. For consistency we used the same control variate, the terminal security price, for each of the three examples.[4] Next we briefly summarize the control variate technique. Let D represent an unbiased simulation estimator of the derivative. That is, $d = E[D]$ where d is the true value of the

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[4] Note that there are always simple control variates available, namely the mean and the variance of the random numbers themselves. We used the terminal security price because it led to a larger reduction in variance.

Table 2   Asian Call Options on Dividend Paying Assets

	Initial Asset Price (S_0)					
	90	(Std. Err.)	100	(Std. Err.)	110	(Std. Err.)
Call Price						
Simulation estimate	0.785	0.025	4.412	0.063	11.794	0.095
Estimate with control	0.772	0.020	4.368	0.034	11.722	0.037
Delta (dp/dS_0)						
Resimulation estimate	0.173	0.004	0.567	0.005	0.873	0.004
Resimulation with control	0.171	0.003	0.564	0.003	0.871	0.003
Pathwise estimate	0.173	0.004	0.567	0.005	0.873	0.004
Pathwise with control	0.171	0.003	0.564	0.003	0.871	0.003
Likelihood ratio estimate	0.177	0.007	0.573	0.013	0.896	0.019
LR estimate with control	0.174	0.006	0.566	0.010	0.886	0.014
Vega ($dp/d\sigma$)						
Resimulation estimate	8.872	0.225	15.190	0.251	8.846	0.344
Resimulation with control	8.747	0.167	15.024	0.149	8.622	0.212
Pathwise estimate	8.871	0.225	15.190	0.251	8.843	0.345
Pathwise with control	8.747	0.167	15.024	0.149	8.618	0.212
Likelihood ratio estimate	9.161	0.962	13.844	2.516	5.338	4.812
LR estimate with control	8.984	0.937	13.511	2.483	4.854	4.777
Gamma (d^2p/dS_0^2)						
Resimulation estimate	0.020	0.005	0.046	0.008	0.025	0.006
Resimulation with control	0.020	0.005	0.046	0.008	0.025	0.006
Pathwise estimate	0.023	0.004	0.044	0.005	0.021	0.003
Pathwise with control	0.023	0.004	0.044	0.005	0.021	0.003
Resim of pathwise delta	0.024	0.007	0.041	0.009	0.016	0.005
Previous est. with control	0.024	0.007	0.041	0.009	0.016	0.005
Likelihood ratio estimate	0.030	0.002	0.042	0.003	0.021	0.004
LR estimate with control	0.030	0.002	0.041	0.003	0.020	0.003
Rho (dp/dr)						
Resimulation estimate	2.349	0.056	8.216	0.077	13.037	0.053
Resimulation with control	2.320	0.043	8.170	0.052	13.014	0.045
Pathwise estimate	2.349	0.056	8.217	0.077	13.036	0.053
Pathwise with control	2.320	0.043	8.170	0.052	13.013	0.045
Likelihood ratio estimate	2.378	0.095	8.189	0.196	13.139	0.316
LR estimate with control	2.334	0.079	8.071	0.134	12.931	0.189
Theta ($-dp/dT$)						
Resimulation estimate	−8.622	0.224	−16.638	0.262	−13.432	0.334
Resimulation with control	−8.507	0.175	−16.489	0.189	−13.253	0.253
Pathwise estimate	−8.622	0.224	−16.640	0.262	−13.429	0.334
Pathwise with control	−8.507	0.175	−16.491	0.189	−13.249	0.253
Likelihood ratio estimate	−8.693	0.523	−16.792	0.978	−13.691	1.491
LR estimate with control	−8.535	0.487	−16.481	0.903	−13.220	1.377

Parameters: $r = 0.1$, $K = 100$, $\delta = 0.03$, $\sigma = 0.25$, and $T = 0.2$
All simulation results based on 10,000 trials. The averaging period is $m = 30$ days.
Resimulation estimates use $h = 0.0001$ except for gamma where $h = 0.05$ is used.

Table 3   Call Options on Dividend Paying Assets with Stochastic Volatility

	Initial Asset Price (S_0)					
	90	(Std. Err.)	100	(Std. Err.)	110	(Std. Err.)
Call Price						
Simulation estimate	1.306	0.039	5.139	0.077	12.286	0.111
Estimate with control	1.285	0.027	5.086	0.032	12.203	0.021
Delta (dp/dS_0)						
Resimulation estimate	0.220	0.005	0.560	0.005	0.846	0.004
Resimulation with control	0.217	0.003	0.556	0.003	0.843	0.003
Pathwise estimate	0.220	0.005	0.560	0.005	0.846	0.004
Pathwise with control	0.217	0.003	0.556	0.003	0.843	0.003
Vega ($dp/d\sigma_0$)						
Resimulation estimate	12.229	0.286	17.604	0.314	11.272	0.411
Resimulation with control	12.058	0.179	17.395	0.157	11.006	0.219
Pathwise estimate	12.227	0.286	17.604	0.314	11.272	0.411
Pathwise with control	12.056	0.179	17.396	0.157	11.006	0.219
Vega1 ($dp/d\xi$)						
Resimulation estimate	0.339	0.025	0.004	0.031	−0.201	0.037
Resimulation with control	0.332	0.023	−0.003	0.030	−0.209	0.035
Pathwise estimate	0.339	0.025	0.004	0.031	−0.201	0.037
Pathwise with control	0.332	0.023	−0.003	0.030	−0.209	0.035
Vega2 ($dp/d\mu$)						
Resimulation estimate	0.302	0.008	0.429	0.009	0.273	0.012
Resimulation with control	0.298	0.005	0.423	0.006	0.266	0.009
Pathwise estimate	0.302	0.008	0.429	0;009	0.273	0.012
Pathwise with control	0.298	0.005	0.423	0.006	0.266	0.009
Gamma (d^2p/dS_0^2)						
Resimulation estimate	0.024	0.006	0.022	0.005	0.021	0.005
Resimulation with control	0.024	0.006	0.022	0.005	0.021	0.005
Pathwise estimate	0.029	0.001	0.034	0.001	0.018	0.001
Pathwise with control	0.028	0.001	0.034	0.001	0.019	0.001
Resim of pathwise delta	0.024	0.007	0.029	0.008	0.018	0.006
Previous est. with control	0.024	0.007	0.029	0.008	0.018	0.006
Rho (dp/dr)						
Resimulation estimate	3.687	0.077	10.159	0.098	16.140	0.075
Resimulation with control	3.644	0.052	10.100	0.061	16.106	0.059
Pathwise estimate	3.686	0.077	10.159	0.098	16.139	0.075
Pathwise with control	3.643	0.052	10.100	0.061	16.105	0.059
Theta ($-dp/dT$)						
Resimulation estimate	−8.996	0.208	−14.191	0.222	−12.037	0.266
Resimulation with control	−8.872	0.130	−14.040	0.099	−11.858	0.126
Pathwise estimate	−8.996	0.208	−14.193	0.222	−12.039	0.266
Pathwise with control	−8.871	0.130	−14.041	0.099	−11.859	0.126

Parameters: $r = 0.1$, $K = 100$, $\delta = 0.03$, $\sigma_0 = 0.25$, $T = 0.2$, $\mu = -0.1$, $\xi = 0.3$, and $\rho = 0.5$.
All simulation results based on 10,000 trials. Time is discretized using $m = 130$ increments.
Resimulation estimates use $h = 0.0001$ except for gamma where $h = 0.05$ is used.

derivative to be estimated. Let S_T represent the simulated terminal price of the security. Since $E[S_T] = e^{(r-\delta)T}S_0$, another unbiased estimator of the derivative is

$$D' = D + \beta(S_T - e^{(r-\delta)T}S_0), \qquad (23)$$

for any β. The parameter β can be chosen to *minimize* the variance of the estimator D'. This problem is $\min_\beta E[D' - d]^2$. An easy computational device for solving this problem is linear regression. Thus, if the estimators D are regressed on S_T, the slope of the regression line solves the minimization problem. The last step of optimizing over β can significantly improve the effectiveness of the control variate technique.[5] The efficiency of the resulting estimator D' depends on the absolute value of the correlation between the original estimator, D, and the control variate, S_T.

5. Conclusions

In this paper two methods for estimating derivatives of security prices using simulation were presented. The first method uses the dependence of the security payoff on the parameter of interest. Differentiating this relationship leads, under appropriate conditions, to an unbiased estimator for the derivative of the security price. Since the dependence of the parameter is identified through the random security payoff, this method is termed the *pathwise method*. The second method is based on *likelihood ratios*. Here the dependence of the underlying probability density on the parameter of interest is exploited to obtain derivative information.

The main advantage of the direct methods over resimulation is increased computational speed. The estimation of n derivatives by the resimulation method requires $n + 1$ simulation runs. With the direct methods, the information from a single simulation can be used to estimate all n derivatives. Solving for the implied value of a parameter given a security price typically requires many simulation runs. The use of direct methods for estimating derivatives can lead to significant computational savings in these cases. Another advantage is that the direct methods give unbiased estimates of deriva-

tives, whereas the estimates obtained by resimulation are generally biased.

To illustrate and compare the methods, derivatives were computed for a path independent model, a path dependent model, and a model with multiple state variables. The computational results indicate that the likelihood ratio method gives significantly larger standard errors than the pathwise method. The pathwise and resimulation methods give nearly identical point estimates and standard errors. Hence, the bias in the resimulation estimates is not a problem of practical significance in the examples we considered. Since the results for the pathwise and resimulation methods are nearly identical, the computational speed advantage of the pathwise method is a strong argument in its favor.

Appendix A

General Conditions for Unbiased Estimators

In this appendix, we discuss general conditions for derivative estimators to be unbiased, giving particular attention to the more delicate case of pathwise estimators.

Let $\{X_n, n \geq 0\}$ be a vector-valued state process recording, for example, the price of an underlying asset, the prevailing interest rate, and any other variables influencing the price of a derivative security. (Our vectors are column vectors.) The process $\{X_n\}$ may be a discretization of a continuous-time process. We take the discrete-time model as our starting point. Suppose the discounted payoff associated with a derivative security is given by $f(X)$, where $X = (X_1, \ldots, X_T)$, T is the maturity, and f is real-valued. Thus, the price of the security is $p = E[f(X)]$.

Now suppose the state process is a function of a scalar parameter θ ranging over an open interval Θ. In other words, each X_n is a random function on Θ. For the existence of pathwise derivatives, we require the following conditions:

(A1) At each $\theta \in \Theta$,

$$X'_n(\theta) \equiv \lim_{h \to 0} \frac{X_n(\theta + h) - X_n(\theta)}{h}$$

exists with probability 1.

(A2) If D_f denotes the set of points at which f is differentiable, then

$$P(X(\theta) \in D_f) = 1, \quad \text{for all } \theta \in \Theta.$$

Under these conditions, the discounted payoff has a pathwise derivative, given by

$$\frac{d}{d\theta} f(X(\theta)) = \sum_{n=1}^{T} [\nabla_{x_n} f(X(\theta))]^t X'_n(\theta), \qquad (24)$$

where $\nabla_{x_n} f$ denotes the vector of partial derivatives of f with respect to the components of X_n, and the superscript t denotes transpose. For

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[5] Although this observation is standard in the simulation literature, e.g., §11.4 of Law and Kelton (1991), it has been substantially underutilized in the finance literature.

this pathwise derivative to be an unbiased estimator of the derivative of p, we require further conditions:

(A3) There exists a constant k_f such that

$$|f(x) - f(y)| \leq k_f \|x - y\|,$$

for all vectors x, y in the domain of f.

(A4) There exist random variables K_n, $n = 1, 2, \ldots$, such that

$$\|X_n(\theta_2) - X_n(\theta_1)\| \leq K_n |\theta_2 - \theta_1|,$$

for all n, and for all $\theta_1, \theta_2 \in \Theta$. For each n, $E[K_n] < \infty$.

Condition (A3) states that f is Lipschitz continuous; condition (A4) states each X_n is almost surely Lipschitz with an integrable modulus K_n. We now have

PROPOSITION 1. *If (A1)–(A4) hold, then at every $\theta \in \Theta$, $dp(\theta)/d\theta$ exists and equals $E[df(X)/d\theta]$.*

PROOF OF PROPOSITION 1. Let $P(\theta) = f(X(\theta))$; then, as already noted, $P'(\theta)$ exists with probability 1 if (A1) and (A2) hold. The Lipschitz property is preserved by composition. Hence, under (A3) and the first part of (A4), P is almost surely Lipschitz continuous; that is, there exists a random variable K_P such that

$$|P(\theta_2) - P(\theta_1)| \leq K_P |\theta_2 - \theta_1|, \quad \forall \theta_1, \theta_2 \in \Theta,$$

with probability 1. It follows that for any θ and $\theta + h$ in Θ, we have

$$\frac{|P(\theta + h) - P(\theta)|}{h} \leq K_P.$$

Moreover, under the second part of (A4), the bound K_P has finite expectation (it is bounded by a linear combination of $K_1, \ldots, K_T$), so we may invoke the dominated convergence theorem to interchange an expectation and the limit as $h \to 0$ to conclude that $p'(\theta)$ exists and equals $E[P'(\theta)]$. □

The same considerations that arise in justifying the interchange of derivative and integral for the likelihood ratio method arise in maximum likelihood estimation. Consequently, these issues have been addressed in the statistical literature, and standard sufficient conditions can be found in statistics texts. Generally speaking, if the density is a reasonably smooth function of the parameter in question, the interchange is permissible. For a more detailed examination of this interchange in the derivative estimation context, see L'Ecuyer (1990).

When a pathwise estimator of a first derivative is Lipschitz continuous, the argument in Proposition 1 can be applied to show that the pathwise *second* derivative is also unbiased. However, we have seen that first derivative estimators often involve indicator functions, making them discontinuous. As a result, pathwise estimators of second derivatives do not lend themselves to a simple, unified treatment along the lines of Proposition 1. The particular type of "smoothing" required to obtain an unbiased second derivative estimator is problem dependent. So, we justify our gamma estimators individually in Appendix C. Closely related approaches are used in other contexts in Fu and Hu (1993) and in Chapter 7 of Glasserman (1991).

Appendix B

Optimal Choice of the Parameter Increment in the Resimulation Method

Let h denote the parameter increment in the resimulation method. There is an apparent tradeoff involved in the choice of h. If h is too small, then the variance in the estimates of the original and perturbed prices can cause a large variance in the estimate of the derivative. If h is too large, then the nonlinearity of the price as a function of the parameter of interest can cause a large bias in the derivative estimate. This tradeoff is discussed next. For more extensive treatments of this topic, see Zazanis and Suri (1993) for the case of independent resimulations, and Glynn (1989) for the case of common random numbers. See also Broadie (1988).

Suppose that the resimulation method is used to estimate the derivative of the security price p with respect to a parameter θ. If the function $p(\theta)$ is twice continuously differentiable, Taylor's theorem implies that the function can be approximated by

$$p(\theta) = p_0 + a(\theta - \theta_0) + b(\theta - \theta_0)^2 + o(\theta - \theta_0)^2,$$

where $p_0 = p(\theta_0)$ and $a = dp/d\theta$ evaluated at $\theta = \theta_0$. Suppose that we wish to estimate a. Let h denote the size of the parameter perturbation and set $\theta_1 = \theta_0 + h$. Let $P(\theta_i) = p(\theta_i) + \epsilon_i$, for $i = 0, 1$, denote the simulation estimator of $p(\theta_i)$. The resimulation estimator of a is $\hat{a} = (P(\theta_1) - P(\theta_0))/h$.[6]

Suppose that the objective is to minimize the mean squared estimation error. Ignoring higher order terms,

$$E[\hat{a} - a]^2 = E\left[\left(bh + \frac{\epsilon_1 - \epsilon_0}{h}\right)^2\right]. \tag{25}$$

For simplicity, suppose that the variances of ϵ_0 and ϵ_1 are equal and denoted by ν^2. Also, let ρ denote the correlation of ϵ_0 and ϵ_1, and suppose it is independent of h.

With these assumptions, the parameter increment h^* that minimizes the mean squared estimation error is

$$h^* = \sqrt[4]{\frac{2\nu^2(1 - \rho)}{b^2}}. \tag{26}$$

This follows by expanding the terms in (25) and minimizing (25) over h. Equation (26) squares with intuition in several regards. As the accuracy of the estimators $P(\theta_i)$ increases (i.e., as ν^2 decreases with additional trials in the simulation) the optimal increment h^* decreases. As b^2 decreases (i.e., as $p(\theta)$ becomes more nearly linear) the optimal increment h^* increases. Finally, h^* decreases as the correlation of the errors approaches one.

[6] In terms of derivative estimation alone, it would be better to use a symmetric interval for the finite difference. That is, estimate the derivative at θ_0 using the estimators $P(\theta_0 - h/2)$ and $P(\theta_0 + h/2)$. However, this approach requires two additional simulations for each derivative estimate instead of one with the approach in the text.

Evaluating (25) at h^* gives

$$E[\hat{a} - a]^2 = 2|b\nu|\sqrt{2(1 - \rho)}.$$

This expression illustrates the importance of using common random numbers for the resimulation. Using different random numbers gives a ρ of zero, but using the same stream of random numbers typically gives a correlation near one, and hence a better derivative estimate.

In our examples, the assumption of equal variances for ϵ_0 and ϵ_1 does not hold precisely, but more importantly, the assumption of a constant ρ does not hold. In many simulation contexts, e.g., many discrete-event systems, the variance of $\epsilon_1 - \epsilon_0$ can be written as $h\sigma_1^2(1 - \rho_1) + o(h)$. The optimal increment h in this case is typically smaller than indicated by (26); see Glynn (1989) for details.

Appendix C

Summary of Estimators

The proofs of the following propositions are generally similar to the derivations in the text. Hence, most of the propositions are stated without proof. Where necessary, sketches of the derivation are given.

PROPOSITION 2 (*European call option derivatives*):

Delta (dp/dS_0): $\quad e^{-\delta T}N(d_1(K))$, $\qquad$ (27)

Vega $(dp/d\sigma)$: $\quad \sqrt{T}e^{-\delta T}S_0 n(d_1(K))$, $\qquad$ (28)

Gamma (d^2p/dS_0^2): $\quad e^{-\delta T}\dfrac{n(d_1(K))}{S_0\sigma\sqrt{T}}$, $\qquad$ (29)

Rho (dp/dr): $\quad KTe^{-rT}N(d_2(K))$, $\qquad$ (30)

Theta $(-dp/dT)$: $\quad -\dfrac{\sigma e^{-\delta T}S_0 n(d_1(K))}{2\sqrt{T}}$

$\qquad\qquad + \delta e^{-\delta T}S_0 N(d_1(K)) - rKe^{-rT}N(d_2(K))$, $\qquad$ (31)

where

$$d_1(x) = [\ln(S_0/x) + (r - \delta + \tfrac{1}{2}\sigma^2)T]/(\sigma\sqrt{T}) = -d(x) + \sigma\sqrt{T},$$

and $d_2(x) = -d(x)$. Also, $N(\cdot)$ is the cumulative distribution function of a standard normal random variable.

PROOF OF PROPOSITION 2. The European call option value is $p = S_0e^{-\delta T}N(d_1(K)) - e^{-rT}KN(d_2(K))$, see, e.g., Black and Scholes (1973) and Merton (1973). The results follow by differentiation. □

PROPOSITION 3 (EUROPEAN OPTION PATHWISE DERIVATIVE ESTIMATORS). *The following are unbiased pathwise estimators of the indicated derivatives of European option prices:*

Delta (dp/dS_0): $\quad e^{-rT}1_{\{S_T \geq K\}}\dfrac{S_T}{S_0}$, $\qquad$ (32)

Vega $(dp/d\sigma)$: $\quad e^{-rT}1_{\{S_T \geq K\}}\dfrac{S_T}{\sigma}(\ln(S_T/S_0) - (r - \delta - \tfrac{1}{2}\sigma^2)T)$, $\qquad$ (33)

Gamma (d^2p/dS_0^2): $\quad e^{-\delta T}\dfrac{n(d_1(K))}{S_0\sigma\sqrt{T}}$, $\qquad$ (34)

Rho (dp/dr): $\quad KTe^{-rT}1_{\{S_T \geq K\}}$, $\qquad$ (35)

Theta $(-dp/dT)$: $\quad re^{-rT}\max(S_T - K, 0)$

$\qquad - 1_{\{S_T \geq K\}}e^{-rT}\dfrac{S_T}{2T}(\ln(S_T/S_0) + (r - \delta - \tfrac{1}{2}\sigma^2)T)$. $\qquad$ (36)

PROOF OF PROPOSITION 3. For each case other than (34), differentiability with probability 1, as required by conditions (A1)–(A2) of Appendix A follows from (3) and (4): equation (4) shows that S_T is a smooth function of its parameters, and equation (3) shows that P is differentiable in S_T except when $S_T = K$, which occurs with probability 0. For conditions (A3)–(A4), notice that addition, multiplication by a constant, and the max operation are all Lipschitz functions. Exponentiation is Lipschitz on bounded intervals and the square root function is Lipschitz away from the origin. In particular, the discounted payoff P is Lipschitz in a neighborhood of each of its arguments (since $\sigma > 0$ and $T > 0$). Integrability of the corresponding moduli is easily verified in each case. The derivation and justification of the gamma estimator are given in §3. □

PROPOSITION 4 (EUROPEAN OPTION LIKELIHOOD RATIO DERIVATIVE ESTIMATORS). *The following are unbiased likelihood ratio estimators of the indicated derivatives of European option prices.*

Delta (dp/dS_0): $\quad e^{-rT}\max(S_T - K, 0)\dfrac{1}{S_0\sigma^2 T}$

$\qquad\qquad \times (\ln(S_T/S_0) - (r - \delta - \tfrac{1}{2}\sigma^2)T)$, $\qquad$ (37)

Vega $(dp/d\sigma)$: $\quad e^{-rT}\max(S_T - K, 0)\left(-d\dfrac{\partial d}{\partial\sigma} - \dfrac{1}{\sigma}\right)$, $\qquad$ (38)

Gamma (d^2p/dS_0^2): $\quad e^{-rT}\max(S_T - K, 0)\dfrac{d^2 - d\sigma\sqrt{T} - 1}{S_0^2\sigma^2 T}$, $\qquad$ (39)

Rho (dp/dr): $\quad e^{-rT}\max(S_T - K, 0)\left(-T + \dfrac{d\sqrt{T}}{\sigma}\right)$, $\qquad$ (40)

Theta $(-dp/dT)$: $\quad e^{-rT}\max(S_T - K, 0)\left(r + d\dfrac{\partial d}{\partial T} + \dfrac{1}{2T}\right)$, $\qquad$ (41)

where in (38)–(41)

$$d = d(S_T) = (\ln(S_T/S_0) - (r - \delta - \tfrac{1}{2}\sigma^2)T)/(\sigma\sqrt{T}),$$

in (38)

$$\partial d/\partial\sigma = (\ln(S_0/S_T) + (r - \delta + \tfrac{1}{2}\sigma^2)T)/(\sigma^2\sqrt{T})$$

and in (41)

$$\partial d/\partial T = (-\ln(S_T/S_0) - (r - \delta - \tfrac{1}{2}\sigma^2)T)/(2\sigma T^{3/2}).$$

PROPOSITION 5 (ASIAN OPTION PATHWISE DERIVATIVE ESTIMATORS). *The following are unbiased pathwise estimators of the indicated derivatives of Asian option prices*:

Delta (dp/dS_0):
$$e^{-rT}1_{\{\bar{S}\geq K\}}\frac{\bar{S}}{S_0}, \qquad (42)$$

Vega $(dp/d\sigma)$:
$$e^{-rT}1_{\{\bar{S}\geq K\}}\frac{1}{m\sigma}\sum_{i=1}^{m}S_i$$
$$\times (\ln(S_i/S_0) - (r - \delta + \tfrac{1}{2}\sigma^2)t_i), \qquad (43)$$

Gamma (d^2p/dS_0^2):
$$e^{-rT}\left(\frac{K}{S_0}\right)^2 mg(S_{m-1}, w_m, \Delta t_m), \qquad (44)$$

Rho (dp/dr):
$$1_{\{\bar{S}\geq K\}}e^{-rT}\left(\frac{1}{m}\sum_{i=1}^{m}S_i t_i - T\right), \qquad (45)$$

Theta $(-dp/dT)$:
$$re^{-rT}\max(\bar{S} - K, 0) - 1_{\{\bar{S}\geq K\}}e^{-rT}\frac{\bar{S}}{2t_1}$$
$$\times (\ln(S_1/S_0) + (r - \delta - \tfrac{1}{2}\sigma^2)t_1), \qquad (46)$$

where in (44) $\Delta t_i = t_i - t_{i-1}$, $w_m = m(K - \bar{S}) + S_m$,

$$g(u, \nu, t) = n(d(u, \nu, t))/(\nu\sigma\sqrt{t}), \quad and$$

$$d(u, \nu, t) = (\ln(\nu/u) - (r - \delta - \tfrac{1}{2}\sigma^2)t)/(\sigma\sqrt{t}).$$

In Table 2 in the text, a hybrid (biased) estimator of gamma is also used. This hybrid estimator, based on a resimulation of the pathwise delta estimator, is defined by

Gamma (d^2p/dS_0^2):
$$\frac{1}{h}\left(e^{-rT}1_{\{\bar{S}(S_0+h)\geq K\}}\frac{\bar{S}(S_0 + h)}{S_0 + h}\right.$$
$$\left. - e^{-rT}1_{\{\bar{S}(S_0)\geq K\}}\frac{\bar{S}(S_0)}{S_0}\right). \qquad (47)$$

PROOF OF PROPOSITION 5. The derivations of vega, theta, and gamma are sketched. Note that S_i can be written as $S_0 \prod_{j=1}^{i} X_j$ where $\ln(X_j)$ is normally distributed with mean $(r - \delta - \tfrac{1}{2}\sigma^2)\Delta t_j$ and variance $\sigma^2\Delta t_j$. To compute $d\bar{S}/d\sigma$, the intermediate terms $dS_i/d\sigma$ are needed. Using

$$\frac{d(\prod_{j=1}^{i} X_j)}{d\sigma} = \sum_{j=1}^{i}\left(\prod_{k\neq j} X_k\right)\frac{dX_j}{d\sigma},$$

$dS_i/d\sigma$ can be written as

$$(S_i/\sigma)(\ln(S_i/S_0) - (r - \delta + \tfrac{1}{2}\sigma^2)t_i).$$

The formula for vega now follows from arguments similar to those in the text.

For theta, recall that the derivative with respect to the maturity means the derivative with respect to t_1, the time until averaging begins. With this understanding, the derivation is essentially the same as in the European case.

By the same argument used in §3 for the European option, gamma can be written as the product of $e^{-rT}(K/S_0)^2$ and the density of $\bar{S}$ at $\bar{S}$

$= K$. There is no closed form expression for this density, but it can be estimated in the simulation. Conditioning on $S_1, \ldots, S_{m-1}$, we get, for any x,

$$P(\bar{S} \leq x) = E[P(\bar{S} \leq x \mid S_1, \ldots, S_{m-1})]$$
$$= E\left[G\left(mx - \sum_{j=1}^{m-1} S_j\right)\right],$$

where G is the cumulative lognormal distribution of S_m/S_{m-1}. Differentiating both sides and setting $x = K$, we find that an unbiased estimator of the required density value is $mg(S_{m-1}, w_m, \Delta t_m)$. $\square$

Alternative estimators of gamma are obtained through the argument above by conditioning on $\{S_j, j \neq i\}$, $i = 1, \ldots, m - 1$. The ith such estimator of the density of $\bar{S}$ at K is

$$m\frac{g(S_{i-1}, w_i, \Delta t_i)g(w_i, S_{i+1}, \Delta t_{i+1})}{g(S_{i-1}, S_{i+1}, \Delta t_i + \Delta t_{i+1})}.$$

Averaging these m unbiased estimators gives another estimator of gamma:

$$e^{-rT}\left(\frac{K}{S_0}\right)^2\sum_{i=1}^{m-1}\frac{g(S_{i-1}, w_i, \Delta t_i)g(w_i, S_{i+1}, \Delta t_{i+1})}{g(S_{i-1}, S_{i+1}, \Delta t_i + \Delta t_{i+1})} + g(S_{m-1}, w_m, \Delta t_m).$$

Though theoretically this estimator should have smaller standard error than (44), empirically we found no significant difference.

PROPOSITION 6 (ASIAN OPTION LIKELIHOOD RATIO DERIVATIVE ESTIMATORS). *The following are unbiased likelihood ratio estimators of the indicated derivatives of Asian option prices*:

Delta (dp/dS_0):
$$e^{-rT}\max(\bar{S} - K, 0)\frac{1}{S_0\sigma^2\Delta t_1}$$
$$\times (\ln(S_1/S_0) - (r - \delta - \tfrac{1}{2}\sigma^2)\Delta t_1), \qquad (48)$$

Vega $(dp/d\sigma)$:
$$e^{-rT}\max(\bar{S} - K, 0)\sum_{i=1}^{m}\left(-d_i\frac{\partial d_i}{\partial\sigma} - \frac{1}{\sigma}\right), \qquad (49)$$

Gamma (d^2p/dS_0^2):
$$e^{-rT}\max(\bar{S} - K, 0)\frac{d_1^2 - d_1\sigma\sqrt{\Delta t_1} - 1}{S_0^2\sigma^2\Delta t_1}, \qquad (50)$$

Rho (dp/dr):
$$e^{-rT}\max(\bar{S} - K, 0)\left(-T + \sum_{i=1}^{m}\frac{d_i\sqrt{\Delta t_i}}{\sigma}\right), \qquad (51)$$

Theta $(-dp/dT)$:
$$e^{-rT}\max(\bar{S} - K, 0)\left(r + d_1\frac{\partial d_1}{\partial T} + \frac{1}{2\Delta t_1}\right), \qquad (52)$$

where $\Delta t_i = t_i - t_{i-1}$, *in* (49)–(52)

$$d_i = (\ln(S_i/S_{i-1}) - (r - \delta - \tfrac{1}{2}\sigma^2)\Delta t_i)/(\sigma\sqrt{\Delta t_i}),$$

in (49)

$$\partial d_i/\partial\sigma = (\ln(S_{i-1}/S_i) + (r - \delta + \tfrac{1}{2}\sigma^2)\Delta t_i)/(\sigma^2\sqrt{\Delta t_i}),$$

and in (52) $\partial d_1/\partial T$ *is given by*

$$(-\ln(S_1/S_0) - (r - \delta - \tfrac{1}{2}\sigma^2)\Delta t_1)/(2\sigma\Delta t_1^{3/2}).$$

PROPOSITION 7 (PATHWISE DERIVATIVE ESTIMATORS OF OPTION DE-RIVATIVES WITH STOCHASTIC VOLATILITY). *Let* $t_i = (i/m)T$ *and let* S_i, σ_i *represent the simulated asset price and volatility, respectively, at time* t_i. *In this discrete time model, the following are unbiased pathwise estimators of the indicated derivatives of Asian option prices.*

$$\text{Delta } (dp/dS_0):\quad e^{-rT}1_{\{S_T\geq K\}}\frac{S_T}{S_0}, \tag{53}$$

$$\text{Vega } (dp/d\sigma_0):\quad e^{-rT}1_{\{S_T\geq K\}}\frac{S_T}{\sigma_0}\sum_{i=1}^{m}\left(1 - \frac{(1+(r-\delta)dt)S_{i-1}}{S_i}\right), \tag{54}$$

$$\text{Vega1 } (dp/d\xi):\quad e^{-rT}1_{\{S_T\geq K\}}\frac{S_T}{\xi}\sum_{i=1}^{m}\left(1 - \frac{(1+(r-\delta)dt)S_{i-1}}{S_i}\right)$$
$$\times\left(\sum_{k=1}^{i-1}\left[1 - \frac{(1+\mu dt)\sigma_{k-1}}{\sigma_k}\right]\right), \tag{55}$$

$$\text{Vega2 } (dp/d\mu):\quad e^{-rT}1_{\{S_T\geq K\}}S_T dt\sum_{i=2}^{m}\left(1 - \frac{(1+(r-\delta)dt)S_{i-1}}{S_i}\right)$$
$$\times\left(\sum_{k=1}^{i-1}\frac{\sigma_{k-1}}{\sigma_k}\right), \tag{56}$$

$\text{Gamma } (d^2p/dS_0^2):$

$$e^{-rT}\left(\frac{K}{S_0}\right)^2 n\left(\frac{KS_m/(S_T S_{m-1}) - (1+(r-\delta)dt)}{\sigma_{m-1}\sqrt{dt}}\right), \tag{57}$$

$$\text{Rho } (dp/dr):\quad e^{-rT}1_{\{S_T\geq K\}}\left(-T(S_T - K) + S_T dt\sum_{i=1}^{m}\frac{S_{i-1}}{S_i}\right), \tag{58}$$

$$\text{Theta } (-dp/dT):\quad e^{-rT}1_{\{S_T\geq K\}}\left(r(S_T - K) - \frac{dS_T}{dT}\right), \tag{59}$$

where $dt = T/m$ *and* $n(\cdot)$ *represents the standard normal density function. In* (59) *the derivative* dS_T/dT *can be evaluated recursively using*

$$\frac{dS_i}{dT} = \frac{dS_{i-1}}{dT}\left(\frac{S_i}{S_{i-1}}\right) + S_{i-1}\left(\frac{r-\delta}{m} + \frac{d\sigma_{i-1}}{dT}Z_i\sqrt{dt} + \frac{\sigma_{i-1}Z_i}{2\sqrt{mT}}\right),\quad \text{and} \tag{60}$$

$$\frac{d\sigma_i}{dT} = \frac{d\sigma_{i-1}}{dT}\left(\frac{\sigma_i}{\sigma_{i-1}}\right) + \sigma_{i-1}\left(\frac{\mu}{m} + \frac{\xi W_i}{2\sqrt{mT}}\right), \tag{61}$$

with $dS_0/dT = 0$ *and* $d\sigma_0/dT = 0$. *In* (60) *and* (61) Z_i *and* W_i *are the correlated standard normal random variables used in the simulation.*

In Table 3 in the text, a hybrid (biased) estimator of gamma is also used. This hybrid estimator, based on a resimulation of the pathwise delta estimator, is defined by

$$\text{Gamma } (d^2p/dS_0^2):\quad \frac{1}{h}\left(e^{-rT}1_{\{\bar{S}(S_0+h)\geq K\}}\frac{S_T(S_0+h)}{S_0+h}\right.$$
$$\left. - e^{-rT}1_{\{\bar{S}(S_0)\geq K\}}\frac{S_T(S_0)}{S_0}\right). \tag{62}$$

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