A NEW GENERALIZATION OF THE GOLDEN RATIO

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ABSTRACT

We propose a generalization of the golden section based on division in mean and extreme ratio. The associated integer sequences have many interesting properties.

1. GENERALIZED GOLDEN RATIOS

There have been many generalizations of the number known as *golden ratio* or *golden section*, $\phi = \frac{1+\sqrt{5}}{2}$. Examples are G.A. Moore's golden numbers [10] and S. Bradley's nearly golden sections [5] (see also [7] and [9]). A generalization that has been considered by several authors are the positive roots of $x^{k+1} - x^k - 1 = 0$; see [12] and [14]. In this paper, a similar generalization is proposed. It is based on the original definition of ϕ, division of a line segment in mean and extreme ratio.

Let G be a point dividing the segment $\overline{AB}$ in parts of length $a = |AG|$ and $b = |GB|$; suppose $a > b$. The division is *mean and extreme* if the ratio of the larger to the smaller part equals the ratio of the whole segment to the larger part:

$$\frac{a}{b} = \frac{a+b}{a}.$$

Given a positive integer k, we consider divisions satisfying

$$\left(\frac{a}{b}\right)^k = \frac{a+b}{a}.$$

For $k > 1$, we have not one but two ratios: $\varphi_k = \frac{a}{b}$ and $\phi_k = \frac{a+b}{a} = 1 + \frac{1}{\varphi_k}$. These numbers will be called the k-th *lower* and *upper golden ratio*, respectively. Obviously, $(\varphi_k)^k = \phi_k$. It is also evident that φ_k is a root of the polynomial $p_k(x) = x^{k+1} - x - 1$ and ϕ_k is a root of the polynomial $P_k(x) = x(x-1)^k - 1$.

Proposition 1.1: *For every positive integer k, the polynomials $p_k(x)$ and $P_k(x)$ have a unique positive root. If k is even this is the only real root, and if k is odd the polynomials have another negative root.*

Proof: The equation $p_k(x) = 0$ can be rewritten as $x^k - 1 = \frac{1}{x}$. Thus, real roots correspond to intersections of the hyperbola $y = \frac{1}{x}$ and the graph of the power function translated one unit downwards, $y = x^k - 1$. Similarly, real roots of P_k correspond to intersections of the hyperbola and the graph of the power function translated one unit to the right, $y = (x-1)^k$. The claims follow from elementary properties of the functions involved. □

Therefore, φ_k is the unique positive root of p_k and ϕ_k is the unique positive root of P_k. The only instance when φ_k and ϕ_k coincide is $k = 1$, when both are equal to the ordinary golden ratio ϕ. The second lower golden ratio φ_2 has been called *plastic number* by the Benedictine monk and architect Dom Hans van der Laan [1]. This is the smallest Pisot-Vijayaraghavan number (see [4]). Its square, ϕ_2, is also a cubic Pisot-Vijayaraghavan number. In Table 1, we list decimal approximations to the first five lower and upper golden ratios. As k grows, the lower golden ratios tend to 1 and the upper golden ratios tend to 2.

k	φ_k	ϕ_k
1	1.6180339887	1.6180339887
2	1.3247179572	1.7548776662
3	1.2207440846	1.8191725134
4	1.1673039783	1.8566748839
5	1.1347241384	1.8812714616

Table 1: Lower and upper golden ratios.

Proposition 1.2: $\lim\limits_{k\to\infty} \varphi_k = 1$, $\lim\limits_{k\to\infty} \phi_k = 2$.

Proof: By direct computation, p_k is strictly increasing on $\left[1, \sqrt[k+1]{3}\right]$, attains a negative value at $x = 1$ and a positive value at $x = \sqrt[k+1]{3}$. Hence, p_k has a unique zero in this interval, i.e. $\varphi_k \in \left(1, \sqrt[k+1]{3}\right)$. The proposition follows from $\lim\limits_{k\to\infty} \sqrt[k+1]{3} = 1$ and $\phi_k = 1 + \frac{1}{\varphi_k}$. □

2. ASSOCIATED INTEGER SEQUENCES

The connection between the golden ratio and Fibonacci numbers is well known. We can define integer sequences associated with the generalized golden ratios in a similar manner. The k-th *lower Fibonacci sequence* $f_n^{(k)}$ is defined by $f_1^{(k)} = f_2^{(k)} = \ldots = f_{k+1}^{(k)} = 1$ and the linear recurrence with characteristic polynomial p_k:

$$f_n^{(k)} = f_{n-k}^{(k)} + f_{n-k-1}^{(k)}.$$

The k-th *upper Fibonacci sequence* $F_n^{(k)}$ satisfies the same initial conditions and the linear recurrence with characteristic polynomial P_k. By the binomial theorem, we get

$$F_n^{(k)} = \sum_{i=1}^{k} \binom{k}{i} (-1)^{i+1} F_{n-i}^{(k)} + F_{n-k-1}^{(k)}.$$

Of course, both $f_n^{(1)}$ and $F_n^{(1)}$ are just the Fibonacci numbers. The second lower Fibonacci sequence has been called the *Padovan sequence* in [13]:

$$(f_n^{(2)}) = (1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, \ldots).$$

This is sequence number A000931 in N. Sloane's *Encyclopedia of Integer Sequences* [11]. Another interesting sequence satisfying the same recurrence with different initial conditions is the

Perrin sequence (Sloane's A001608), giving a necessary condition for primality [2]. The second upper Fibonacci sequence is Sloane's A005251:

$$(F_n^{(2)}) = (1, 1, 1, 2, 4, 7, 12, 21, 37, 65, 114, 200, 351, 616, 1081, \dots).$$

Among other combinatorial interpretations, $F_n^{(2)}$ is the number of compositions of n without 2's [6] and the number of binary strings of length $n-3$ without isolated ones [3]. Notice that $F_{n+1}^{(2)} = f_{2n-1}^{(2)}$.

The third lower Fibonacci sequence is listed in [11] as A079398:

$$(f_n^{(3)}) = (1, 1, 1, 1, 2, 2, 2, 3, 4, 4, 5, 7, 8, 9, 12, 15, 17, 21, 27, 32, \dots).$$

Upper Fibonacci sequences are currently listed up to $k = 5$. Here are the first few values of $F_n^{(3)}$, Sloane's A003522:

$$(F_n^{(3)}) = (1, 1, 1, 1, 2, 5, 11, 21, 37, 64, 113, 205, 377, 693, 1266, \dots).$$

De Villiers [14] considered sequences defined by the recurrence $L_n^{(k)} = L_{n-1}^{(k)} + L_{n-k-1}^{(k)}$. When equipped with Fibonacci-like initial conditions, $L_1^{(k)} = \ldots = L_{k+1}^{(k)} = 1$, these are the *Lamé sequences of higher order* (according to [11]). De Villiers gave a partial proof that ratios of consecutive members tend to the positive root of $x^{k+1} - x^k - 1 = 0$, generalizing a famous property of the Fibonacci numbers. The proof was later completed by S. Falcon [8]. Not surprisingly, ratios of consecutive members of the lower and upper Fibonacci sequences tend to the corresponding golden ratios.

Theorem 2.1: $\displaystyle\lim_{n\to\infty} \frac{f_{n+1}^{(k)}}{f_n^{(k)}} = \varphi_k, \ \lim_{n\to\infty} \frac{F_{n+1}^{(k)}}{F_n^{(k)}} = \phi_k.$

Proof: The polynomials p_k, P_k and their derivatives are relatively prime. Therefore, p_k and P_k have $k+1$ distinct complex roots each and formulae for the corresponding integer sequences are of the form $a_n = C_0 z_0^n + \ldots + C_k z_k^n$. Here, $z_0, \ldots, z_k$ are the roots of p_k or P_k and $C_0, \ldots, C_k$ are constants. The quotient of two consecutive sequence members can be expressed as

$$\frac{a_{n+1}}{a_n} = \frac{C_0 z_0^{n+1} + \ldots + C_k z_k^{n+1}}{C_0 z_0^n + \ldots + C_k z_k^n} = \frac{C_0 z_0 + C_1 z_1 \left(\frac{z_1}{z_0}\right)^n + \ldots + C_k z_k \left(\frac{z_k}{z_0}\right)^n}{C_0 + C_1 \left(\frac{z_1}{z_0}\right)^n + \ldots + C_k \left(\frac{z_k}{z_0}\right)^n}.$$

Suppose $|z_0| > |z_i|$ for $i = 1, \ldots, k$. Then, $\left(\frac{z_i}{z_0}\right)^n \to 0$ as $n \to \infty$ and $\frac{a_{n+1}}{a_n} \to z_0$, provided $C_0 \neq 0$. Thus, it remains to be shown that the coefficients $C_0, \ldots, C_k$ are not zero and φ_k, ϕ_k are greater than the absolute values of the remaining roots of p_k and P_k.

The coefficients $C_0, \dots, C_k$ satisfy the system of linear equations

$$\begin{bmatrix} z_0 & z_1 & \cdots & z_k \\ z_0^2 & z_1^2 & \cdots & z_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ z_0^{k+1} & z_1^{k+1} & \cdots & z_k^{k+1} \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_k \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Let A be the square matrix on the left. By Cramer's rule we have

$$C_i = \frac{1}{\det A} \begin{vmatrix} z_0 & \cdots & 1 & \cdots & z_k \\ z_0^2 & \cdots & 1 & \cdots & z_k^2 \\ \vdots & & \vdots & & \vdots \\ z_0^{k+1} & \cdots & 1 & \cdots & z_k^{k+1} \end{vmatrix}.$$

The Vandermonde determinant in the numerator is not zero because the roots are all distinct and 1 is neither a root of p_k nor of P_k.

Finally, let $z = x + i\,y \neq \varphi_k$ be a root of p_k and denote its absolute value by $r = |z| = \sqrt{x^2 + y^2}$. By Proposition 1.1, z is either the unique negative root (for odd k) or else $y \neq 0$; in both cases $x < r$. Taking the absolute value of $p_k(z) = 0$ we have:

$$|z|^{k+1} = |z+1| = \sqrt{(x+1)^2 + y^2} < \sqrt{x^2 + y^2 + 2r + 1}.$$

Equivalently, $r^{k+1} < \sqrt{r^2 + 2r + 1} = r + 1$, i.e. $p_k(r) < 0$. The polynomial p_k is strictly increasing on $[1, +\infty)$ and $p_k(\varphi_k) = 0$. Therefore, $p_k(x) > 0$ for all $x > \varphi_k$ and we conclude $r < \varphi_k$. Similarly, if $z = x + i\,y \neq \phi_k$ is a root of P_k, we get

$$1 = |z| \cdot |z-1|^k = r\sqrt{r^2 - 2x + 1}^{\,k} > r(r-1)^k \implies P_k(r) < 0.$$

Again, $P_k(x) > 0$ for all $x > \phi_k$ and $r < \phi_k$ follows. This completes the proof. $\square$

Corollary 2.2: $\displaystyle\lim_{n\to\infty} \frac{f_{n+k}^{(k)}}{f_n^{(k)}} = \phi_k$.

Proof: By the preceding theorem, consecutive ratios of the k-th lower Fibonacci sequences tend to φ_k so we have

$$\frac{f_{n+k}^{(k)}}{f_n^{(k)}} = \frac{f_{n+k}^{(k)}}{f_{n+k-1}^{(k)}} \cdot \frac{f_{n+k-1}^{(k)}}{f_{n+k-2}^{(k)}} \cdots \frac{f_{n+1}^{(k)}}{f_n^{(k)}} \to \varphi_k \cdot \varphi_k \cdots \varphi_k = (\varphi_k)^k = \phi_k. \quad \square$$

Just like ordinary Fibonacci numbers, their upper "cousins" can be expressed as sums of binomial coefficients. We will need the following lemma.

Lemma 2.3: *For any* $k \le l \le m$, $\sum_{j=0}^{k}(-1)^j\binom{k}{j}\binom{m-j}{l} = \binom{m-k}{l-k}$.

Proof: Let M be a set of m elements and suppose a subset of k elements is given. The right side enumerates all l-element subsets of M containing the given k elements. On the other hand, $\binom{k}{j}\binom{m-j}{l}$ is the number of l-subsets avoiding at least j of the k given elements. The sum on the left equals the binomial coefficient on the right by inclusion-exclusion. □

Proposition 2.4: $F_{n+1}^{(k)} = \sum_{i\ge 0}\binom{n-i}{k\,i}$

Proof: Obviously, $\sum_{i\ge 0}\binom{n-i}{k\,i} = 1$ for all $n \le k$. The recurrence for the upper Fibonacci numbers can be rewritten as

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}F_{n+k+1-j}^{(k)} = F_n^{(k)}.$$

By substituting appropriate sums of binomial coefficients we get

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}\sum_{i\ge 0}\binom{n+k-j-i}{k\,i} = \sum_{i\ge 0}\binom{n-1-i}{k\,i} = \sum_{i\ge 1}\binom{n-i}{k(i-1)}.$$

Equivalently,

$$\sum_{i\ge 1}\left[\sum_{j=0}^{k}(-1)^j\binom{k}{j}\binom{n+k-i-j}{k\,i} - \binom{n-i}{k(i-1)}\right] = 0.$$

The terms in the square brackets are all zero by Lemma 2.3 for $m = n+k-i$ and $l = k\,i$. Therefore, the considered sums satisfy the the initial conditions and the recurrence for the upper Fibonacci sequence. □

Members of the Lamé sequences can also be expressed as sums of binomial coefficients [11]:

$$L_{n+1}^{(k)} = \sum_{i=0}^{\lfloor n/k\rfloor}\binom{n-k\,i}{i}.$$

It would be of interest to find a similar formula for the lower Fibonacci sequences and to generalize other known properties of Fibonacci numbers.

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