SPECIAL INTEGER SEQUENCES CONTROLLED BY THREE PARAMETERS

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1. INTRODUCTION

The positive integers h, n, and k are used as parameters to postulate a set of rules for generating a family of sequences of positive integers. It is shown that some of the sequences are directly related to sums of the k^{th} powers of roots of selected n^{th} degree polynomials in which the coefficient of the $(n - h)^{th}$ power is zero. The remaining sequences are the Lucas-like sequences described in a previous paper [1] plus a transition sequence.

2. FIRST-TYPE SEQUENCE

For a given n, the k^{th} member of a sequence is u_{kn}. For each h, n has the values specified by $n \geq h + 1$. There are, in general, four types of behavior within a sequence. A general sequence is formularized in (1) with boundaries between types of behavior indicated by xxxxx, ooooo, or _____.

For the special case $h = 1$, there are no values above the xxxxx divider. By interpreting a summation as zero when its upper limit is zero, it is seen that the first term (i. e., the $k = 1$ term) for $h = 1$ appears between the xxxxx and ooooo dividers and is zero. For $h \geq 2$ there are always some terms for each type of behavior, and the first term of a sequence is always one. Some examples are given in Table 1.

Table 1

k	h=1, n=2	h=1, n=6	h=3, n=7	h=5, n=8
1	0	0	1	1
2	0000000 2	0000000 2	3	3
3	0	3	xxxxxxx 4	7
4	2	6	0000000 11	15
5	0	10	21	xxxxxxx 26
6	2	17	42	0000000 57
7	0	21	78	113
8	2	38	139	223

(1)

$$\left.\begin{aligned}
u_{1n} &= 2^1 - 1 \\
&\cdots\cdots \\
u_{kn} &= 2^k - 1, \text{ (general term)} \\
&\cdots\cdots \\
u_{h-1,n} &= 2^{h-1} - 1,
\end{aligned}\right\} (1 \le k \le h - 1)$$

xxxxxxxxxxx

$$u_{hn} = \sum_{b=1}^{h-1} u_{bn} , \qquad (k = h)$$

ooooooooooo

$$\left.\begin{aligned}
u_{h+1,n} &= \left(\sum_{b=1}^{h} u_{bn}\right) - u_{1n} + h + 1 , \\
&\cdots\cdots\cdots\cdots \\
u_{kn} &= \left(\sum_{b=1}^{k-1} u_{bn}\right) - u_{k-h,n} + k \text{ (general term)} \\
&\cdots\cdots\cdots\cdots \\
u_{nn} &= \left(\sum_{b=1}^{n-1} u_{bn}\right) - u_{n-h,n} + n
\end{aligned}\right\} (h + 1 \le k \le n)$$

$$\left.\begin{aligned}
u_{n+1,n} &= \left(\sum_{b=1}^{n} u_{bn}\right) - u_{n+1-h,n} \\
&\cdots\cdots\cdots\cdots \\
u_{kn} &= \left(\sum_{b=k-n}^{k-1} u_{bn}\right) - u_{k-h,n} \text{ (general term)} \\
&\cdots\cdots\cdots\cdots
\end{aligned}\right\} k \ge n + 1$$

It is interesting to note that there are $h - 1$ terms prior to a xxxxx divider and n terms prior to a _____ divider. Inspection of (1) shows that for $h \ge 2$ the first $h - 1$ terms follow the pattern $1, 3, 7, 15, 31, \cdots, 2^k - 1, \cdots$. For values of $k > h$, it is seen from (1) that u_{kn} is found from a sum which includes u_{kn} 's in an order which would be consecutive except for an always excluded $u_{k-h,n}$ term. Behavior of the first-type sequences is included in tables in the Appendix for $h = 1(1)5$, $n = 1(1)11$, and $k = 1(1)11$.

3. A USE OF THE FIRST-TYPE SEQUENCE

For selected h and n, the k^{th} term of a first-type sequence is the same as $S_k^{(n)}$, the sum of the k^{th} powers of the roots of

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n , \tag{2}$$

if the choices $a_0 = 1$, $a_h = 0$, and all other a's = -1 are made. Verification over a limited range can be made by direct comparison of Table 1 of [1] and the corresponding table of the Appendix. The interpretation is, of course, that $S_k^{(n)} = u_{kn}$ for a given h.

4. SECOND-TYPE SEQUENCE

The first-type sequence applied for $n \geq h + 1$ and the u_{kn} 's were identically the $S_k^{(n)}$'s in that range. If for $2 \leq n \leq h$ the $S_k^{(n)}$'s are calculated and interpreted as u_{kn} 's, the u_{kn} 's so determined are members of a second-type sequence. The tables of the Appendix include second-type sequences.

For $n \leq h - 1$, (2) does not have an a_hx^{n-h} term, and does not have the missing term resulting from $a_n = 0$. Since the Lucas-like sequences of [1] are found from (2) with no missing terms, the second-type sequences are the Lucas-like sequences for $n \leq h - 1$.

For $n = h - 1$ and $n = h$, the second-type sequences are the same since setting $a_h = 0$ in each case produces equations (2) differing only by a root factor $(x - 0)$ which contributes nothing to the sum of powers of roots. The sequence for $n = h > 2$ accordingly is equal to the Lucas-like sequence obtained for $n = h - 1$. Alternatively, it is seen that the sequence for $n = h > 2$ is related to the second-type sequences. This is demonstrated in (3) which is applicable for $n = h > 2$ only.

$$\left.\begin{array}{l} u_{1n} = 2^1 - 1 \\ \cdots\cdots \\ u_{kn} = 2^k - 1 \quad \text{(general term)} \\ \cdots\cdots \\ u_{h-1,n} = 2^{h-1} - 1 \end{array}\right\} \quad (1 \le k \le h - 1)$$

xxxxxxxxxxxxx

(3)

$$u_{hn} = \sum_{b=1}^{h-1} u_{bn} \qquad (k = h)$$

ooooooooooooo

____

$$\left.\begin{array}{l} u_{h+1,n} = \left(\displaystyle\sum_{b=1}^{n} u_{bn} \right) - u_{1n} \\ \cdots\cdots\cdots\cdots \\ u_{kn} = \left(\displaystyle\sum_{b=k-n}^{k-1} u_{bn} \right) - u_{k-n,n} \quad \text{(general term)} \\ \cdots\cdots\cdots\cdots \end{array}\right\} \quad (k \ge h + 1)$$

Comparison of (3) with (1) indicates that (3) is essentially (1) with the 000000000 and _________ boundaries coalesced. Thus, it is seen that a second-type sequence for $n = h > 2$ is a transition between Lucas-like sequences and a first-type sequence.

5. APPENDIX

Table 2 h = 1

k / n	1	2	3	4	5	6	7	8	9	10	11
1	0	0	0	0	0	0	0	0	0	0	0
2	0	2	2	2	2	2	2	2	2	2	2
3	0	0	3	3	3	3	3	3	3	3	3
4	0	2	2	6	6	6	6	6	6	6	6
5	0	0	5	5	10	10	10	10	10	10	10
6	0	2	5	11	11	17	17	17	17	17	17
7	0	0	7	14	21	21	28	28	28	28	28
8	0	2	10	22	30	38	38	46	46	46	46
9	0	0	12	30	48	57	66	66	75	75	75
10	0	2	17	47	72	92	102	112	112	122	122
11	0	0	22	66	110	143	165	176	187	187	198

First-Type Sequences

Second-Type Sequence

Table 3 h = 2

k / n	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1	1
3	1	1	4	4	4	4	4	4	4	4	4
4	1	1	5	9	9	9	9	9	9	9	9
5	1	1	6	11	16	16	16	16	16	16	16
6	1	1	10	16	22	28	28	28	28	28	28
7	1	1	15	29	36	43	50	50	50	50	50
8	1	1	21	39	67	73	81	89	89	89	89
9	1	1	31	66	114	130	139	148	157	157	157
10	1	1	46	111	188	226	246	256	266	276	276
11	1	1	67	179	313	386	430	452	463	474	485

First-Type Sequences

Second-Type Sequences

Table 4 h = 3

k / n	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1	1
2	1	3	3	3	3	3	3	3	3	3	3
3	1	4	4	4	4	4	4	4	4	4	4
4	1	7	7	11	11	11	11	11	11	11	11
5	1	11	11	16	21	21	21	21	21	21	21
6	1	18	18	30	36	42	42	42	42	42	42
7	1	29	29	50	64	71	78	78	78	78	78
8	1	47	47	91	115	131	139	147	147	147	147
9	1	76	76	157	211	238	256	265	274	274	274
10	1	123	123	278	383	443	473	493	503	513	513
11	1	199	199	485	694	815	881	914	936	947	958

Second-Type Sequences (n = 2, 3) First-Type Sequences (n = 4–11)

Table 5 h = 4

k/n	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1	1
2	1	3	3	3	3	3	3	3	3	3	3
3	1	4	7	7	7	7	7	7	7	7	7
4	1	7	11	11	11	11	11	11	11	11	11
5	1	11	21	21	26	26	26	26	26	26	26
6	1	18	39	39	45	51	51	51	51	51	51
7	1	29	71	71	85	92	99	99	99	99	99
8	1	47	131	131	163	179	187	195	195	195	195
9	1	76	241	241	304	340	358	367	376	376	376
10	1	123	442	442	578	648	688	708	718	728	728
11	1	199	814	814	1090	1244	1321	1365	1387	1398	1409

Second-Type Sequences (n = 2–4) First-Type Sequences (n = 5–11)

Table 6 $h = 5$

k/n	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1	1
2	1	3	3	3	3	3	3	3	3	3	3
3	1	4	7	7	7	7	7	7	7	7	7
4	1	7	11	15	15	15	15	15	15	15	15
5	1	11	21	26	26	26	26	26	26	26	26
6	1	18	39	51	51	57	57	57	57	57	57
7	1	29	71	99	99	106	113	113	113	113	113
8	1	47	131	191	191	207	215	223	223	223	223
9	1	76	241	367	367	403	421	430	439	439	439
10	1	123	443	708	708	788	828	848	858	868	868
11	1	199	815	1365	1365	1530	1618	1662	1684	1695	1706

Second-Type Sequences — First-Type Sequences

6. REFERENCE

1. D. C. Fielder, "Certain Lucas-Like Sequences and their Generation by Partitions of Numbers," Fibonacci Quarterly, Vol. 5, No. 4, Nov., 1967, pp. 319-324.

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ERRATA

SCOTT'S FIBONACCI SCRAPBOOK

In the equations on p. 176, please arrange all the exponents in ascending order. Also on p. 176, please change the sign in the line beginning with $P_4(x)$ to a plus instead of minus. On p. 191 (continuation of Scott's article), please make the line beginning with $P_5(x)$ read as follows:

$$P_5(x) = 3125 + 7768x - 15851x^2 - 9463x^3 + 1976x^4 + 243x^5$$

On page 166, please make the following corrections: In $P_4(x)$, change the next-to last number to $2689x^6$. In $P_5(x)$, change the last number on the first line to read: $594,362x^5$. In $P_6(x)$, change the last number on the first line to read: $85,906,862x^4$, and the following number to $21,282,070x^5$. In $P_7(x)$, please change the last number of the first line to read: $3,730,909,778x^3$, and the following number to $2,311,372,054x^4$.

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