Pascal's and Huygens's game-theoretic foundations for probability

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Blaise Pascal and Christiaan Huygens developed game-theoretic foundations for the calculus of chances – foundations that replaced appeals to frequency with arguments based on a game's temporal structure. Pascal argued for equal division when chances are equal. Huygens extended the argument by considering strategies for a player who can make any bet with any opponent so long as its terms are equal.

These game-theoretic foundations were disregarded by Pascal's and Huygens's 18th century successors, who found the already established foundation of equally frequent chances more conceptually relevant and mathematically fruitful. But the game-theoretic foundations can be developed in ways that merit attention in the 21st century.

1. The calculus of chances before Pascal and Fermat

We are often told that probability theory began with an exchange of letters in 1654 between Blaise Pascal (1623-1662) and Pierre Fermat (1607-1665). As Florence Nightingale David put it,

> The name of Blaise Pascal is always linked with that of Fermat as one of the "joint discoverers" of the probability calculus.[1]

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[1] [1], p. 75.

We can trace this attribution back to Laplace, who told his students at the École Normale in 1795 that probability theory

> owes its birth to two French geometers of the 17th century.[2]

Laplace repeated these words in 1812, in the first edition of his *Théorie analytique*,[3] but he tempered them two years later in the history of probability theory with which he concluded his *Essai philosophique*:

> For quite a long time, people have ascertained the ratios of favourable to unfavourable chances in the simplest games; stakes and bets were fixed by these ratios. But before Pascal and Fermat, no one gave principles and methods for reducing the matter to calculation, and no one had solved problems of this type that were even a little complicated. So we should attribute to these two great geometers the first elements of the science of probabilities …[4]

Many of Laplace's successors have found the nuances unnecessary. Lacroix, for example, began his 1816 probability textbook with this unqualified attribution:

> The probability calculus, invented by Pascal and Fermat, has never since ceased exciting the interest and exercising the wisdom of their most illustrious successors …[5]

Similar unqualified statements by mathematicians and historians of mathematics abound, throughout the 19th and 20th centuries and up to the present day. But Laplace was surely correct when he conceded that people had been counting chances and using the counts to fix stakes and bets long before Pascal and Fermat.

Counting chances

People have been making finely balanced dice for millennia, and they have probably been counting the chances for throws of these dice for just as

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2 Apparently not published at the time, Laplace's lecture was reproduced on pp. 146-177 of Volume XIV of his complete works [2]. The words translated here come at the end of the lecture. Except when otherwise noted, all translations are mine.

3 [3], p. 3.

4 [4], p. 89.

5 [5], p. iii.

long. But the earliest documentary evidence for such counting appears to be the Latin poem *De Vetula*, probably written around 1250 by a teacher of the quadrivium (arithmetic, geometry, astronomy, and music) at the University of Paris.[6]

A long poem, touching on philosophical, religious, and scientific topics, *De Vetula* begins by warning its readers against the temptations of erotic love and gambling. In the case of gambling, the author warns that a gambler faces ruin even if he knows how to count chances, then proceeds to count them anyway for the sum of the points on three dice. There are 216 chances, he explains, all of equal force and frequency. But the sum of the points can range from 3 to 18, and these 16 possibilities have unequal force and frequency. There are 108 chances that the sum will be between 3 and 10, distributed unequally:

Sum of points	3	4	5	6	7	8	9	10	Total
# of chances	1	3	6	10	15	21	25	27	108

There are another 108 chances that the sum will be between 11 and 18, distributed similarly:

Sum of points	18	17	16	15	14	13	12	11	Total
# of chances	1	3	6	10	15	21	25	27	108

David Bellhouse has called *De Vetula* a "medieval bestseller". It was often quoted. Nearly 60 manuscript copies survive. The first printed edition appeared in about 1475. Not everyone who reproduced it understood how the author counted chances. But some did, including the editors of editions printed in 1479, 1534, and 1662.

There are at least two other surviving documents in which mathematicians counted the chances for dice before 1654: a book by Cardano, who died in 1576, and a letter by Galileo, who died in 1642. Neither appeared in print in its author's lifetime. Cardano's *Liber de Ludo Aleae* was published in his collected works in 1663,[7] and Galileo's letter appeared in his collected works in 1718.[8] Both Cardano and Galileo counted the chances for the sum

6 [6], [7], [8], [9].

7 Geralamo Cardano, *Liber de Ludo Aleae*, in [10], volume 1, pp. 262-276. English translation in [11], pp. 182-241.

8 [12], pp. 591-594. English translation [1], pp. 192-195.

of points on three dice. As Bellhouse has pointed out, Cardano's presentation suggests that he may have been influenced directly by *De Vetula*, whereas Galileo obtains the counts in a different way.[9]

As mathematics developed, mathematicians' ability to count chances improved. Galileo mentions that the number of equally frequent chances is multiplied by 6 every time a die is added to the throw. There are 216 equal chances in the case of three dice because $6 \times 6 \times 6 = 216$. The author of *De Vetula* had not mentioned this.

Fixing stakes and bets

The whole point of counting chances is to use them to fix stakes and bets. The author of *De Vetula* does not bother to explain how this is done, but readers adept in mathematics would have known what to do: use the rule of three.

The universities of medieval Europe prepared young men for careers in the priesthood, law and medicine. To learn practical mathematics, you went elsewhere – to teachers who prepared young men to work in trade. We know what these teachers taught, because countless of their manuals – *commercial arithmetics*, we call them – have survived. This being a lecture in honour of George Sarton, I pause to recall Sarton's interest in these manuals. As he pointed out in 1933, they were being written in both Arabic and in Spanish in Spain in the 11th century.[10] They spread throughout Europe as trade developed.[11]

The rule of three was the main tool of the commercial arithmetics. After learning how to add, subtract, divide and multiply, merchants and their clerks need to understand proportions. If you buy 15 bushels of wheat for 10 shillings, what price should you charge someone else for 3 bushels? For us, this is a matter of algebra: $15/3 = 10/x$, and so $x = 2$ shillings. But al-Khwarizmi's 9th-century algebra was all in words, and the medieval commercial arithmetics still had only words. Algebra with symbols emerged only in the Renaissance. It was largely developed by the authors of commercial arithmetics – the Italian abacus masters and the German

9 [13].
10 [14], [15].
11 See for example [16].

reckoning masters. But even in the 19^{th} century, commercial arithmetics emphasized the non-symbolic rule of three, deploying it in problem after problem in which you find an unknown fourth number in a proportion from three that are known, problems about trading in goods, dividing profits, changing currencies, pricing alloys, etc., etc. Occasionally, for fun, an author might throw in a problem about a game.

Here are two questions that could have been answered by anyone who was adept at the rule of three and could count the chances for three dice.

Q1. Three dice are to be thrown repeatedly until either a 9 or a 15 appears. Player A bets on 9 and Player B bets on 15. Player A puts 5 shillings on the table. How much should Player B put on the table?

Q2. What should Player A pay in order to win 80 shillings if he throws an 11 on a single throw of three dice?

Permitting ourselves a bit of algebra rather than trying to imitate a 13^{th}-century abacus teacher's use of the rule of three, we can answer these questions as follows.

A1. To answer Q1, we recall that there are 25 chances of throwing a 9 and only 10 chances of throwing a 15. The chances have equal frequency. So Player B wins 10 times for every 25 times Player A wins. Player A has put 5 shillings on the table for Player B to win. If we write x for the amount Player B puts on the table, then Player B wins 10×5 shillings every time Player A wins $25 \times x$. This is fair if $x = 2$ shillings.

A2. To answer Q2, suppose Player B is the counterparty. Player A gives x to Player B, and Player B gives back 80 shillings if Player A throws an 11. Player A has 27 chances of getting the 80 shillings. Player B has 216 chances of getting x. So Player A gets 27×80 shillings every time Player B gets $216 \times x$, and this is fair if $x = 10$ shillings.

These answers deploy the notions of frequency and fairness. Frequency was basic to everyone's understanding of chances for dice. Fairness comes along with the rule of three. Commercial arithmetics always sought the fair price. What actually happens is another matter; the merchant will surely ask for a bit more.

Laplace's assertion that no one before Pascal and Fermat gave principles and methods for calculating stakes and bets seems to be correct so far as the surviving public record is concerned. But we do find explicit arguments for proportionality in Cardano's 16th-century *Liber de Ludo Aleae*. Concerning bets on a throw of two dice, where there are 36 equally frequent chances, Cardano writes as follows:

> If, therefore, someone should say, "I want an ace, a deuce, or a trey," you know that there are 27 favourable throws, and since the circuit is 36, the rest of the throws in which these points will not turn up will be 9; the odds will therefore be 3 to 1. Therefore, in 4 throws, if fortune be equal, an ace, deuce, or trey will turn up 3 times and only one throw will be without any of them; if, therefore, the player who wants an ace, deuce, or trey were to wager three ducats and the other player one, then the former would win three times and would gain three ducats, and the other once and would win three ducats; therefore in the circuit of 4 throws they would always be equal. So this is the rationale of contending on equal terms; if, therefore, one of them were to wager more, he would strive under an unfair condition and with loss; but if less, then with gain. Similarly, if the 4 be included, there will be 32 favourable throws, and the number of remaining throws will be only 4. Therefore, the player will place a stake eight times as great as his opponent, because the proportion 32 to 4 is eightfold, and similarly for the other cases …[12]

The qualification "if fortune be equal" is important here. As Bellhouse has emphasized, Cardano's discourse emphasized fairness, not exact prediction.[13]

2. The division problem

Among the documents that Pascal left behind was a memorandum in Latin dated 1654, setting out his agenda for mathematical research and listing treatises he plans to complete.[14] It is addressed to an informal academy of Paris mathematicians, a group whose regular meetings Pascal was attending. By all accounts, this group descended from the equally informal scientific academy that Marin Mersenne had organized in 1635. Pascal's

12 Pp. 200-201 of [11].
13 [13].
14 Mesnard provides the Latin text, a commentary, and a translation into French on pp. 1021-1035 of Volume II of [17].

father Etienne Pascal had been part of Mersenne's circle, and Blaise had first become known as a mathematician after his father brought him into the circle as a teenager. We do not know exactly when in 1654 Pascal wrote the memorandum, but Jean Mesnard, the most assiduous of his many biographers, has argued persuasively that it was written before Pascal's correspondence with Fermat. In the memorandum, Pascal describes one of the topics on which he plans to write as follows:

> A field of research that is completely new and concerns a matter that is completely unexplored, namely structure of chances in games subject to chance, what we call in French *faire les partys des jeux*, where the uncertainty of fate is so well overcome by the rigor of calculation that each of two players can see themselves assigned exactly what they have coming. This must be sought all the more vigorously by reasoning, because there is so much less possibility to find it by experience. In fact, the uncertain outcome of a random event should be attributed more to the chance of contingency than to the necessity of nature. This is why the question has remained unsettled. But now, even if it has been a rebel to experience, it could not escape from the empire of reason. We have reduced it to an art with so much surety, thanks to mathematics, that having gained part of mathematics' certitude, it can now advance audaciously and, by virtue of the union thus achieved between mathematical demonstrations and the uncertainty of chance, and by the reconciling of these apparent opposites, it can take both names and lay claim to the surprising title *Geometry of Chance*.

This paragraph's sense of excitement is palpable; Pascal believes that he has solved a problem others had tried and failed to solve. This problem is new as a field of mathematical research but so familiar to his countrymen that it has a French name. The French noun *parti*, here spelled *party*, can be translated as *share* or as *division into shares*, and so we can translate *faire les partys des jeux* as *divide into shares in games*.

Departing from established usage, I will call the problem of how to *faire les partys* the *division problem*. Since the early 18th century, it has usually been called *le problème des partis* in French and *the problem of points* in English. But these names can be a source of confusion when we try to understand what Pascal actually wrote in his letters to Fermat.

Pascal's solution of the division problem

We learn more about what Pascal meant by *faire les partys* in the first five pages of another short document that he left behind, printed but not published and bearing the title *Usage du triangle arithmétique pour detérminer les parties qu'on doit faire entre deux jouers qui jouent en plusieurs parties*.[15] This title can be translated as *Using the arithmetic triangle to determine the divisions one should make between two players who play in several rounds*. Here I will refer to it simply as Pascal's *Usage*.

By "play in several rounds", Pascal meant that the stakes are won by the first player who wins a specified number of rounds. If the players agree to stop when neither has yet won the specified number, how should the stakes be divided?

Pascal reasons backwards from situations where the appropriate division is clear. Suppose, for example, that Players A and B have each put 32 pistoles on the table. Player A is one round short of winning the entire stakes, and Player B is two rounds short. If the players were to play one more round, the division would be clear:

- If Player A wins the round, he gets all 64 pistoles.
- If Player B wins the round, the two players are even, both being one round short of winning. So they should split the 64 pistoles evenly, each getting back the 32 pistoles he put up.

Player A is thus certain of getting at least 32 pistoles and has an equal chance of getting the other 32. Pascal argues that he can therefore claim the first 32 and half of the second 32, for a total of 48, leaving 16 for Player B.

Having found what each player is entitled to when Player A is one round short and Player B is two rounds short, we can then find what each is entitled to when Player A is one round short and Player B is three rounds short. In this case, another round would either give all 64 pistoles to Player A or put the players in the situation just analysed (Player A one round short and Player B two rounds short), where Player A is entitled to 48 pistoles. So Player A is entitled to (1) the 48 he will have in either case, and (2) half the remaining 16, for a total of 56, leaving only 8 for Player B.

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15 The treatise is reproduced in Volume II of [17] and in other editions of Pascal's works.

As Pascal explains in great detail, at this level of formality, we can reason backwards in this way to find the entitlements of the two players no matter how many how rounds each lacks. He then mentions that there are also two other ways of solving the problem: using combinations and using the arithmetic triangle. He then proceeds to explain how the arithmetic triangle enables us to obtain the answers more quickly.

We know, from Pascal's letter to Fermat dated 29 July 1654, that Pascal mastered this use of the arithmetic triangle only after that date. But from the claim he made in his earlier memorandum to his Paris colleagues, we may assume that he had discovered his method of backward recursion before beginning his correspondence with Fermat. Having understood how he could use the arithmetic triangle in the course of the correspondence, he folded his proposed *Geometry of Chance* into his *Usage*.[16]

Published antecedents

There is a slight but interesting difference between the way Pascal describes the division problem to his Paris colleagues and Fermat and the way he describes it later, in his *Usage*. In the memorandum, he writes about games *subject to chance*. In the letter of 29 July to Fermat, he writes about the two players having an *equal chance* (*le hasard est égal*). Here he could be talking not only about dice games but also about ball games and other competitions that involve both skill and chance. Such games are subject to chance, and when players play on equal terms it is not unusual to say that each has the same chance as the other, even if they do not have the same skill. It is also not unusual for players who disagree about who is more skilful to think it fair that they should bet on even terms. But in his *Usage*, Pascal specifies that he is considering *games of pure chance*.

This difference is of some significance, because previous solutions of the division problem, in handwritten commercial arithmetics and in printed

[16] This speculation rests on Mesnard's conclusion that Pascal wrote his memorandum before his correspondence with Fermat. A. W. F. Edwards has challenged this conclusion on the grounds that Pascal solved the division problem only after his letter of 29 July; see [18], p. 86, reprinted in [19]. This overlooks the fact that the argument in the first five pages of Pascal's *Usage* fully solves the problem. The letter of 29 July shows Pascal struggling to obtain the more efficient and elegant solution that he later obtains using the arithmetic triangle, but it does not refute the hypothesis that he had already solved the problem.

books beginning with Pacioli's *Summa* at the end of the 15th century, had considered games where skill enters – ball games and archery competitions. Perhaps the authors also had dice games in mind, not mentioning them in order to avoid any hint of impiety, but in any case they apparently thought that their arguments applied to games where skill also enters. Their solutions of the division problem usually involved some application of the rule of three. The rule of three can be applied in various ways (in particular, do we consider the number of rounds won or the number lacking?), and so different authors obtained different answers. None of them obtained Pascal's answer, and historians of probability usually express this by saying that their answers were all wrong.[17] But it is also reasonable to conclude, with Tartaglia, that there is no single right answer.

It is also reasonable to conjecture that Pascal was aware of some of the previous efforts to solve the division problem. Would Mersenne not have known about the published work of Pacioli, Cardano, and Tartaglia? Pascal's comment about the question remaining unsettled may be a reference to their disagreements.

Unpublished antecedents

Although none of the previously published treatments of the division problem obtained Pascal's solution, we have learned in recent decades that two unpublished manuscripts by Italian abacus masters, both writing around 1400, did obtain his solution. The first, a fragment noticed in the National Central Library of Florence in 1985 by Laura Toti Rigatelli and subsequently studied by several authors, used an intricate argument to arrive at Pascal's answer for the case where one player is one round short and the other is two rounds short. The second, a commercial arithmetic noticed in the Vatican Apostolic Library in 2003 by Raffaella Franci, develops Pascal's method fully, even for more than two players. Both of these manuscripts have been discussed thoroughly by Norbert Meusnier.[18]

The author of the Vatican commercial arithmetic cautions his students not to divulge his method for solving the division problem but to study it and

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[17] See for example [20], pp. 34-36.

[18] [21]. As Meusnier notes, the Florence manuscript is concerned with games of chess, supporting the hypothesis that the division problem dates back at least to Arabic sources.

stand ready to use it, perhaps to dazzle town leaders or merchants who might employ them to teach. This evidence of a secret tradition centuries before Pascal raises tantalizing but unanswerable questions. How widely was the division problem discussed by teachers of commercial arithmetic in Pascal's time? Had the Vatican manuscript's method survived in an oral tradition? As Ivo Schneider has noted, commercial arithmetic did constitute an oral tradition.[19]

Pascal was not one to cite predecessors. As A. W. F. Edwards has noted,

> Pascal was … a little forgetful about his sources. Practically everything in the *Traité* except the solution of the important "Problem of Points" will have been known to Mersenne's circle by 1637. It seems likely that Pascal absorbed most of this as a young man, and then, more than a decade later, his correspondence with Fermat stimulated him to compose the *Traité*, which he did in the space of a few weeks. The evidence is that, with the passage of time, he had lost most of the details whilst retaining the outline. Just as a lecturer often lectures best when, after careful preparation, he forgets his lecture notes, so Pascal poured forth his mature view of the Arithmetical Triangle and its uses, uncluttered with peripheral detail.[20]

It is conceivable, if unlikely, that Pascal's solution of the division problem is also something that he had picked up in his youth and then forgotten. All we can say with confidence is that Pascal believed in 1654 that it was his own new discovery.

3. Pascal's game-theoretic foundation

In his memorandum to his Paris colleagues, Pascal was concerned with the problem of dividing stakes between two players. This question comes up in his letters to Fermat, but the questions he has posed to Fermat appear to involve a more subtle kind of division, which for clarity I will call *apportionment* rather than *division*. How do we apportion a player's gains to the successive rounds of a multi-round game?

Pascal repeatedly mentions that he and Fermat have different methods for solving questions of apportionment. Fermat was using the venerable

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19 [22], pp. 269-279.
20 [19], p. 58.

method of counting chances, which he wielded with a mathematical power unmatched by any predecessor. Pascal was using backward recursion.

In a game involving multiple rounds or multiple dice, the chances we count can also be called combinations (*combinaisons* in French), as each chance tells how all the rounds or dice come out. Pascal sometimes called Fermat's method the method of combinations. In his first surviving letter, dated 29 July, he mentions that he too had first used combinations but claims that his own method is quicker, at least in some cases. In his second surviving letter, dated 24 August, he makes a more aggressive case for his own method, claiming that it is

- more universal, applicable to any kind of apportionment under any imaginable conditions, and
- more fundamental, carrying its demonstration in itself.

As the correspondence continues, Fermat appears to convince Pascal that the method of combinations is also universal and computationally efficient. On 27 October 1654, in his final letter to Fermat that year, Pascal writes,

> I admire your method for apportionment, all the more because I understand it quite well. It is entirely yours, having nothing in common with mine, and arrives easily at the same end.

But Pascal does not retract the claim that his method is more fundamental, and from a philosophical point of view, this is the most interesting aspect of his contribution. He vindicates the claim he made in his memorandum by giving an argument for his method of backward recursion that relies on reason alone, not on experience. Because backward recursion arrives at the same end as the established method of combinations, this is also a justification of that established method.

Because Pascal's method reasons about the play of the game rather than about frequencies, we may call it *game-theoretic*.

Enter the Chevalier de Méré

The legend of Pascal's and Fermat's invention of probability was embellished in 1837 by Siméon-Denis Poisson, who began his book on probability with this sentence:

> A problem about games of chance proposed to an austere Jansenist by a man of the world was the origin of the calculus of probabilities.[21]

Jansen was a Dutch theologian, and Pascal was the Jansenist. The man of the world was Antoine Gombaud, the Chevalier de Méré. Many authors have concluded that Gombaud introduced Pascal to the problem of division we discussed earlier. It is also possible, and perhaps more likely, that Gombaud's only posed some particular questions about apportionment.

By the early 1650s, Pascal was a close friend of the wealthy and powerful Duke of Roannez. Gombaud, a nobleman of modest means, was occasionally part of the Duke's entourage. At the age of 61 (in 1668, after Pascal's death), he began to publish his letters and essays and became well known as a stylist and moralist, participating in the 17th-century French debate concerning what it means to be an honourable man (*honnête homme*). He made a great virtue of having good manners and pleasing others. He practiced these virtues, and he claimed to have taught Pascal to enjoy himself. He also claimed credit for mathematical discoveries concerning chance. It is possible, though perhaps unlikely, that Gombaud introduced Pascal to the whole topic of calculating chances.

At the beginning of his letter of 29 July, Pascal mentions that he and Fermat had been discussing two questions of apportionment that Gombaud had proposed: apportionment for dice (*les partis des dés*), and apportionment for rounds (*les partis des parties*). Fermat, he acknowledges, has answered the questions using combinations, but at this point Pascal thinks his own method is quicker. What were the questions? How were they related to what I have been calling the division problem, the problem of how to *faire les partys* that Pascal discussed in his memorandum to the Paris mathematicians and solved in the first five pages of his *Usage*? And how are they solved by Fermat's method and by Pascal's method?

In the second section of his letter of 29 July, Pascal explains how to *faire les partys* in the case where two players play to win three rounds, but he does not stop there. After explaining how to find the value to a player of each possible position (as he also did in his *Usage*), he then finds, by subtraction, how this value changes when the player wins a round. Gombaud's question about apportionment for rounds, it seems, concerned

21 [23]

not the value of a *position* in the game but the value of a *round* in the game. How much of his opponent's money does a player gain by winning the round? To use the language of later centuries, the question is not about *expectations* but about *changes in expectations*.

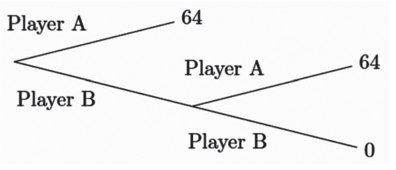
What exactly was Gombaud's other question, the question about apportionment for dice? Not having the previous letters between Pascal and Fermat, we cannot be certain. But we do have an undated fragment of one previous letter from Fermat, and it suggests some possibilities. In this fragment, Fermat says that Pascal has asked about a player who has undertaken to get a six in 8 throws and has already lost the first three. How much should he be compensated for not making his next throw? The answer depends on whether the compensation is taken out of the stake on the table, or whether that stake will all remain for him to try to win with one of his remaining throws. But Pascal must have made a slip, because the answer he gave Fermat, 125/1296 of the stake, is not correct in either case. Fermat finds a rather different question that does have 125/1296 as its correct answer.

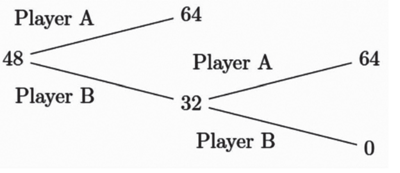
Carrying its demonstration in itself

Whereas Fermat delighted in solving problems, Pascal was more interested in getting to the bottom of things. What are the true principles? What is the real starting point?

On the first two pages his *Usage*, Pascal explains that his method is based on two fundamental principles. First, a player should take any portion of the stakes that will be his regardless of whether he wins or loses. Second, if *the game is one of pure chance*, there is as much chance for the one player as the other to win a certain sum, and they want to stop playing, then they should divide the sum equally.

Pascal makes his assertion that his method is more fundamental than the method of combinations in his letter of 24 August, in the course of explaining how he had defended Fermat's use of the method of combinations to his Paris colleague Gilles de Roberval, a teacher of mathematics at the *Collège royal*. To understand Roberval's objection to Fermat's method, consider again the classic case where Player A is one round short of winning and Player B is two rounds short. If they are playing for 64 pistoles, as in Pascal's presentation of the problem in his letter of 29 July, then we can use diagrams to picture Player A's possible gains and Pascal's argument:[22]





As indicated in the diagram to the left, Player A wins the 64 pistoles if he wins the first round; otherwise they play a second round and Player A may win either 64 or 0. As indicated in the diagram to the right, Pascal concluded that Player A's position is worth 32 pistoles right after he loses the first round and therefore 48 pistoles at the outset. Fermat solved the problem in a different way; he supposed that two rounds are played no matter how the first comes out, so that the four equally frequent chances are

> Player A wins the first round, Player A wins the second round;
> Player A wins the first round, Player B wins the second round;
> Player B wins the first round, Player A wins the second round;
> Player B wins the first round, Player B wins the second round.

Because Player A wins the 64 pistoles in three out of the four equally frequent chances, he is entitled to three-fourths of the 64 pistoles at the outset. Pascal reports to Fermat that Roberval objected to the fiction that the players would play two rounds no matter how the first came out. He then reports what he said to Roberval, including the following:

> I responded to him that I relied not so much on this method of combinations, which was not really appropriate for the problem, as on my other universal method, which misses nothing and carries its demonstration in itself, and which finds precisely the same division as the method of combinations …

In other words, the method of combinations is correct only because its results agree with Pascal's method of backward recursion.

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22 Pascal did not draw any such diagram; historically the first appearance of such a diagram seems to be in an unpublished note written by Huygens in 1676. See pp. 151-155 of Volume XIV of [24] and pp. 380-384 of [25].

The method of combinations does not carry its demonstration in itself, because its counting of chances relies on experience. To see the full force of Pascal's argument, we need to notice that the appeal to experience becomes less and less convincing as the number of rounds becomes greater and greater. Do we really have enough experience to know that the 6^{10} ways 10 throws of a die can come out have equal frequency?

4. Huygens's game-theoretic foundation

Although neither Pascal nor Fermat published their work on games of chance, the problems they had discussed soon became widely known through the work of Christiaan Huygens (1629-1695). Son of the prominent Dutch diplomat and poet Constantijn Huygens, Christiaan Huygens was steeped in French culture, but his first visit to Paris was delayed by the turmoil of the times until 1655. During that visit, he learned something about Pascal's and Fermat's ideas from their Parisian colleagues.

From what he heard in Paris, Huygens saw an opportunity to apply the new understanding of algebra that he had learned from Descartes through his teacher Francis van Schooten, and this led him to write an account of calculation in games of chance that Van Schooten could publish as an appendix to his forthcoming textbook on algebra. He drafted it in Dutch in 1656. Van Schooten translated it into Latin for the Latin version of his textbook, which appeared in 1657. The Dutch version appeared in 1660.[23]

By casting the matter in terms of algebra, Huygens deepened Pascal's foundational argument, making it more game-theoretic. Instead of relying on the principle that chance gives contending players equal claims, Huygens's argument relies merely on the players' willingness to contend on equal terms.

What did Huygens learn in Paris?

What did Huygens learn from the Paris mathematicians about the problems Pascal and Fermat had discussed? In an insightful article published in

23 Discussion between Huygens and Van Schooten concerning the translation is preserved in letters published in [26].

1982,[24] Ernest Coumet called attention to three letters written by Huygens that cast light on this question. In a letter to Van Schooten dated 20 April 1656, Huygens wrote:

> Here is what you wanted concerning games of chance … You can judge the difficulty of this material from the fact, among others, that Pascal, a young man with the most penetrating mind, said that he had never encountered anything so obscure, and that nothing had ever required more effort from him. For his part, he certainly went deeply into the questions I consider, or most of them, as did Fermat. But what principles did they rely on? I think no one yet knows.

On May 6, Huygens wrote again to Van Schooten:

> It would be appropriate to put at the beginning, as a preface, a letter from me giving some explanations about the material itself and who first undertook to study it, along with what I learned in France about Pascal's discoveries in this domain. Very little I suppose, but just the same I don't think I can conceal it.

Then on July 21, Huygens wrote to the English mathematician John Wallis:

> I have recently used demonstrations of this type [by algebra] in a treatise on the use of calculation in matters of chance, which Van Schooten proposed to publish with his own work, now being published. I came by the opportunity in France, where mathematicians had asked me questions like this: In how many throws can one expect to get a six with a die like those now usually used? Or to get a double six with two dice? And many more of the same type, for whose solution it was not at all easy to find the first principles.

The tone of these passages suggests that calculation in games of chance was not a surprising topic for mathematicians in the 1650s. Huygens did not learn much that was new in Paris. We also see that Huygens, like Pascal, was a seeker after first principles.

We may surmise that the Paris mathematicians who posed the questions to Huygens could answer some or all of them, and that they remembered that Pascal had a way of justifying the answers that went deeper than counting

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[24] [27], reprinted on pp. 437-452 of [28].

chances, but that they had never fully understood Pascal's arguments. Perhaps Pascal never fully explained everything he could do with his two principles.

From April 1656 to March 1657, Huygens corresponded with Pierre Carcavy and the Paris mathematicians Roberval and Claude Mylon. The correspondence with Carcavy put him in indirect touch with Pascal and Fermat, who both provided additional questions that he added to his treatise with answers but without solutions. Coumet saw in this correspondence a further attempt on Huygens's part to learn Pascal's and Fermat's first principles, ultimately unsuccessful because he did not ask his questions directly, not wanting to reveal how much or little he himself already understood.

Huygens's preface did acknowledge that renowned French mathematicians had worked on his topic. He added that

> though they have tried to solve many a difficult question by corresponding with each other, they have concealed their own mode of invention. I, therefore, was obliged to examine everything from the beginning to the end and am not yet sure that the point whence we started was the same.

This passage can be taken as a claim by Huygens he did not learn anything from the Parisians about how to *faire les partys*, but Huygens's letters to Van Schooten and Wallis support the skepticism about such a claim that has been expressed by Coumet, Edwards, and Schneider. Coumet asks whether we may be misunderstanding Huygens's words. Did his 17th century audience read "mode of invention" (*manier van uytvinding*) as merely a way of finding an answer or, as the sentence following it might suggest, something deeper?

We may also ask whether Huygens was really unaware of what the authors of commercial arithmetic had said about how to *faire les partys*, Ivo Schneider argues that the form of *De Ratiocinniis* (formal propositions, with full explanations, followed by problems with numerical answers but no explanations) suggests familiarity with the work of the reckoning masters.[25]

[25] [29], p. 182.

The first sentence of *De Ratiocinniis*, in its Latin version at least, suggests that the treatise is concerned with games that depend only on chance. Here is the sentence in Latin:

> Etsi lusionum, quas sola sors moderator, incerti solent ese eventos, attamen in his, quanto quis ad vincendum quam perdendum propior sit, certam Semper habet determinatiionem.

Here it is in Dutch:

> Al-hoewel in de spelen, daer alleen het geval plaets heeft, de uytkomsten onseecker zijn, soo heeft nochtans de kansse, die yemandt heeft om te winnen of te verliesen, haere seeckere bepaling.

The editors of Huygens's complete works translate this somewhat archaic Dutch into French in a way that makes it agree with the Latin:

> Quoique dans les jeux de hasard pur les résultats soient incertains, la chance qu'un joueur a de gagner ou de perdre a cependant une valeur déterminée.[26]

The French is easily translated into English:

> Although outcomes in games of pure chance are uncertain, the chance a player has to win or lose nevertheless has a definite value.

To the extent that he was concerned with games of pure chance, Huygens was following Pascal. But whereas Pascal emphasized that he was considering only games of pure chance in order to justify his second principle, Huygens makes an argument that applies equally well to the mixed games that had been considered by authors in the tradition of the commercial arithmetics.

Using algebra

Huygens invented his own first principles, and they went deeper than Pascal's. A concise and insightful explanation of Huygens's principles was provided by Hans Freudenthal in 1980.[27] Here is Freudenthal's translation from the Dutch of Huygens's first proposition.

26 [24], p. 60.
27 [30]

PROPOSITION I. If I have the same chance to get a or b it is worth as much to me as $(a + b)/2$.

In order not only to prove but also to discover this rule, I put x for what the chance is worth to me. Hence having x I must be able to arrive at the same chance by an equitable game. Let it be the game which I play against another with stake x, where the other is also staking x; and let it be agreed that the one who wins shall give a to the one who loses. This game is equitable, and it appears that by this I have an equal chance to win a, that is, even if I lose the game, or $2x - a$ if I win, because then I get the stakes $2x$ from which I must give the other a. Suppose that $2x - a$ were as much as b, then I would have the same chance for a and b. So I put $2x - a = b$, and it follows that $x = (a + b)/2$ for the value of my chance. The proof of this is easy, because having $(a + b)/2$, I can venture against another who will also stake $(a + b)/2$, with the stipulation that the one who wins the game shall give a to the other. Therefore I will have an equal chance to get a, that is to say if I lose, or b if I win, because then I take $a + b$, which is the stake, and from this I give him a.

Huygens begins with the principle of fairness that the players must be treated the same. If two players both put up $(a + b)/2$, and the winner gets a and the loser gets b, then the two players are being treated the same.

Huygens's proof of his first proposition is a nice illustration of the new role of algebra in the 17th century. As Fermat had learned from Vieta and Van Schooten had learned from Descartes, you can use algebraic equations to discover solutions to geometric or physical problems, but to achieve certainty you must translate this discovery into a proof in the style of the Euclid and the other ancients.[28] In Huygens's proposition, as in geometry, the synthetic proof has a constructive character. It says that $(a + b)/2$ is the right value because I can use this amount to reconstruct my position. In the contemporary language of finance, it is the cost of hedging the position.[29]

From his first proposition, Huygens moved quickly to his third proposition:

PROPOSITION III. If the number of chances I have for a is p, and the number of chances I have b is q, then assuming that every chance can happen as easily, it is worth to me as much as $(pa + qb)/(p + q)$.

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28 [31,32]

29 [33]

Here is the synthetic version of Huygens's proof: a fair arrangement where I risk $(pa + qb)/(p + q)$ to get p chances for a and q chances for b. Consider a game where I and $p + q - 1$ other players each have an equal chance of winning. Each player puts up $(pa + qb)/(p + q)$ and the winner takes it all; this is evidently fair. I make a fair side bet with each of q of my opponents: if one of us wins, he will give the other b. I also make a fair side bet with each of my remaining $p - 1$ opponents: if one of us wins, he will give the other a. If one of the q opponents wins, I end up with b. If one of the $p - 1$ opponents wins, I end up with a. If I win, I get the $(pa + qb)/(p + q)$ put up by each other $p + q$ players, myself included, but I pay b to q opponents and a to $p - 1$ opponents, netting

$$(p + q)\,(pa + qb)/(p + q) - qb - (p - 1)a = a.$$

So I have p chances for a and q chances for b.

Here Huygens has done something left undone by Pascal. He has derived from first principles a general rule for calculating from equally possible chances how stakes should be fixed. In the writings Pascal left behind, we see this done only for the case where there are only two equal chances; Pascal called this a "lemma" in his *Usage*.

Ivo Schneider has raised an objection to Huygens's argument. Huygens's fundamental principle is that players should be treated alike. But here one player gets to set the bets. He arranges side bets with many players, and as a result his position is different from that of the others.[30] As this objection illustrates, Huygens's notion of fairness is not defined with mathematical precision. We cannot say that Huygens has a game-theoretic foundation that meets the standards of rigor of modern game theory, in which the rules of play are clearly specified.

Huygens also uses algebra in his last proposition, which I paraphrase as follows:

> PROPOSITION XIV. Player A and Player B take turns throwing two dice. Player A wins if he throws 7 points before Player B throws 6 points. If Player B throws first, what is the ratio of their chances?

Whenever it is Player A's turn to throw, he has 6 chances out of 36 to win on that throw; whenever it is Player B's turn, he has 5 chances out of 36 to

30 [29].

win on that throw. Huygens wrote a for the stakes for which they are playing; let us simplify by setting $a = 1$. Huygens wrote x for the value of Player A's chance at the outset and y for the value of his chance if and when he gets to throw again, after Player B has lost his first throw. At the outset, Player B has 5 chances to win and 31 chances to put Player A in the position where the value of his chance is y. So

$$x = (5/36) \times 0 + (31/36) \times y.$$

If Player B loses his first throw, then Player A has 6 chances of winning on his first throw and 5 chances of returning to x. So

$$y = (6/36) \times 1 + (30/36) \times x.$$

Solving the two equations, we find that $x = 30/61$. So the ratio of Player A's chance to Player B's is 30 to 31.

This argument may have seemed a little intricate at the time, but it is an impressive advance on what medieval mathematicians could do. Player A first throws two dice, and if he loses Player B throws two dice again. So solving the problem by the rule of three requires somehow considering the 1296 chances for the result of throwing four dice.

5. Back to frequency

Laplace got it right in 1814. The calculus of games of chance, in a mathematically rudimentary form, goes back centuries if not millennia before Pascal and Fermat. Born from the experience of dice players, this calculus had always been a calculus of frequencies. So it is not surprising that Pascal's and Huygens's game-theoretic foundations quickly disappeared, pushed aside with little ado by the deeply entrenched concept of equally frequent chances.

Huygens's immediate successors in the development of the calculus of probability were Montmort, De Moivre, and Bernoulli. Each, in his own way, favoured and developed Fermat's method of combinations, not because it was Fermat's method, but because it was everyone's method.

Montmort

Pierre Rémond de Monmort (1678-1719) published his own book on games of chance in French in 1708.[31] Montmort explains that he learned the elements of the subject from Pascal's *Usage*, but he relies primarily on the method of combinations. He ignores Pascal's and Huygens's foundational arguments, returning to the rule of three to argue that if a player has m chances out of $m + n$ to get A, his expectation (*sort*) is $mA/(m + n)$.[32] Accused by De Moivre of following Huygens (because he had used some algebra), he denied having learned anything important from Huygens, dismissing what he called Huygens's "lemma" as mere common sense.[33] (Here he was probably referring to Huygens's Proposition III, perhaps also confusing it with Pascal's lemma.)

De Moivre

Abraham De Moivre (1667-1754) published a far-reaching article in Latin in 1711.[34] He began with two principles. First, if two players contend for the sum a, and p out of $p + q$ chances favor the event that the first player wins, then his expectation is worth $pa/(p + q)$. Second, multiplication is used to find the numbers of chances for events that have no dependence on each other. These are essentially the medieval principles, updated by explicit reference to multiplication and to the concept of an event. De Moivre later explained that he had learned the elements from Huygens, but that he was determined to use combinations rather than Huygens's (algebraic) method.

Bernoulli

Jacob Bernoulli (1655-1705) worked on probability well before Montmort and De Moivre, but his book on the topic was published after his death, in 1713.[35] The book begins by reproducing Huygens's treatise with commen-

31 [34,35]
32 [34], pp. 3-4; [35], pp. 75-76.
33 [35], p. xxx.
34 [36,37]. See also [38,39].
35 [40,41]

tary. But Bernoulli then turns to the method of combinations. His commentary on Huygens's first three propositions suggests that he does not find Huygens's constructive argument necessary. He suggests that it can be replaced by reasoning that is "more popular" and "more adapted to common comprehension", using merely the assumption that two players together are sure to win the entire stakes and should be treated equally. He then notes that Huygens's rules are analogous to the rules for mixtures used in business mathematics.[36]

6. Conclusion

George Sarton saw the history as essential to a scientist's understanding of his subject. As he once wrote,

> to understand and to appraise at its just value what one possesses, it is well to know what the people possessed who came before us; this is as true in the domain of science as it is in daily life. It is his historical knowledge that discloses to the scientist his precise attitude toward the problems with which he has to grapple, and that enables him to dominate them.[37]

Marie-France Bru and Bernard Bru have made the same point with these words:

> To penetrate to the reasons of things, look at how they have gradually been revealed in the course of time, in their progression and in their ruptures …[38]

Since the time of Laplace, successive students of probability have pursued this historical method in their quest for a clearer understanding of probability. Here are some thoughts about how the preceding perspectives on Pascal, Fermat, and Huygens can help us with contemporary puzzlements.

Conceptual revolution?

The legend of Pascal and Fermat was embellished yet further in the 1970s, when philosophers and historians reviewed the history of science in search for examples of conceptual change. Most famously, Ian Hacking argued

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36 [41], pp. 134, 138.
37 Quoted by Stadler [42], p. 74.
38 [8], p. 287.

that the correspondence between Pascal and Fermat marked the emergence of a dual concept of probability, combining belief and frequency. In 1975, Hacking wrote:

> Probability, as we now conceive it, came into being about 1660. It was essentially dual, on the one hand having to do with degrees of belief, on the other, with devices tending to produce stable long-run frequencies.[39]

This thesis of a conceptual revolution for probability has been widely repeated and further embellished, in both scholarly and popular contexts. Here, for example, is an assessment offered by Keith Devlin, a widely read writer on mathematics:

> The Pascal-Fermat correspondence showed that it is possible to use mathematics to see into the future.[40]

The history recounted in this article suggests a greater conceptual continuity. In the case of dice at least, Pascal and Fermat connected frequency with betting on the future in the same way as the author of *De Vetula* had 400 years earlier, and we have every reason to suppose that dice players had been making the same connection for millennia.

The advances that we see in Pascal's and Fermat's reasoning, then in Huygens's treatise and the following work by Montmort, De Moivre, and Bernoulli, are primarily advances in mathematics, not conceptual changes. These scholars' increasing facility with numbers made it possible for the first time to fix stakes and bets in games that were, as Laplace put it, even a little complicated. Perhaps Pascal's and Fermat's most important contribution was to offer to Huygens the more difficult problems that he stated at the end of his treatise, with answers but without explanations. Montmort, De Moivre, and Bernoulli all began their work on probability by solving these problems.

The arguments advanced by Pascal and Huygens did contain the seeds of a conceptual revolution, one that retained the role of fairness but replaced frequency with reasoning about the structure of the game. But this was an aborted revolution, because the connection between frequency and betting was so firmly entrenched.

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39 [43], p. vi.
40 [44], p. 164.

The most important conceptual development spurred by Pascal and Huygens was the ambition that their mathematical successes awoke, in Bernoulli and his 18th century successors, to extend their calculations from games of chance to other problems of uncertainty, thus making the calculus of chances a calculus of probability. But this is another story.

Modernizing the game-theoretic foundation

In the 20th century, mathematical probability became pure mathematics. Attribution of meaning to its terms is now an exercise undertaken after the theory is first developed by pure reason, without any intrusion of ideas about fairness or frequency.

The pure mathematics of probability can be developed either measure-theoretically or game-theoretically. The measure-theoretic development is an abstract generalization of the counting of chances; probabilities and the corresponding expected values being taken as given.[41] The game-theoretic development is an abstract generalization of Huygens's picture of a player who is allowed to construct betting strategies.[42] In both developments, frequencies enter the picture through Bernoulli's theorem and its many generalizations. In the measure-theoretic development, these theorems say that basic probabilities (which play the role of the classical equal chances) will be approximated by frequencies with high probability. In the game-theoretic development, they say that the player has a strategy that multiplies the capital he risks by a large factor if the approximation fails. The measure-theoretic development can then be connected with frequencies in the world through the presumption that easily specified events with high probability will happen, while the game-theoretic development makes the same connection through the presumption that simple betting strategies will not succeed.

The modern game-theoretic formulation begins with a game in the sense of modern game theory, defining players, rules for play, and a rule for who wins. This takes us away from Huygens's and Pascal's notion of fairness as symmetric treatment of players. As Schneider's objection to Huygens

[41] Usually cited in this connection is Andrei Kolmogorov's *Grundbegriffe der Wahrscheinlichkeits-rechnung*, [45]. See also [46].

[42] See [47].

shows, the notion of a strategy for betting fits awkwardly with such symmetry. The game we need has instead one player who gives odds or prices, another who is allowed to choose how to gamble at those odds or prices and can therefore construct strategies, and another who decides outcomes.[43]

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