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THE COMPUTATION OF LOGARITHMS BY HUYGENS

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In 1627, after nearly three decennia of arduous computation, the first complete 10-place table of logarithms for the basis 10 was published [1]. It allowed to find logarithms up to 10 places at most. Huygens, born in 1669, indicated in 1661 how one could compute the logarithm of *any* number, with *any* accuracy wished for *any* basis. In fact he approximated the logarithm by a *rational function.*

Huygens refused to apply calculus and restricted himself to the use of infinitesimals. Indeed his geometrical methods were, for most of the then actual problems, much more powerful than calculus could provide. Whenever Huygens wished to compute some mathematical function or a quantity he reduced the problem to funding the area below a curve. Thus for a logarithm one has the relation for an increment D of the argument x and looks for the function $y(x)$ such that

$$yD = \mathrm{Log}\,(x+D) - \mathrm{Log}\,x = \mathrm{Log}\,(1+D/x).$$

A difficulty in determining the derivative of a logarithm was the discontinuity at $n=0$ of the variant

$$(1+n)^{1/n},$$

which for *n tending* to zero approaches a limit, which is different from what one obtains in *putting* $n=0$, leading to an infinite power of 1, and thus to unity. Huygens plotted a curve showing negative powers of 2 and — if one sees his manuscripts — drew with emphasis a tangent to this curve at the point (0, 1), which turning the drawing over 90°, shows a logarithmic curve with a tangent at (1, 0). Thus we have

$$\mathrm{Log}\,(1+D/x) = MD/x,$$

and dividing by the infinitesimal D, the curve searched for is found:

$$xy = M.$$

Computing the area below such a curve leads to a logarithm with *as yet* an unknown basis. One has to divide by $\mathrm{Log}\,B$ for this same unknown basis in order to obtain logarithms for the basis B. Huygens chooses $M=1$ for his "fundamental logarithms", which are then natural logarithms.

Huygens determined several times an area by means of the properties of a centre of gravity. For the equilateral hyperbola

$$x^2-y^2=1$$

the abscissa X of the centre of gravity of a segment with area F, delimited by the line of points with abscissa x, leads to

$$XF=\Sigma\, 2yxDx=\Sigma\, 2y^2Dy=(2/3)\,y^{3/2}=(2/3)(x^2-1)^{3/2},$$

as is evident from Archimede's result in the "Squaring the parabola".

On the other hand, the area of the triangle with the same basis and the same "vertex" is

$$T=y(x-1)=(x^2-1)^{3/2}/(x+1)$$

and finally

$$3XF=2(x+1)T.$$

The *exact* position of the centre of gravity leads to an *exact* value of the logarithm. Huygens remarks that the centre of gravity will be not much different from the centre of gravity of a parabola having the same basis and the same vertex as the segment of the hyperbola, and using the known position for the centre of gravity of a parabola, dividing the sagitta in a ratio 3:2, one has to compute the area T of the triangle, multiply that into known constants and to subtract it from the area of a trapezium in order to have the approximate value of the logarithm. This leads to the formula in modern symbols

$$\operatorname{Log} n \sim \frac{n-1}{n}\;\frac{36n-n^2-1+12(n+1)\sqrt{n}}{18(n+1)+24\sqrt{n}}.$$

Putting here $n=1+x$ and approximating the square root by a (part of a) binomial series, this rational function of n leads to [2]

$$\operatorname{Log}(1+x)=x-x^2/2+x^3/3-x^4/4+x^5/5-x^6/6+(3199/3200(x^7/7)$$
$$-(799/800)(x^8/8)$$

from which it is clear that for small x the error is about

$$-x^7/22400+x^8/6400.$$

Huygens wished to have Briggs' logarithms and therefore didn't need to consider the basis of his logarithms he computed in the areas, as he simply had to divide all values by the logarithm of 10.

Remark 1. Just as Huygens explicitly indicated in other solutions of the problems, he could have shown that his computation yields always too small values. Making the points of intersection with the line $x=a$ for $y_1^2=x^2-1$ and $y^2=2p(x-1)$ the same one finds $2p=(a+1)$ and then follows $y^2/y_1^2=(a+1)(x-1)/(x^2-1)=(a+1)/(x+1)>1$, from which it is clear that the centre of gravity of the hyperbola is nearer to the basis than that of the parabola, which means that Huygens' formula always gives *too small* a value of the logarithm.

Remark 2. The accuracy of the formula can easily be checked. Taking $n=1.21$ there is *no rounding off* of intermediary values and the result is

$$\text{Log } 1.21 = 15.264459/80.0778 = 0.19062035902 \ldots$$

Again taking $n = 1.1$, with *one* square root follows

$$\text{Log } 1.1 = 0.0953101798012 \ldots$$

and the double of this value deviates by 5.8×10^{-10} from Log 1.21 which was computed.

One can easily understand Huygens' enthusiasm for this method of computing logarithms and his announcing that his method is "much shorter than those which were applied till now" ... and he indicates that one needs at most 6 square roots ... It is, however, clear that for the logarithms this is *much more* than is needed:

$n = 2$, $\text{Log } 2 = 121.9116882/175.8822510 = 0.6931437$, error 3.4×10^{-6} obtained with only *one* square root.

From the fourth and eighth roots, respectively, 1.1892071551 and 1.090507733 one finds a quarter of Log 2 as 0.1733867949, from which Log 2 = 0.69314717968 ..., error 8.8×10^{-10}. This shows that Huygens' indication as to the number of roots needed for the computation of Logarithms is by far too high.

We think that this indication arose by the *inverse* problem, the computation of "antilogarithms". If one has given a quantity $\text{Log } a_1$ and wishes to determine a_1, one can choose a simple value near to a_1, say $a_1 = a + D$, and then

$$u = \text{Log}(a + D) - \text{Log } a = \text{Log}(1 + D/a) \sim D/a - (D/a)^2/2$$

leads, following Halley's well-known procedure by solving a quadratic [3] equation, to

$$D \sim a(u + u^2/2 + u^3/8 \ldots).$$

For the determination of the basis of Huygens' logarithms one chooses e. g.*

$\sqrt{n} = 1.01575$, thus $n = 1.03174\ 80625$ and n^2, also known with all digits, exactly to be 1.06450406447250390625, and then the formula of Huygens yields

$$\text{Log } 1.03174\ 80625 = 0.03125\ 45117\ 71396\ 8924 \ldots$$

$$= 0.03125 + 4.5117713968924 \times 10^{-6}.$$

Therefore the 32nd root of *Euler's e* is found to be

$$1.0317480625 - 4.65501139714 \times 10^{-6} + 1.050117 \times 10^{-11}$$

$$= 1.03174940749910403 \ldots, \text{ error } 1.26 \times 10^{-15}.$$

For this computation and with this accuracy one has to consider the 32nd roots, which is just *one root less* than Huygens' cautiously indicated for logarithms ... whereas it holds true for the supplementary computation of antilogarithms, because the logarithm of the 64th root of 10, viz.

* This first approximation in obtained by the square root of $1 + 0.03125 + (0.03125)^2/2 \sim 1.015745$.

1.036632928 . . . leads to an error of 3.9×10^{-15}, giving still six places more than Huygens wished to guarantee . . . and splitting off factors 10 one never has to consider quantities greater than 10.

REFERENCES

1. E. de Decker. Tweede Deel van de Nievwe Tel-Konst. Gouda, 1627.
2. E. M. Bruins. (comp.) Janus LXV, 1978, p. 100; Janus LXVII, 1980, p. 256.
3. E. Halley discussed this method in detail. *Phil. Trans.*, 1694.

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