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The existence of minimal immersions of 2-spheres

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In this paper we develop an existence theory for minimal 2-spheres in compact Riemannian manifolds. The spheres we obtain are conformally immersed minimal surfaces except at a finite number of isolated points, where the structure is that of a branch point. We obtain an existence theory for harmonic maps of orientable surfaces into Riemannian manifolds via a complete existence theory for a perturbed variational problem. Convergence of the critical maps of the perturbed problem is sufficient to produce at least one harmonic map of the sphere into the Riemannian manifold. A harmonic map from a sphere is in fact a conformal branched minimal immersion.

We prove the existence of minimizing harmonic maps in two cases. If N is a compact Riemannian manifold with $\pi_2(N) = 0$, then every homotopy class of maps from a closed orientable surface M to N contains a minimizing harmonic map (Theorem 5.1). This has also been shown by Lemaire [L4] and Schoen and Yau [Sch-Y]. If $\pi_2(N) \neq 0$, then there exists a generating set for $\pi_2(N)$ consisting of conformal branched minimal immersions of spheres which minimize energy and area in their homotopy classes (Theorem 5.9). Our main result is the proof of the existence of a conformal branched minimal immersion of a sphere when the universal covering space of N is not contractible (Theorem 5.8). When $\pi_2(N) = 0$ this cannot be minimizing for the energy in the single homotopy class of maps. An important tool that is developed is a regularity theorem due to Morrey in the case that the harmonic map is minimizing [MO1]. In our version, Theorem 3.6, a harmonic map with finite energy from the punctured disk into N is C^∞ and harmonic in the entire disk.

The outline of the paper is as follows: The first section contains a discussion of the properties of harmonic maps from any compact orientable surface M into a Riemannian manifold N. The second section describes the properties of the perturbed problem. Section Three contains the main *a*

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priori estimate needed for convergence and for proving our regularity theorem, Theorem 3.6. Section Four describes the convergence properties of the perturbed problem, and finally in Section Five there is a collection of results on harmonic maps and minimal spheres.

The difficulties which will arise in our construction of harmonic maps are best illustrated by the case $M = S^2$. Here we are trying to parametrize conformally by the standard sphere geometric objects representing minimal spheres. There are several difficulties which are obvious. Firstly, the conformal parametrization by the standard sphere is not unique. The group of conformal transformations of S^2 is the group of linear fractional transformations, which is not compact, so that the set of critical maps of the energy integral on $C^1(S^2, N)$ must be noncompact. In some way it is necessary to make a choice of parametrization. This problem is solved by our perturbation technique. The perturbed integral is not quite invariant under conformal transformations of the sphere, and prefers a parametrization which is carried over when the limit of the critical maps of the perturbed integral is computed.

Furthermore, once we have one minimal sphere we have many. Given $s: S^2 \to N$ harmonic and $f: S^2 \to S^2$ any meromorphic function, then $s \circ f: S^2 \to N$ is harmonic and will also be found by our techniques. This is a similar problem to the one arising from the fact that coverings of a closed geodesic also count as closed geodesics, although there is really only one geometric object represented. We can assume that it will be harder to count the number of primitive minimal spheres than it is to count the number of primitive closed geodesics. The most severe difficulty, however, seems to be of the following type:

Assume that $\pi_1(N) = 0$ and $\pi_2(N)$ has at least two generators γ_1 and γ_2, and try to minimize the energy and area of the image of maps $s: S^2 \to N$ in every connected component of the mapping space $C^1(S^2, N)$. Conceivably we may find a sphere with minimal energy and area in each of the connected components of $C^1(S^2, N)$ corresponding to γ_1 and γ_2. A natural candidate for the minimal area map from $S^2 \to N$ in the component of $C^1(S^2, N)$ corresponding to $\gamma_1 + \gamma_2$ is a map with image consisting of the two spheres already found connected by a one-dimensional bridge. Of course there is a map of S^2 into this object, but not a conformal one. Thus we would not expect to be able to minimize the energy in this particular component. Minimal spheres connected by minimizing geodesics are likely to arise in any method of trying to establish a Morse theory for minimal spheres, but these objects cannot be conformally parametrized by spheres.

One can argue that one shouldn't expect to find minimal spheres corresponding to every element in $\pi_2(N)$, but only to some of them, and $\gamma_1 + \gamma_2$ is the wrong one to choose. However, it is very hard to construct a convergence scheme for producing critical maps which sometimes converge and sometimes diverge, and which can be shown to converge at least once. The existence of peculiar representatives of minimal spheres makes it more difficult to find the nice ones.

A rough description of our technique for finding harmonic maps follows. We find the critical maps of a perturbed energy integral for $\alpha > 1$ (E_1 is the usual energy integral plus a constant):

$$E_\alpha(s) = \int_M (1 + |ds|^2)^\alpha d\mu ,$$

and check the convergence of these maps as $\alpha \to 1$. E_α satisfies Ljusternik-Schnirelman theory and a Morse theory if $\alpha > 1$. In what sense do the critical maps s_α of E_α approximate harmonic maps? The conformal invariance of the unperturbed energy integral and the approximate conformal invariance of E_α for α near 1 come into play here. For α near 1, the map s_α is near a (possibly trivial) harmonic map $s_0: M \to N$ except on a finite number of small disks $D_i \subset M$ whose radii go to zero as $\alpha \to 1$. Each of these disks D_i should be thought of as expanded conformally to cover almost all of S^2, and on each D_i, s_α is near a harmonic map $s_i: S^2 \to N$. Actually, the process can repeat, with s_α near s_i on D_i, except on a finite set of small disks $D_{ij} \subset D_i$, each of which is conformally almost S^2, and on D_{ij} where s_α is near a harmonic map $s_{ij}: S^2 \to N$. This process repeats. There is a bound on the number of harmonic maps produced in this way which is given by the bound on the energy. There is evidence that in the limit the object connecting the image of the different harmonic maps $s_0: M \to N$, $s_i: S^2 \to N$, $s_{ij}: S^2 \to N$ and so forth should be geodesics. The domain for a map into the limit as $\alpha \to 1$ should be M connected to a sequence of spheres by curves. One can see the difficulty of keeping track of the entire set of limits of s_α as $\alpha \to 1$.

One can show that this complicated type of convergence actually occurs by looking at radially symmetric critical maps of $\bar{E}_\alpha(s) = \int_M |ds|^{2\alpha} d\mu$ on $C^1(M, S^2)$ for M a disk or S^2. Computations along these lines are found in the master's thesis of G. Schwarz [S].

The perturbed integral $\bar{E}_\alpha(s) = \int_M |ds|^{2\alpha} d\mu$ is in many ways easier to deal with than the perturbed integral $E_\alpha(s) = \int_M (1 + |ds|^2)^\alpha d\mu$ because of the invariance of the former integral under expansion from small disks to large disks, which makes explicit calculations easier. However, at some point we need the uniform ellipticity of the Euler-Lagrange equations for the perturbed integral. Hence our choice of $E_\alpha(s) = \int_M (1 + |ds|^2)^\alpha d\mu$.

The technique developed in this paper makes essential use of a conformal parametrization of a minimal sphere. Therefore the method cannot be extended to cover high dimensional minimal volume problems. Nor will the method extend to yield existence theorems for harmonic maps from manifolds of dimension larger than 2. A corresponding theory for a conformally invariant integral is available for domain manifolds of dimension greater than 2, but the images of the critical maps in this case will not be nearly as interesting as minimal surfaces. It is, however, possible to approach the existence question for minimal surfaces which are not spheres, with or without boundary, using some of the results in this paper: see [S-U] and and [Sch-Y].

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1. Harmonic maps from surfaces

Let M denote a compact orientable surface with a given conformal structure and N a C^∞ Riemannian manifold without boundary of dimension greater than or equal to 2. We shall find it technically convenient to assume that $N \subset \mathbf{R}^k$ is a C^∞ isometric imbedding. From the Nash imbedding theorem we know that such an imbedding can always be constructed for sufficiently large k. Assume from now on that M has been given a Riemannian metric compatible with its conformal structure and that this metric induces the measure $d\mu$ on M. Let $L_1^p(M, N)$ be the Sobolev space of maps $s: M \to N$ whose first derivatives lie in L^p.

Definition 1.1. A map $s \in L_1^2(M, \mathbf{R}^k) \cap C^0(M, N)$ is *harmonic* if it is a critical point of the energy integral $E(s) = \int_M |ds|^2 d\mu$. If $N \subset \mathbf{R}^k$ has second fundamental form A, then the Euler-Lagrange equations have the form

$$\Delta s + A(s)(ds, ds) = 0\ . \tag{1}$$

LEMMA 1.2 [MO1], [U]. *If s is harmonic then $s \in C^\infty(M, N)$.*

LEMMA 1.3 [E-S2]. *E is a conformal invariant of M.*

LEMMA 1.4 [E-S2]. *If s is a conformal immersion, then s is harmonic if and only if $s(M)$ is a minimally immersed surface.*

Let ϕ be the quadratic differential which in a local isothermal parameter $z = x + iy$ on M is defined by

$$\phi = \{|s_x|^2 - |s_y|^2 + 2i(s_x, s_y)\}dz^2 .$$

We call s *weakly conformal* if $\phi \equiv 0$.

LEMMA 1.5 [C-G], [L1]. *If s is harmonic then ϕ is holomorphic.*

For the definition of *branched immersion* and the proof of the next theorem we refer the reader to Gulliver-Osserman-Royden [G-O-R].

THEOREM 1.6. *That s is harmonic and weakly conformal implies s is a branched immersion.*

COROLLARY 1.7. *If $s: S^2 \to N$ is harmonic and dimension $N \geqq 3$, then s is a C^∞ conformal branched minimal immersion (see also* [C-G]).

Proof. The result follows from Lemmas 1.2, 1.4 and 1.5 and Theorem 1.6, together with the fact that there are no nontrivial holomorphic quadratic differentials on S^2.

The situation for surfaces of genus larger than zero is complicated by the fact that they have many possible conformal structures. The following theorem gives a sufficient condition for a harmonic map from such a surface to also be a minimal immersion.

THEOREM 1.8. *If s is a critical map of E both with respect to variation of s and the conformal structure on M, then s is a conformal branched minimal immersion.*

Proof. First we show that s is critical with respect to all variations in the metric. Let $g(t)$ be a variation of the metric $g = g(0)$ on M. Every such variation arises from a composition of

(a) the pull-back of g by a C^∞ family $\sigma(t)$ of orientation preserving diffeomorphisms of M,

(b) a smooth curve in the Teichmüller space for the genus of M, and

(c) a family of conformal changes in the metric.

See e.g., [EA-E]. By hypothesis and Lemma 1.3 it is sufficient to show that s is critical for variations of type (a). To this end let $d\mu(t)$ be the measure on M induced by $\sigma^*(t)g$ and let u be the variation of s given by $u = d/dt(s \circ \sigma(t)^{-1})|_{t=0}$. Then, identifying any metric on TM with the canonically induced metric on T^*M, we have

$$\frac{d}{dt}\int_M (\sigma^*(t)g)(ds, ds)d\mu(t)\Big|_{t=0}$$

$$= \frac{d}{dt}\int_M g\big(d(s \circ \sigma(t)^{-1}), d(s \circ \sigma(t)^{-1})\big)d\mu\Big|_{t=0}$$

$$= dE_s(u) = 0 ,$$

since s is harmonic in the metric g.

Now we show that s is weakly conformal by proving that the holomorphic quadratic differential ϕ of Lemma 1.5 vanishes identically on any isothermal chart U. Let z be a local isothermal (with respect to g) parameter on the coordinate chart U. Let $g(t) = (g_{ij}(t, z))$ be a variation of g supported in U and $(g^{ij}(t, z)) = (g_{ij}(t, z))^{-1}$. We can assume that $(g_{ij}(t, z)) = (\delta_{ij})$ when $t = 0$ or for z near ∂U and that $g_{11}(t, z) = g_{22}(t, z) = 1$ for all (t, z): Then

$$\frac{d}{dt}\int_M g(t)(ds, ds)d\mu(t)\Big|_{t=0}$$

$$= \frac{d}{dt}\int_U \left(|s_x(z)|^2 + |s_y(z)|^2 + 2g^{12}(t, z)\right)(s_x(z), s_y(z))\sqrt{1 - (g^{12}(t, z))^2}\,dxdy\,\Big|_{t=0}$$

$$= 2\int_U (s_x(z), s_y(z))\frac{\partial}{\partial t}\, g^{12}(t, z)\,\Big|_{t=0} dxdy = 0 ,$$

by the first part of the proof. Since $\partial/\partial t(g^{12}(t, z))$ can be chosen arbitrarily, $(s_x(z), s_y(z)) = 0$ for z on an open subset of U. The same argument applied to the rotated isothermal coordinates $e^{i\pi/4}z$ yields

$$(s_x(z) + s_y(z), s_x(z) - s_y(z)) = |s_x(z)|^2 - |s_y(z)|^2 = 0$$

for z on an open subset of U. Since its real and imaginary parts vanish on an open set, $\phi(t) = 0$ on all of U. The result follows from Lemmas 1.4 and 1.5 and Theorem 1.6.

2. Properties of the perturbed problem

We approximate the integral E, whose critical points are harmonic maps, by a slightly different integral. For convenience, choose a measure on M so that the area of M equals 1. Let

$$E_\alpha(s) = \int_M \left(1 + \sum_{i=1}^k (ds^i(x), ds^i(x))\right)^\alpha d\mu = \int_M (1 + |ds(x)|^2)^\alpha d\mu .$$

For $\alpha = 1$, $E_1(s) = 1 + E(s)$ has harmonic maps as critical points. For $\alpha > 1$, E_α is well-behaved. The Sobolev space of maps

$$L_1^{2\alpha}(M, N) = \{s \in L_1^{2\alpha}(M, R^k)\colon s(x) \in N\} \subset C^0(M, N)$$

is a C^2 separable Banach manifold for $\alpha > 1$. This, and the following theorem, plus several other basic smoothness theorems can be found in Palais [P3].

THEOREM 2.1. *E_α is C^2 on the Banach manifold $L_1^{2\alpha}(M, N)$ and satisfies the Palais-Smale condition* (C) *in a complete Finsler metric on $L_1^{2\alpha}(M, N)$*

provided that N is compact.

The Palais-Smale condition (C) is as strong a condition on a functional as one can expect to get, but we shall use the following results only:

THEOREM 2.2. (*Palais* [P2]). *If f is a C^2 function on a complete, separable C^2 Finsler manifold L, which satisfies the Palais-Smale condition* (C) *with respect to the Finsler structure, then*

(a) *f takes on its minimum in every component of L,*

(b) *if there are no critical values of f in the interval $[a, b]$, then there exists a deformation retraction*

$$\rho: f^{-1}(-\infty, b] \longrightarrow f^{-1}(-\infty, a] .$$

Now we give a regularity theorem for critical maps of E_α. Although the theorem is true for $\alpha > 1$, we give here a simple proof for $\alpha - 1$ small. We use the same technique later to get local estimates.

PROPOSITION 2.3. *The critical maps of E_α in $L_1^{2\alpha}(M, N)$ are C^∞ if $a > 1$.*

Proof. The Euler-Lagrange equations for a critical map $s: M \to N \subset R^k$ can be written

$$d^*(1 + |ds|^2)^{\alpha-1}ds + (1 + |ds|^2)^{\alpha-1}A(s)(ds, ds) = 0 .$$

Here A is the second fundamental form of the imbedding $N \subset R^k$. By Sobolev, for dim $M = 2$, s is of Hölder class $C^{1-1/\alpha}(M, N) \subset C^0(M, N)$. By Morrey [MO1], Theorem 1.11.1, $ds \in L_1^2(M, N)$. We may now differentiate and rewrite the Euler-Lagrange equations

$$\Delta s + (\alpha - 1)\frac{(d^2s, ds)ds}{1 + |ds|^2} + A(s)(ds, ds) = 0 . \tag{2}$$

If $\alpha - 1$ is small, we have nice inverses for the linear operator $\Delta_s: L_2^4(M, N) \to L_0^4(M, N)$ where

$$\Delta_s u = \Delta u + (\alpha - 1)\frac{(d^2u, ds)ds}{(1 + |ds|^2)} . \tag{3}$$

It follows that $s \in L_2^4(M, N) \subset C^1(M, N)$. The regularity of the non-linear equation (2) is now proved by treating the equation as a linear equation in s with coefficients which are Hölder continuous (although they depend formally on s). The smoothness of solutions follows from Theorem 5.63 of [MO1].

Theorems 2.2 and 2.3 are nearly sufficient to prove the existence of non-trivial maps of E_α for $\alpha > 1$. The only difficulty is that $N_0 = \{s: M \to N: s(M) = y \in N\} \cong N$ is the set of trivial critical maps of E_α for $\alpha \geqq 1$ on which E_α takes its absolute minimum value 1. This submanifold of minima will have to be treated in a fashion similar to the submanifold of trivial minima in the geodesic problem. We shall be using the fact that the homotopy type is the same for all mapping spaces, from $C^0(M, N)$ to $L_1^{2\alpha}(M, N)$ to $C^\infty(M, N)$ (see [P1]).

PROPOSITION 2.4. *Let* $\alpha > 1$. *In every connected component of* $L_1^{2\alpha}(M, N)$ *the minimum value of* E_α *is taken on at some map* $s_\alpha \in C^\infty(M, N)$, *which also minimizes* E_α *in its connected component in* $C^\infty(M, N)$. *There exists a* B *independent of* α *such that* $\min E_\alpha \leqq (1 + B^2)^\alpha$ *in that component.*

Proof. Since E_α satisfies the Palais-Smale condition (C), it takes on its minimum in every component of $L_1^{2\alpha}(M, N)$. Proposition 2.3 implies that the critical maps lie in $C^\infty(M, N)$. In each component we locate a differentiable map u, and let $B = \max_{x \in M} |du(x)|$. Then $\min E_\alpha \leqq E_\alpha(u) \leqq (1 + B^2)^\alpha$ in that component.

We need now to analyze the structure of the submanifold $N_0 \subset L_1^{2\alpha}(M, N)$ of trivial maps to points in N. Recall that at $y \in N_0 \subset L_1^{2\alpha}(M, N)$, we can interpret $T_y L_1^{2\alpha}(M, N) = L_1^{2\alpha}(M, T_y N)$ and $T_y N_0 = \{a\colon da = 0\}$. Then in a weak L^2 sense we construct a normal bundle to N_0,

$$\mathfrak{N} = \bigcup_{y \in N_0} \mathfrak{N}_y \subset TL_1^{2\alpha}(M, N)|N_0 ,$$

$$\mathfrak{N}_y = \left\{v \in L_1^{2\alpha}(M, T_y N)\colon \int_M v\,d\mu = 0\right\} .$$

We use the exponential map exp: $TN \to N$ to define

$$e\colon TL_1^{2\alpha}(M, N) \longrightarrow L_1^{2\alpha}(M, N)$$

by the formula $e(s, v)(x) = \exp(s(x), v(x))$.

LEMMA 2.5. $e|\mathfrak{N} \to L_1^{2\alpha}(M, N)$ *is a diffeomorphism from a neighborhood of the zero section of* $\mathfrak{N}$ *to a neighborhood of* $N_0 \subset L_1^{2\alpha}(M, N)$.

Proof. $de_{(y,0)}(a, v)(x) = \exp_y(a, v(x))$ for $a \in T_y N$, $v \in \mathfrak{N}_y \subset L_1^{2\alpha}(M, T_y N)$. We choose $\mathfrak{N}_y$ so that $de_{(y,0)}\colon T_y N \oplus \mathfrak{N}_y \to T_y L_1^{2\alpha}(M, N)$ is an isomorphism. The result is a direct application of the implicit function theorem.

THEOREM 2.6. *Given* $\alpha > 1$, *there exists a* $\delta > 0$ *depending on* α *and a deformation retraction*

$$\sigma\colon E_\alpha^{-1}[1, 1 + \delta] \longrightarrow E_\alpha^{-1}(1) = N_0 .$$

Proof. Since $(1 + \lambda)^\alpha - 1 - \lambda^\alpha \geqq 0$ for $\lambda \geqq 0$, if $s \in E_\alpha^{-1}[1, 1 + \delta]$, $\int_M |ds|^{2\alpha} d\mu < \delta$, and $\max \left| s - \int_M s \right| \leqq c_\alpha \delta^{1/2\alpha}$ where c_α is the norm of the Sobolev imbedding $L_1^{2\alpha}(M, R^k) \subset C^0(M, R^k)$. The metric topology of inclusion $L_1^{2\alpha}(M, N) \subset L_1^{2\alpha}(M, R^k)$ and the intrinsic topology are the same. We may conclude that if δ is sufficiently small and $s \in E_\alpha^{-1}[1, 1+\delta]$, then there exist $y \in N$, $v \in \mathfrak{N}_y$ such that $s = e(y, v)$. Lemma 2.5 also implies that both $\|v\|_\infty$ and $\int_M |dv|^{2\alpha}$ may be made arbitrarily small by choosing δ sufficiently small.

Finally, we consider the candidate for a retraction $\sigma\colon E_\alpha^{-1}[1, 1+\delta] \times [0, 1] \to L_1^{2\alpha}[M, N]$. Define $\sigma(s, t) = e(y, tv)$ for $s = e(y, v)$. Then $\sigma(s, 1) = s$, $\sigma(s, 0) = y \in N$ and σ is continuous if δ is sufficiently small.

For simplicity, we denote $\sigma(s, t) = u = \exp(y, tv)$ and $d/dt\, \sigma(s, t) = d\exp_{(y,tv)} \cdot v$. Since $d\exp_{(y,0)}$ is the identity, we calculate

$$dv = 1/t\, du + 0(\|v\|_\infty)\cdot du\ ,$$

$$d/dt\, E_\alpha(\sigma(s, t)) = 2\alpha\int_M (1 + |du|^2)^{1-\alpha}(dv, du)d\mu$$

$$\geqq 2\alpha/t\int_M (1 + |du|^2)^{1-\alpha}|du|^2(1 - t\|v\|_\infty)d\mu\ .$$

Consequently, if δ and therefore $\|v\|_\infty$ are sufficiently small, $d/dt\, E_\alpha(\sigma(s, t)) \geqq 0$ and $\sigma(\cdot, 0)$ is a retraction.

Let $z_0 \in M$ and $q_0 \in N$ be chosen as base points for M and N. $\Omega(M, N)$ will denote the space of base point-preserving maps from M to N. The map $p\colon C^0(M, N) \to N$ defined by $p(s) = s(z_0)$ is a fibration with fiber $\Omega(M, N)$.

THEOREM 2.7. *If $\Omega(M, N)$ is not contractible, then there exists a $B > 0$ such that for all $\alpha > 1$, E_α has a critical value in the interval $(1, (1 + B^2)^\alpha)$.*

Proof. The fibration $p\colon C^0(M, N) \to N$ has a section $N \to N_0 \subset C^0(M, N)$ defined by mapping $q \in N$ to the constant map $s(M) = q$. Therefore the exact homotopy sequence splits and

$$\pi_k(C^0(M, N)) = \pi_k(N) \oplus \pi_k(\Omega(M, N))\ .$$

If $C^0(M, N)$ is not connected, apply Proposition 2.4 in a connected component not containing N_0. Ootherwise choose a non-zero homotopy class $\gamma \in \pi_k(\Omega(M, N))$. Note that $\gamma\colon S^k \to \Omega(M, N) \subset C^0(M, N)$ has its image lying in $C^0(M, N)$ and is not homotopic to any map $\tilde{\gamma}\colon S^k \to N_0$. Let $B = \max_{y \in S^k, x \in M} |d\gamma(y)(x)|$. Then $E_\alpha(\gamma(y)) \leqq (1 + B^2)^\alpha$ for all $y \in S^k$.

Suppose that E_α has no critical value in the interval $(1, (1 + B^2)^\alpha)$. Then by Theorem 2.2 there exists a deformation retraction $\rho\colon E_\alpha^{-1}[1, (1 + B^2)^\alpha] \to E_\alpha^{-1}[1, 1 + \delta]$ for all $\delta > 0$. Choose δ as in Theorem 2.6. Composing ρ with the deformation retraction σ of Theorem 2.6 produces the deformation retraction

$$\sigma\circ\rho\colon E_\alpha^{-1}[1, (1 + B^2)^\alpha] \longrightarrow E_\alpha^{-1}(1) = N_0\ .$$

But $\sigma\circ\rho\circ\gamma\colon S^k \to N_0$ is homotopic to γ, a contradiction.

PROPOSITION 2.8. *If $M = S^2$ and the universal covering space of N is not contractible, then there exists a B and a critical map of E_α with values in the range $(1, (1 + B^2)^\alpha)$ for $\alpha > 1$.*

Proof. If the covering space of N is not contractible, $\pi_{k+2}(N) = \pi_k(\Omega(S^2, N)) \neq 0$ for some $k \geqq 0$. Now apply Theorem 2.7.

3. Estimates and extensions

In this section we discuss local properties. The theorems and definitions on previous pages should be interpreted locally where necessary. First we note the difference in the roles of M and N. In the theory of harmonic maps the curvature of N plays an important role. In this paper the estimates on this curvature are in terms of the second fundamental form A of the isometric imbedding $N \subset \mathbf{R}^k$, although with some extra work one could show that they depend only on the sectional curvature of N. The curvature and topology of M do not play a role in these estimates. To see this, cover M by small disks of radius R on which the metric differs from the ordinary Euclidean metric by terms of order ε. When we expand these disks conformally to be of unit size the integral becomes $E_\alpha(s) = R^{2(1-\alpha)} \int_D (R^2 + |ds|^2)^\alpha d\mu$ where D is the unit disk, on which the induced metric still differs from the Euclidean metric by ε, but the curvature now differs by terms of order εR^2. In fact, the smaller the original disk, the nearer to Euclidean is the metric on the expanded disk. For this reason, a priori estimates are uniform in $\alpha \geqq 1$, $0 < R \leqq 1$ and the Laplacian Δ close to the flat Laplacian $\partial^2/\partial x^2 + \partial^2/\partial y^2$ for critical maps of $E_\alpha(s) = R^{2(1-\alpha)} \int_D (R^2 + |ds|^2)^\alpha d\mu$.

The two expressions $E_\alpha(s) = \int_D (1 + |ds|^2)^\alpha d\mu = R^{2(1-\alpha)} \int_D (R^2 + |ds|'^2)^\alpha d\mu'$ are the same integral in coordinate patches with different parámetrizations. Because the factor $R^{2(1-\alpha)}$ does not affect the Euler-Lagrange equations, we often omit it. However, we try to make estimates in terms of the energy $E(s|D)$ rather than $E_\alpha(s|D)$ because this conformal factor is confusing to deal with. The Euler-Lagrange equations (2) and (3) appear in the following forms after this conformal dilation:

$$d^*(R^2 + |ds|^2)^{1-\alpha} ds + (R^2 + |ds|^2)^{1-\alpha} A(ds, ds) = 0 \tag{4}$$

or

$$\Delta s + 2(\alpha - 1)(d^2 s, ds) ds (R^2 + |ds|^2)^{-1} + A(ds, ds) = 0 \,. \tag{5}$$

PROPOSITION 3.1. *Let $s\colon D \to N$ be a critical point of E_α. If $\alpha - 1 > 0$ is sufficiently small depending on $\infty > p > 1$, for all smaller disks $D' \subset D$,*

$$\|ds\|_{D',1,p} \leqq k(p, D', \|s\|_{D,0,4})\|ds\|_{(D,0,4)} .$$

Proof. Let φ be a smooth function with support in D which is 1 on D', and choose a base point in R^k so that $\int s = 0$. Then $\|ds\|_{D,0,p}$ can be used as a norm for $\|ds\|_{D,1,p}$. Multiplying (5) by φ and putting terms from commuting differentiation with multiplication by φ on the right gives us

$$|\Delta(\varphi s) + 2(\alpha - 1)(d^2(\varphi s), ds)ds(R^2 + |ds|^2)^{-1} + A(d(\varphi s), ds)| \leqq k(\varphi)(|ds| + |s|) .$$

Here the size of $k(\varphi)$ depends on two derivatives of φ, $\|A\|_{0,\infty}$ and $\|s\|_{0,\infty}$. For all p we get an estimate of the form

$$(6) \quad \|\Delta(\varphi s)\|_{0,p} \leqq 2(\alpha - 1)\|\varphi s\|_{2,p} + \|A\|_{0,\infty}\| |d(\varphi s)| |ds| \|_{0,p} + k(\varphi)\|s\|_{1,p} .$$

Let $c(p)$ be the norm of Δ^{-1} as a map from $L_0^p \to (L_2^p \cap L_{1,0}^2)$ on the disk ([AG]). Then from (6) we get

$$(7) \quad c(p)^{-1}\|\varphi s\|_{2,p} \leqq 2(\alpha - 1)\|\varphi s\|_{2,p} + \|A\|_{0,\infty}\| |d(\varphi s)| |ds| \|_{0,p} + k(\varphi)\|s\|_{1,p} .$$

Now let $p = 2$. For $2(\alpha - 1) < c(2)^{-1}$ we get

$$(c(2)^{-1} - 2(\alpha - 1))\|\varphi s\|_{2,2} \leqq \|A\|_{0,\infty}\|\varphi s\|_{1,4}\|s\|_{1,} + k(\varphi)\|s\|_{1,2}$$

which provides a bound on $\|s\|_{D'',2,2}$ where $D'' = \{x \in D: \varphi(x) = 1\}$. By Sobolev, this gives a bound on $\|s\|_{D'',1,p}$ for all p. Repeat (7) for any p with φ now having support in D''. If $c(p)^{-1} > 2(\alpha - 1)$ we get a bound on $\|s\|_{2,p}$ in the interior of D''. These estimates are uniform as $\alpha \to 1$.

We were more careful with our estimates in the proof of Proposition 3.1 than we really needed to be. We shall use inequality (7) in the proof of the main estimate of this paper. One should note that this is the standard sort of estimate for regularity in perturbation theory. It says that if s is close enough to a constant map in a disk, we can get uniform estimates which are not otherwise available.

MAIN ESTIMATE 3.2. *There exists $\varepsilon > 0$ and $\alpha_0 > 1$ such that if $s: D \to N$ is a smooth critical map of E_α, $E(s) < \varepsilon$ and $1 \leqq \alpha < \alpha_0$, then there is an estimate uniform in $1 \leqq \alpha < \alpha_0$,*

$$\|ds\|_{D',1,p} < C(p, D')\|ds\|_{0,2}$$

for $D' \subset D$ any smaller disk.

Proof. Clearly we need only get a bound on $\|ds\|_{D'',0,4}$ for any smaller disk $D'' \subset D$ and apply Proposition 3.1 to this. Again we assume $\int s = 0$ and use (7) with $p = 4/3$. We estimate the bad quadratic term $\| |ds| |d(\varphi s)| \|_{0,4/3}$, using Minkowski's inequality, by $\|ds\|_{0,2}\|d(\varphi s)\|_{0,4}$. The inclusion $L_2^{4/3}(D, \mathbf{R}^k) \subset L_1^4(D, \mathbf{R}^k)$ is exact for the Sobolev imbedding theorems because $1/2 - 1/(4/3) + 1/4 = 0$, and we use $\|d(\varphi s)\|_{0,4} \leqq k'\|s\|_{2,4/3}$. We have now put

(7) in the form

$$(8)\qquad \big(c(4/3)^{-1} - 2(\alpha - 1)\big)\,||\varphi s||_{2,4/3} \leqq k'||A||_{0,\infty}||ds||_{0,2}||\varphi s||_{2,4/3} + k(\varphi)||ds||_{0,4/3}\,.$$

Certainly $||ds||_{0,4/3} \leqq \sqrt{E(s)}$. If in addition $\big(c(4/3)^{-1} - 2(\alpha - 1) - k'||A||_{0,\infty}||ds||_{0,2}\big) > 0$, we get an estimate on $||\varphi s||_{2,4/3} \geqq (k')^{-1}||\varphi s||_{1,4}$. Note that the form on the estimate for $E(s) < \varepsilon$ is

$$\sqrt{E(s)} < 1/2\big[\big(c(4/3)^{-1} - 2(\alpha - 1)\big)\big(k'||A||_{0,\infty}\big)^{-1}\big]$$

and ε is small only if the second fundamental form is large.

Naturally the above estimate can be made globally on M without the boundary terms containing $k(\varphi)$. If the norms are norms on M, $\varphi \equiv 1$, and $c(4/3)$ is the norm of the inverse of the Laplace operator $L^{4/3}(M) \to L_2^{4/3}(M)$ $\left(\text{assuming} \int_M \Delta^{-1}u\,d\mu = 0\right)$ and k' is the norm of the imbedding $L_2^{4/3}(M) \to L_1^4(M)$, equation (8) has the global form

$$\big(c(4/3)^{-1} - 2(\alpha - 1)\big)||s||_{2,4/3} \leqq k'||A||_{0,\infty}\sqrt{E(s)}\,||s||_{2,4/3}\,.$$

Clearly, if $\sqrt{E(s)}$ is too small, this has no solutions except $s \equiv \int s$. We use this later.

THEOREM 3.3. *There exists $\varepsilon > 0$ and $\alpha_0 > 1$ such that if $E(s) < \varepsilon$, $1 \leqq \alpha < \alpha_0$ and s is a critical map of E_α, then $s \in N_0$ and $E(s) = 0$.*

In our definition of harmonic maps, we assumed that the maps were continuous and satisfied the Euler-Lagrange equations in a weak sense. It followed from regularity theorems that the harmonic maps were smooth. Here we prove a slightly stronger theorem. We assume that $s\colon D - \{0\} \to N$ is harmonic, and that the energy $\int_D |ds|^2 d\mu < \infty$. In this case s is a weak solution of the Euler-Lagrange equation in $L_1^2(D, N)$. We will prove that in such a case s is smooth. In the case that s is a strict minimum, this is proved by another method by Morrey [MO1], Section 4.3. He directly uses the minimizing properties of $E(s)$. Notice that due to the conformal equivalence of $D - \{0\}$ with $\mathbf{R}^2 - D$, this theorem can be interpreted as a theorem on the growth at infinity of harmonic maps. We use the Main Estimate 3.2 derived in Section 3 for a more general equation. $D(x_0, R)$ denotes the disk of radius R and center x_0; $D(R)$ the disk of radius R and center the origin; $D = D(1)$. We choose isothermal coordinates centered at the origin of the disk. Since $\int_D |ds|^2 d\mu < \infty$, $\lim_{R\to 0} \int_{D(R)} |ds|^2 d\mu = 0$. By a conformal expansion to $D(2)$ with $\int_{D(R)} |ds|^2 d\mu < \varepsilon$, we can assume $\int_{D(2)} |ds|^2 d\mu < \varepsilon$. We choose ε later.

LEMMA 3.4. *There exists $\varepsilon > 0$, such that if $\int_{D(2)} |ds|^2 d\mu < \varepsilon$, then there exists a constant c such that*

$$|ds(x)|\,|x| < c\left(\int_{D(2|x|)} |ds|^2 d\mu\right)^{1/2} \text{ for } x \in D\,.$$

Proof. Choose ε from the Main Estimate 3.2 with $p = 4$ and $\alpha = 1$. For $x_0 \in D$, define $\tilde{s}(x) = s(x_0 + |x_0|x)$. Then

$$\int_D |d\tilde{s}|^2 d\mu = \int_{D(x_0,|x_0|)} |ds|^2 d\mu < \int_{D(2)} |ds|^2 d\mu < \varepsilon\,.$$

We may apply 3.2 to $\tilde{s}: D \to N$, which is also a harmonic map. By the Sobolev imbedding theorem and 3.2 we have

$$\max_{x \in D(1/2)} |d\tilde{s}(x)| \leqq \tilde{c}\,||ds||_{D(1/2),1,4} \leqq \tilde{c}c\,||d\tilde{s}||_{0,2}\,.$$

If we transform this into an inequality on s, we get

$$|ds(x_0)|\,|x_0| = |d\tilde{s}(0)| \leqq \tilde{c}c(4)\,||ds||_{D(x_0,|x_0|)} \leqq c\,||ds||_{D(2|x_0|),0,2}$$

which is what we wanted to show.

LEMMA 3.5. *Let $s: D - \{0\} \to N \subset \mathbf{R}^k$ be a smooth harmonic map such that $E(s) < \infty$. Then*

$$\int_0^{2\pi} |s_\theta(z)|^2 d\theta = r^2 \int_0^{2\pi} |s_r(z)|^2 d\theta\,.$$

Proof. Let $\varphi(z) = w(z)dz^2$ be the holomorphic quadratic differential defined preceding Lemma 1.5. From Lemma 3.4 we get $|w(z)| \leqq 2|ds(z)|^2 \leqq c'|z|^{-2}$. Therefore $w(z)$ has a pole of order at most two at $z = 0$. Since $\int_D |w(z)|d\mu \leqq 2\int_D |ds|^2 d\mu < \infty$, the order of the pole is at most one. A direct computation shows that

$$\text{Re}\, w(z)z^2 = |s_\theta(z)|^2 - |z|^2 |s_r(z)|^2$$

in polar coordinates. From Cauchy's theorem,

$$0 = \text{Im}\int_{|z|=r} w(z)zdz = \text{Re}\int_{0\,|z|=r}^{2\pi} (w(z)z^2)d\theta = \int_0^{2\pi} (|s_\theta(r,\theta)^2| - r^2|s_r(r,\theta)|^2)d\theta\,.$$

This holds for all $|z| = r \leqq 1$.

THEOREM 3.6. *If $s: D - \{0\} \to N$ is harmonic with finite energy, then s extends to a smooth harmonic map $s: D \to N$.*

Proof. We can assume that $\int_{D(2)} |ds|^2 d\mu < \varepsilon$ (by a conformal expansion), where ε is at least as small as the ε chosen in Lemma 3.4. We approximate s by a function q which is piecewise linear in $\log r$ and depends only on the radial coordinate. Let $q(2^{-m}) = 1/2\pi \int_0^{2\pi} s(2^{-m}, \theta)d\theta$. Then $q(r)$ is harmonic for

r between 2^{-m+1} and 2^{-m}, $m \geqq 1$. Now for $2^{-m} \leqq r \leqq 2^{-m+1}$,

$$|q(r) - s(r, \theta)| \leqq |q(2^{-m}) - q(2^{-m+1})| + |s(r, \theta) - q(2^{-m+1})| .$$

Since

$$\begin{aligned} &\max\{|s(x) - s(y)|: 2^{-m} \leqq |x|, |y| \leqq 2^{-m+1}\} \\ &\leqq 2^{-m+3} \max\{|ds|(x): 2^{-m} \leqq |x| \leqq 2^{-m+1}\} \\ &\leqq c2^3\left(\int_{|x| \leqq 2} |ds|^2 d\mu\right)^{1/2} . \end{aligned}$$

we can assume

$$(9) \qquad |q(r) - s(r, \theta)| \leqq 2^4 c \left(\int_{|x| \leqq 2^{-m+1}} |ds|^2 d\mu\right)^{1/2} \leqq 2^4 \varepsilon^{1/2} .$$

Here we have used the main estimate incorporated into Lemma 3.4. Now we estimate the L_1^2 norm of the difference between q and s.

$$(10) \qquad \int_D |dq - ds|^2 d\mu = \sum_{m=1}^{\infty} r \int_\theta (q(r) - s(r, \theta))\cdot(s_r(r, \theta) - q'(r))d\theta \Big|_{2^{-m}}^{2^{-m+1}} - \int_D (q - s)\Delta(q - s)d\mu .$$

The integral in θ of the boundary term containing $q'(r)$ disappears because q is the average of s at 2^{-m}. The terms with $s_r(r, \theta)$ cancel with succeeding and preceding terms, since $s_r(r, \theta)$ is continuously defined, as are q and s. One checks by a limiting argument and (9) that as $m \to \infty$ the limit converges and no boundary term is necessary at 2^{-m} as $m \to \infty$. But $\Delta(q - s) = -A(s)(ds, ds)$. So we estimate $-\int_D (q - s)\Delta(q - s)d\mu$ by

$$\|A\|_{0,\infty}\|q - s\|_{0,\infty}\|ds\|_{0,2}^2 \leqq \|A\|_{0,\infty}2^4c\sqrt{\varepsilon}\,\|ds\|_{0,2}^2 .$$

Choose $\|A\|_{0,\infty}2^4 c\sqrt{\varepsilon} < \delta$. Then from (9) and (10) we have derived

$$\int_{D(1)} |d(s - q)|^2 d\mu \leqq \left(\int_{r=1} |s - q|^2 d\right)^{1/2} \left(\int_{r=1} |s_r|^2 d\theta\right)^{1/2} + \delta\int_{D(1)} |ds|^2 d\mu .$$

We may replace the left side by $1/2 \int_{D(1)} |ds|^2 d\mu = \int_{D(1)} |s_\theta|^2 d\mu$. We note this can only decrease the left side by Lemma 3.5 and the fact that q does not depend on θ. Because q is the average value of s,

$$\left(\int_{r=1} |s - q|^2 d\theta\right)^{1/2} \leqq \left(\int_{r=1} |s_\theta|^2 d\theta\right)^{1/2} = \left(\int_{r=1} |s_r|^2 d\theta\right)^{1/2} = \left(\frac{1}{2}\int_{r=1} |ds|^2 d\theta\right)^{1/2}$$

for the right-hand side. Finally, we obtain the estimate $(1 - 2\delta) \int_{D(1)} |ds|^2 d\mu \leqq \int_{r=1} |ds|^2 d\theta$. If we translate this inequality by expansion and contraction into a disk of any radius, we get for $r \leqq 1$,

$$(1 - 2\delta)\int_{D(r)} |ds|^2 d\mu \leqq r \int_{r=1} |ds|^2 d\theta \ .$$

This inequality integrates to yield $\int_{D(r)} |ds|^2 d\mu \leqq r^{1-2\delta} \int_D |ds|^2 d\mu$. See for example [L-U, Section 9.7]. Apply Lemma 3.4 one last time to get, for $0 < |x_0| < 1/2$,

$$|ds(x_0)|\,|x_0| \leqq c\,|2x_0|^{(1-2\delta)/2}\left(\int_D |ds|^2 d\mu\right)^{1/2} .$$

This now implies $s \in L_1^{2\alpha}(D, N)$ for $\alpha > 1$, and the proof of (2.3) applied to equation (1) gives regularity.

4. Convergence properties of critical maps of the perturbed problem

In the previous section we obtained existence of critical maps of the perturbed integrals and some uniform estimates. Our main result of this section, Theorem 4.7, is that as $\alpha \to 1$, either these converge to a harmonic map, or there is a minimal sphere acting as an obstruction. This statement sounds a bit circular when $M = S^2$, but it works. The technique is based on the type of estimates used in regularity theorems for elliptic operators; see for example reference [GI-M].

LEMMA 4.1. *Let s_α, for $\alpha \to 1$, be a sequence of critical maps for E_α, with $E_\alpha(s_\alpha) \leqq B$. Then there exists a subsequence $\{\beta\} \subset \{\alpha\}$ such that $s_\beta \to s$ weakly in $L_1^2(M, \mathbf{R}^k)$ and* $\lim_{\beta \to 1} E(s_\beta) \geqq E(s)$.

Proof. This is just the weak compactness of the unit ball in $L_1^2(M, \mathbf{R}^k)$. Because we have not assumed any minimizing properties for the sequence s_α, we do not know much about $s \in L_1^2(M, \mathbf{R}^k)$ except $s(x) \in N$ for almost all $x \in M$. We do not, for example, know that s is continuous. It can certainly happen that $\lim_{\beta \to 1} E(s_\beta) > E(s)$ and $s \in N_0$. Recall that N_0 is the set of trivial maps to a point.

As in the previous section, we assume M has been covered by disks, with the disks of half the radius covering M, and the metric on these disks uniformly close to the flat metric. These disks can be as small as we want, say of radius $R = 2^{-m}$, and we make the additional assumption that each point of M is contained in at most h disks, where h is uniform as $2^{-m} \to 0$. If we expand these small disks to unit size, the integral appears on the unit disks D in the form $E_\alpha(s) = \int_D (R^2 + |ds|^2)^\alpha d\mu$ where R is the radius of the disk. Define $\tilde{E}_\alpha = E_\alpha - R^{2\alpha}\int_D d\mu$.

LEMMA 4.2. *Let s_α: $D(R) \to N$ be a sequence of critical maps of E_α for* *a sequence $\alpha \to 1$, which is weakly convergent in $L_1^{2\alpha}(D(R), \mathbf{R}^k)$. Then there exists $\varepsilon > 0$ such that if $E(s_\alpha) < \varepsilon$ then $s_\alpha \to s$ in $C^1(D(R/2), N)$ and $s: D(R/2) \to N$ is a smooth harmonic map.*

Proof. From the conformal invariance we can assume $D(R) = D$. Choose ε from the Main Estimate 3.2 with $p = 4$ and $D' = D(1/2)$. We have a uniform estimate $\|ds_\alpha\|_{D(1/2),1,4} \leqq C(4, D(1/2))\varepsilon$, since $\alpha \to 1$. From the compact Sobolev imbedding $L_2^p(D(1/2), \mathbf{R}^k) \subset C^1(D(1), \mathbf{R}^k)$, it follows that the set of limit points of s_α in $C^1(D(1/2), \mathbf{R}^k)$ is compact, and from the weak convergence $s_\alpha \to s$ in $L_1^{2\alpha}(D, \mathbf{R}^k)$, $s_\alpha \to s$ in $C^1(D(1/2), \mathbf{R}^k)$. Since convergence is in C^1, the form of the Euler-Lagrange equations (4) shows s is harmonic.

PROPOSITION 4.3. *Let $U \subset M$ be an open set and $s_\alpha: U \to N \subset \mathbf{R}^k$ be a sequence of critical maps of E_α for $\alpha \to 1$, $s_\alpha \to s$ weakly in $L_1^2(U, \mathbf{R}^k)$, $E_\alpha(s_\alpha) < B$. Let $U_m = \{x \in U: D(x, 2^{-m+1}) \subset U\}$. Then there exists a subsequence $\{\alpha(l)\} \subset \{\alpha\}$ and a finite number of points $\{x_{1,m}, \cdots, x_{l,m}\}$, where l depends on B and N but not on m, such that*

$$s_{\alpha(l)} \longrightarrow s \text{ in } C^1\left(U_m - \bigcup_{i=1}^{l} D(x_i, 2^{-m-1}), N\right).$$

Proof. Cover U_m by disks $D(x_i, 2^{-m}) \subset U$ such that each point $x \in U$ is covered at most h times and the disks of half the radius cover U_m. Then $\sum_i \int_{D(x_i, 2^{-m+1})} |ds_\alpha|^2 d\mu < Bh$ and for each α there are at most Bh/ε disks on which $\int_{D(x_i, 2^{-m+1})} |ds_\alpha|^2 d\mu > \varepsilon$, where ε is the constant from Lemma 4.2. We claim a subsequence $\{\alpha(l)\} \subset \{\alpha\}$ can be selected to converge to s in $C^1(D(x_i, 2^{-m-1}), N)$ except on $l < hB/\varepsilon + 1$ disks. Suppose we have shown $s_{\alpha(k)} \to s$ in $C^1(D(x_i, 2^{-m-1}), N)$ for $i = 1, 2, \cdots, k$ disks and there are more than Bh/ε disks on which the convergence fails. Then there must be at least one disk $D(y, 2^{-m})$ in the remaining disks, and a subsequence $\{\alpha(k+1)\} \subset \{\alpha(k)\}$ such that for $\alpha = \alpha(k+1)$,

$$\int_{D(y, 2^{-m})} |ds_\alpha|^2 d\mu < \varepsilon .$$

Then $\tilde{s}_\alpha(x) = s_\alpha(x + y)$ is a critical map of E_α on $C^1(D(2^{-m}), N)$ and $E(\tilde{s}_\alpha(x)) < \varepsilon$. From Lemma 4.2, $\tilde{s}_\alpha \to s$ in $C^1(D(2^{-m-1}), N)$, which is the same thing as $s_\alpha \to s$ in $C^1(D(y, 2^{-m-1}), N)$. We repeat this procedure until there are $l < hB/\varepsilon + 1$ disks remaining.

THEOREM 4.4. *Let $U \subset M$ be an open set and $s_\alpha: U \to N$ be a critical map of E_α and $E(s_\alpha) < B$, $\alpha \to 1$ and $s_\alpha \to s$ weakly in $L_1^2(U, \mathbf{R}^k)$. Then there exists a subsequence $\{\beta\} \subset \{\alpha\}$ and a finite number of points $\{x_1, \cdots, x_l\}$, where l does not depend on U such that $s_\beta \to s$ in $C^1(U - \{x_1, \cdots, x_l\}, N)$.*

Moreover, $s\colon U \to N$ *is a smooth harmonic map.*

Proof. We use the preceding proposition to construct a series of subsequences $\{\alpha(m)\} \subset \{\alpha(m-1)\} \subset \{\alpha\}$ with $\alpha(m) \to 1$ and $s_{\alpha(m)} \to s$ in $C^1\big(U_m - \bigcup_{i \leq l} D(x_{i,m}, 2^{-m}), N\big)$ for each integer m. Here $l < Bh/\varepsilon + 1$. Choose a diagonal subsequence β of the sequences $\{\alpha(m)\}$. Then $s_\beta \to s$ in

$$C^1\big(\textstyle\bigcup_m (U_m - \bigcup_{i \leq l} D(x_{i,m}, 2^{-m}), N\big) = C^1\big(U - \bigcap_m (\bigcup_{i \leq l} D(x_{i,m}, 2^{-m}), N\big) = C^1\big(U - \{x_1, \cdots, x_l\}, N\big)\,.$$

We have constructed $s \in C^1\big(U - \{x_1, \cdots, x_l\}, N\big)$. Because s is a weak limit in $L_1^2(U, \mathbf{R}^k)$, we have $E(s) \leqq \lim_{\alpha \to 1} E(s_\alpha) < B$, and we can apply Theorem 3.6 to get $s\colon U \to N$ smooth and harmonic.

We have no assurance that s is not trivial when $U = M$, or that the convergence can be extended over the points $\{x_1, \cdots, x_l\}$ in the theorem. However in some cases we can directly argue that the convergence $s_\alpha \to s$ in the C^1 topology.

LEMMA 4.5. *Suppose that the hypotheses of Theorem* 4.4 *are true and there exists* $\delta > 0$ *such that* $\max_{x \in D(x_i, \delta)} |ds_\alpha(x)| \leqq B < \infty$. *Then* $s_\alpha \to s$ *in* $C^1\big(D(x_i, \delta), N\big)$.

Proof. In a small enough disk $D(x_i, R) \subset D(x_i, \delta)$, $\int_{D(x_i, R)} |ds_\alpha|^2 d\mu \leqq \pi R^2 B^2 < \varepsilon$. Choose $\pi R^2 B^2 < \varepsilon$. Then we may apply Lemma 4.2 to get $s_\alpha \to s$ in $C^1\big(D(x_i, R/2)\big)$.

THEOREM 4.6. *Let* s_α *be a sequence of critical maps of* E_α *for* $\alpha \to 1$, $E_\alpha(s_\alpha) < B$ *and* $s_\alpha \to s$ *in* $C^1\big(M - \{x_1, \cdots, x_l\}, N\big)$ *but not in* $C^1\big(M - \{x_2, \cdots, x_l\}, N\big)$. *Then there exists a harmonic map* $\tilde{s}\colon S^2 \to N$ *which is not a map to a point such that*

$$\tilde{s}(S^2) \subset \textstyle\bigcap_{m \to \infty} \big(\bigcap_{\alpha \to 1} \bigcup_{\beta \leqq \alpha} s_\beta\big(D(x_1, 2^{-m})\big)\big)\,.$$

Moreover $E(s) + E(\tilde{s}) \leqq \overline{\lim}_{\alpha \to 1} E(s_\alpha)$.

Proof. Let $b_\alpha = \max_{x \in D(x_1, 2^{-m})} |ds_\alpha(x)|$ and $x_\alpha \in D(x_1, 2^{-m})$ be a point a which the maximum b_α is taken on. By choosing a subsequence, by Lemm 4.5 we may assume $\lim_{\alpha \to 1} b_\alpha = \infty$. Moreover, $\lim_{\alpha \to 1} x_\alpha = x_1$, because $s_\alpha \to$ in $C^1\big(M - \{x_1, \cdots, x_l\}, N\big)$. Define $\tilde{s}_\alpha(x) = s(x_\alpha + b_\alpha^{-1} x)$. Then $\tilde{s}_\alpha\colon D(0, 2^{-m} b_\alpha) \to N$ is a critical map of E_α and $|d\tilde{s}_\alpha(x)| \leqq 1$ for $x \in D(0, 2^{-m} b_\alpha)$. We have chose $|d\tilde{s}_\alpha(0)| = 1$. Note that the disks on which the maps $\tilde{s}_\alpha$ are defined have rad going to ∞ as $\alpha \to 1$ and $b_\alpha \to \infty$, and the metric on these disks converge to the Euclidean metric. Using Theorem 4.4 and Lemma 4.5, for any $R < \infty$ we can find $\tilde{s}_\alpha \to \tilde{s}$ in $C^1\big(D(R), N\big)$, where $\tilde{s}\colon D(R) \to N$ is smooth and harmoni Since $|d\tilde{s}(0)| = 1$, $\tilde{s}$ cannot be a map to a point. By a diagonal argumen for a subsequence $\tilde{s}_\beta \to \tilde{s}$ in $C^1(\mathbf{R}^2, N)$,

$$\begin{aligned} E(\tilde{s}) + E(s \,|\, M - D(x_1, 2^{-m})) &\leqq \overline{\lim}_{\beta\to1} \{E(\tilde{s}_\beta \,|\, D(0, 2^{-m}b_\alpha)) + E(s_\beta \,|\, M - D(x_1, 2^{-m}))\} \\ &\leqq \overline{\lim}_{\beta\to1} E(s_\beta) \ . \end{aligned}$$

By letting $m \to \infty$, we have $E(\tilde{s}) + E(s) \leqq \overline{\lim}_{\beta\to1} E(s_\beta)$. But $\mathbf{R}^2 = S^2 - \{p\}$ conformally, and $E(\tilde{s}) < \infty$, so from Theorem 3.6, $\tilde{s}$ extends to a map $\tilde{s}\colon S^2 \to N$.

THEOREM 4.7. *Let s_α be a sequence of critical maps of E_α for $\alpha \to 1$, and $s_\alpha \to s$ weakly in $L_1^2(M, \mathbf{R}^k)$. Then either $s_\alpha \to s$ in $C^1(M, N)$, or there exists a non-trivial harmonic map $\tilde{s}\colon S^2 \to N$ with $\tilde{s}(S^2) \subset \bigcap_{\alpha\to1} \overline{\bigcup_{\beta<\alpha} s_\beta(M)}$. Moreover $E(s) + E(\tilde{s}) \leqq \overline{\lim}_{\alpha\to1} E(s_\alpha)$.*

5. Applications and results

In this section we state and prove the final results using the convergence schemes developed in Section 4. In what follows, ε is a uniform constant depending on the second fundamental form of the imbedding $N \subset \mathbf{R}^k$ and is assumed to be the minimum of the constants appearing in the Main Estimate 3.2, Theorem 3.3 and Lemma 4.2. Theorems 5.1, 5.2 and 5.5 yield existence of minimizing harmonic maps. Theorems 5.1 and 5.2 have been obtained independently by Lemaire [L4] and Schoen and Yau [Sch-Y] by other methods. Note that some hypothesis like $\pi_2(N) = 0$ is necessary in Theorem 5.1 in view of the examples of Eells and Wood in [E-W].

THEOREM 5.1. *If N is compact and $\pi_2(N) = 0$, then there exists a minimizing harmonic map in every homotopy class of maps in $C^0(M, N)$.*

Proof. Let $s_\alpha\colon M \to N$ be a minimizing map for E_α in a fixed homotopy class with $E_\alpha(s_\alpha) < (1 + B^2)^\alpha$ as in Proposition 2.4. By Theorem 4.4, with $M = U$, we can choose a subsequence $\beta \to 1$ such that $s_\beta \to s$ in $C^1(M - \{x_1, \cdots, x_l\}, N)$ with $s\colon M \to N$ harmonic. We claim $s_\beta \to s$ in $C^1(M, N)$.

Center a small ball about x_i in M of radius ρ, where ρ is small enough so $x_j \notin D(\rho)$ for $j \neq i$. We will choose ρ later. Define a modified function $\hat{s}_\beta\colon D(\rho) \to N$ which agrees with s_β on the boundary of $D(\rho)$ and with s in the center. Let η be a smooth function which is 1 on $r \geqq 1$ and 0 on $r \leqq 1/2$. Let exp be the exponential map on N.

$$\hat{s}_\beta(x) = \exp_{s(x)}(\eta(|x|/\rho) \exp^{-1}_{s^{-1}(x)} \circ s_\beta(x)) \ . \tag{11}$$

Then $s_\beta \to s$ in $C^1(\operatorname{supp} \eta(|x|/\rho)) \cap D(\rho), N)$ and we have $\hat{s}_\beta \to s$ in $C^1(D(\rho), N)$. Recall that for $s \in L_1^{2\alpha}(M, N)$, $\tilde{E}_\alpha(s) = \int_M (1 + |ds|^2)^\alpha d\mu - 1$, which implies

(12) $$\lim_{\beta\to 1} \tilde{E}_\beta(\hat{s}_\beta) = E(s \mid D(\rho)) .$$

By assumption, $\pi_2(N) = 0$ and s_β and $\hat{s}_\beta$ are homotopic. Since s_β is a minimizing function for E_β in its homotopy class, $E_\beta(s_\beta | D(\rho)) \leqq E_\beta(\hat{s}_\beta | D(\rho))$. Apply (12) and

$$\overline{\lim}_{\beta\to 1} E_\beta(s_\beta \mid D(\rho)) \leqq E(s \mid D(\rho)) \leqq \rho^2\pi \,\| s \|_{1,\infty}^2 .$$

If we initially choose ρ so $\rho^2\pi \,\| s \|_{1,\infty}^2 < \varepsilon/2$, we can apply Lemma 4.2 to get $s_\beta \to s$ in $C^1(D(\rho), N)$, since $E_\beta(s_\beta | D(\rho)) < \varepsilon$ for β sufficiently close to 1. We may conclude $s_\beta \to s$ in $C^1(M, N)$. Since s_β minimizes $\tilde{E}_\beta$, s must minimize E in the same homotopy class.

A free homotopy class of (unbased) maps from M to N induces a map between $\pi_1(M)$ and $\pi_1(N)$. The following theorem has exactly the same proof as Theorem 5.1 which we do not repeat. When we replace s_β by $\hat{s}_\beta$ in $D(\rho)$, we do not change the map on the fundamental group.

THEOREM 5.2. *Every conjugacy class of homomorphisms from $\pi_1(M)$ into $\pi_1(N)$ is induced by a minimizing harmonic map from M into N.*

If $\pi_2(N) = 0$, then Theorem 5.2 implies Theorem 5.1, since in this case conjugacy classes of homomorphisms from $\pi_1(M)$ into $\pi_1(N)$ are canonically identified with the components of $C^0(M, N)$.

In the next lemma we use the construction (11) in the proof of Theorem 5.1 to relate the minimal values of E in the free homotopy classes of $C^0(S^2, N)$ to the structure of $\pi_2(N)$ acted on by $\pi_1(N)$. Each $\gamma \in \pi_2(N)$ determines a free homotopy class of maps from S^2 into N and two elements γ and γ' in $\pi_2(N)$ determine the same free homotopy class if and only if they belong to the same orbit $\pi_1(N)\gamma = \pi_1(N)\gamma'$ under the usual action of $\pi_1(N)$ on $\pi_2(N)$; i.e., the set $\pi_0 C^0(S^2, N)$ of free homotopy classes of maps is in natural one-to-one correspondence with the set of orbits $\pi_1(N)\gamma \subset \pi_2(N)$. We denote by $\Gamma \in \pi_0 C^0(S^2, N)$ the free homotopy class corresponding to $\pi_1(N)\gamma$. For $\Gamma \in \pi_0 C^0(S^2, N)$, γ will denote any element of $\pi_2(N)$ such that $\pi_1(N)\gamma$ corresponds to Γ, and we shall write $\gamma \in \Gamma$. Note that for $\alpha \in \pi_1(N)$ and $\gamma_1, \gamma_2 \in \pi_2(N)$ we have $\alpha(\gamma_1 + \gamma_2) = \alpha\gamma_1 + \alpha\gamma_2$. Moreover, given $\Gamma_i = \pi_1(N)\gamma_i$ for $i = 1, 2, 3$, with $\gamma_1 + \gamma_2 = \gamma_3$, then, since $\alpha\gamma_1 + \alpha\gamma_2 = \alpha\gamma_3$ we have $\pi_1(N)\gamma_3 \subset \pi_1(N)\gamma_1 + \pi_1(N)\gamma_2$. This last relation is the substitution for addition in free homotopy classes. Define

$$\begin{aligned} \sharp\Gamma &= \min\{E(s) : s \in \Gamma \cap L_1^\infty(S^2, N)\} \\ &= \lim_{\alpha\to 1}\{\min \tilde{E}_\alpha(s) : s \in \Gamma \cap L_1^\infty(S^2, N)\} . \end{aligned}$$

Note that $\sharp\Gamma = 0$ if and only if Γ is trivial and that $\sharp\Gamma > \varepsilon$ otherwise.

LEMMA 5.3. *Let $s_\alpha : S^2 \to N$, $\alpha \to 1$, be a sequence of non-trivial critical* *maps of* E_α *with* $s_\alpha \to s$ *in* $C^1(S^2 - \{p\}, N)$. *Then* $s\colon S^2 \to N$ *is not a map to a point.*

Proof. Let (ϕ, θ) be spherical angles on S^2 with $S^+ = \{(\phi, \theta)\colon 0 \leqq \phi \leqq \pi/2\}$ and $S^- = \{(\phi, \theta)\colon \pi/2 \leqq \phi \leqq \pi\}$. Let p be at the north pole $\phi = 0$. Relabel $s_\alpha = v$. We compute the variation of $\tilde{E}_\alpha$ at v in the direction u, which is obtained by differentiating the family $v \circ \sigma_t$. Here $\sigma_t\colon S^2 \to S^2$ is a family of conformal transformations which depend only on the polar angle ϕ. The formula for this variation is $u(\phi, \theta) = v_\phi(\phi, \theta)\sin\phi$. Since v is critical for $\tilde{E}_\alpha$,

$$0 = d\tilde{E}_{\alpha,v}(u) = 2\alpha \int_{S^2} (1 + |dv|^2)^{\alpha-1}(dv \cdot du)d\mu .$$

We evaluate $(dv \cdot du)d\mu$ as

$$\begin{aligned}&[(v_\phi \cdot (v_\phi \sin\phi)_\phi) + \sin^{-2}\phi(v_\theta \cdot (v_\phi \sin\phi)_\theta)]\sin\phi d\theta d\phi \\ &\qquad = \left[\frac{1}{2}|dv|_\phi^2 \sin^2\phi + |dv|^2 \sin\phi\cos\phi\right] d\theta d\phi .\end{aligned}$$

Putting this expression into the integral gives

$$\begin{aligned}0 &= \int_0^{2\pi}\int_0^{\pi} [((1 + |dv|^2)^\alpha - 1)_\phi \sin^2\phi + 2\alpha(1 + |dv|^2)^{\alpha-1}|dv|^2 \sin\phi\cos\phi] d\phi d\theta \\ &= 2\int_{S^2} [-(1 + |dv|^2)^\alpha + 1 + \alpha(1 + |dv|^2)^{\alpha-1}|dv|^2]\cos\phi d\mu\end{aligned}$$

where the last equality is obtained from integration by parts. To estimate the integral, use Taylor's theorem, for real $\lambda > 0$ in the expression

$$\begin{aligned}\frac{1 + \alpha(1 + \lambda)^{\alpha-1}\lambda - (1 + \lambda)^\alpha}{(1 + \lambda)^{\alpha-2}} &= (1 + \lambda)^{2-\alpha} + \alpha(1 + \lambda)\lambda - (1 + \lambda)^2 \\ &= (\alpha - 1)\int_0^1 (\alpha - 2(1 + t\lambda)^{-\alpha} + 2)(1 - t)dt\lambda^2 .\end{aligned}$$

The extremal values of the integral are $\alpha/2$ and 1 at $\lambda = 0$ and $\lambda = \infty$ respectively. It follows that the integrand we obtained over S^2

$$\alpha(1 + |dv|^2)^{\alpha-1}|dv|^2 + 1 - (1 + |dv|^2)^\alpha \sim (1 - \alpha)(1 + |dv|^2)^{\alpha-2}|dv|^2 .$$

Divide the integral up into integration over S^+ and S^-.

$$(13) \quad \alpha/2\int_{S^+} (1 + |dv|^2)^{\alpha-2}|dv|^2 \cos\phi d\mu \leqq \int_{S^-} (1 + |dv|^2)^{\alpha-2}|dv|^2(-\cos\phi)d\mu .$$

Recall $v = s_\alpha$, $\alpha \to 1$, and $s_\alpha \to s$ in $C^1(S^2 - \{p\}, N)$. If s is trivial, by Theorem 3.3, s_α cannot approach s in $C^1(S^2, N)$. From Theorem 4.6 we see that there exist expansions $\tilde{s}_\alpha$ of s_α near p which converge to $\tilde{s}\colon S^2 \to N$. Then

$$\frac{1}{2}E(\tilde{s}) \leqq \lim_{\alpha\to 1} E(s_\alpha|D_\alpha) \leqq \lim_{\alpha\to 1}\int_{S^+}|ds_\alpha|^2\cos\phi d\mu$$
$$\leqq \lim_{\alpha\to 1}\left(-\int_{S^-}(1+|ds_\alpha|)^{\alpha-1}|ds_\alpha|^2\cos\phi\right)d\mu = -\int_{S^-}|ds|^2\cos\phi d\mu\ .$$

Here we use (13) in the last inequality. If $ds = 0$, $\tilde{s}$ is trivial, which is impossible.

LEMMA 5.4. *Let* $\Gamma\in\pi_0 C^0(S^2, M)$. *Then either* Γ *contains a minimizing harmonic map* s *or for all* $\delta > 0$ *there exist non-trivial free homotopy classes* $\Gamma_1 = \pi_1(N)\gamma_1$ *and* $\Gamma_2 = \pi_2(N)\gamma_2$ *such that* $\Gamma = \pi_1(N)\gamma \subset \pi_1(N)\gamma_1 + \pi_1(N)\gamma_2$ *and* $\sharp\Gamma_1 + \sharp\Gamma_2 < \sharp\Gamma + \delta$.

Proof. By Proposition 2.3 and Theorem 4.4 we can find a sequence $\alpha\to 1$ and maps $s_\alpha\in\Gamma$ which are minimizing for $\tilde{E}_\alpha$ in $\Gamma\cap L_1^{2\alpha}(S^2, N)$ and which converge weakly to s in $L_1^2(S^2, R^k)$. In fact $s_\alpha\to s$ in $C^1(S^2 - \{x_1, \cdots, x_l\}, N)$. We can assume that $\lim_{m\to\infty}\overline{\lim}_{\alpha\to 1}\tilde{E}_\alpha(s_\alpha|D(x_1, 2^{-m}))\geqq\varepsilon$, for if not we can apply Lemma 4.2 to remove the singularity x_1 of the convergence of s_α to s. If we can remove all the x_i in this fashion, $s_\alpha\to s$ in $C^1(S^2, N)$ and $s\in\Gamma$ is a harmonic map which minimizes E in Γ. Assuming we cannot do this, pick a small disk $D(\rho)$ around x_1 and use construction (11) to define $\hat{s}_\alpha$: $D(\rho)\to N$. Let

$$u_\alpha(x) = \begin{cases} s_\alpha(x) & x\in S^2 - D(\rho)\\ \hat{s}_\alpha(x) & x\in D(\rho)\ ,\end{cases}$$
$$v_\alpha(x) = \begin{cases} \hat{s}_\alpha\circ f(x) & x\in S^2 - D(\rho)\\ s_\alpha(x) & x\in D(\rho)\ .\end{cases}$$

Here f: $S^2 - D(\rho)\to D(\rho)$ is the conformal reflection leaving the boundary of $D(\rho)$ fixed. Let Γ_1 and Γ_2 be the free homotopy classes of u_α and v_α respectively. Then $\pi_1(N)\gamma\subset\pi_1(N)\gamma_1 + \pi_1(N)\gamma_2$. From (12) and the conformality of f

$$\lim_{\alpha\to 1}\tilde{E}_\alpha(u_\alpha) = \lim_{\alpha\to 1}\tilde{E}_\alpha(s_\alpha|S^2 - D(\rho)) + E(s|D(\rho))$$

and

$$\lim_{\alpha\to 1}\tilde{E}_\alpha(v_\alpha) = \lim_{\alpha\to 1}\tilde{E}_\alpha(s_\alpha|S^2 - D(\rho)) + E(s|D(\rho))\ .$$

If $\rho^2\pi\,||s||_{1,\infty}^2\leqq\delta/6$, we can choose α close enough to 1 so that

$$\tilde{E}_\alpha(u_\alpha)\leqq\tilde{E}_\alpha(s_\alpha|S^2 - D(\rho)) + \delta/3$$
$$\tilde{E}_\alpha(v_\alpha)\leqq\tilde{E}_\alpha(s_\alpha|D(\rho)) + \delta/3$$

and

$$\sharp\Gamma_1 + \sharp\Gamma_2\leqq\tilde{E}_\alpha(u_\alpha) + \tilde{E}_\alpha(v_\alpha)\leqq\tilde{E}_\alpha(s_\alpha) + 2/3\delta < \sharp\Gamma + \delta\ .$$

We assume $\delta < \varepsilon/2$. It is automatically true that $\tilde{E}_\alpha(v_\alpha)\geqq\tilde{E}_\alpha(v_\alpha|D(\rho))\geqq\varepsilon$, so $\sharp\Gamma_1 \leqq \tilde{E}_\alpha(u_\alpha) \leqq \sharp\Gamma + \delta - \varepsilon < \sharp\Gamma$. Then $\Gamma_1 \neq \Gamma$ and $\Gamma_2 \neq 0$. We now need only show $\tilde{E}_\alpha(u_\alpha) > \delta$ to get $\Gamma_1 \neq 0$. If we have chosen α close enough to 1,

$$\tilde{E}_\alpha(u_\alpha) > \tilde{E}_\alpha(s_\alpha \mid S^2 - D(\rho)) \geqq \tilde{E}(s \mid S^2 - D(\rho)) - \delta/6 > E(s) - \delta/3 .$$

If s is not trivial, we are finished as $E(s) > \varepsilon$ from Theorem 3.3. If s is trivial, it was shown in the preceding lemma that there must be at least a second point $x_2 \neq x_1$ where the convergence $s_\alpha \to s$ fails. Then Lemma 4.2 gives us that $\lim_{\alpha\to 1} \tilde{E}_\alpha(s_\alpha \mid D(x_2, \rho)) > \varepsilon$ and $\tilde{E}_\alpha(u_\alpha) > \tilde{E}_\alpha(s_\alpha \mid D(x_2, \rho))$ for small ρ. In either case, $\tilde{E}_\alpha(u_\alpha) > \delta$ and $\Gamma_1 \neq 0$.

THEOREM 5.5. *There exists a set of free homotopy classes $\Lambda_i \subset \pi_0 C^0(S^2, N)$ such that elements $\{\lambda \in \Lambda_i\}$ form a generating set for $\pi_2(N)$ acted on by $\pi_1(N)$, and each Λ_i contains a minimizing harmonic map $s_i\colon S^2 \to N$.*

Proof. Let Λ_i be the homotopy classes containing minimizing harmonic maps. Let $P \subset \pi_2(N)$ be the subgroup generated by elements $\{\lambda \in \Lambda_i\}$. Suppose the inclusion is proper. Pick a class Γ with elements $\gamma \in \Gamma$, $\gamma \notin P$ such that if $\sharp\Gamma' \leqq \sharp\Gamma - \varepsilon/2$, then the elements $\{\gamma' \in \Gamma'\} \subset P$.

By assumption Γ does not contain a minimizing harmonic map, so there exist Γ_1 and Γ_2 with $\pi_1(N)\gamma \subset \pi_1(N)\gamma_1 + \pi_1(N)\gamma_2$ and $\sharp\Gamma_1 + \sharp\Gamma_2 < \sharp\Gamma + \varepsilon/2$. But Γ_1 and Γ_2 are not trivial so $\sharp\Gamma_j \geqq \varepsilon$ for $j = 1, 2$. All this implies $\sharp\Gamma_j < \sharp\Gamma - \varepsilon/2$. By assumption the sets $\pi_1(N)\gamma_j$ are both in P, so

$$\pi_1(N)\gamma \subset \pi_1(N)\gamma_1 + \pi_1(N)\gamma_2 \subset P .$$

In the next two theorems we look into the situation where the harmonic maps are not necessarily minimizing but may be saddle points. Our first result treats the case where there are no obstructions to our convergence technique.

THEOREM 5.6. *Let $\varepsilon_0 = \min E(s)$ for $s\colon S^2 \to N$, s harmonic and not a map to a point and $\varepsilon_0 = \infty$ if this set of harmonic maps is empty. Then $E \mid E^{-1}[0, \varepsilon_0)$ satisfies a Morse theory for $M \neq S^2$, and for $M = S^2$ $E \mid E^{-1}[0, 2\varepsilon_0)$ satisfies a Morse theory.*

Proof. We apply the technique of Morse theory by perturbations E_α from E. The set $\{s_\alpha\colon s_\alpha$ is critical for E_α, $\tilde{E}_\alpha(s_\alpha) \leqq \delta < \varepsilon_0\}$ is compact, since by Theorem 4.6, $s_\alpha \to s$ in $C^1(M, N)$ unless there is a minimal harmonic map $\tilde{s}\colon S^2 \to N$ with image lying in the Hausdorff limit set of $s_\alpha(M)$. In this case $\lim_{\alpha\to 1} \tilde{E}_\alpha(s_\alpha) \geqq E(\tilde{s}) + E(s) \geqq \varepsilon_0$. If $M = S^2$, by Lemma 5.3, $E(\tilde{s}) \geqq \varepsilon_0$. Therefore the results of [U] are valid.

THEOREM 5.7. *If the universal covering space of N is not contractible, then there exists a non-trivial harmonic map $s\colon S^2 \to N$.*

Proof. We apply Theorem 2.8 to get a critical map s_α of $\tilde{E}_\alpha$ with $\varepsilon < \tilde{E}_\alpha(s_\alpha) < B$. Then from Theorem 4.4, there is a subsequence, which we also denote by s_α, such that $s_\alpha \to s$ in $C^1(S^2 - \{x_1, \cdots, x_l\}, N)$, where $\alpha \to 1$ and s is harmonic. If s is not a map to a point, we are finished. If s is a point, since $\tilde{E}_\alpha(s_\alpha) > \varepsilon$, s_α must fail to converge to s at some points. From Theorem 4.7 there exists a harmonic map $\tilde{s}$ with $\tilde{s}(S^2) \subset \bigcap_\alpha \overline{\bigcup_{\beta<\alpha} s_\alpha(S^2)}$. Recall the result of Corollary 1.7: the image of a harmonic map from S^2 to N is a conformal branched minimal immersion. This, together with the preceding theorem, yields the main theorem on the existence of minimal spheres stated in the introduction. Note that the hypothesis on the universal covering space of N cannot be dropped, for if N has non-positive curvature then every harmonic map $s: S^2 \to N$ is constant.

THEOREM 5.8. *If the universal covering space of N is not contractible, then there exists a non-trivial C^∞ conformal branched minimal immersion $s: S^2 \to N$.*

Combination of Corollary 1.7 and Theorem 5.5 yields the final result on minimal spheres.

THEOREM 5.9. *There exists a set of free homotopy classes $\Lambda_i \in \pi_0 C^0(S^2, N)$ such that elements $\{\lambda \in \Lambda_i\}$ generate $\pi_2(N)$ acted on by $\pi_1(N)$ and each Λ_i contains a conformal branched immersion of a sphere having least area among maps of S^2 into N which lie in Λ_i.*

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REFERENCES

[AG] S. AGMON, The L_p approach to the Dirichlet problem, Ann. Scuola. Norm. Sup. Pisa **13** (1959), 405–448.

[AR] N. ARONSZAJN, A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order, J. Math. Pures Appl. **9** (1957), 235–249.

[C-G] S. S. CHERN and S. I. GOLDBERG, On the volume-decreasing property of a class of real harmonic mappings, Am. J. Math. **97** (1975), 133–147.

[EA-E] C. J. EARLE and J. EELLS, A fibre bundle description of Teichmüller theory, J. Diff. Geom. **3** (1969), 19–43.

[E] J. EELLS, A setting for global analysis, Bull. A.M.S. **72** (1966), 751–807.

[E-L] J. EELLS and L. LEMAIRE, A report on harmonic maps, Bull. London Math. Soc. **10** (1978), 1–68.

[E-S1] J. EELLS and J. H. SAMPSON, Variational theory in fiber bundles, Proc. of the U.S. Japan Seminar in Diff. Geo. (1965), 22–33.

[E-S2] ———, Harmonic mappings of Riemannian manifolds, Amer. J. Math. **86** (1964), 109–160.

[E-W] J. EELLS and J. C. WOOD, Restrictions on harmonic maps of surfaces, Topology **15** (1976), 263–266.

[F] A. Friedman, *Partial Differential Equations,* Holt, Rinehart, and Winston, Chicago (1969).

[G-O-R] R. D. Gulliver, R. Osserman, H. L. Royden, A theory of branched immersions of surfaces, Amer. J. Math. **95** (1973), 750-812.

[GI-M] E. Giusti and M. Miranda, Sulla regolarita delle soluzioni deboli di una classe di sistemi ellittici quasi-lineari, Arch. Rat. Mech. Anal. **31** (1968/69), 173-184.

[H-W] P. Hartman and A. Wintner, On the local behavior of solutions of nonparabolic partial differential equations, Amer. J. Math. **75** (1953), 449-476.

[L-U] O. Ladyzhenskaya and N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York (1968).

[L1] L. Lemaire, Applications harmoniques des surfaces Riemanniennes, thesis, Université Libre de Bruxelles (1975).

[L2] ———, Applications harmoniques de surface, CR Paris A **280** (1975), 897-899.

[L3] ———, Applications harmoniques de variétés produits, Comm. Math. Helv. **52** (1977), 11-24.

[L4] ———, Applications harmoniques des surfaces Riemanniennes, J. Diff. Geom. **13** (1978), 51-87.

[MI] J. Milnor, *Morse Theory*, Princeton University Press (1963).

[M01] C. B. Morrey, *Multiple Integrals in the Calculus of Variations*, Springer (1966).

[M02] ———, The problem of Plateau on a Riemannian manifold, Ann. of Math. **49** (1948), 807-851.

[P1] R. S. Palais, Homotopy theory of infinite dimensional manifolds, Topology **5** (1966), 1-16.

[P2] ———, Ljusternik-Schnirelman theory on Banach manifolds, Topology **5** (1966), 115-132.

[P3] ———, *Foundations of Global Non-Linear Analysis*, Benjamin, New York (1968).

[S] G. Schwartz, Master's thesis, M.I.T. (1969).

[S-U] J. Sacks and K. Uhlenbeck, Minimal immersions of compact Riemannian surfaces, preprint.

[Sch-Y] R. Schoen and S.-T. Yau, Existence of incompressible minimal surfaces and the topology of three dimensional manifolds with non-negative scalar curvature, Ann. of Math. **110** (1979), 127-142.

[U] K. Uhlenbeck, Morse theory by perturbation methods with applications to harmonic maps, Trans. A.M.S. (to appear).

[W] J. C. Wood, Ph.D. thesis, University of Warwick, England (1974).

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