SOME RESULTS IN PARTITIONS, PLANE PARTITIONS, AND MULTIPARTITIONS

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ABSTRACT. In this paper we explore various properties of partitions and multipartitions, including various restricted sets of each. Results involving regular partitions include proofs of various well-known identities using binary representation and a generating function for diagonal partitions. Some findings involving multipartitions include an extension of an algebraic construction of multipartitions, a combinatorial proof of a recursive relationship for the partition function, and a bijection involving tri-conjugate shell multipartitions. Furthermore, we discovered several congruences for movable multipartitions and extended a nice proof of the Ramanujan congruences to include multipartition functions and several prime powers. Finally, we close with the several programs and functions written in Java, which were invaluable for our investigations.

1. INTRODUCTION TO PARTITIONS

Definition 1.1. *A partition, λ, of a non-negative integer n, is a non-increasing sequence, $\lambda_1, \lambda_2, \ldots, \lambda_k$, such that $|\lambda| = \sum_{i=1}^{k} \lambda_i = n$. We also call λ_i the parts of the partition, λ.*

For example, we have that $(6,4,3,3,1,1,1,0,0,\ldots)$ is a partition of 19, since

$$6+4+3+3+1+1+1 = 19.$$

Generally, we omit the trailing zeros in the representation of a partition, giving us in this case just $(6,4,3,3,1,1,1)$. We use $\emptyset$ to represent the empty, or zero, partition. We denote the partition function, $p(n)$, to be the number of partitions of n. The following table gives the first few values of $p(n)$.

n	p(n)	partitions, λ, of n
0	1	$\emptyset$
1	1	1
2	2	2, 1+1
3	3	3, 2+1, 1+1+1
4	5	4, 3+1, 2+2, 2+1+1, 1+1+1+1
5	7	5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1+1, 1+1+1+1+1

TABLE 1. Values of $p(n)$ and partitions of n

Sometimes it can be helpful to represent a partition in a graphical manner rather than as simply a sequence of numbers. One way to accomplish this is through the use of Ferrers diagrams.

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Definition 1.2. *The Ferrers diagram of a partition $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ of n is the left-justified array of dots obtained by having λ_1 dots in the first (top) row, λ_2 dots in the second row, and so on through λ_k dots in the final (bottom) row.*

Example 1.3. *The Ferrers diagram of the partition (6, 4, 3, 3, 1, 1, 1) is:*

$$
\begin{array}{llllll}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & & \\
\bullet & \bullet & \bullet & & & \\
\bullet & \bullet & \bullet & & & \\
\bullet & & & & & \\
\bullet & & & & & \\
\bullet & & & & &
\end{array}
$$

We can also extend the concept of a partition into higher dimensions.

Definition 1.4. *A k-component multipartition, Λ, of a non-negative integer n is a k-tuple of partitions $(\lambda^1, \lambda^2, \ldots, \lambda^k)$, where each λ^i is a partition, and $|\Lambda| = \sum_{i=1}^{k} |\lambda^i| = n$. We say that λ^i is the ith component of the multipartition. Furthermore, we say that λ^i_j is the jth part of the ith component of the multipartition Λ.*

For example (3+2, 1+1, $\emptyset$, 2+2+1) and (4, 3, 2+2, 1) are both valid 4-partitions of 12. It should be noted that the order of the components within the multipartition are important, and that (3+2, $\emptyset$, 1+1) is not the same as (1+1, 3+2, $\emptyset$).

We define the multipartition function, $P_k(n)$, to be the number of k-component multipartitions of n.

We can visualize multipartitions by drawing the Ferrers diagram for each component, then simply stacking them on top of each other. This idea of three-dimensional visualization gives rise to a special kind of multipartition.

Definition 1.5. *A k-component movable multipartition (or plane partition) is a multipartition $\Lambda = (\lambda^1, \lambda^2, \ldots, \lambda^k)$ for some $k \in \mathbb{N}$ such that:*

(1) $\lambda^i_j \geq \lambda^i_{j+1}$
(2) $\lambda^i_j \geq \lambda^{i+1}_j$.

This definition of movable multipartitions comes from the work of Furno and Waters [FW07]. An example of a movable multipartition would be (3+2+2, 3+2, 1+1).

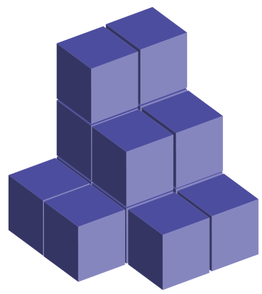
This definition ensures that parts within the same component are non-increasing, and that the jth part of each component is no larger than the one in the previous component. Graphically, this means that each successive Ferrers diagram fits nicely onto the one below it, with no pieces hanging over the "edge" of the level beneath it.

Because of these nice properties, we can represent plane partitions in a very special way. A plane partition of n can be represented as a two-dimensional array of integers whose entries sum to n and whose rows are non-increasing from left to right and whose columns non-increasing from top to bottom. We can then extend this array into three dimensions by thinking of the (i, j)th entry of the array as a stack of boxes whose height equals the integer entry. The following example illustrates this.

Example 1.6. *The following are all ways to describe the same plane partition of 14:*

$$(3+3+2, 2+2, 1+1)$$

$$\begin{pmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & \end{pmatrix}$$



We denote the plane partition function, $PL(n)$, to be the number of plane partitions of n.

A useful and important tool to study partitions, as well as multipartitions and plane partitions, are generating functions. A generating function for a sequence is a formal power series whose nth coefficient corresponds to the nth term of the sequence.

Theorem 1.7 (Euler)**.** *The generating function for $p(n)$ has the following infinite product form:*

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

Proof. We begin by expanding the right hand side of this equation. We know that

$$\frac{1}{1-q^n} = (1 + q^n + q^{2\cdot n} + q^{3\cdot n} + \dots).$$

Thus, we can say that

$$\begin{aligned} \prod_{n=1}^{\infty} \frac{1}{1-q^n} &= (\frac{1}{1-q})(\frac{1}{1-q^2})\cdots \\ &= (1 + q + q^{2\cdot 1} + q^{3\cdot 1} + \dots)(1 + q^2 + q^{2\cdot 2} + q^{3\cdot 2} + \dots)\cdots. \end{aligned}$$

From this expansion, we see that the coefficient of the q^n term will be the number of ways in which we can pick powers of q from these series to add up to n. Each choice of powers that sum to n corresponds to a unique partition of n. Thus the coefficient of q^n is equal to $p(n)$. $\square$

Similar infinite product representations exist for both multipartitions and plane partitions.

Theorem 1.8. *The generating function for $P_k(n)$ has the following form* [And08].

$$\sum_{n=0}^{\infty} P_k(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^k}$$

The generating function for $PL(n)$ has the following form [Mac04].

$$\sum_{n=0}^{\infty} PL(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}$$

2. Summary of Results

The first few sections of this paper deal with regular partitions, from our original definition above. In these sections we consider viewing partitions as binary numbers. In doing this we see slightly different versions of some basic partition identities and structures. We also examine a special type of partition that we refer to as an n-diagonal partition. We prove a recurrence about the generating function of these partitions as well as find a generating function for the generating functions.

The next sections deal with multipartitions and plane partitions. We first look at the algebraic structure of certain types of multipartitions using structures known as special and numerical monoids, as studied recently by Furno and Waters [FW07]. Following this, we offer a combinatorial proof of a recursive relationship given by Gandhi [Gan63]. We then examine a special type of plane partition, which we refer to as "Tri-Conjugate Shell Multipartitions," proposing and proving a representation of a generating function for these partitions.

The following sections contain results about congruences occurring different types of plane partition and multipartition functions. First, we conjecture and prove several congruences about restricted plane partition functions. We also examine the periodic nature of these congruences. Next, we prove a family of congruences for restricted multipartition functions using modular form theory. We also investigate multipartition function congruences involving prime powers using modular forms. These last two sections involve extending a proof of Lachterman, Schayer, and Younger [LSY08].

Finally, we show in the last section java code that was used to investigate certain types of partitions and plane partitions. One function in particular gives the number of partitions that can fit inside a given partition. Another looks at the number of subpartitions of the pyramidal plane partition of size n, defined in section 10.1. Finally, the third enumerates the plane partitions of n with at most k components.

3. Binary Representation of a Partition

Recall from before the notion of using a Ferrers diagram to represent a partition. It is from these diagrams that the idea for representing a partition as a binary string arises. Starting at the top right corner of the diagram, we can outline the right edge in the following way. We represent a move down the diagram with a 1, and a move to the left across the diagram with a 0. The following example illustrates this process.

Example 3.1. *We can represent the partition $(4,3,3,2,1)$ with the Ferrers diagram below:*

$$\begin{array}{llll} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & & \\ \bullet & & & \end{array}$$

Then, following the procedure outlined above, we see that the binary representation of this partition is (1, 0, 1, 1, 0, 1, 0, 1, 0).

We also note that we can disregard any leading 0's and trailing 1's in the sequence as they do not give any additional information about the partition.

Theorem 3.2. *For every nonempty partition λ, there exists a unique binary representation, b_λ, of λ.*

Proof. Let λ_a, λ_b be nonempty partitions with identical binary representations, that is, $b_{\lambda_a} = b_{\lambda_b}$. Since the two binary strings are identical, we can say that b_{λ_a} and b_{λ_b} must trace out identical Ferrers Diagrams, with rows $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$ for some $m \in \mathbb{N}$. We must then have $\lambda_a = \lambda_b = \{\lambda_1, \lambda_2, \dots, \lambda_m\}$. Thus, every nonempty partition has a unique binary representation. □

We now define the function $\theta : P \to \mathbb{N}$, where P is the set of all nonempty partitions, by

$$\theta(\lambda) = \frac{d(b_\lambda)}{2},$$

where $d(b_\lambda)$ takes the binary string b_λ to its decimal equivalent. Given any partition $\lambda \in P$, we know that b_λ is a binary string with no leading 0's or trailing 1's. Thus, the last bit of b_λ must be 0. It then follows that $d(b_\lambda)$ must be even, and thus $\theta(\lambda) \in \mathbb{N}$ for all λ.

Theorem 3.3. *The function θ is a bijection between P and $\mathbb{N}$.*

Proof. First, we show that θ is one-to-one. Let $\lambda_a, \lambda_b \in P$, with $\lambda_a \neq \lambda_b$. By the theorem above, we know that the binary representations of these two partitions, b_{λ_a} and b_{λ_b} are distinct. Thus, we must have

$$d(b_{\lambda_a}) \neq d(b_{\lambda_b}),$$

and

$$\frac{d(b_{\lambda_a})}{2} \neq \frac{d(b_{\lambda_b})}{2}.$$

Thus, we have shown that if $\lambda_a \neq \lambda_b$, then it must follow that $\theta(\lambda_a) \neq \theta(\lambda_b)$, and θ is one-to-one.

Finally, we show that θ is onto. Let $n \in \mathbb{N}$. It is obvious that $2n$ is even, and that the binary representation of $2n$ must have a 1 as the leading bit and a 0 as the trailing bit. Thus, this binary representation of $2n$ also represents the Ferrers Diagram for some partition $\lambda \in P$, which can be drawn as per the guidelines given above. We can then say that $\theta(\lambda) = n$, and thus, θ is onto.

Thus, θ is a well defined function from P to $\mathbb{N}$ that is one-to-one and onto, and we can conclude that θ is a bijection. □

We now examine some basic partition identities with respect to this binary representation.

Theorem 3.4. *The number of partitions of n with largest part at most k is equivalent to the number of partitions of n with at most k parts.*

Proof. Traditionally, this is proved using the idea of conjugate partitions. The conjugate of a partition is given by reading the number of dots in the successive columns of the Ferrers diagram of the partition. We see that the conjugate of any partition of n with at most k parts must have no parts larger than k, since none of the columns can have more than k dots. Thus, the conjugation operation creates a one-to-one correspondence between the two sets of partitions. □

We now give a proof using the binary representation:

Proof. We first note that the largest part of a given partition λ can be found by counting the number of 0's in its binary representation. Similarly, the number of parts of λ can be found by counting the number of 1's. For the binary representation b_λ, we define $(b_\lambda)^*$ to be the binary string obtained by reversing the order of b_λ and flipping each of the bits

$$(1,0,1,1,0)^* = (1,0,0,1,0).$$

It is clear that if b_λ is a valid binary representation (i.e. no leading 0's or trailing 1's), that $(b_\lambda)^*$ must also be a valid binary representation. In fact, we can say that $(b_\lambda)^*$ traces the same Ferrers Diagram using the following guidelines. Starting in the bottom left corner, move right for each 1, and up for each 0. Thus, b_λ and $(b_\lambda)^*$ both represent partitions of the same number.

Thus, if we take λ to be a partition with largest part at most k, then b_λ will contain no more than k 0's. It would then follow that $(b_\lambda)^*$ must contain no more than k 1's, and thus be a representation of a partition with at most k parts. We also note that $((b_\lambda)^*)^* = b_\lambda$, and thus we have a bijection between partitions of n with largest part at most k and those with at most k parts. □

Theorem 3.5. *The number of partitions of n into all odd parts is equal to the number of partitions of n into distinct parts*

The following proof of this theorem appears in [And94]:

Proof. Consider a partition having only odd parts. Let f_i denote the number of times that i appears as a part. We can then write

$$n = f_1 \cdot 1 + f_3 \cdot 3 + f_5 \cdot 5 + \cdots + f_{2M-1} \cdot (2M-1).$$

We can then say that each f_i can be uniquely represented as a sum of distinct powers of 2, giving:

$$n = (2^a + 2^b + \cdots + 2^c) \cdot 1 + (2^e + 2^f + \cdots + 2^g) \cdot 3 + \cdots + (2^r + 2^s + \cdots + 2^t) \cdot (2M-1),$$

and so

$$n = 2^a + 2^b + \cdots + 2^c + 3 \cdot 2^e + 3 \cdot 2^f + \cdots + 3 \cdot 2^g + \cdots + (2M-1)2^r + (2M-1)2^s + \cdots + (2M-1)2^t.$$

This last expression is clearly a partition of n into distinct parts.

We then take a partition of n into distinct parts. We can write each part as an odd number times a power of 2. We then collect these terms into groups based on this odd factor. We then factor out these odd factors and sum the resulting powers of 2, arriving at a partition of n into only odd parts. Thus, we have established a one-to-one correspondence and proved the theorem. □

Using the binary representation, we can achieve the same result in the following way:

Proof. We first consider the binary representation of a partition with all distinct parts. This tells us that the binary representation cannot contain any consecutive 1's. Thus, if λ is a partition with all distinct parts, then b_λ must contain at least as many 0's as 1's.

Now, we look at the binary representation of a partition with only odd parts. In this case, we can say that, with the exception of the final 0, all 0's must occur in pairs (otherwise we could have even parts).

We now define a procedure for transforming the binary representation of a partition of n with all distinct parts into one with all odd parts. Let λ be a partition of n of this type. Starting at the left of the binary string, determine, for each 1, how many 0's follow it in the string, and denote this number z. If z is even, remove the 1 in question, and insert two new 1's into the string between zeros $z/2$ and $z/2+1$ counting from the right this time. We note that this process does not change the number being partitioned. If there is a 1 is followed by z copies of 0, then it represents a row of z dots on the Ferrers diagram. Thus, replacing it with two rows of $z/2$ dots does not change the size of the partition. As an example, the string

$$(1,0,1,0,1,0,0,0)$$

would become

$$(1,0,0,1,0,1,1,0,0).$$

Repeat this process until the string contains no 1's with an even number of 0's following it. The previous example would end up as

$$(1,0,0,1,0,0,1,1,1,1,0).$$

This clearly fits our description of a partition with only odd parts.

Likewise, we can define a transformation in the other direction to take partitions with only odd parts to those with distinct parts. We first identify all pairs of consecutive 1's in the binary representation. For each pair, count the number of 0's to the right of their location, call this number z. Now, remove the pair of 1's and insert a new 1 between 0's $2z$ and $2z+1$ if you are counting from right to left. Again, we note that this does not change the size of the partition. We are removing two rows of length z and replacing them with one row of length $2z$. If there are not $2z$ 0's in the binary string, simply add the necessary number to the left end of the string. Using the same example as above, we see that the binary string

$$(1,0,0,1,0,0,1,1,1,1,0)$$

would become

$$(1,0,0,1,0,1,1,0,0).$$

We apply this same process until the binary string contains no pairs of consecutive 1's. The example string would end up as

$$(1,0,1,0,1,0,0,0).$$

□

Definition 3.6. *We define the* Durfee Square *as the largest square that can be embedded in the Ferrers Diagram. We can say that the size of the Durfee Square for a partition λ is the largest integer i such that $\lambda_i \geq i$.*

Example 3.7. *The partition*

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & & \\ \bullet & & & \end{array}$$

has a Durfee square of size 3.

We can find an analogue of the Durfee Square in the binary representation of a partition.

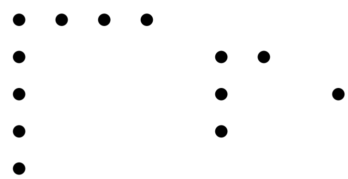
Theorem 3.8. *The size of the Durfee Square of a partition is equal to the number of nested pairs of 1's and 0's in the binary representation of the partition*

As a quick example, the binary representation of the partition above is: $(1,0,1,1,0,1,0,1,0)$. The nested pairs of 1's and 0's in this binary string are:

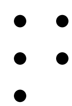
$$(\underline{1},0,\underline{1},\underline{1},\underline{0},1,\underline{0},1,\underline{0}).$$

Thus, there are three nested pairs and the partition has a Durfee Square of size 3.

We can see this equivalence in the following way. We can separate the partition above into three nested shells in the following way.



If we take the binary string beginning with the second 1 from the left and second 0 from the right, that is, the string within the second pair of nested 1's and 0's, we are left with $(1,1,0,1,0)$. This is equivalent to the following partition.



Which is also equivalent to the original partition with the outermost shell stripped away. If we take the process one step further, to the third 1 from the left and the third 0 from the right, we are left with the string $(1,0)$. It is easy to see that this is equivalent to the original partition with the two outermost shells stripped away.

Thus, each time we move to the next nested pair of 1's and 0's, we strip away the next outermost shell of the original partition. Since the number of these shells in a given partition is equal to the size of the Durfee Square, and also equivalent to the number of nested pairs of 1's and 0's, we can see that the theorem must hold.

4. Diagonal Partitions

4.1. **Introduction.** We also studied several variations on regular partitions such that there is some restriction on the partition. For example, here we consider partitions such that the ith part is less than or equal to $n-i$ for a fixed n. This case study will also allow us to demonstrate how combinatorial and generating function methods can be combined to study partitions.

Definition 4.1. *A n-diagonal partition is partition $(\lambda_1,\cdots,\lambda_k)$ such that $\lambda_i \geq \lambda_{i+1}$ and $\lambda_i \leq n-i$ for $1 \leq i \leq k$. Let $d_n(m)$ denote the number of n-diagonal partitions of m. Let*

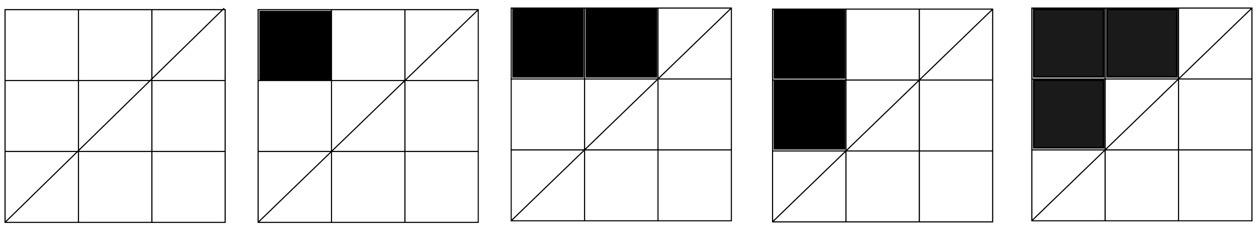
$$g_n(q) = \sum_{m\geq 0}^{\infty} d_n(m)q^m$$

be the generating function for $d_n(m)$. Note that $d_n(m) = 0$ if $m > \frac{1}{2}n(n-1)$ so that $g_n(q)$ is a polynomial with finitely many terms.

We call these partitions n-diagonal partitions because they are precisely the partitions such that their Ferrers diagram representation does not cross the diagonal of an $n \times n$ square. For example,

$$g_3(q) = 1 + q + 2q^2 + q^3$$

since the only partitions that do not cross the diagonal of a 3×3 square are:



Note that $g_0(q) = 1$ and $g_1(q) = 1$. Also note that $g_n(1)$ equals the total number of n-diagonal partitions, which are in bijection with the paths that move one unit up or to the right at a time from the lower-left-hand corner of a $n \times n$ square to the upper-right-hand corner of an $n \times n$ square and stay above the diagonal. The number of such paths is precisely the nth Catalan number C_n, which is equal to

$$\frac{1}{n+1}\binom{2n}{n}.$$

Therefore, we have

$$g_n(1) = C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Now we will study $g_n(q)$ by finding a recursive relationship for these polynomials and then use it to find a generating function for $g_n(q)$. In particular, we will prove the following two theorems:

Theorem 4.2. *$g_n(q)$ satisfies the following recursive relationship:*

$$g_n(q) \quad = \quad \begin{bmatrix} 2n \\ n \end{bmatrix} - \sum_{k=0}^{n-1} q^{(k+1)(n-k)} \begin{bmatrix} 2k+1 \\ k \end{bmatrix} g_{n-k-1}(q).$$

and

Theorem 4.3. *If $f(z) = \sum_{n=0}^{\infty} q^{-n^2/2} g_n(q) z^n$, then*

$$f(z) = \frac{\sum_{n=0}^{\infty} q^{-n^2/2} \begin{bmatrix} 2n \\ n \end{bmatrix} z^n}{1 + zq^{1/2} \sum_{n=0}^{\infty} q^{-n^2/2} \begin{bmatrix} 2n+1 \\ n \end{bmatrix} z^n}.$$

4.2. **Recursive Relationship.** To find a recursive relationship for $g_n(q)$, we count the number of n-diagonal partitions (the partitions that do not cross the diagonal) by counting the total number of partitions in an $n \times n$ square and then subtracting the number of partitions that do cross the diagonal.

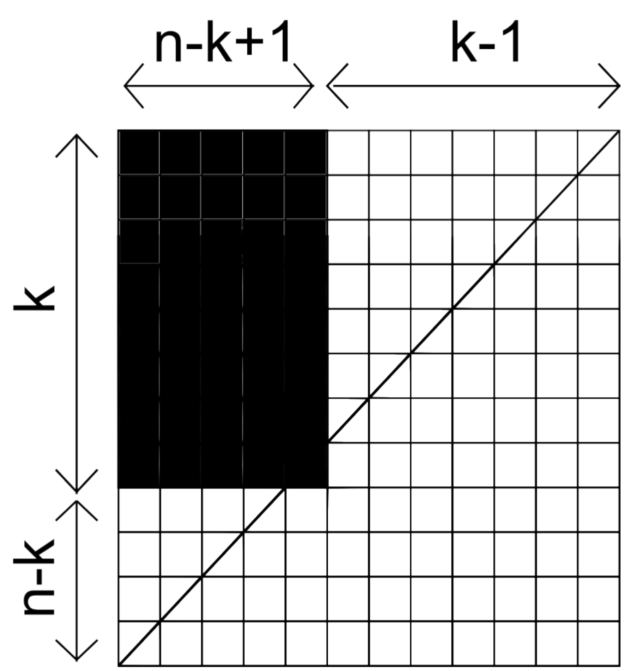
Using standard notation, we let

$$\begin{bmatrix} n+m \\ n \end{bmatrix} = \frac{(q)_{n+m}}{(q)_n (q)_m}$$

denote the Gaussian polynomial, the generating function for the number of partitions with less than m parts, each part less than n. Such partitions can be visualized as those that fit inside an $n \times m$ rectangle. Therefore, the generating function for the total number of partitions in the $n \times n$ square is

$$\begin{bmatrix} 2n \\ n \end{bmatrix}.$$

Now we need to count the number of partitions inside an $n \times n$ square that cross the diagonal. We note that every partition that goes over the diagonal has a last part that goes over the diagonal and we group these partitions based on which part crosses last. Suppose the kth part of a partition is the last that goes over the diagonal. We want to find the generating function for these partitions. We have the following situation in this case:



Since it goes over the diagonal, the kth part must have size at least $n-k+1$. This means that the first k parts all have size at least $n-k+1$. Besides the condition that all parts must be less than n (since we are counting partitions in a $n \times n$ square that go over the diagonal), there are no more restrictions on the first k parts. Therefore the generating function for partitions formed by the first k parts is

$$q^{k(n-k+1)} \begin{bmatrix} k+k-1 \\ k \end{bmatrix}.$$

Also, by definition, the last $n-k$ parts do not cross the diagonal. Therefore, the generating function for these part of the partition is $g_{n-k}(q)$ as defined before.

Therefore, the generating function for all partitions such that the kth part is the last part that crosses the diagonal is

$$q^{k(n-k+1)} \begin{bmatrix} 2k-1 \\ k \end{bmatrix} g_{n-k}(q).$$

If we sum up over k for $1 \leq k \leq n$, we get the generating function for all partitions in an $n \times n$ square that cross the diagonal. As noted before, the generating function for the n-diagonal partitions is the generating function for all partitions in an $n \times n$ square minus the generating function for all partitions in an $n \times n$ square that cross the diagonal. Therefore we have the recursive relationship for $n \geq 1$:

$$\begin{aligned} g_n(q) &= \begin{bmatrix} 2n \\ n \end{bmatrix} - \sum_{k=1}^{n} q^{k(n-k+1)} \begin{bmatrix} 2k-1 \\ k \end{bmatrix} g_{n-k}(q) \\ &= \begin{bmatrix} 2n \\ n \end{bmatrix} - \sum_{k=1}^{n} q^{k(n-k+1)} \begin{bmatrix} 2k-1 \\ k-1 \end{bmatrix} g_{n-k}(q) \\ &= \begin{bmatrix} 2n \\ n \end{bmatrix} - \sum_{k=0}^{n-1} q^{(k+1)((n-k-1)+1)} \begin{bmatrix} 2k+1 \\ k \end{bmatrix} g_{n-k-1}(q). \end{aligned}$$

which proves Theorem 4.2.

4.3. **Generating function for g_n.** We can use the recursive relationship in Theorem 4.2 to find

$$f(z) = \sum_{n=0}^{\infty} q^{-n^2/2} g_n(q) z^n,$$

a modified form of the generating function for g_n. Noting that

$$(k+1)((n-k-1)+1) = \frac{(n+1)^2 - (k+1)^2 - ((n-k-1)+1)^2}{2},$$

we can write the above relationship as

$$\begin{aligned} &q^{-(n+1)^2/2} g_n(q) \\ &\qquad = q^{-(n+1)^2/2} \begin{bmatrix} 2n \\ n \end{bmatrix} - \sum_{k=0}^{n-1} \left(q^{-(k+1)^2/2} \begin{bmatrix} 2k+1 \\ k \end{bmatrix} \right) \left(q^{-((n-k-1)+1)^2/} g_{n-k-1}(q) \right). \end{aligned}$$

Since the above relationship holds for $n \geq 1$, it is equivalent to

$$\sum_{n=1}^{\infty} q^{-(n+1)^2/2} g_n(q) z^n$$
$$= \sum_{n=1}^{\infty} q^{-(n+1)^2/2} \left[\begin{array}{c} 2n \\ n \end{array} \right] z^n - z \left(\sum_{n=0}^{\infty} q^{-(n+1)^2/2} g_n(q) z^n \right) \left(\sum_{n=0}^{\infty} q^{-(n+1)^2/2} \left[\begin{array}{c} 2n+1 \\ n \end{array} \right] z^n \right).$$

Noting that

$$q^{-(0+1)^2/2} g_0(q) z^0 = q^{-(0+1)^2/2} \left[\begin{array}{c} 2 \times 0 \\ 0 \end{array} \right] z^0 = q^{-1/2},$$

we can index all sums from 0 and get

$$\sum_{n=0}^{\infty} q^{-(n+1)^2/2} g_n(q) z^n$$
$$= \sum_{n=0}^{\infty} q^{-(n+1)^2/2} \left[\begin{array}{c} 2n \\ n \end{array} \right] z^n - z \left(\sum_{n=0}^{\infty} q^{-(n+1)^2/2} g_n(q) z^n \right) \left(\sum_{n=0}^{\infty} q^{-(n+1)^2/2} \left[\begin{array}{c} 2n+1 \\ n \end{array} \right] z^n \right).$$

Now we can solve for $f(z)$ and get

$$f(z) = \sum_{n=0}^{\infty} q^{-(n+1)^2/2} g_n(q) z^n = \frac{\sum_{n=0}^{\infty} q^{-(n+1)^2/2} \left[\begin{array}{c} 2n \\ n \end{array} \right] z^n}{1 + z \sum_{n=0}^{\infty} q^{-(n+1)^2/2} \left[\begin{array}{c} 2n+1 \\ n \end{array} \right] z^n}.$$

We can simply this formula by letting z be zq, in which case the formula becomes

$$f(z) = \sum_{n=0}^{\infty} q^{-n^2/2} g_n(q) z^n = \frac{\sum_{n=0}^{\infty} q^{-n^2/2} \left[\begin{array}{c} 2n \\ n \end{array} \right] z^n}{1 + z q^{1/2} \sum_{n=0}^{\infty} q^{-n^2/2} \left[\begin{array}{c} 2n+1 \\ n \end{array} \right] z^n},$$

which proves Theorem 4.3.

We expanded the above equation in Mathematica and multiplied the coefficient of z^n by $q^{n^2/2}$ to get $g_n(q)$. Below is a list of $g_n(q)$ for $n \leq 10$.

$$\begin{aligned}
g_0(q) &= 1\\
g_1(q) &= 1\\
g_2(q) &= 1+q\\
g_3(q) &= 1+q+2q^2+q^3\\
g_4(q) &= 1+q+2q^2+3q^3+3q^4+3q^5+q^6\\
g_5(q) &= 1+q+2q^2+3q^3+5q^4+5q^5+7q^6+7q^7+6q^8+\\
&\quad 4q^9+q^{10}\\
g_6(q) &= 1+q+2q^2+3q^3+5q^4+7q^5+9q^6+11q^7+14q^8+\\
&\quad 16q^9+16q^{10}+17q^{11}+14q^{12}+10q^{13}+5q^{14}+q^{15}\\
g_7(q) &= 1+q+2q^2+3q^3+5q^4+7q^5+11q^6+13q^7+18q^8+\\
&\quad 22q^9+28q^{10}+32q^{11}+37q^{12}+40q^{13}+44q^{14}+\\
&\quad 43q^{15}+40q^{16}+35q^{17}+25q^{18}+15q^{19}+6q^{20}+q^{21}\\
g_8(q) &= 1+q+2q^2+3q^3+5q^4+7q^5+11q^6+15q^7+20q^8+\\
&\quad 26q^9+34q^{10}+42q^{11}+53q^{12}+63q^{13}+73q^{14}+85q^{15}+\\
&\quad 96q^{16}+106q^{17}+113q^{18}+118q^{19}+118q^{20}+115q^{21}+\\
&\quad 102q^{22}+86q^{23}+65q^{24}+41q^{25}+21q^{26}+7q^{27}+q^{28}\\
g_9(q) &= 1+q+2q^2+3q^3+5q^4+7q^5+11q^6+15q^7+22q^8+\\
&\quad 28q^9+38q^{10}+48q^{11}+63q^{12}+77q^{13}+97q^{14}+\\
&\quad 116q^{15}+139q^{16}+162q^{17}+190q^{18}+215q^{19}+245q^{20}+\\
&\quad 268q^{21}+293q^{22}+314q^{23}+331q^{24}+338q^{25}+338q^{26}+\\
&\quad 326q^{27}+303q^{28}+268q^{29}+219q^{30}+167q^{31}+112q^{32}+\\
&\quad 63q^{33}+28q^{34}+8q^{35}+q^{36}\\
g_{10}(q) &= 1+q+2q^2+3q^3+5q^4+7q^5+11q^6+15q^7+22q^8+\\
&\quad 30q^9+40q^{10}+52q^{11}+69q^{12}+87q^{13}+111q^{14}+\\
&\quad 138q^{15}+171q^{16}+207q^{17}+249q^{18}+295q^{19}+348q^{20}+\\
&\quad 405q^{21}+466q^{22}+531q^{23}+598q^{24}+665q^{25}+734q^{26}+\\
&\quad 801q^{27}+862q^{28}+918q^{29}+958q^{30}+990q^{31}+1003q^{32}+\\
&\quad 995q^{33}+959q^{34}+901q^{35}+813q^{36}+704q^{37}+574q^{38}+\\
&\quad 434q^{39}+301q^{40}+182q^{41}+92q^{42}+36q^{43}+9q^{44}+q^{45}
\end{aligned}$$

Note that if $m \leq n$, then the coefficient of q^m in $g_n(q)$ will be just $p(m)$ since in this situation all the partitions of m will fit into an $n \times n$ square without crossing the diagonal. Also note that the coefficient of the highest q-term in each generating function is a triangular number $\frac{1}{2}n(n-1)$, which corresponds to the condition that the entire region above the diagonal is filled in by the partition.

5. Algebraic Structure

5.1. **Introduction.** A natural extension from the exploration of partitions is the study of multipartitions. In [FW07], Furno and Waters studied 'friendly' multipartitions (fmmp's), whose $i+1$st component is the ith component 'plus' some partition. This construction is quite natural and can be thought of as a partition of partitions. Their use of monoids allowed them to extend the natural numbers to unrestricted partitions to friendly multipartitions with relative ease. We extend their ideas to include new restricted sets of partitions and friendly multipartitions, by use of numerical monoids. Specifically, we generalize their results to subsets of natural numbers, and introduce the concepts of numerical monoids.

5.2. **Definitions and Examples.** We cite many sections verbatim from [FW07], simply generalizing some hypotheses to allow more freedom in the later sections.

Definition 5.1. *Let $M=(S,+,\leq,|\,|)$. Then M is a* special monoid *iff:*
1. $+$ is an associative, commutative binary operation on S, with an identity element 0.
2. $\leq$ is a partial ordering on S.
3. For all $a,b\in S$, $a\leq a+b$
4. $a\leq b$ implies that there exists a unique $c\in S$ such that $a+c=b$.
5. $|\,|$: $S\to\mathbb{N}^0$ satisfies the the property $|a+b|=|a|+|b|$.

By $\mathbb{N}^0$, we mean the whole numbers, $\mathbb{N}\cup\{0\}$. If, for two elements a,b there is a unique c such that $a+c=b$, then we will sometimes write $c=b-a$ even though a may not have an inverse.

We now define the construction of Furno and Waters that takes one special monoid to another containing it.

Definition 5.2. *If M is a special monoid, then define $O(M)$ to be the set*

$$\{\mu\in(m_1,m_2,\ldots,m_k,0,\ldots)|\ \mu_{i+1}\leq\mu_i\}$$

together with $+$ being the usual component-wise addition of sequences. We say $\mu^1\leq\mu^2$ if and only if there exists μ^3 such that $\mu^1+\mu^3=\mu^2$ (when μ^3 exists, it must be unique), and let $|\mu|=\sum_i|\mu_i|$.

Clearly, for any special monoid M, we have that $O(M)$ is also a special monoid. Also, we see that $O(M)$ contains M in the same sense that $M[x]$ contains M.

Example 5.3. *The whole numbers $\mathbb{N}^0$ are a special monoid with $|\,|$ defined by $|n|=n$. Let P be the set of partitions, then $P=O(\mathbb{N}^0)$. Let F be the set of fmmp's, then $F=O(P)=O^2(\mathbb{N}^0)$.*

Definition 5.4. *If M is a special monoid, define $N(M;n)$ to be the number of elements $\mu\in M$ with $|\mu|=n$. Observe that such a number may be zero.*

We recall that $\lambda\vdash n$ denotes that λ is a partition of n, and we introduce the notation $\lambda_i\in\lambda$ to denote the value of the λ_i part of λ.

Lemma 5.5. *If $N(M;n)$ exists for all $n \in S \subseteq \mathbb{N}$, then $N(O(M);n)$ also exists for all $n \in S$. In this case both M and $O(M)$ are countable.*

Proof. We know that $|\mu| = \sum_i |\mu_i|$. Thus for any element μ of $O(M)$ that contributes $N(O(M);n)$, the numbers $|\mu_i|$ must partition n with parts from S. Therefore,

$$N(O(M);n) = \sum_{\lambda \vdash n} \prod_{\lambda_i \in \lambda} N(M;\lambda_i).$$

The first part of our lemma follows because the partitions of n are finite, every partition has finitely many parts, and $N(M;m)$ is always finite. Let M be a special monoid with $N(M,n)$ always finite, then for each n, index the set $\{\mu \in M \mid |\mu| = n\}$ with natural numbers $m(\mu)$. Map each μ to the prime power $p_n^{m(\mu)}$. This is an injection from M to $\mathbb{N}$. $\square$

Lemma 5.6. *There is a bijection φ from $O(M)$ to $M[x]$ with $|\mu| = \sum_i i|\varphi(\mu)_i|$.*

Proof. Let $\mu \in O(M)$, then define $\varphi(\mu) = \sum_i (\mu_i - \mu_{i+1})x^i$. Then

$$\begin{aligned}
\sum_i i|\varphi(\mu)_i| &= \sum_i i|\mu_i - \mu_{i+1}| \\
&= \sum_{i=0}^{d} i|\mu_i| - \sum_{i=0}^{d} i|\mu_{i+1}| \\
&= \sum_{i=0}^{d} i|\mu_i| - \sum_{i=1}^{d+1} (i-1)|\mu_{i+1}| \\
&= \sum_{i=0}^{d} |\mu_i| \\
&= |\mu|.
\end{aligned}$$

In our sum, d was the degree of μ, so when we telescoped the sum, we used that $\mu_{d+1} = 0$. By definition of $O(M)$, $\mu_i - \mu_{i+1}$ exists uniquely, so our function is well defined. It is clear that elements in $O(M)$ are in bijection with their 'difference sequences' in $M[x]$. $\square$

Theorem 5.7. *If $N(M;n)$ exists for all $n \in S \subseteq N$, then*

$$\sum_{n=0}^{\infty} N(O(M);n)q^n = \prod_{j=0}^{\infty} \sum_{n=0}^{\infty} N(M;n)q^{jn}.$$

Proof. By our lemma, $N(M;n)$ exists and $O(M)$ is countable, so index all elements $\mu \in O(M)$ with 'upstairs' indices in $\mathbb{N}$. Then

$$\begin{aligned}
\sum_{n=0}^{\infty} N(O(M);n)q^n &= \sum_i q^{|\mu^i|} \\
&= \sum_i q^{\sum_j j|\varphi(\mu^i)_j|}.
\end{aligned}$$

Since φ is a bijection, we may instead sum over all elements $\sigma \in M[x]$, again indexed upstairs with natural numbers.

$$\sum_{n=0}^{\infty} N(O(M);n)q^n = \sum_i q^{\sum_j j|\sigma^i_j|}$$

$$= \sum_i q^{|\sigma_1^i|}(q^2)^{|\sigma_2^i|}(q^3)^{|\sigma_3^i|}\cdots(q^d)^{|\sigma_d^i|}$$
$$= \prod_{j=1}^{\infty}\sum_i (q^j)^{|m_i|}.$$

In the previous line we have indexed the elements of m, of M with upstairs indices in $\mathbb{N}$. This step is justified by the following bijection between terms on the last two lines:

$$q^{|\sigma_1^i|}(q^2)^{|\sigma_2^i|}\cdots(q^d)^{|\sigma_d^i|} \mapsto q^{|m^{i_1}|}(q^2)^{|m^{i_1}|}\cdots(q^d)^{|m^{i_d}|},$$

where m^{i_j} is the element of M equal to σ_j^i. This is a bijection because each element in $M[x]$ is a finite sequence in M. Notice that each term is mapped to one with the same power of q. Thus we conclude

$$\sum_{n=0}^{\infty} N(O(M);n)q^n = \prod_{j=0}^{\infty}\sum_{n=0}^{\infty} N(M;n)q^{jn}.$$

□

5.3. **Numerical Monoids.** We study, exclusively, the case where $S \subseteq \mathbb{N}^0$, which are known as numerical monoids. We can define and characterize them as follows:

Definition 5.8. *If $n_1 < n_2 < \cdots < n_t$ are natural numbers, then set*

$$\langle n_1, n_2, \ldots, n_t\rangle = \{x_1n_1 + x_2n_2 + \ldots x_tn_t| \text{ each } x_i \in \mathbb{N}^0\}.$$

The set $\langle n_1, n_2, \ldots, n_t\rangle$ is an additive submonoid of $\mathbb{N}^0$ called a numerical monoid.

It is clear that every numerical monoid is isomorphic to some minimal (i.e., least number of generators) numerical monoid we call a primitive numerical monoid where every generator is coprime.

Theorem 5.9. *If a special monoid is generated by a single element, say $S =< a >\subseteq \mathbb{N}$, then*

$$\sum_{n=1}^{\infty} N(O(S);n)q^n = \prod_{j=1}^{\infty}\frac{1}{(1-q^{ja})}.$$

Proof.

$$\begin{aligned}\sum_{n=1}^{\infty} N(O(S);n)q^n &= \prod_{j=1}^{\infty}\sum_{n=0}^{\infty} N(S;n)q^{nj}\\ &= \prod_{j=1}^{\infty}\sum_{a|n}^{\infty} N(S;n)q^{nj}\\ &= \prod_{j=1}^{\infty}\sum_{a|n} q^{nj}\\ &= \prod_{j=1}^{\infty}(1+q^{ja}+q^{2ja}+q^{3ja}+\ldots)\\ &= \prod_{j=1}^{\infty}\frac{1}{(1-q^{ja})}.\end{aligned}$$

□

We note in general that if $S =< a, b, \ldots, c >\subseteq \mathbb{N}$, then

$$\begin{aligned}\sum_{n=1}^{\infty} N(O(S);n)q^n &= \prod_{j=1}^{\infty}\sum_{n=0}^{\infty} N(S;n)q^{nj}\\ &= \prod_{j=1}^{\infty}\sum_{n\in S} N(S;n)q^{nj}\\ &= \prod_{j=1}^{\infty}\sum_{n\in S} q^{nj}\\ &= \prod_{j,k,\ldots,s\geq 0}' \frac{1}{(1-q^{ja+kb+\cdots+sc})}.\end{aligned}$$

Where the modified product in the final line indicates that at least one term of $j,k,\ldots,s$ is required to be non-zero.

So, by letting $S = \langle 1\rangle$, we note that we have

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty}\frac{1}{(1-q^n)},$$

and

$$\sum_{n=0}^{\infty} fmmp(n)q^n = \prod_{n=1}^{\infty}\frac{1}{(1-q^n)^{d(n)}},$$

where $d(n)$ is the number of divisors of n. These results were proven explicitly in [FW07].

We see that, in general, if $S = \langle a\rangle$, then forming $O(S)$ will construct a generating function for partitions with all parts divisible by a. The next natural extension, $O^2(S)$, will generate a subset of the friendly movable multipartitions where every component only has parts divisible by a.

6. Combinatorial Proof of a Partition Recursive Relation

6.1. **Introduction.** Partitions can be investigated with generating functions or with combinatorial methods. In this section, we give a combinatorial proof of a well-known partition recursive relation, which has already been proven by Gandhi using generating functions[Gan63].

Recall that $P_r(n)$ satisfies

$$\sum_{n=0}^{\infty} P_r(n)q^n = \prod_{n=1}^{\infty}\frac{1}{(1-q^n)^r}$$

for all integers r. Note that for $r\geq 1$, $P_r(n)$ is the number of r-component multipartitions of n. Later we will give a combinatorial interpretation for $P_r(n)$ when $r<0$. Gandhi described the following recursive relation for $P_r(n)$: for $n\geq 1$ and all integer r

$$P_r(n) = \frac{r}{n}\sum_{k=1}^{n} P_r(n-k)\sigma(k),$$

where $\sigma(k) = \sum_{d|k} d$ [Gan63]. In this section, we will prove Gandhi's recursive relation combinatorially for all integer r (trivially for $r = 0$).

6.2. **Proof Using Generating Functions.** We first describe Gandhi's proof, which uses generating functions. Taking the log of both sides of the generating function for $P_r(n)$, we get

$$\log\left(\sum_{n=0}^{\infty} P_r(n)q^n\right) = \log\left(\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^r}\right) = -r\sum_{n=1}^{\infty} \log(1-q^n).$$

Now differentiating with respect to q and multiplying by q, we get

$$\frac{\sum_{n=0}^{\infty} nP_r(n)q^n}{\sum_{n=0}^{\infty} P_r(n)q^n} = r\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}.$$

Noting that

$$\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} = (q+q^2+q^3+\cdots)+2(q^2+q^4+q^6+\cdots)+3(q^3+q^6+q^9+\cdots)+\cdots$$

is the generating function for $\sigma(n)$, we have

$$\sum_{n=0}^{\infty} nP_r(n)q^n = r\left(\sum_{n=0}^{\infty} P_r(n)q^n\right)\left(\sum_{n=1}^{\infty} \sigma(n)q^n\right),$$

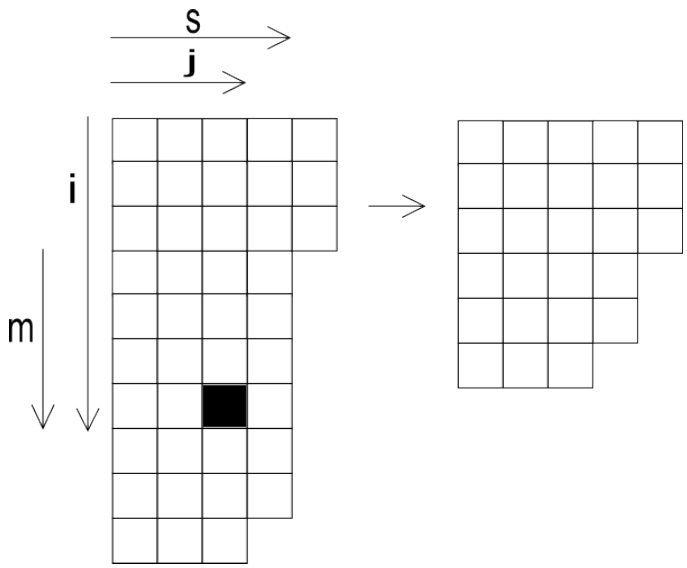
which immediately leads to Gandhi's recursive relation. Note that this proof holds for all r but here we consider only integer r.

6.3. **Combinatorial Proofs.** Now we present a combinatorial proof of Gandhi's recursive relation. We first present a proof for the case $r = 1$, the regular partitions. In this case, the recursive relation takes the form

$$np(n) = \sum_{k=1}^{n} p(n-k)\sigma(k).$$

We can interpret the left-hand-side of this relation as having n copies of each partition π of n. Since n is the number of blocks in the Ferrers diagram of π, we can associate to each copy of π a unique block of the Ferrers diagram of π and to each block a single copy. Suppose that π is associated with the (i, j) block of its Ferrers diagram (jth block in the ith part) and suppose that this block lies in the mth occurrence of parts of size s. Transform this partition by removing the first m occurrences of parts of size s. If we transform all copies of π in this way, we remove m parts of size s from s copies of π since we do this whenever $1 \le i \le s$. Note that if we transform all copies of all partitions of n, we get $np(n)$ resulting partitions of varying sizes.

For example, in the figure below, the partition on the left is associated with the $(7,3)$ block of its Ferrers diagram and $m = 4, s = 4$ since the associated block lies in the 4th occurrence of parts of size 4. We remove the first $m = 4$ occurrences of parts of size $s = 4$ from the partition on the left (the bolded region) and get the partition on the right. Note that we would remove $m = 4$ parts of size $s = 4$ from the partition on the left even if this partition were associated with any of $(1,7),(2,7),(3,7),(4,7)$ - so that we removed $m = 4$ parts of size $s = 4$ from $s = 4$ copies of the partition on the left.



To get Gandhi's recursive relation, we count the number of resulting partitions in a different way. Of the resulting partitions, consider a fixed partition π of size $n-k$, which may occur more than once. We first count how many times π occurs. π could only have been formed by removing the first m parts of the same size from one of the original partitions such that the total number of blocks removed is k. If the divisors of k are $\{d_j\}$, then this can only be done by removing the first d_j parts of size k/d_j from one of the original partitions. We also know that we remove m parts of size s from s copies of a fixed partition of n. Therefore, π occurs k/d_j times if it is formed by removing d_j parts of size k/d_j from a partition of n. Thus we see that π occurs $k/d_1 + k/d_2 + \cdots + k/d_i = \sigma(k)$ times, where $d_1, \cdots, d_i$ are all divisors of k. But since π was arbitrary, we see that every partition of $n-k$ occurs $\sigma(k)$ times. Therefore, partitions of $n-k$ occur $\sigma(k)p(n-k)$ times. This holds for all $1 \le k \le n$ and thus the total number of resulting partitions is $\sigma(1)p(n-1) + \cdots + \sigma(n)p(0)$. But we noted earlier that the number of resulting partitions is equal to $np(n)$. Thus we have

$$np(n) = \sigma(1)p(n-1) + \cdots + \sigma(n)p(0),$$

which is Gandhi's recursive relation for the case $r = 1$.

We now prove Gandhi's recursive relation for the case $r > 1$, the case of the r-component multipartitions. In this case, we have the recursive relation

$$nP_r(n) = r\sum_{k=1}^{n} P_r(n-k)\sigma(k)$$

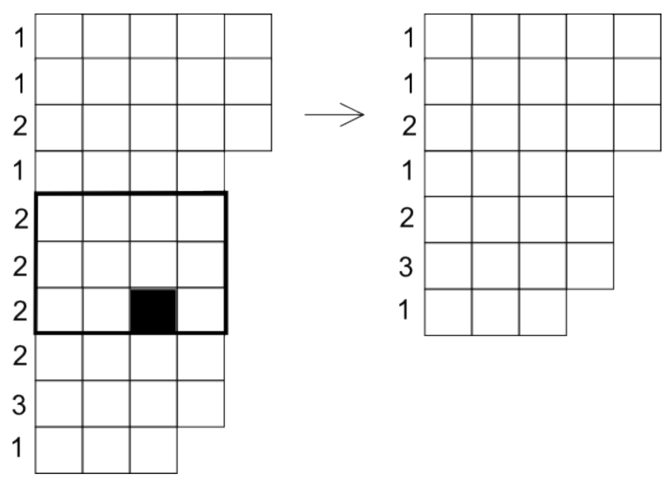
for $r > 1$. To make the combinatorial proof clearer, we introduce r-colored partitions.

Definition 6.1. *A r-colored partition λ of n is a partition of n such that are each part of λ can come in r colors and that the order of colors does not matter among parts of the same size; that is, it only matters how many parts of a certain size are of a certain colors.*

Note that there is an obvious bijection between the set of r-component multipartitions of n and the set of r-colored partitions of n. Therefore, we can consider r-colored partitions instead of r-component multipartitions.

Now we proceed as before and interpret the left-hand-side of the recursive relation as having n copies of each r-colored partition π of n. Since the order of the colors does not matter among parts of the same size, we can order the parts of the same size based on their color. We shall treat parts of the same size but of different colors as different. As before, we associate each copy of π to a unique block of its Ferrers diagram. Also as before, we transform these copies by removing parts of the same size but since we treat parts of different colors as different, we remove blocks of the same size *and* of the same color. Suppose that a partition π is associated with the (i,j) block of its Ferrers diagram and that this block lies in the mth occurrences of parts of size s *and* a fixed color c. Remove the first m occurrences of parts of size s and color c. If we transform all copies of π in this way, we remove m parts of size s from s copies of π as before. If we transform all copies of all r-colored partitions of n, we get $nP_r(n)$ resulting r-colored partitions of varying sizes.

For example, the figure below shows two 3-colored partitions (with colors 1,2, and 3). The 3-colored partition on the left is again associated with the $(7,3)$ block of its Ferrers diagram. But here $m=3, s=4, c=2$ since the associated block lies in the 3rd occurrence of parts of size 4 and color 2 since we treat parts of different colors as different. We remove the first $m=3$ occurrences of parts of size 4 and color 2 (the bolded region) from the 3-colored partition on the left to get the 3-colored partition on the right.



To get Gandhi's recursive relation, we count the number of resulting r-colored partitions in a different way. A fixed r-colored partition π of $n-k$ would occur $\sigma(k)$ times as before if we ignore the color of the original partitions from which π was formed. The removed parts were all of a single color and we have r choices for the color of the removed blocks. Thus π occurs is $r\sigma(k)$ times if we consider color. Since this is true for all r-colored partitions of size $n-k$, such partitions occur $r\sigma(k)P_r(n-k)$ times. Since this is true for $1 \le k \le n$, the total number of resulting partitions is

$$r\left(\sigma(1)P_r(n-1)+\cdots+\sigma(n)P_r(0)\right).$$

But we noted earlier that the total number of resulting partitions is equal to $nP_r(n)$. Thus we have

$$nP_r(n) = r(\sigma(1)P_r(n-1)+\cdots+\sigma(n)P_r(0)),$$

which is Gandhi's recursive relation for the $r > 1$ case.

We now prove Gandhi's recursive relation for the case $r < 0$. In this case, we have

$$\sum_{n=0}^{\infty} P_{-r}(n)q^n = \prod_{n=1}^{\infty}(1-q^n)^r$$

for $r > 0$. In order to interpret $P_{-r}(n)$ combinatorially we define $P_{-r,e}$ and $P_{-r,o}$ as the sets of r-colored partitions of n such that no two parts of the same size have the same color and that the total number of parts is even, odd, respectively. Also, let

$$P_{-r,e}(n) = \sum_{\pi|-n, \pi\in P_{-r,e}} 1$$

and

$$P_{-r,o}(n) = \sum_{\pi|-n, \pi\in P_{-r,o}} 1.$$

Then, we have $P_{-r}(n) = P_{-r,e}(n) - P_{-r,o}(n)$.

In the case $r < 0$, Gandhi's recursive relation becomes

$$nP_{-r,e}(n) - nP_{-r,o}(n) = r\sum_{k=1}^{n} P_{-r,o}(n-k)\sigma(k) - r\sum_{k=1}^{n} P_{-r,e}(n-k)\sigma(k)$$

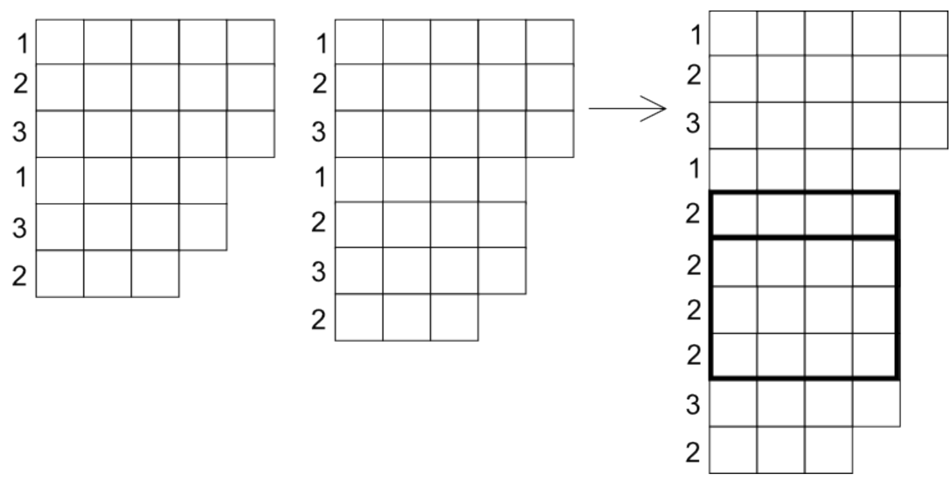
for $r > 0$. In this case, we start by interpreting the right-hand-side of the recursive relation instead of the left-hand-side. The term $rP_{-r,o}(n-k)\sigma(k)$ can be interpreted as having $r\sigma(k)$ copies of each $P_{-r,o}$ partition π of $n-k$. We can reverse the operations we performed in the first two combinatorial proofs by transforming the $r\sigma(k)$ copies of π into r-colored partitions of n by adding k blocks to each copy of π in the following way: if the divisors of k are $\{d_j\}$, then we add d_j parts of size k/d_j in any r colors to π - using the same color for all the parts - such that we add d_j parts of size k/d_j to k/d_j copies of π. We can add d_j parts of size k/d_j to k/d_j copies of π because we have $r\sigma(k)$ copies of π. The rest of the terms on the right-hand-side can be interpreted similarly and transformed into r-colored partitions of n in the same way.

The resulting r-colored partitions are all partitions of n but they may not all be in $P_{-r,o}$ or $P_{-r,e}$ - we can call the partitions that are not in $P_{-r,o}$ or $P_{-r,e}$ "bad" partitions since they do not occur in the left-hand-side of the recursive relation. Thus a "bad" partition is a partition of n such that no two parts of the same size have the same color except for exactly one set of parts of the same size and color. Note that a "bad" partition may have several identical copies.

Suppose that π is a "bad" partition with a part of size s and color c repeated $m > 1$ times and that the total number of parts in π is even. Note that π may have several identical copies. Define $\pi - (a\cdot b)_c$ as the r-colored partition achieved by removing b parts of size a of color c from π if this is possible. If m is even, then π could only be formed from a $P_{-r,e}$ partition defined by $\pi - (s\cdot m)_c$ and from a $P_{-r,o}$ partition defined by $\pi - (s\cdot(m-1))_c$. Since in both cases π is formed by adding parts of size s to the original partitions, we see that π occurs s times in both cases. In this way, each "bad" partition of n has half of its copies formed from $P_{-r,o}$ partitions and half of its copies formed from $P_{-r,e}$ partitions. Therefore when we take the difference on the right-hand-side of the recursive relation, the copies of each "bad" partitions cancel out. If m is odd, then π could only be formed from a $P_{-r,e}$ partition defined by $\pi - (s\cdot(m-1))_c$ and from a $P_{-r,o}$ partition defined by

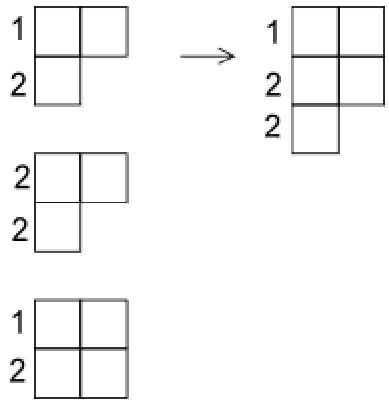
$\pi-(s\cdot m)_c$ and again all copies cancel out. Analogous results hold when the number of parts in π is odd.

For example, in the figure below, the rightmost 3-colored partition π is “bad” since it has parts of size $s=4$ and color $c=2$ repeated $m=4$ times. π has 10 parts in total and can be formed from the leftmost partition, which is $\pi-(4\cdot 4)_2$ and in $P_{-r,e}$, or from the middle partition, which is $\pi-(4\cdot 3)_2$ and in $P_{-r,o}$. The bold region in the rightmost partition indicates the parts that were added to the leftmost and middle partitions. Furthermore, we transform $s=4$ copies of $\pi-(4\cdot 4)_2$ and $\pi-(4\cdot 3)_2$ to π since we add parts of size $s=4$ and similarly for $\pi-(4\cdot 3)_2$.



Furthermore, all n copies of the $P_{-r,e}$ partitions of n are formed when we transform the $P_{-r,o}$ partitions in the right-hand-side to partitions of n. Consider a $P_{-r,e}$ partition π of n. We can determine the $P_{-r,o}$ partitions from which π is formed by deleting each of its parts one at a time. For example, suppose one of these $P_{-r,o}$ partitions is $\pi-(s\cdot 1)_c$. Note that originally we transformed s copies of $\pi-(s\cdot 1)_c$ into π because we added 1 part of size $s/1$ to $s/1$ copies of $\pi-(s\cdot 1)_c$. This is true whenever s is a part of π. Thus, we see that the number of copies of π is equal to the sum of the sizes of its parts, which is just n. Similarly, all n copies of the $P_{-r,o}$ partitions of n are formed from the $P_{-r,e}$ partitions in the right-hand-side.

For example, the figure below shows how the $P_{-2,o}$ partition of $n=5$ on the right side can be formed from the $P_{-2,e}$ partitions on the left. We transform 2 copies of the top left $P_{-2,e}$ partition into the right $P_{-2,o}$ partition (since we add a part of size 2 to this partition), 2 copies of the middle left $P_{-2,e}$ partition into the $P_{-2,o}$ partition (since we add a part of size 2 to this partition), and 1 copy of the bottom left $P_{-2,e}$ partition into the $P_{-2,o}$ partition (since we add a part of size 1 to this partition). Therefore the $P_{-2,o}$ partition on the right has $n=5=2+2+1$ copies.



Thus when we take the difference on the right-hand-side, all that remains is the difference between the $P_{-r,e}$ partitions of n, each with n copies, and the $P_{-r,o}$ partitions of n, also each with n copies. That is, we have

$$nP_{-r,e}(n) - nP_{-r,o}(n) = r\sum_{k=1}^{n} P_{-r,o}(n-k)\sigma(k) - r\sum_{k=1}^{n} P_{-r,e}(n-k)\sigma(k),$$

which is Gandhi's recursive relation for case $r < 0$. Thus we have proven Gandhi's recursive relation for all r.

6.4. **Conclusion.** Since Gandhi's recursive relation is equivalent to the q-series identity

$$\sum_{n=0}^{\infty} nP_r(n)q^n = r\left(\sum_{n=0}^{\infty} P_r(n)q^n\right)\left(\sum_{n=1}^{\infty} \sigma(n)q^n\right),$$

the combinatorial proof of the relation provides another proof of this q-series identity. It is feasible that the above method could provide combinatorial proofs to more difficult q-series identities. It also seems possible that we can use this method to find recursive relations for related generating functions such as

$$\prod_{n=1}^{\infty}\left(\frac{1}{1-q^n}\right)^{f(n)},$$

where $f(n)$ is any integer-valued function, and then prove these relations combinatorially.

7. Tri-Conjugate Shell Multipartitions

7.1. **Introduction.** We construct and study a special subset of movable multipartitions we call "tri-conjugate shell multipartitions." They were studied to a similar extent by MacMahon [Mac04] which we extend by presenting an interesting bijection involving them.

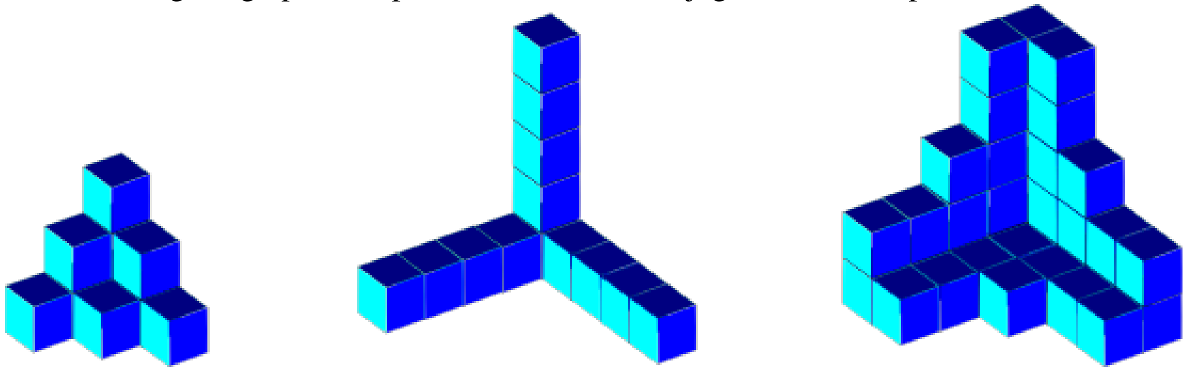
7.2. **Definitions and Examples.**

Definition 7.1. *A tri-conjugate shell multipartition is a movable multipartition, Λ, which satisfies the following criteria:*

- *If τ is any element of $S_{\{x,y,z\}}$, then $\tau \circ \Lambda = \Lambda$. We take here $S_{\{x,y,z\}}$ as the group of all permutations of the coordinate axes of the multipartition.*
- *If $i > 1$ and $j > 1$, then $\Lambda^i_j \leq 1$.*

REMARK. It is clear by the definition that by choosing $\tau = (x\,y)$ that the first component must be self-conjugate. Likewise, by choosing $\tau' = (z\,y)$ or $\tau'' = (z\,x)$ it becomes clear that the inverted partitions on the *yz* and *xz* planes are also self-conjugate.

The following are graphical representations of tri-conjugate shell multipartitions:



We will use $TCS_b(\infty)$ to be all tri-conjugate shell multipartitions which fit in a $b \times b \times b$ box.

Example 7.2. *We have the following:*

- $TCS_0(\infty) = 1$ *by the multipartition* $\emptyset$*.*
- $TCS_1(\infty) = 2$ *by the multipartitions* (1) *and* $\emptyset$*.*
- $TCS_2(\infty) = 4$ *by the shell multipartitions* (1), $(2+1,\,1)$, $(2+2,\,2+1)$, $\emptyset$*.*

We note that $TCS_b(\infty)$ restricts the set of all tri-conjugate shell multipartitions, so we introduce the following notation:

We define $TCS(n)$ to be the number of *unrestricted* tri-conjugate shell multipartitions of the integer n.

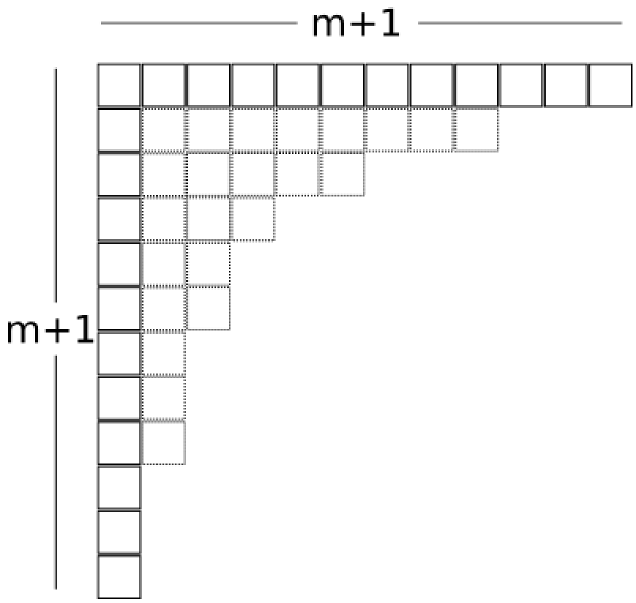
7.3. $TCS_m(\infty)$.

Theorem 7.3. *If $TCS_m(\infty)$ is the number of tri-conjugate shell multipartitions that fit in an $m \times m \times m$ box, then $TCS_m(\infty) = 2^m$ for all non-negative integers m.*

Proof of 7.3. We first observe that in order to count the number of tri-conjugate shell multipartitions in a given cube of arbitrary size, we can utilize the symmetries inherent in the definition, and so it becomes satisfactory to only count the number of self-conjugate partitions in a square of the same arbitrary size. This holds because this square projection corresponds by a clear bijection to a shell partition by simply copying and orienting the partition onto the other axis planes. Now, we prove the conjecture by induction. We observe that $TCS_0(\infty) = 1 = 2^0$. Suppose $S_m(\infty)$ counts the number of shell partitions for all squares $\leq m$. Now, we consider a square of size $(m+1) \times (m+1)$. We consider two cases: whether or or not the full $m+1$ height and width is used. If the full square is not used, then we must fit in an $m \times m$ box, so by the inductive step we have $TCS_m(\infty)$ shell partitions. We now consider the case when the full $(m+1) \times (m+1)$ height and width are met. We first observe that, clearly, the partition

$$(m+1,\, \underbrace{1,\, 1,\, \ldots,\, 1}_{m})$$

is a self-conjugate partition which fills the $(m+1)\times(m+1)$ box, and is, in fact, the minimal self-conjugate partition to do so. Then, there remains an $m\times m$ box in which we can extend our partition with any self-conjugate partition, as the example follows:



Clearly, there are another $TCS_m(\infty)$ partitions which can be "added" to the base partition $(m+1,\ldots,1)$.

So, we have a total of $2TCS_b(m)$ self-conjugate partitions in the $m+1$ case. Hence, using the inductive hypothesis:

$$\begin{aligned} TCS_{m+1}(\infty) &= 2TCS_m(\infty) \\ &= 2(2^m) \\ &= 2^{m+1} \end{aligned}$$

□

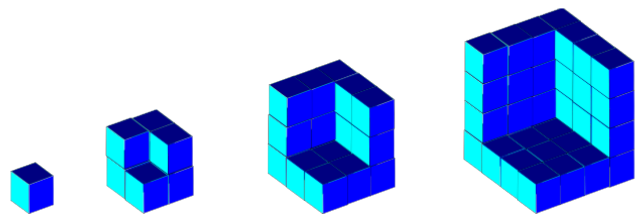
7.4. Generating Function for $TCS(n)$.

Theorem 7.4. *If $TCS(n)$ is the number of shell partitions of n then,*

$$\sum_{n=0}^{\infty} TCS(n)q^n = 1+\sum_{k=0}^{\infty}\frac{q^{3k^2-3k+1}}{(q^3;q^6)_k},$$

where $(q^3;q^6)_k = \prod_{j=0}^{k-1}(1-q^{6j+3})$.

Proof of 7.4. We note that given an arbitrary tri-conjugate shell multipartition, there is a maximal "modified Durfee cube." The following are examples of the first few cases:



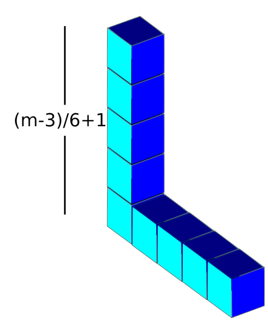
We note that these modified Durfee cubes partition integers of the form $3k^2-3k+1$. Clearly, then, the q^{3k^2-3k+1} term in the generating function sum counts these modified Durfee cubes. So, we are left to decifer the denominator of the infinite sum: $(q^3,\, q^6)_k^{-1}$. The terms of these finite products are of the form:

$$\frac{1}{1-q^3}\cdot\frac{1}{1-q^9}\cdot\dots\cdot\frac{1}{1-q^{6(k-1)+3}} = (1+q^3+q^{3+3}+\dots)(1+q^9+q^{9+9}+\dots)\cdots(1+q^{6(k-1)+3}+\dots)$$

We interpret these infinite sums as hooks which are adjoined to the base, modified Durfee cube. In particular,

$$q^{\underbrace{m+\cdots+m}_{k\ multiples}}$$

corresponds to k multiples of m-hooks adjoined on each side of the modified Durfee cube, where an m-hook has the following form:



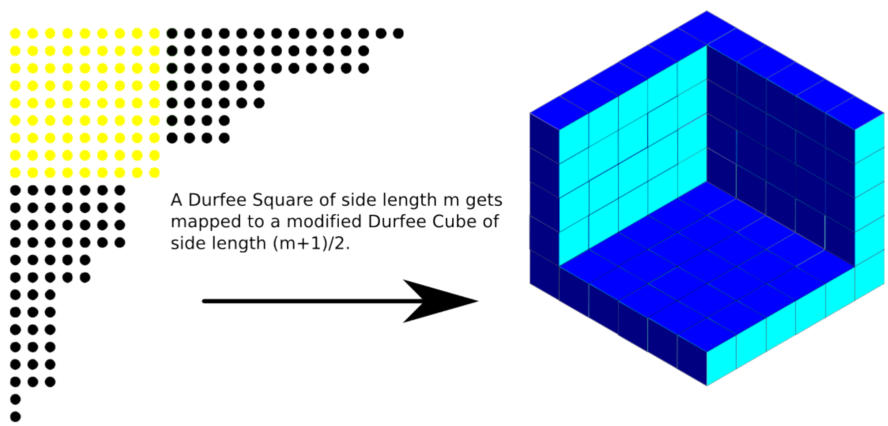
By the restriction of the finite product we note that these hooks maintain a well-defined tri-conjugate shell multipartition. Also, it is obvious that if $j<m$, then the j-hooks must be adjoined to the modified Durfee Cube after the m-hooks.

Since we can construct any tri-conjugate shell multipartition by these modified Durfee cubes and m-hooks, we have verified that the given generating function indeed counts these multipartitions. □

7.5. **Bijection Involving Tri-Conjugate Shell Multipartitions.**

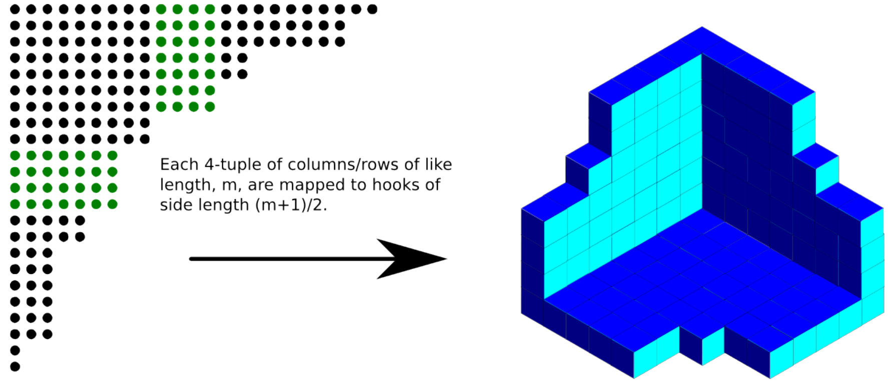
Theorem 7.5. *The number of self-conjugate partitions of* $4n+1$ *with all odd parts is equal to the number of tri-conjugate shell multipartitions of* $3n+1$.

Proof. We will provide constant graphical examples for reference. Given a partition of the first type, consider the Ferrers Diagram which describes it. Then first consider the Durfee Square:

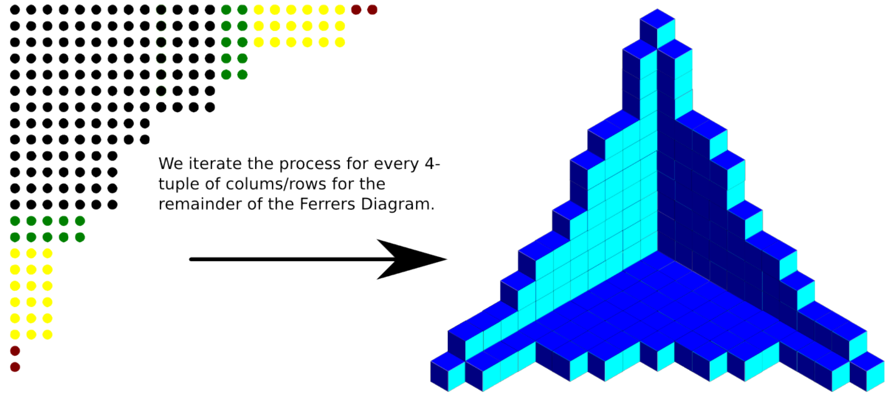


We observe that the Durfee Square describes a modified Durfee Cube under the following transformation: *If the Durfee Square has side length m, then the side length of the Durfee Cube is* $(m+1)/2$.

By the definition of the self-conjugate partitions we consider, there must be a multiple of 4 columns and rows adjoined to the Durfee Square for each integer. We group these together and use the following transformation: *If there are* $4j$ *columns and rows of length k, then there are* j *copies of a hook with side length* $(m+1)/2$ *adjoined to each side of the Durfee Cube*.



We iterate this process for the rest of the Ferrers Diagram, for each group of multiples of 4:



It is easy to check that, by the construction of each transformation, any partition of the first type of $4n+1$ will become a tri-conjugate shell multipartition of $3n+1$. To map in the other direction, the bijection is simply iterated in reverse order.

□

7.5.1. *Further Application of Bijection Method.* We can use a similar bijection proposed in the previous passage to prove another interesting partition identity. We begin by introducing a new restriction on partitions.

Lemma 7.6. *If $B(n)$ is the number of partitions of n where all even integers less than or equal to the largest part appear exactly once, then*

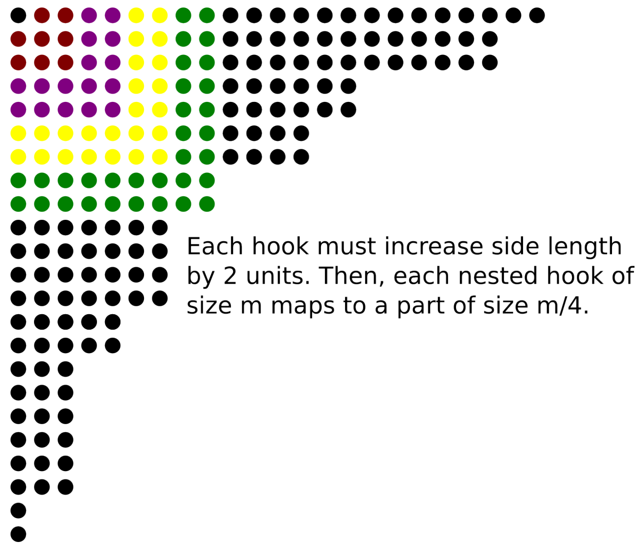
$$\sum_{n=0}^{\infty} B(n)q^n = \sum_{k=0}^{\infty} q^{k(k+1)} \prod_{j=0}^{k} \frac{1}{(1-q^{2j+1})}$$

Proof. Note that the $q^{k(k+1)}$ term corresponds to all of the even terms, since $2+4+\cdots+2k = 2(1+2+\cdots+k) = 2(k(k+1)/2) = k(k+1)$. Then each term of the finite product corresponds to some odd part at most 1 greater than the largest even part with any multiplicity, so the equivalence follows. □

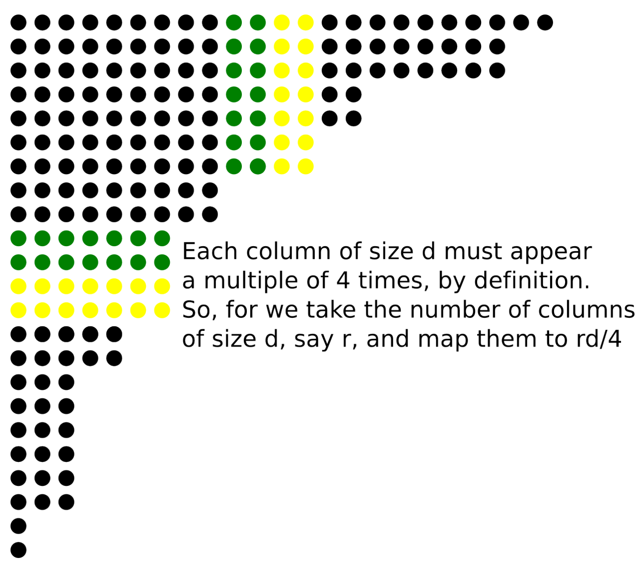
We can now prove the following:

Corollary 7.7 (Corollary to Theorem 7.5)**.** *The number of self-conjugate partitions of $4n+1$ with all odd parts is equal to $B(n)$ for all n.*

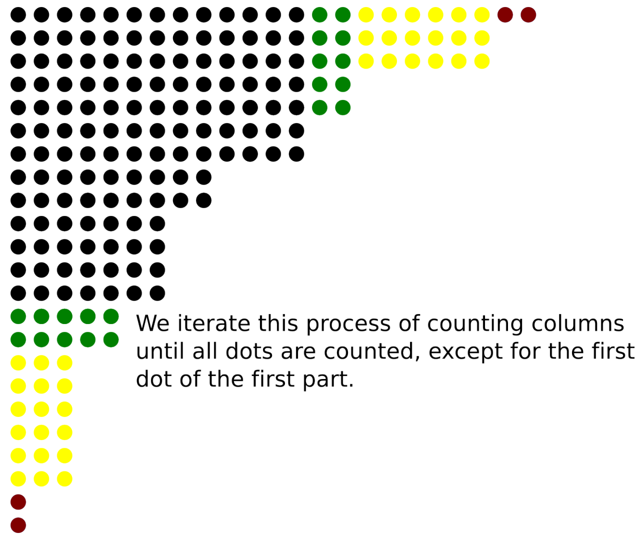
Proof. We will again provide constant graphical examples for reference. Given a partition of the first type, consider the Ferrers Diagram which describes it. Initially ignoring the dot at the origin, consider nested hooks as follows:



We have the following transformation: *If a hook has m dots, then it maps to a part of size $m/4$.* Next, we group pairs of columns (and by conjugation, rows) of dots to map to odd parts:



Formally, we have the following transformation: *If there are $4j$ columns/rows of size r, then there are j copies of a part of size r.*



We note that since the first dot is subtracted from the original partition, and every step involves a division by 4, then this clearly maps partitions of $4n+1$ to n. To map in the other direction, the bijection is simply iterated in reverse order. $\square$

7.6. **Mock Theta Function.** We note that Ramanujan's third-order mock theta function, $\nu(q)$ defined as follows

$$\nu(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_n}.$$

We see that

$$q\nu(q^3) = \sum_{n=0}^{\infty} \frac{q^{3n(n+1)+1}}{(-q^3;q^6)_n} = \sum_{n=0}^{\infty} \frac{q^{3n^2+3n+1}}{(-q^3;q^6)_n}.$$

These terms alternate sign, so the absolute value of the values of the coefficients of this series correspond to the coefficients of the generating function of $TCS(n)$.

7.7. **Conclusion.** The next logical step of these studies are to consider nesting tri-conjugate shell multipartitions to retain all of the same symmetries, however weakening the restrictions imposed in the definition. The difficulty of this problem lies in the amount of freedom and variables apparent in the subsequent "tiers" of tri-conjugate shells. Any generating function extensions we discovered were too unwieldy to reveal any new or useful information. Also, the relationship of the generating function to mock theta functions is interesting, and could present new approaches to similar problems, or reveal interesting results by exploiting tools of modular forms.

8. Periodic Movable Multipartition Congruences

8.1. **Introduction and Statement of Results.** Much work has been completed on congruences of the unrestricted and restricted partition functions as originally presented and proven by Ramanujan [Ram19] using q-series analysis. In a celebrated article [Dys44], Dyson presented a statistic known as the "rank," later proven by Atkin and Swinnerton-Dyer [ASD54], and proposed a statistic he called the "crank," later discovered by Andrews and Garvan [AG88]. These provide combinatorial tools to prove Ramanujan's famous congruences, as well as many generalizations to other primes and multipartitions. Other modern proofs rely on results of modular forms [LSY08]. In this section we consider a new class of movable multipartition congruences proven with a new method founded in work completed by Kwong [Kwo89a]. Recall that a movable multipartition, with k-components (alternatively, a plane partition with k-components) is a multipartition, $\Lambda = (\lambda^1, \lambda^2, \ldots, \lambda^k)$, satisfying the following criterion: $\lambda^i_j \geq \lambda^{i+1}_j$ and $\lambda^i_j \geq \lambda^i_{j+1}$, for all i and j. We let $m_k(n)$ count the number of k-component movable multipartitions of n. Then, the following identity holds [And98]:

$$\mathcal{M}(q) = \sum_{n=0}^{\infty} m_k(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{min(k,n)}}.$$

Specifically, we study the sequence of $m_k(n)$ for $k = p$, where p is a prime. It turns out that the coefficients of the rational polynomial functions we study have periods modulo primes. Therefore, we let $\pi(k)$ be the minimum period of the coefficients of $\frac{1}{(1-q)(1-q^2)^2 \ldots (1-q^{k-1})^{k-1}}$ modulo k. The purpose of this paper is to prove the following:

Theorem 8.1. *Let $s,t < \infty$ and $\beta_i, \gamma_j \in \mathbb{N}$ for all i, j. If*

$$\sum_{i=1}^{s} m_k(kn+\beta_i) \equiv \sum_{j=1}^{t} m_k(kn+\gamma_j) \pmod{k}$$

holds for all $n < \pi(k)$, then it holds for all $n \in \mathbb{Z}$.

These types of congruences were studied by Gandhi [Gan67], and proven using relationships between $m_k(n)$ and regular multipartition functions, for which many congruences were already known. Our approach does not rely on previous results of multipartition congruences, but rather on the cyclic nature of these types of functions modulo prime numbers.

The strength of this theorem rests in the ability to confirm an infinite congruence by a finite number of calculations. At the end of this paper, we list all confirmed congruences of this form known to date, including a seemingly new one modulo 7.

8.2. **Prime Modulus Congruence.** We begin by proving a useful lemma which will become invaluable later.

Lemma 8.2. *If p is prime, then*

$$\frac{1}{(1-q^j)^p} \equiv \frac{1}{(1-q^{jp})} \pmod{p}$$

for all $j \in \mathbb{N}$

Proof. First, assume $p > 2$. Then,

$$\begin{aligned}
\frac{(1-q^j)^p}{(1-q^{jp})} &= \frac{(1-\binom{p}{1}q^j+\binom{p}{2}q^{2j}-\cdots+(-1)^p q^{jp})}{(1-q^{jp})} \\
&\equiv \frac{(1+(-1)^p q^{jp})}{1-q^{jp}} \pmod{p} \\
&\equiv (1-q^{jp})(1+q^{jp}+q^{2jp}+\dots) \pmod{p} \\
&\equiv 1 \pmod{p}
\end{aligned}$$

Now, assume that $p = 2$. Then,

$$\begin{aligned}
\frac{(1-q^j)^2}{(1-q^{2j})} &= \frac{(1-2q^j+q^{2j}}{(1-q^{2j})} \\
&\equiv \frac{1+q^{2j}}{1-q^{2j}} \pmod{p} \\
&\equiv (1+q^{2j})(1+q^{2j}+q^{4j}+\dots) \pmod{p} \\
&\equiv 1+2q^{2j}+2q^{4j}+\dots \pmod{p} \\
&\equiv 1 \pmod{p}
\end{aligned}$$

Thus, for all prime, p,

$$\frac{(1-q^j)^p}{(1-q^{jp})} \equiv 1 \pmod{p}$$

Since $0 \neq (1-q^j)^p \pmod{p}$, has an inverse modulo p, we can divide both sides of the equation by $(1-q^j)^p$, thus proving the lemma.

□

We now define several concepts of periodicity in order to properly introduce a theorem proven by Kwong [Kwo89b].

8.3. **The Periodicity of the Partition Function Modulo Primes.**

Definition 8.3. *If $A(x) \in \mathbb{Z}[\![x]\!]$ generates an infinite integer sequence $\{a_n\}_{n\geq 0}$, then a* period of $A(x)$ modulo M *is a positive integer π such that for some nonnegative integer Γ,*

$$a_{n+\pi} \equiv a_n \pmod{M} \textit{ for } n \geq \Gamma$$

We say that the smallest such π is the *minimum period*, and $A(x)$ is *purely periodic* if $\Gamma = 0$.

Definition 8.4. *For a given integer α and any prime p, we define $e(\alpha)$ to be the integer satisfying:*

$$p^{e(\alpha)} | \alpha \textit{ and } p^{e(\alpha)+1} \nmid \alpha$$

The prime number p is always clear from the context of the use.

Definition 8.5. *We define $F(M)$ to be*

$$F(M) := \prod_{n=0}^{M-1} \frac{1}{(1-q^n)^n}$$

We note that $\sum_{n=0}^{\infty} m_p q^n = F(p) \prod_{j=p}^{\infty} \frac{1}{(1-q^j)^p}$

We can now state the theorem proven by Kwong, followed by a corollary:

Theorem 8.6 ([Kwo89b], Theorem 14)**.** *Let S be a multiset of positive integers, L be the p-free part of $lcm\{\alpha|\alpha \in S\}$, and b be the least integer such that*

$$p^b \geq \sum_{\alpha \in S} p^{e(\alpha)}$$

Then, $\{p(n;S) \pmod{p^N}\}_{n\geq 0}$ is purely periodic with minimum period $p^{N+b-1}L$.

Corollary 8.7. *The sequence $\{F(p) \pmod{p}\}_{n=0}$ is periodic.*

Proof. We first note that

$$F(p) := \frac{1}{(1-q)(1-q^2)^2 \dots (1-q^{p-1})^{p-1}}$$

We let a multiset be defined by $S := \{i_j | 1 \leq i \leq p-1, 1 \leq j \leq i\}$. We let $|i_j| = i$ for all j, so by this construction, it is clear that our choice of S implies that $F(p) = p(n;S)$. Hence, it follows that $\{F(q) \pmod{p}\}$ is purely periodic. □

Using Theorem 8.6, it is then possible to calculate the minimum period of each $F(p)$. If we let $\pi(p)$ denote the minimum period of $F(p)$, then we can calculate the following:

Example 8.8.
- $\pi(3) = 3^1 \cdot 2 = 6$
- $\pi(5) = 5^2 \cdot 12 = 300$
- $\pi(7) = 7^2 \cdot 60 = 2940$

It is then clear from the work above, including Lemma 8.2, that

$$\sum_{n=0}^{\infty} m_p q^n \equiv \left(\sum_{k=0}^{\infty} \sum_{n=0}^{\pi(p)-1} \alpha_n q^{\pi(p)k+n} \right) \left(\prod_{j=p}^{\infty} \frac{1}{(1-q^{pj})} \right) \pmod{p} \tag{1}$$

8.4. **Further Generalizations and Proof of Main Theorem.**

Proof of Theorem 8.1. We prove a basic case where the congruence holds between just two coefficients. The extension to any finite number of coefficients follows easily by induction.

By the hypotheses of the theorem, we can assume that the equivalence

$$m_p(pn+\beta) \equiv m_p(pn+\gamma) \pmod{p}$$

holds for all $n < \pi(p)$.

So, it must hold that $\alpha_\beta \equiv \alpha_\gamma \pmod{p}$, $\alpha_{p+\beta} \equiv \alpha_{p+\gamma} \pmod{p}, \dots, \alpha_{p(\pi(p)-1)+\beta} \equiv \alpha_{p(\pi(p)-1)+\gamma}$ $\pmod{p}$, where α_n is the coefficient of the n^{th} term of the q-series expansion of $\mathcal{M}(x)$.

In particular, these assumptions imply that the proof of the theorem is satisfied by showing that every α_n is equivalent to a linear combination of α_{k_i}'s, where each $k_i < \pi(p)$ and each k_i has the same residue modulo p. Since, if this holds, then $\alpha_{pn+\beta} \equiv \alpha_{pn+\gamma}$ for any n, which proves the theorem.

To show this fact, we consider the action of "choosing" q's from the infinite product and the infinite sum in (1) to "build" the exponent of q. From the infinite product in (1), we observe that it is only possible to collect terms of the form q^{pj}, so any non-zero residue modulo p attributed to the exponent of q must come from the choice of a single term from the infinite sum on the right hand side of (1). So, consider an arbitrary selection of q^{pj_i} and a single $q^{pn+\beta}$ such that the sum of the exponents is $pm+\beta$. By Corollary 8.7, it is clear that $\alpha_{pn+\beta} \equiv \alpha_{pn'+\beta} \pmod p$ for some $n' < \pi(p)$. Also, it is clear that $\alpha_{j_i} \equiv 1 \pmod p$ for all i. Hence,

$$\prod_k \alpha_k \equiv \alpha_{pn'+\beta} \pmod p$$

where k ranges over all of the exponents of the q's chosen. Since any "construction" of q's which contribute to the coefficient of the $q^{pm+\beta}$ term will follow this same pattern, it is obvious that the term selected from the infinite sum must have residue β modulo p. Therefore, the coefficient of an arbitrary $q^{pm+\beta}$ term is equivalent to some linear combination of these $\alpha_{pn+\beta}$ for $n < \pi(p)$, which was to be shown.

$\square$

8.5. **Proof of Modulo 3 Cases of Main Theorem.** The motivation for the following proof of the modulo 3 cases of Theorem 9.8 is now clear. We present it with a clever characterization of the coefficients into two classes. This simply disguises the underlying cyclic principle driving the proof, however such a characterization would be incredibly useful to extend this proofing method to larger prime powers, as it eliminates tedious calculations.

Proof of Modulo 3 Cases. We first note that

$$\sum_{n=0}^{\infty} m_3(n) q^n = \frac{1}{(1-q)(1-q^2)^2} \prod_{n=3}^{\infty} \frac{1}{(1-q^n)^3}$$

We wish to examine the expansion of $\frac{1}{(1-q)(1-q^2)^2}$. Note that

$$\frac{1}{(1-q)(1-q^2)^2} = (1+q+q^2+\dots)(1+2q^2+3q^4+4q^6+\dots)$$

We wish to classify the coefficient for an arbitrary αq^N. We observe that we essentially "choose" one term from each infinite sum, say q^j from the first sum and βq^n from the second sum such that $j+n=N$. In particular, for any N, we can "choose" all terms from the second infinite sum, say βq^n, such that $n \le N$ and will find an appropriate term q^{N-n} from the first sum. So, it is easy to calculate that the coefficient of q^N must necessarily be:

$$\alpha = \begin{cases} \frac{(N+2)(N+4)}{8} & \text{if } N \in 2\mathbb{N} \\ \frac{(N+1)(N+3)}{8} & \text{if } N \in 2\mathbb{N}+1 \end{cases}$$

We check all cases of $N = 6n+k$ for $0 \le k < 6$ modulo 3:

$$\text{If } N = 6n, \text{ then } \alpha = \frac{(6n+2)(6n+4)}{8} \equiv \frac{36}{8}n^2 + \frac{36}{8}n + 1 \equiv 1 \pmod 3$$

likewise,

$$\begin{aligned}
&\text{If } N = 6n+1, \text{ then } \alpha \equiv 1 \pmod 3\\
&\text{If } N = 6n+2, \text{ then } \alpha \equiv 0 \pmod 3\\
&\text{If } N = 6n+3, \text{ then } \alpha \equiv 0 \pmod 3\\
&\text{If } N = 6n+4, \text{ then } \alpha \equiv 0 \pmod 3\\
&\text{If } N = 6n+5, \text{ then } \alpha \equiv 0 \pmod 3
\end{aligned}$$

Hence, we can conclude that

$$\frac{1}{(1-q)(1-q^2)^2} \equiv \sum_{j=0}^{\infty} (1 \cdot (q^{6j}) + 1 \cdot (q^{6j+1})) \pmod 3$$

Therefore,

$$\sum_{n=0}^{\infty} m_3(n) q^n \equiv \sum_{j=0}^{\infty} \left(1 \cdot (q^{6j}) + 1 \cdot (q^{6j+1})\right) \left(\prod_{k=0}^{\infty} \frac{1}{(1-q^{3k})}\right) \pmod 3$$

by (1).
We let

$$A(q^3) := \left(\sum_{n=0}^{\infty} 1 \cdot (q^{6j})\right) \left(\prod_{k=0}^{\infty} \frac{1}{(1-q^{3k})}\right) = \sum_{n=0}^{\infty} \alpha_{3n} q^{3n}$$

So,

$$\sum_{n=0}^{\infty} m_3(n) q^n \equiv A(q^3) + qA(q^3) = \sum_{n=0}^{\infty} \alpha_{3n} q^{3n} + \sum_{n=0}^{\infty} \alpha_{3n} q^{3n+1} \pmod 3$$

Since the power series are not supported by any power of q congruent to 2 modulo 3, then (3) follows.

Also, given an arbitrary q^{3n} term, it is clear that the coefficient of it and q^{3n+1} are the same modulo 3, in particular, they are both congruent to α_{3n}, thus proving (4).

□

8.6. **Conclusion.** It is clear that these results are heavily limited by the computational capabilities available. We conjecture that these type of congruences hold for an infinite number of primes and for powers of primes. However, beyond these small primes, it is beyond our current capabilities to compute the necessary calculations. As such, we propose that our results can be further simplified by finding more truncated and feasible forms of the generating functions, or more powerful tools to deal with what is the essential problem of classifying the coefficients of $F(p)$. Also, strengthening Lemma 8.2 could help extend our proof to prime powers. An interesting problem lies in whether any more congruences of the form (3) exist. It seems, to the author, at this point, that such types will not exist beyond this exceptional case, however a proof of confirmation or contradiction to this claim has yet to be examined. Thus far, $m_3(pn+\alpha) \not\equiv 0 \pmod p$ for any prime $p \leq 113$ or any

α. Also, it is worthwhile realizing that the proof of (3) and (4) relies on a clever counting of the coefficients which are then reduced modulo 6. New tools would be invaluable in this end to create more efficient ways to find congruences and prove them.

8.7. **Congruences.** We list every congruence we have confirmed using this theorem and current computational capabilities.

8.7.1. *Proven Congruences.*

Theorem 8.9. *If $m_k(n)$ is defined by the number of k-component movable multipartitions, then the following hold:*

$$m_2(2n+1) \equiv m_2(2n) \pmod 2 \tag{2}$$

$$m_3(3n+2) \equiv 0 \pmod 3 \tag{3}$$

$$m_3(3n+1) \equiv m_3(3n) \pmod 3 \tag{4}$$

$$m_5(5n+2) \equiv m_5(5n+4) \pmod 5 \tag{5}$$

$$m_5(5n+1) \equiv m_5(5n+3) \pmod 5 \tag{6}$$

$$m_7(7n+2) + m_7(7n+3) \equiv m_7(7n+4) + m_7(7n+5) \tag{7}$$

for all $n \in \mathbb{N}$

8.7.2. *Conjectured Congruence.* The following also seems to hold:

Conjecture 8.10.

$$m_4(4n+1) \equiv m_4(4n+2) + m_4(4n+3) \pmod 4 \tag{8}$$

9. Congruences and Modular Forms

9.1. **Introduction and Statement of Results.** One of the most surprising and fascinating results in the study of partitions is the fact that the partition function $p(n)$ satisfies certain congruence properties. Ramanujan was the first key figure in the study partition congruences and he discovered and proved the following classical congruences: for all $n \in \mathbb{Z}$,

$$p(5n+4) \equiv 0 \pmod 5 \tag{9}$$

$$p(7n+5) \equiv 0 \pmod 7 \tag{10}$$

$$p(11n+6) \equiv 0 \pmod{11} \tag{11}$$

Many proofs of these congruences exist, relying on a variety of different methods such as q-series manipulation, combinatorial interpretation, or the theory of modular forms, including Hecke operators. In 2003, Lachterman, Schayer, and Younger [LSY08] presented a new proof utilizing modular forms but relying on little more than basic modular form theory and without using Hecke operators. This paper proposes to show the adaptability of their proof by generalizing it to infinite families of multipartitions, which also encapsulate the classical Ramanujan congruences.

Following the notation of Kiming and Olsson[KO92], we say that there is a congruence at (ℓ^m, r, a) if for all $n \in \mathbb{N}$,

$$P_r(\ell^m n + a) \equiv 0 \pmod{\ell^m}.$$

Our main result in this section is the following theorem:

Theorem 9.1. *Let $\ell \geq 5$ be a prime, m a positive integer. Let $0 \leq r < \ell^m$ such that $r \equiv -4 \pmod{\ell^{m-1}}$, and define $\delta_{r,\ell,m} := r\left(\frac{\ell^{2m}-1}{24}\right)$. Suppose that for each $1 \leq k < m$,*

$$P_r(\ell^k n - \delta_{r,\ell,k}) \equiv 0 \pmod{\ell^k}$$

for all $n \geq 0$. Then

$$P_r(\ell^m n - \delta_{r,\ell,m}) \equiv 0 \pmod{\ell^m} \tag{12}$$

for all $n \geq 0$ if and only if (12) holds for the particular values $0 \leq n < \frac{\ell^m+2\ell+2}{24}$.

This theorem will allow us to prove congruences for modulo primes other than $\ell = 5, 7, 11$ as well as modulo some prime powers. For example, we will prove the congruence

$$p(25n+24) \equiv 0 \pmod{25}.$$

9.2. **Proof of Results.** We prove several small results using [LSY08] as a guide and citing several theorems verbatim. First, we will prove the following lemma, which will be used repeatedly throughout. This lemma will allow us to prove congruences modulo prime powers.

Lemma 9.2. *Let $\mathbb{Z}[[q]]$ be the set of formal power series with integer coefficients. Suppose ℓ is a prime, $m \geq 1$ is an integer, and $f(q) \in \mathbb{Z}[[q]]$. Then there exist $f_0, \cdots, f_m \in \mathbb{Z}[[q]]$ such that*

$$f(q)^{\ell^m} = f_0(q^{\ell^m}) + \ell f_1(q^{\ell^{m-1}}) + \ell^2 f_2(q^{\ell^{m-2}}) + \cdots + \ell^{m-1} f_{m-1}(q^{\ell}) + \ell^m f_m(q).$$

Remark. Note that Lemma 9.2 implies that if $f(q)^{\ell^m} = \sum_n c_n q^n$, then $\ell^k \nmid n$ implies $\ell^{m-k+1} \mid c_n$.

Proof. We will prove this lemma using induction on m. For the case $m = 1$, suppose that $f(q) = \sum_{n \geq 0} a_n q^n$. Then

$$f(q)^{\ell} = \sum_{k_0+k_1+\cdots=\ell, k_i \geq 0} \binom{\ell}{k_0, k_1, \cdots} (a_0 q^0)^{k_0} (a_1 q^1)^{k_1} \cdots . \tag{13}$$

Note that only finitely many k_i are non-zero since $k_0 + k_1 + \cdots = \ell$ and $k_i \geq 0$. Therefore $\binom{\ell}{k_0,k_1,\cdots}$ makes sense and note that $\ell \mid \binom{\ell}{k_0,k_1,\cdots}$ unless $k_i = \ell$ for some i and $k_j = 0$ for $j \neq i$. Therefore, each term in right-hand-side sum of (5) is either divisible by ℓ or is of the form $(a_i q^i)^{\ell}$ for some i. Therefore, we have

$$f(q)^{\ell} = g_0(q^{\ell}) + \ell g_1(q)$$

for some $g_0, g_1 \in \mathbb{Z}[[q]]$, which proves the $m = 1$ case.

Now suppose that the lemma holds for some $m \geq 1$. Then

$$\begin{aligned} f(q)^{\ell^{m+1}} &= (f_0(q^{\ell^m}) + \ell f_1(q^{\ell^{m-1}}) + \ell^2 f_2(q^{\ell^{m-1}}) + \cdots + \ell^{m-1} f_{m-1}(q^{\ell}) + \ell^m f_m(q))^{\ell} \\ &= \sum_{a_0=\ell, a_0 \geq 1} \binom{\ell}{a_0} f_0(q^{\ell^m})^{a_0} \\ &+ \sum_{a_0+a_1=\ell, a_1 \geq 1} \binom{\ell}{a_0, a_1} f_0(q^{\ell^m})^{a_0} (\ell f_1(q^{\ell^{m-1}}))^{a_1} \\ &\vdots \\ &+ \sum_{a_0+a_1+\cdots+a_m=\ell, a_m \geq 1} \binom{\ell}{a_0, a_1, \cdots, a_m} f_0(q^{\ell^m})^{a_0} (\ell f_1(q^{\ell^{m-1}}))^{a_1} \cdots (\ell^m f_m(q))^{a_m}. \end{aligned}$$

From the case $m = 1$, we see that the first sum is

$$f_0(q^{\ell^m})^{\ell} = g_0(q^{\ell^{m+1}}) + \ell g_1(q^{\ell^m})$$

for some $g_0, g_1 \in \mathbb{Z}[[q]]$. Now consider the rest of the sums. Note that the kth sum (with index $a_0 + \cdots + a_k = \ell, a_k \geq 1, k \geq 1$) is divisible by ℓ^k since $a_k \geq 1$. Also, $\ell \mid \binom{\ell}{a_0,a_1,\cdots,a_k}$ unless $a_k = \ell$ as before. So each term in the kth sum except for the $a_k = \ell$ term is divisible again by ℓ. But the $a_k = \ell$ term is divisible by $\ell^{k\ell}$ and therefore by ℓ since $k \geq 1$. So the kth sum is divisible by ℓ^{k+1}. Furthermore, the kth sum is a power series in $q^{\ell^{m-k}}$ since it is the product of power series in $q^{\ell^{m-j}}$ for $j \leq k$. Thus the kth sum is of the form $\ell^{k+1} g_{k+1}(q^{\ell^{m-k}})$, where $g_{k+1} \in \mathbb{Z}[[q]]$. Therefore we have

$$f(q)^{\ell^{m+1}} = g_0(q^{\ell^{m+1}}) + \ell g_1(q^{\ell^m}) + \ell^2 g_2(q^{\ell^{m-1}}) + \cdots + \ell^{m+1} g_{m+1}(q),$$

which proves the $m+1$ case. The lemma follows by induction. □

Lemma 9.2 allows us to find an infinite class of congruences modulo prime powers once we know one set of congruences. We prove this in the following lemma:

Lemma 9.3. *There is a congruence at (ℓ^k, r, a) for $k \leq m$ if and only if there is a congruence at $(\ell^k, r + \ell^m, a)$ for $k \leq m$.*

Remark. Lemma 9.3 is a generalization of a lemma of Kiming and Olsson[KO92], which is the special case $m = 1$: there is a congruence at (ℓ, r, a) if and only if there is a congruence at $(\ell, r + \ell, a)$.

Proof. From Lemma 9.2 we have that

$$\left(\prod_{n \geq 1} \frac{1}{1-q^n}\right)^{\ell^m} = f_0(q^{\ell^m}) + \ell f_1(q^{\ell^{m-1}}) + \ell^2 f_2(q^{\ell^{m-1}}) + \cdots + \ell^{m-1} f_{m-1}(q^{\ell}) + \ell^m f_m(q)$$

for some $f_i \in \mathbb{Z}[[q]]$. Now let $f_i(q^{\ell^{m-i}}) = \sum_n c_i(n\ell^{m-i}) q^{n\ell^{m-i}}$. Note that

$$\left(\prod_{n \geq 1} \frac{1}{1-q^n}\right)^{r+\ell^m} = \left(\prod_{n \geq 1} \frac{1}{1-q^n}\right)^{r} \left(\prod_{n \geq 1} \frac{1}{1-q^n}\right)^{\ell^m}$$

implies that

$$\sum_n P_{r+\ell^m}(n)q^n = \left(\sum_n P_r(n)q^n\right)\left(\sum_n c_0(\ell^m n)q^{\ell^m n} + \ell\sum_n c_1(\ell^{m-1}n)q^{\ell^{m-1}n} + \cdots + \ell^m \sum_n c_m(n)q^n\right).$$

Since the sums on the right-hand-side are supported only on powers of l, we have

$$\begin{aligned} P_{r+\ell^m}(n\ell^m + a) &= \sum_{i+j=n} P_r(\ell^m i + a)c_0(\ell^m j) \\ &+ \ell \sum_{i+j=n\ell} P_r(\ell^{m-1} i + a)c_1(\ell^{m-1} j) \\ &\vdots \\ &+ \ell^{m-1} \sum_{i+j=n\ell^{m-1}} P_r(\ell i + a)c_{m-1}(\ell j) \\ &+ \ell^m \sum_{i+j=n\ell^m} P_r(i+a)c_m(j). \end{aligned}$$

Consider the kth sum (with index $i+j=n\ell^k, k \geq 0$). By the hypothesis, we have a congruence at (ℓ^{m-k}, r, a) for $k \leq m$, which means that $P_r(\ell^{m-k}i + a) \equiv 0 \pmod{\ell^{m-k}}$. Noting that the kth sum has a ℓ^k factor in front, we see that the kth sum is divisible by $\ell^{m-k} \cdot \ell^k = \ell^m$. Since this is true for all k, we have $P_{r+\ell^m}(n\ell^m + a) \equiv 0 \pmod{\ell^m}$, which means we have a congruence at $(\ell^m, r+\ell^m, a)$.

Now we proceed by induction on m. The case $m = 1$ follows immediately from the fact that we have a congruence at $(\ell, r+\ell, a)$. Assume that the theorem holds for some $m \geq 1$; we will prove the $m+1$ case. Thus we start with congruences at (ℓ^k, r, a) for $k \leq m+1$ and need to prove congruences at $(\ell^k, r+\ell^{m+1}, a)$ for $k \leq m+1$. Since there are congruences at (ℓ^k, r, a) for $k \leq m$ and the theorem holds for m case by the induction hypothesis, there are congruences at $(\ell^k, r+\ell^m, a)$ for $k \leq m$. Thus there are congruences at $(\ell^k, r+n\ell^m, a)$ for $k \leq m$ for any n. If $n = \ell$, we get congruences at $(\ell^k, r+\ell^{m+1}, a)$ for $k \leq m$. To prove the $m+1$ case of the theorem, we need the $(\ell^{m+1}, r+\ell^{m+1}, a)$ congruence, which follows from the previous discussion. Thus the $m+1$ case holds and the theorem follows by induction. The only if case is proved analogously. □

Lemma 9.3 implies that if we want to find all r such that there are congruences at (ℓ^k, r, a) for all $k \leq m$, it is sufficient to check $0 \leq r < \ell^m$. Therefore, we can assume that $r < \ell^m$ in such a situation, which explains the condition $r < \ell^m$ in Theorem 9.1. It would be interesting to see whether Lemma 9.3 can be strengthened to a statement of the form: there is a congruence at (ℓ^m, r, a) if and only if there is a congruence at $(\ell^m, r+\ell^m, a)$, without requiring congruences at (ℓ^k, r, a) for $k < m$. We briefly investigated this situation but in fact could not find any situation in which there is a congruence at (ℓ^m, r, a) but not a congruence at (ℓ^{m-1}, r, a) so that strengthening the lemma would seem to have no application. This raises the question of whether it is possible for there to be congruence at (ℓ^m, r, a) but not a congruence at (ℓ^{m-1}, r, a). Clearly, the opposite situation in which there to be congruence at (ℓ^{m-1}, r, a) but not a congruence at (ℓ^{m-1}, r, a) is possible. For example, there is a congruence at $(5, 4, -1)$ but not at $(25, 4, -1)$.

To prove Theorem 0.1, we utilize an elegant and simple result from modular form theory due to Choie, Kohnen, Ono[CKO05]. We define $const(f)$ to be the constant term in the q-series expansion of $f(z)$ and define $\tilde{E}_k$, for all positive even integers k, by

$$\tilde{E}_k = \begin{cases} 1, & k \equiv 0 \pmod{12} \\ E_{14}, & k \equiv 2 \pmod{12} \\ E_4, & k \equiv 4 \pmod{12} \\ E_6, & k \equiv 6 \pmod{12} \\ E_4^2, & k \equiv 8 \pmod{12} \\ E_4E_6, & k \equiv 10 \pmod{12}. \end{cases}$$

Also recall that M_k, the set of modular forms of weight k, forms a finite-dimensional vector space over $\mathbb{C}$ and that

$$\dim(M_k) = \begin{cases} \lfloor \frac{k}{12} \rfloor & \text{if } k \equiv 2 \pmod{12} \\ \lfloor \frac{k}{12} \rfloor + 1 & \text{if } k \not\equiv 2 \pmod{12}. \end{cases}$$

Theorem 9.4 (Choie, Kohnen, Ono). *If $f \in M_{12n+14}$ and $g \in M_k$, then*

$$const\left(\frac{f \cdot g}{\Delta^{n+m(k)}\tilde{E}_k}\right) = 0$$

where $m(k) := dim(M_k)$.

Now we can use Lemma 9.2, Lemma 9.3, and Theorem 9.4 to prove Theorem 9.1 following the proof of Lachterman, Schayer, and Younger.

9.3. **Proof of Main Theorem.** Let $\delta_{r,\ell,m} = r\left(\frac{\ell^{2m}-1}{24}\right) = \frac{r\ell^{2m}-r}{24}$; we use just δ when the situation is unambiguous. Note that $\delta \in \mathbb{Z}$ since $\ell^2 \equiv 1 \pmod{24}$ for any prime $\ell \geq 5$ and $2m$ is even. Recall that $\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24}$ and let $\sum_n \tau_{r,m}(n)q^n := \Delta(z)^{\delta}$. We prove the following technical proposition, a generalization of a proposition of Lachterman, Schayer, and Younger[LSY08], which is helpful in our proof:

Proposition 9.5. *There are congruences at $(\ell^k, r, -\delta)$ for all $k \leq m$ if and only if*

$$\tau_{r,m}(\ell^k n) \equiv 0 \pmod{\ell^k}$$

for all n and all $k \leq m$, where $\delta = r\left(\frac{\ell^{2m}-1}{24}\right)$. Also if $P_r(\ell^j i - \delta) \equiv 0 \pmod{\ell^k}$ for all $j \leq k$ and $i \leq n$, then $\tau_{r,m}(\ell^k n) \equiv 0 \pmod{\ell^m}$.

Proof. Here we first prove the "only if" part of the claim. This is the only part of the proposition that we shall actually need. Using the previous definitions, we get that

$$\sum_n \tau_{r,m}(n)q^n := \Delta^{\delta} = \frac{q^{\delta}}{\prod_{n\geq 1}(1-q^n)^r}\left(\prod_{n\geq 1} 1-q^n\right)^{r\ell^{2m}} = \left(\sum_n P_r(n-\delta)q^n\right)\left(\prod_{n\geq 1} 1-q^n\right)^{r\ell^{2m}}.$$

From Lemma 9.2, we have

$$\left(\prod_{n\geq 1} 1-q^n\right)^{r\ell^{2m}} \equiv \sum_i c_0(\ell^k i)q^{\ell^k i} + l\sum_i c_1(\ell^{k-1}i)q^{\ell^{k-1}i} + \cdots + \ell^{k-1}\sum_i c_{k-1}(\ell i)q^{\ell i} \pmod{\ell^k}$$

for all $k \leq m$. Thus

$$\tau_{r,m}(\ell^k n) \equiv \sum_i P_r(\ell^k i-\delta)c_0(\ell^k(n-i)) + \cdots + \ell^{k-1}\sum_i P_r(\ell i-\delta)c_{k-1}(\ell(n-i)) \pmod{\ell^k}$$

and since there are congruences at $(\ell^j, r, -\delta)$ for all $j \leq k$, we have $\tau_{r,m}(\ell^k n) \equiv 0 \pmod{\ell^k}$ for all $k \leq m$, proving the "only if" part of the claim. Note that this proof also shows that if $P_r(\ell^j i-\delta) \equiv 0 \pmod{\ell^k}$ for all $j \leq k$ and $i \leq n$, then $\tau_{r,m}(\ell^k n) \equiv 0 \pmod{\ell^m}$.

Now we prove the "if" part of our claim. Considered a fixed m. We have that $\tau_{r,m}(\ell^k n) \equiv 0 \pmod{\ell^k}$ for all n and all $k \leq m$ and need to prove that there are congruences at $(\ell^k, r, -\delta)$ for all $k \leq m$. We shall prove this using induction on $k \leq m$. Suppose that there are congruences at $(\ell^j, r, -\delta)$ for all $j < k$ and need to prove that there is a congruence at $(\ell^k, r, -\delta)$. Note that

$$\begin{aligned}
\tau_{r,m}(\ell^k n) &\equiv \sum_i p(\ell^k i-\delta)c_0(\ell^k(n-i)) + \cdots + \ell^{k-1}\sum_i p(\ell i-\delta)c_{k-1}(\ell(n-i)) \pmod{\ell^k} \\
&\equiv \sum_i p(\ell^k i-\delta)c_0(\ell^k(n-i)) \pmod{\ell^k} \\
&\equiv p(\ell^k n-\delta) + \sum_{i<n} p(\ell^k i-\delta)c_0(\ell^k(n-i)) \pmod{\ell^k},
\end{aligned}$$

where the second equality comes from the fact that we have congruences at $(\ell^j, r, -\delta_m)$ for all $j < k$ and the third equality from the fact that $c_0(0)$ is just the first term in the expansion of

$$\left(\prod_{n\geq 1} 1-q^n\right)^{r\ell^{2m}},$$

which is 1. Now if $p(\ell^k i-\delta) \equiv 0 \pmod{\ell^k}$ for all $i < n$, then since $\tau_{r,m}(\ell^k n) \equiv 0 \pmod{\ell^k}$, we have $p(\ell^k n-\delta) \equiv 0 \pmod{\ell^k}$. By induction on n, we have that $p(\ell^k n-\delta) \equiv 0 \pmod{\ell^k}$ for all n and therefore there is a congruence at $(\ell^k, r, -\delta)$. Therefore by induction on k, we have congruences at $(\ell^k, r, -\delta)$ for all $k \leq m$, which proves the "if" part of the claim. □

We are now able to prove Theorem 9.1.

Proof of Theorem 9.1. We now refer to Theorem 0.4. Let $g = \Delta(z)^\delta$ so that $k = 12\delta, \tilde{E}_k = 1$, and $m(k) = \delta+1$. Let $t = \ell^m n - \delta - 1$ so that $t + m(k) = \ell^m n$. Let $w = 12t+14$ and $k \equiv w \pmod{\ell-1}$, where $k \in \{0, 4, 6, \cdots, \ell+1\}$. Let $f = E_k(z)^{\ell^m} E_{\ell^{m-1}(\ell-1)}(z)^s$, where $s = \frac{w-kl^m}{l^{m-1}(l-1)}$. Since $k \equiv w$

$\pmod{\ell - 1}$, we have $w - k\ell^m \equiv 0 \pmod{\ell - 1}$. Also $w - k\ell^m \equiv 0 \pmod{\ell^{m-1}}$ because

$$\begin{aligned} w &= 12t + 14 \\ &= 12(\ell^m n - \delta - 1) + 14 \\ &\equiv -12\delta + 2 \pmod{\ell^{m-1}} \\ &\equiv -12\left(\frac{r\ell^{2m} - r}{24}\right) + 2 \pmod{\ell^{m-1}} \\ &\equiv \frac{r}{2} + 2 \pmod{\ell^{m-1}} \\ &\equiv -\frac{4}{2} + 2 \pmod{\ell^{m-1}} \\ &\equiv 0 \pmod{\ell^{m-1}} \end{aligned}$$

where the second to last equality comes from the fact that $r \equiv -4 \pmod{\ell^{m-1}}$. Since ℓ^{m-1} and $\ell - 1$ are coprime, we have $w - kl^m \equiv 0 \pmod{\ell^{m-1}(\ell - 1)}$ and thus s is an integer.

Also, if $n \geq \frac{\ell^m + 2\ell + 2}{24}$, then $s \geq 0$ since

$$\begin{aligned} w - k\ell^m &= 12t + 14 - k\ell^m \\ &= 12(\ell^m n - \delta - 1) + 14 - k\ell^m \\ &= 12\ell^m n - 12\delta + 2 - k\ell^m \\ &\geq 12\ell^m n - r\left(\frac{\ell^m - 1}{2}\right) - k\ell^m \\ &\geq 12\ell^m n - \frac{1}{2}\ell^m \cdot \ell^m - (l+1)\ell^m \\ &\geq 0 \end{aligned}$$

where the second inequality is due to the fact that $k \leq \ell + 1$ and because we can assume that $r < \ell^m$ by Lemma 9.3.

Now let $f(z) = E_k(z)^{\ell^m} E_{\ell^{m-1}(\ell-1)}(z)^s$, where $E_k(z)$ is the normalized Eisenstein series of weight k as before. Since $E_{\ell^{m-1}(\ell-1)}(z) \equiv 1 \pmod{\ell^m}$ by the Staudt-Claussen theorem [IR90], we have $f_{t,\ell}(z) \equiv E_k(z)^{\ell^m} \pmod{\ell^m}$. Note that since s is an integer and $s \geq 0$, $f = E_k(z)^{\ell^m}(z) E_{\ell^{m-1}(\ell-1)}(z)^s$ is a modular form. Also note that $f(z) \in M_w$ as required since $w = k\ell^m + \ell^{m-1}(\ell-1)\frac{w - k\ell^m}{\ell^{m-1}(\ell-1)}$. Thus we can apply Theorem 9.4 to $f(z) = E_k(z)^{\ell^m} E_{\ell^{m-1}(\ell-1)}(z)^s$ and $g(z) = \Delta(z)^\delta$:

$$\begin{aligned} \operatorname{const}\left(\frac{f \cdot g}{\Delta^{t+m(k)}\tilde{E}_k}\right) &= \operatorname{const}\left(\frac{E_k^{\ell^m} E_{\ell(\ell-1)}^s \Delta^\delta}{\Delta^{n\ell^m}}\right) \\ &\equiv \operatorname{const}\left(\frac{E_k^{\ell^m} \Delta^\delta}{\Delta^{nl^m}}\right) \pmod{\ell^m}. \end{aligned}$$

By Lemma 9.2,

$$\frac{E_k^{\ell^m}}{\Delta^{n\ell^m}} \equiv \sum_i \alpha_0(\ell^m i) q^{\ell^m i} + \ell \sum_i \alpha_1(\ell^{m-1} i) q^{\ell^{m-1} i} + \cdots + \ell^{m-1} \sum_i \alpha_{m-1}(\ell i) q^{\ell i} \pmod{\ell^m}.$$

Thus the formula for the constant term becomes

$$\sum_i \tau_{r,m}(\ell^m i)\alpha_0(-\ell^m i) + \ell \sum_i \tau_{r,m}(\ell^{m-1} i)\alpha_1(-\ell^{m-1} i) + \cdots + \ell^{m-1} \sum_i \tau_{r,m}(\ell i)\alpha_{m-1}(-\ell i) \equiv 0 \pmod{\ell^m}.$$

Since we assumed that there are congruences at $(\ell^k, r, -\delta)$ for $k < m$, we have that for $k < m$ $\tau_{r,m}(\ell^k i) \equiv 0 \pmod{\ell^k}$ for all i by Proposition 9.5. Also note that the first non-zero term in the expansion of $\frac{E_k^{\ell^m}}{\Delta^{n\ell^m}}$ is $q^{-n\ell^m}$ and that $\tau_{r,m}(i) = 0$ if $i < 0$. Thus the constant term becomes

$$\sum_{i=0}^{n} \tau_{r,m}(\ell^m i)\alpha_0(-\ell^m i) \equiv 0 \pmod{\ell^m}.$$

Note that this formula holds only when $n \geq \frac{\ell^m+2\ell+2}{24}$. By the assumptions of Theorem 9.1, we have $p(\ell^m n - \delta) \equiv 0 \pmod{\ell^m}$ for $n < \frac{\ell^m+2\ell+2}{24}$. Since we also have congruences at $(\ell^k, r, -\delta)$ for $k < m$, then by Proposition 9.5 we have $\tau_{r,m}(\ell^m n) \equiv 0 \pmod{\ell^m}$ for $n < \frac{\ell^m+2\ell+2}{24}$. Now we prove by induction that $\tau_{r,m}(l^m n) \equiv 0$ for all $n \geq \frac{\ell^m+2\ell+2}{24}$. If we have $\tau_{r,m}(\ell^m i) \equiv 0 \pmod{\ell^m}$ for all $i < n$, where $n \geq \frac{\ell^m+2\ell+2}{24}$, then $\tau_{r,m}(\ell^m n)\alpha_0(-\ell^m n) \equiv 0 \pmod{\ell^m}$. But since $\alpha_0(-\ell^m n) = 1$ because $\alpha_0(-\ell^m n)$ is just the first term in the q-expansion of $\frac{E_k^{\ell^m}}{\Delta^{n\ell^m}}$), we have $\tau_{r,m}(\ell^m n) \equiv 0 \pmod{\ell^m}$. Then Theorem 0.1 follows by induction on n. □

9.4. **Versatility of Theorem 9.1.** An obvious question to ask is exactly what types of congruences Theorem 9.1 is able to prove. Using a theorem of Kiming and Olsson, we shall see that that Theorem 9.1 can be used to prove almost all congruences that have $m = 1$. We note that Kiming and Olsson define an *exceptional* congruence to be one of the following form:

$$P_r(\ell n + a) \equiv 0 \pmod{\ell}$$

where $r \notin \{\ell - 1, \ell - 3\}$.

They proved the following theorem.

Theorem 9.6 ([KO92], Theorem 1)**.** *Let $\ell \geq 5$ be a prime number. Suppose there is an exceptional congruence for l where*

$$P_r(\ell n + a) \equiv 0 \pmod{\ell}.$$

Then r is odd and $24a \equiv r \pmod{\ell}$.

Remark. Theorem 9.1 provides a proof for every exceptional congruence. We observe that our theorem provides a proof of every exceptional congruence by setting $\delta_{\ell,r} := -a = \frac{r\ell^2 - r}{24}$. Note that in this case r is arbitrary since the condition $r \equiv -4 \pmod{\ell^{m-1}}$ is trivial for $m = 1$. In particular, the classical Ramanujan congruences hold for $\delta_{5,1}$, $\delta_{7,1}$, and $\delta_{11,1}$.

We also find that certain choices of $\delta_{\ell,r}$ generate non-exceptional cases. Specifically, every prime ℓ has $\frac{l+1}{2}$ congruences for $r = \ell - 3$ [And08], and we form the following lemma:

Remark. For some choice of $\delta_{\ell,r}$, our theorem proves a non-exceptional case of $P_{\ell-3}(\ell n + a) \equiv 0 \pmod{\ell}$

Proof. We recall a weakened result of Gandhi:

Theorem 9.7 ([Gan63], (IV)). *If $\ell \geq 5$ is prime and $a = \frac{\ell^2-1}{8}$, then $P_{\ell-3}(\ell n+a) \equiv 0$.*

Since $a \equiv \frac{-1}{8} \pmod{\ell}$, we need to show that $\delta_{\ell,r}$ satisfies the same equivalence where $r = \ell-3$. By the definition, $\delta_{\ell,\ell-3} = \frac{(\ell-3)\ell^2-(\ell-3)}{24} \equiv \frac{3}{24} \equiv \frac{1}{8} \pmod{\ell}$, which was to be shown. □

Hence, we have proven the following:

Theorem 9.1 encompasses all exceptional congruences and at least one non-exceptional congruence of the form $P_{\ell-3}(\ell n+a) \equiv 0 \pmod{\ell}$ for all primes ℓ.

In this sense, Theorem 9.1 can be used to prove almost all congruences that have $m = 1$. We also investigated whether Theorem 9.1 could prove an equally high portion of congruences when $m > 1$ (modulo prime powers) but found that this was not the case, which is due to the fact that if $m > 1$, then $r \equiv -4 \pmod{\ell^{m-1}}$ is no longer a trivial equality. This condition was necessary so that $s = \frac{w-kl^m}{l^{m-1}(l-1)}$ would be an integer, which would let $f(z) = E_k^{l^m}(z)E^s_{l^{m-1}(l-1)}(z)$ be a modular form, which would let us apply Theorem 9.4. One possible approach is considering $E_{\ell-1}(q^{\ell^{m-1}})^s$ instead of $E_{\ell^{m-1}(\ell-1)}(q)^s$, which allows s be a integer without placing such stringent conditions on r. However, f is not longer invariant under all of $SL_2(\mathbb{Z})$ but under a subgroup of this group. In this case, it may still be possible to use a weaker form of Theorem 9.4 that is sufficient for our purposes. Another issue concerning proving congruences modulo prime powers was that we were only able to actually prove congruences modulo prime powers for $m = 2$; however, this was most likely due to low computing capabilities rather than a weakness of Theorem 9.4 for prime powers.

In attempting to prove congruences, we also realized that a key strength of Theorem 9.1 concerns the following situation: if this theorem can be applied to prove a congruence at (ℓ^m, r, a), then we must know that there are congruences at (ℓ^k, r, a) for $k < m$ and in fact Theorem 9.1 can be applied to prove these other congruences. Thus, all congruences at (ℓ^k, r, a) for $k \leq m$ can be proven using just Theorem 9.1, without having to use other results proving the congruences at (ℓ^k, r, a) for $k < m$.

9.5. **Conclusion.** Although the original proof by Lachterman, et. al. extends very nicely to this multipartition generalization, there are still other congruences which could potentially be proven using similar techniques of modular forms without the use of Hecke operators. For example, we believe a proof of Ramanujan's congruences modulo prime powers

$$p(5^m n - \delta_{5,m}) \equiv 0 \pmod{5^m} \tag{14}$$

$$p(7^m n - \delta_{7,m}) \equiv 0 \pmod{7^{\lfloor \frac{m}{2} \rfloor+1}} \tag{15}$$

$$p(11^m n - \delta_{11,m}) \equiv 0 \pmod{11^m} \tag{16}$$

where $\delta_{\ell,k} = \frac{\ell^{2m}-1}{24}$, is possible using similar techniques and some form of induction. Here we were only able to prove the cases $\ell = 5$ and $k = 1,2$. Also, since there are congruences for every integer coprime to 6, as proven by Ahlgren, Ono [AO01], it seems reasonable to attempt a proof of other infinite classes of congruences using similar methods as those presented here.

We conclude with all infinite families of congruences that can be proven with Theorem 9.1 for $\ell \leq 13$ and $m \leq 2$.

9.6. **All Applicable Congruences for Small Primes.**

Theorem 9.8. *For all $n \in \mathbb{Z}$ and all $r \in \mathbb{Z}$,*

$$P_{2+5r}(5n+3) \equiv 0 \pmod{5} \tag{17}$$

$$P_{1+5r}(5n+4) \equiv 0 \pmod{5} \tag{18}$$

$$P_{1+7k}(7n+5) \equiv 0 \pmod{7} \tag{19}$$

$$P_{4+7k}(7n+6) \equiv 0 \pmod{7} \tag{20}$$

$$P_{8+11r}(11n+4) \equiv 0 \pmod{11} \tag{21}$$

$$P_{1+11r}(11n+6) \equiv 0 \pmod{11} \tag{22}$$

$$P_{3+11r}(11n+7) \equiv 0 \pmod{11} \tag{23}$$

$$P_{5+11r}(11n+8) \equiv 0 \pmod{11} \tag{24}$$

$$P_{7+11r}(11n+9) \equiv 0 \pmod{11} \tag{25}$$

$$P_{10+13r}(13n+8) \equiv 0 \pmod{13} \tag{26}$$

and

$$P_{11+5^2r}(5^2n+14) \equiv 0 \pmod{5^2} \tag{27}$$

$$P_{6+5^2r}(5^2n+19) \equiv 0 \pmod{5^2} \tag{28}$$

$$P_{1+5^2r}(5^2n+24) \equiv 0 \pmod{5^2} \tag{29}$$

$$P_{95+11^2r}(11^2n+9) \equiv 0 \pmod{11^2}. \tag{30}$$

$$P_{7+11^2r}(11^2n+86) \equiv 0 \pmod{11^2} \tag{31}$$

$$P_{29+11^2r}(11^2n+97) \equiv 0 \pmod{11^2}. \tag{32}$$

$$P_{51+11^2r}(11^2n+108) \equiv 0 \pmod{11^2}. \tag{33}$$

$$P_{73+11^2r}(11^2n+119) \equiv 0 \pmod{11^2}. \tag{34}$$

9.7. **Introduction and Purpose.** The following programs and functions were written in Java for the purpose of counting and printing various forms of partitions and plane partitions. They were also used in attempting to search for congruences in partition functions.

Regular partitions are represented as arrays of integers, with the ith element of the array representing the ith part, or row, of the partition.

Plane partitions are represented as two-dimensional arrays of integers, with the (i, j)th element of the array representing the jth part of the ith component of the partition. In this way, the plane partition is being stored more like a general multipartition rather than a plane partition.

The first program was written to answer the question: How many different partitions could be placed within the "footprint" of a specified partition. In other words, given a partition

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n),$$

how many other partitions $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$ for some $m \leq n$ satisfy

$$\pi_i \leq \lambda_i \text{ for all } 1 \leq i \leq m$$

.

The second program examines the number of plane partitions that can fit within the pyramidal plane partition of size n, which we define as:

$$\begin{array}{cccccc} n & n-1 & \ldots & 2 & 1 \\ n-1 & n-2 & \ldots & 1 & \\ \vdots & \vdots & & & \\ 2 & 1 & & & \\ 1 & & & & \end{array}$$

The third program counts the number of plane partitions of an integer n with at most k components.

9.8. **Common Functions.** First, we have functions used to print out partitions and plane partitions.

```
public static void printPartition(int partition[]){
   for (int i = 0; i < partition.length-1; i++){
      System.out.print(partition[i]+" + ");
   }
   System.out.println(partition[partition.length-1]);
}
```

For a given partition π, this function simply prints the string: $\pi_1 + \pi_2 + \cdots + \pi_n$.

```
public static void printPlane(int partition[][]){
   int toPrint[][]= new int[n][n];
   for (int i = 0; i < n; i++){
      for (int j = 0; j < n; j++){
         for (int k = 0; k < partition[i][j]; k++){
            toPrint[j][k]++;
         }
      }
```

```
        }
        for (int i = 0; i < n; i++){
            for (int j = 0; j < n; j++){
                System.out.print(toPrint[i][j]+"  ");
            }
            System.out.println("");
        }
    }
```

This function first converts from the multipartition form of representing the data to the traditional plane partition diagram and then prints it out. That is, it would take the multipartition

$$(4+3+3+1, 3+2+1+1, 2)$$

and print

$$\begin{matrix} 3 & 3 & 2 & 1 \\ 2 & 2 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 0 & 0 \end{matrix}$$

These two functions were mainly used for debugging purposes to ensure that the programs were generating the correct partitions for the given problems.

```
    public static int sum(int partition[]){
        int sum = 0;
        for (int i = 0; i < partition.length; i++){
            sum += partition[i];
        }
        return sum;
    }
```

As the name implies, this function simply sums the parts of the given partition. Another, slightly different version was used to count the partial sum of a partition up to a specified row in order to improve the running time of the code.

9.9. **Program 1.** Here is the code for the main recursive function for the first of the three programs, intended to count the number of partitions that can fit within the outline of a given partition.

```
public static int recurse1(int partition[], int level){
    int sum = 0;
    if (partition.length == level+1){
        for (int i = partition[level]; i >= 0; i--){
            partition[level]=i;
            printPartition(partition);
            sum++;
        }
    }
    else{
        int temp[] = new int[partition.length];
```

```
      for (int i = 0; i < level; i++){
         temp[i]=partition[i];
      }
      for (int j = 0; j <= partition[level]; j++){
         temp[level]=partition[level]-j;
         for (int k = level; k < partition.length; k++){
            temp[k]=Math.min(temp[level], partition[k]);
         }
         sum += recurse1(temp,level+1);
      }
   }
   return sum;
}
```

This function takes as its inputs an array of integers representing a partition and an integer, *level*, that denotes the current row of the partition that is being examined. It returns an integer, *sum*, that represents the number of partitions that can be fit within the partition specified by the first parameter.

At each calling of this function, sum is initialized to 0. The if statement checks to see if we are in the terminating recursive state, when the specified level is the last row of the partition. If this is the case, then we set sum to be equal to the value of the smallest part of the given partition plus 1, realizing that each of the numbers $i \in \{0, \dots, partition[level]\}$ as the last part of the partition will be valid. The loop here is not necessary for this purpose, but was used for printing each of the possible partitions during debugging.

If we are not in the terminating case, we recurse as follows. First, create a new temporary array to hold a partition. Copy the values of the parts we have already determined to this new array, i.e. rows 0 through $level - 1$. For the part specified by the variable *level*, we iterate down from the current value to 0. In each of these iterations, we construct the remaining rows of the array by taking the minimum of their current value and the current value of the row we are iterating through. Doing this ensures that the resulting array will represent a valid partition. We then call the recursive function again, using the newly constructed array and moving down a level. We add the result of this call to the current value of sum.

9.10. **Program 2.** Here is the code for the main recursive function of the second program. It is used to count the number of subpartitions of the the pyramidal plane partition of height *n*.

```
public static int recurse2(int partition[][], int row, int level){
   int sum = 0;
   if (row == n-level-1 || partition[level][row+1]==0){
      for (int i = partition[level][row]; i >= 0; i--){
         partition[level][row] = i;
         if (level == n-1){
            sum++;
            printPlane(partition);
         }
         else{
```

```
            int nextLevel[] = new int[n];
            for (int j = 0; j < n; j++){
               nextLevel[j] = Math.min(partition[level][j],Math.max(n-level-j-1,0));
            }
            if (isPartition(nextLevel)){
               int temp[][] = new int[n][n];
               for (int j = 0; j <=level; j++){
                  for (int k = 0; k < n; k++){
                     temp[j][k] = partition[j][k];
                  }
               }
               for (int j = 0; j < n; j++){
                  temp[level+1][j]=nextLevel[j];
               }
               sum+=recurse2(temp,0,level+1);
            }
            else {
               sum++;
               printPlane(partition);
            }
         }
      }
   }
   else{
      int temp[][] = new int[n][n];
      for (int i = 0; i < level; i++){
         for (int j = 0; j < n; j++){
            temp[i][j] = partition[i][j];
         }
      }
      for (int i = 0; i < row; i++){
         temp[level][i] = partition[level][i];
      }
      for (int i = 0; i <= partition[level][row]; i++){
         temp[level][row] = partition[level][row]-i;
         for (int j = row; j < n; j++){
            temp[level][j] = Math.min(partition[level][j], temp[level][row]);
         }
         sum += recurse2(temp,row+1,level);
      }
   }
   return sum;
}
```

This function takes three variables as input, a two-dimensional array of integers representing the plane partition as described in the introduction, an integer, *row*, representing the row of the current component of the plane partition we are examining, and an integer, *level*, representing the current component, or vertical layer of the partition. Like before, we return a single integer representing the number of plane partitions that can fit within the pyramidal plane partition above the base specified in the *partition* variable. It should also be noted that n is a global variable representing the dimensions of the pyramidal plane partition.

The main recursive loop, within the outermost else statement, is almost identical to the one in the first program. The only major difference is that we must first copy the values from each component less than *level* to the temporary array, as well as the values of the rows of the current component less than *row*. Once this temporary array is created, we recurse to the next row of the current component.

However, the termination case is significantly different. Once we reach the last row of the current component, we again iterate through all possible values of this row, from the current value down to 0. For each of these iterations, there are two possibilities. If the current *level* is n, the highest we can go, then we add the appropriate values to sum and exit this level of the recursion. Otherwise, we create the largest possible partition that can fit atop the current component and still be within the pyramidal plane partition. This is accomplished for each row of the next component by taking the minimum of the current component's row and the maximum that that row of the next component could possibly be and still be within the pyramid. If this new component is a valid partition, we construct a new array containing the current plane partition with the newly constructed component appended to it. We then recurse up to the next component, setting *row* back to 0 and incrementing *level*.

Running this program on my laptop allowed me to generate the first few terms of the sequence, for n from 0 to 6 in a short amount of time. When 7 was attempted, the program took a very long time to complete and the sum variable overflowed at some point during execution. The first terms are: 1, 2, 9, 96, 2498, 161422, 26217833.

9.11. **Program 3.** Here is the code for the main recursive function of the third program. This counts the number of plane partitions of the number n with largest part less than or equal to k.

```
public static int recurse3(int partition[], int row, int level, int prevSum){
   recurses++;
   int sum = 0;
   if (row+1 == partition.length || partition[row+1]==0){
      for (int i = partition[row]; i >= 0; i--){
         partition[row]=i;
         int tempsum = sum(partition)+prevSum;
         if (tempsum==n){
            sum++;
         }
         else if (tempsum < n && level < k-1){
            sum += recurse3(partition,0,level+1,tempsum);
         }
      }
```

```
    }
    else{
        for (int i = 0; i <= partition[row]; i++){
            int temp[] = new int[n];
            for (int j = 0; j < row; j++){
                temp[j] = partition[j];
            }
            temp[row]=partition[row]-i;
            if (partialSum(temp,row)+prevSum <= n){
                for (int j = row+1; j < partition.length; j++){
                    temp[j] = Math.min(temp[row], partition[j]);
                }
                sum += recurse3(temp,row+1,level,prevSum);
            }
        }
    }
    return sum;
}
```

This function takes four inputs. The first is an array of integers representing the current level of the plane partition. The second integer represents the row of the current level that is being examined. The third integer represents the current component of the plane partition, to ensure that it does not exceed the maximum height. The fourth integer represents the sum of the previous levels of the plane partition to ensure that we do not exceed the number we are partitioning. The function returns the number of plane partitions of n that have largest part less than or equal to k. It should be noted that n and k are global variables representing the number we are partitioning and the maximum height of the plane partition, respectively.

The user of the program specifies the values of n and k in the main program body by inputting them into the terminal. The program then generates an outline of a regular partition into which all regular partitions of n could fit using the following code:

```
for (int i = 0; i < n; i++){
    partition[i]=(int)(n/(i+1));
}
```

This ensures that we get all possible bases for the plane partition. This outline is then passed to the recursive function with $row, level,$ and $prevSum$ all equal to 0.

The main part of the recursive function, within the outermost else statement remains mostly unchanged. I added a single if statement to check the size of the current partition. If the partial sum, that is the sum of the rows 0 to *row* plus the size of the previous components, exceeds n, then there is no need to continue recursing, as any partition generated from the existing one would be larger than n. The addition of this single statement saves a considerable number of recursive steps for larger values of n. For example, with $n = k = 10$, the program as written executes the recursive function 5517 times. Without it, the recursive function is called 73520 times.

The termination case of this function is much cleaner than that of program two. This is because of the decision to only keep track of the current component of the plane partition rather than the entire thing. Again, when we get to the final row of the current component, we iterate down through all of the possibilities from the current value to 0. The sum of the current component is added to the sum of the previous components. If this value is equal to n, then we increment *sum*. Otherwise, if this value is less than n, and we are not already in the k^{th} component, we recurse up to the next level. Since we are not worried about the bounding plane as in program two, we simply start with a copy of the current level as the initial value of the next level.

The initial results of this program were used to generate a table of values for small n and k.

$n\backslash k$	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	2	3	3	3	3	3	3	3	3	3
3	3	5	6	6	6	6	6	6	6	6
4	5	10	12	13	13	13	13	13	13	13
5	7	16	21	23	24	24	24	24	24	24
6	11	29	40	45	47	48	48	48	48	48
7	15	45	67	78	83	85	86	86	86	86
8	22	75	117	141	152	157	159	160	160	160
9	30	115	193	239	263	274	279	281	282	282
10	42	181	319	409	457	481	492	497	499	500

We have also used this program to search for some basic congruences between plane partitions of different forms. For example, we let $M_4(n)$ be the number of plane partitions of n with largest part less than or equal to 4. By modding the results of the program by four, we discovered that $M_4(4n+1) \equiv M_4(4n+2)+M_4(4n+3) \pmod 4$.

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References

[AG88] George E. Andrews and F. G. Garvan. Dyson's crank of a partition. *Bull. Amer. Math. Soc. (N.S.)*, 18(2):167–171, 1988.

[And94] George E. Andrews. *Number theory*. Dover Publications Inc., New York, 1994. Corrected reprint of the 1971 original [Dover, New York; MR0309838 (46 #8943)].

[And98] George E. Andrews. *The theory of partitions*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1998. Reprint of the 1976 original.

[And08] George E. Andrews. A survey of multipartitions: congruences and identities. In *Surveys in number theory*, volume 17 of *Dev. Math.*, pages 1–19. Springer, New York, 2008.

[AO01] Scott Ahlgren and Ken Ono. Congruences and conjectures for the partition function. In *q-series with applications to combinatorics, number theory, and physics (Urbana, IL, 2000)*, volume 291 of *Contemp. Math.*, pages 1–10. Amer. Math. Soc., Providence, RI, 2001.

[ASD54] A. O. L. Atkin and P. Swinnerton-Dyer. Some properties of partitions. *Proc. London Math. Soc. (3)*, 4:84–106, 1954.

[CKO05] Y. Choie, W. Kohnen, and K. Ono. Linear relations between modular form coefficients and non-ordinary primes. *Bull. Lond. Math. Soc.*, 37:335–341, 2005.

[Dys44] F.J. Dyson. Some guesses in the theory of partitions. *Eureka (Cambridge)*, 8:10–15, 1944.

[FW07] J. Furno and P. Waters. Investigations regarding partitions and multipartitions. *Proceedings of Oregon State University Mathematics REU*, pages 31–60, 2007.

[Gan63] J. M. Gandhi. Congruences for $p_r(n)$ and Ramanujan's τ function. *Amer. Math. Monthly*, 70:265–274, 1963.

[Gan67] J. M. Gandhi. Some congruences for k line partitions of a number. *Amer. Math. Monthly*, 74:179–181, 1967.

[IR90] K. Ireland and M. Rosen. *A Classical Introduction to Modern Number Theory*. Springer, Berlin, 2nd edition, 1990.

[KO92] Ian Kiming and Jørn B. Olsson. Congruences like Ramanujan's for powers of the partition function. *Arch. Math. (Basel)*, 59(4):348–360, 1992.

[Kwo89a] Y. H. Harris Kwong. Minimum periods of binomial coefficients modulo M. *Fibonacci Quart.*, 27(4):348–351, 1989.

[Kwo89b] Y. H. Harris Kwong. Minimum periods of partition functions modulo M. *Utilitas Math.*, 35:3–8, 1989.

[LSY08] Samuel Lachterman, Rhiannon Schayer, and Brendan Younger. A new proof of the Ramanujan congruences for the partition function. *Ramanujan J.*, 15(2):197–204, 2008.

[Mac04] Percy A. MacMahon. *Combinatory analysis. Vol. I, II (bound in one volume)*. Dover Phoenix Editions. Dover Publications Inc., Mineola, NY, 2004. Reprint of ıt An introduction to combinatory analysis (1920) and ıt Combinatory analysis. Vol. I, II (1915, 1916).

[Ram19] S. Ramanujan. Some properties of p(n); the number of partitions of n. *Proc. Cambridge Philos. Soc.*, 19:207–210, 1919.

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