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The Largest Small Hexagon

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The problem of determining the largest area a plane hexagon of unit diameter can have, raised some 20 years ago by H. Lenz, is settled. It is shown that such a hexagon is unique and has an area exceeding that of a regular hexagon of unit diameter by about 4%.

INTRODUCTION

Given any n-gon of unit diameter in the plane, it is natural to inquire how large an area it can have. In 1922, Reinhardt [8] showed that for odd n, the obvious configuration is optimal, namely, among all n-gons of diameter 1, the regular n-gon has the maximum area. It was noted, however, that this is not the case for n even.[1] In particular, H. Lenz [7] raised the question of finding the largest area, denoted by F_6, a plane hexagon of unit diameter can have.[2] In this note we answer this question. It is shown that F_6 satisfies a 10th degree irreducible polynomial over the integers and is approximately equal to 0.674981. Furthermore, there is a unique unit diameter hexagon with area F_6.

THE DIAMETER GRAPH $D(X)$

For a subset X of the Euclidean plane $\mathbb{E}^2$, let $CH(X)$ denote the convex hull of X. If $\bar{u}$ is an arbitrary unit vector and t is a real number, we let $X + t\bar{u}$ denote the set $\{\bar{x} + t\bar{u}: \bar{x} \in X\}$.

FACT 1. If A and B are bounded convex subsets of $\mathbb{E}^2$ then the real function $G(t)$ defined by

$$G(t) = \text{area } CH(A \cup (B + t\bar{u})),$$

is convex.

[1] For $n = 4$, the square with diagonal 1 has the maximum possible area of $\frac{1}{2}$ but it is not the unique quadrilateral with this property (cf. [9]).

[2] A theorem of Blaschke [4] guarantees the existence of a hexagon with area F_6.

This is proved in [2] and holds more generally in $\mathbb{E}^n$.

For $X \subseteq \mathbb{E}^2$ finite, define a graph (see [5] for standard graph theory terminology) $D(X)$, called the *diameter graph* of X, as follows. The *vertices* of $D(X)$ are the points of X. A pair of vertices $\{x_1, x_2\}$ is an *edge* of $D(X)$ if the Euclidean distance $d(x_1, x_2)$ between x_1 and x_2 is equal to $\operatorname{diam}(X)$, the diameter (i.e., $\sup_{x,y \in X} d(x, y)$) of X.

FACT 2. Suppose $X \subseteq \mathbb{E}^2$ with $D(X)$ disconnected. Then there exists $X' \subseteq \mathbb{E}^2$ such that:

(i) $\operatorname{diam}(X') = \operatorname{diam}(X)$;

(ii) area $CH(X') \geqslant$ area $CH(X)$;

(iii) $D(X')$ is connected.

The proof is a simple application of Fact 1, using induction on the number of components of $D(X)$. Thus, in searching for a 6 element set X_6 with area $CH(X_6) = F_6$, we may restrict ourselves to those X with $D(X)$ connected. In fact, it will be seen that the unique extremal X_6 has this property. We note that $D(X_6)$ has at least 5 edges (since $D(X_6)$ is connected) and at most 6 edges (by a result of Erdös [1]). It is also not difficult to see that if $x_1, x_2, y_1, y_2 \in X$ with $d(x_1, x_2) = d(y_1, y_2) = 1 = \operatorname{diam}(X)$ then the closed line segments joining x_1, x_2 and y_1, y_2 intersect in exactly one point, which is either a common end point or a common interior point.

LINEAR THRACKLES

A graph G is said (by John Conway) to have a *linear thrackleation* G^* if G can be represented as a graph G^* with the following properties:

(a) The vertices of G^* are points in $\mathbb{E}^2$;

(b) The edges of G^* are straight line segments connecting certain pairs of vertices of G^*;

(c) Any two edges of G^* have exactly one common point, which is either a common vertex of each edge or an interior point of each edge.

By the facts noted in the preceding section, it follows that $D(X_6)$ has a linear thrackleation. Hence, by a result of Woodall [11], $D(X_6)$ must be one of the 10 graphs shown in Fig. 1.

In Fig. 2 we illustrate linear thrackleations for the 10 cases shown in Fig. 1.

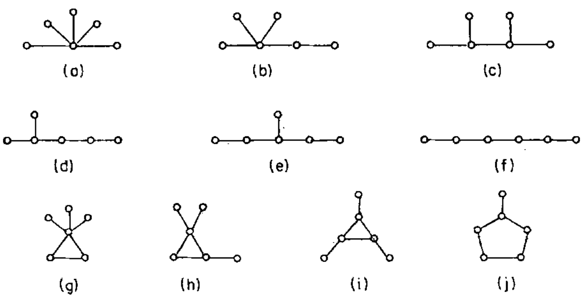


FIG. 1. Possibilities for $D(X_6)$.

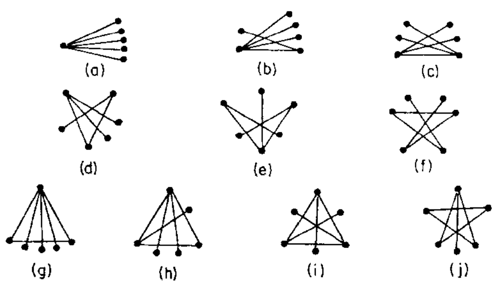


FIG. 2. Linear thrackleations for the 10 cases shown in Fig. 1.

DETERMINATION OF F_6

The problem of determining F_6 is now reduced to an examination of the 10 cases mentioned in the preceding section, as the 6 points X in each case continuously vary without changing the corresponding $D(X)$. A critical fact to notice is that for a fixed diameter graph $D(X)$ it is sufficient to examine the *local* maxima of area $CH(X)$. This follows from the fact that the constraints on the $x_i \in X$ are all of the form $0 < d(x_i, x_j) < 1$. Any maximum of area $CH(X)$ which is not a local maximum would either be found for $|X| < 6$ (when $d(x_i, x_j) = 0$) or a different diameter graph (when $d(x_i, x_j) = 1$).

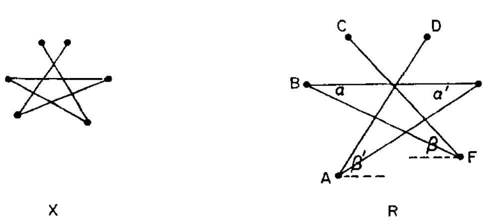


FIGURE 3.

In all cases except (f) and (j), elementary geometrical considerations show that area $CH(X_6) < 0.64$. For case (f), the following argument, due to J. J. Schäffer and W. Noll [10], shows that no local maximum occurs.

In Fig. 3, let B and E have coordinates $(-\frac{1}{2}, 0)$ and $(\frac{1}{2}, 0)$, respectively, where all line segments have unit length. Then the coordinates of the remaining points are given by

$$\begin{aligned}
A &= (\tfrac{1}{2} - \cos\alpha', -\sin\alpha'),\\
F &= (-\tfrac{1}{2} + \cos\alpha, -\sin\alpha),\\
D &= (\tfrac{1}{2} - \cos\alpha' + \cos\beta', -\sin\alpha' + \sin\beta'),\\
C &= (-\tfrac{1}{2} + \cos\alpha - \cos\beta', -\sin\alpha + \sin\beta).
\end{aligned}$$

Thus,

$$\begin{aligned}
2\,\text{area}\, R = {} & 2\sin(\alpha + \alpha') - \sin\alpha - \sin\alpha' + \sin(\beta + \beta')\\
& + \sin\beta + \sin\beta' - \sin(\alpha' + \beta) - \sin(\alpha + \beta').
\end{aligned}$$

Letting $u = (\alpha + \alpha')/2$, $v = (\alpha - \alpha')/2$ and considering area R as a function of the variables u, v, β, β', an easy calculation shows

$$2(\partial^2(\text{area}\, R)/\partial v^2) = \sin\alpha + \sin\alpha' + \sin(\alpha + \beta') + \sin(\alpha' + \beta).$$

However, we must have $0 < \alpha, \alpha' < \pi/3$, $0 < \beta, \beta' < \pi$ and in this case $(\partial^2(\text{area}\, R)/\partial v^2 > 0$. Hence, no local maximum occurs for case (f).

We are left with the last, but not least (in fact, most), case which is (j). This is shown in Fig. 4.

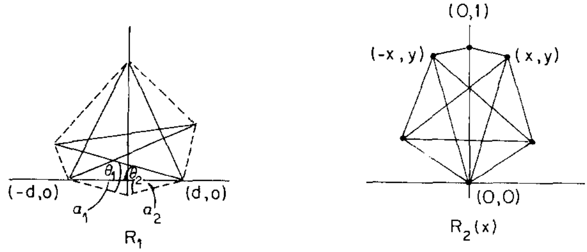


FIGURE 4.

It is immediate that in order to maximize area R_1, it is necessary that $\alpha_1 = \alpha_2$. It is slightly less immediate (but equally true) that it is also necessary that $\theta_1 = \theta_2$. (The details are not particularly interesting and are omitted.) Thus, we are left with the symmetric configuration $R_2(x)$. Here, there is just one degree of freedom and area $R_2(x)$ can be written as

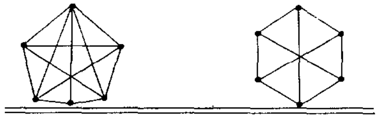
$$\text{area}\, R_2(x) = (\tfrac{1}{2} - x)(1 - x^2)^{1/2} + x(1 + (1 - (x + \tfrac{1}{2})^2)^{1/2}).$$

This expression has a unique local maximum for $x \in (0, \frac{1}{2})$ which can be obtained by setting the first derivative of area $R_2(x)$ equal to zero and solving. The resulting root $x_0 = 0.343771...$ has a minimal polynomial $P(x)$ of degree 10. The corresponding expression area $R(x_0) = 0.674981...$ also has a minimal polynomial $Q(x)$ of degree 10 which can be found choosing the appropriate factor for the 40th degree resultant of $P(x)$ and the rationalized expression for area $R(x)$ (this was done quite cleverly by S. C. Johnson; see [6]). $Q(x)$ is given by

$$\begin{aligned} Q(x) = {} & 4096x^{10} + 8192x^9 - 3008x^8 - 30848x^7 \\ & + 21056x^6 + 146496x^5 - 221360x^4 \\ & + 1232x^3 + 144464x^2 - 78488x + 11993. \end{aligned}$$

This is the minimal polynomial for $F_6 = 0.674981....$

We note that if a configuration X has area $CH(X) = F_6$, $\operatorname{diam}(X) \leqslant 1$ and $D(X)$ *not* connected, then by a suitable translation (from Fact 1), we could obtain an X' with area $CH(X') = F_6$, $\operatorname{diam}(X') \leqslant 1$ and $D(X')$ not equal to the graph of case (j), which is impossible. Hence, the optimal configuration X_6 is unique. We show it in Fig. 5.



X_6 area = 0.67498...

Regular hexagon area = 0.64952...

FIG. 5. Comparison of X_6 and regular hexagon.

We remark that the value of F_6 has found recent application in some work of Guy and Selfridge [3].

CONCLUDING REMARKS

In principle, the preceding techniques may be applied to the determination of F_{2m} in general. However, it is not hard to show that for $m \geqslant 2$ there are exactly

$$\frac{1}{8m} \sum_{\substack{d \mid m \\ d \text{ odd}}} \Phi(d)\, 4^{m/d} + 4^{m-2} + 2^{m-1} - 1,$$

distinct diameter graphs $D(X_{2m})$ to consider, where Φ denotes the familiar totient function of Euler. Most of these, however, could be trivially eliminated from consideration as serious candidates for an optimal solution.

It is conjectured that the optimal solution has the diameter graph $D(X_{2m})$ which consists of a circuit of length $2m - 1$ together with a single edge from one vertex (generalizing case (j)). However, at present, it is not even known that an optimal configuration must have an axis of symmetry.

It would also be interesting to look at the corresponding questions in $\mathbb{E}^r$ although, no doubt, some new ideas will be needed.

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