MECCANO MATH I*

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*With few if any applications.

1. Introduction: Meccano

A curious branch of mathematics is inspired on the meccano set that I used to play with. What I will be using here is only one set of meccano parts, strips, with holes in them at fixed, integral distances apart (measured in half-inches as it happened to be). They can have any number of holes, and they can be bolted together using nuts and bolts, but a single bolt will always allow a strips to rotate. See Fig. 1.

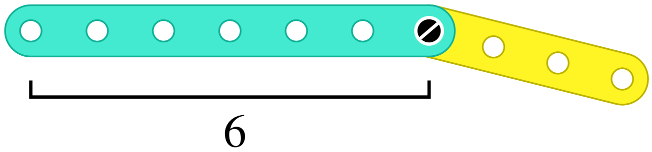

Figure 1: Example of meccano strips, here having lengths 6 and 3, which implies that their numbers of holes are 7 and 4. They can be attached to one another at any of the holes, using bolts.

Although we limit ourselves to constructions in a single plane, we can produce many different figures. Suppose that the strips and the bolts were machined with infinite precision, and as many of them could be combined as we wish. Which figures can one produce, which distances and angles can we produce accurately? What is the minimal number of strips needed? There is a bunch of interesting questions one can ask.

Our strips will be assumed to be strictly two-dimensional. As many of them as we want may be completely or partly overlapping. The bolts are assumed to take no space either. Of course, this is not quite what they would be in the real world, and indeed, our more 'elegant' solutions will be those where strips overlapping the bolts of other strips are avoided. In practice, however, this requirement is difficult to be made precise, so it will often be ignored, such as in our final 'solutions' for the heptagon, Figs. 8 and 9.

In the figures here, a color code will be employed helping us to recognize strips of different lengths. The following chapters show that one can basically construct all of the fundamental figures that can be generated using the axioms of *Euclidean geometry*. Usually, Euclidean geometry refers to a *compass* and a *straightedge* as the only legitimate tools[1]. Here, I replace compass and straightedge by *Meccano strips*. One striking discovery: *The meccano strips are more versatile than compass and straightedge!* For example: Compass and straightedge do not allow me to divide an arbitrary angle by three. With the meccano strips, this can be done, as I will show, in Section 5.

2. Rational strips

Having strips that only have holes at integral distances apart, may seem to be a big restriction, but, using several strips, I can easily construct new units that behave as strips

[1] See "Euclidean geometry" in Wikipedia: http://en.wikipedia.org/wiki/Euclidean_geometry

where the holes can be chosen at *any rational position* on the strip. A rational number is a number that can be written as

$$R = p/q \ , \tag{2.1}$$

where p and q are integers (preferrably without a common factor larger than one).

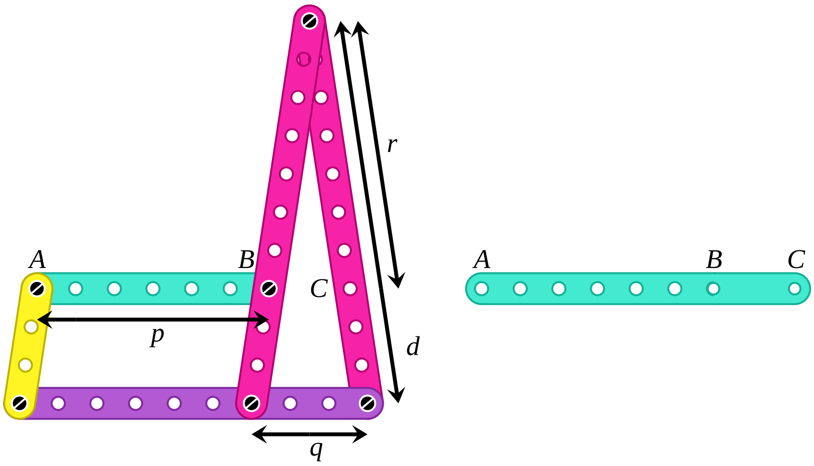

Figure 2: Construction of a generalized strip that effectively has holes on the positions $A,\ A+1,\ \cdots\ B,$ and C, where the distance BC is the rational number $q\,r/d$. At the right is the strip that one effectively obtains.

In Fig. 2 we show how this can be done. In this construction, we find that the points $A,\ B$ and C can be used as if they form a strip of length $p+\frac{q\,r}{d}$. It will be clear that we can choose $p,\ q,\ r$ and d to be any set of numbers, and the strip can be extended to contain such rational holes at several places, by joining a few of these constructions together. Indeed, the construction was done in such a way that the piece (AB) can be used to extend the rational strip and have it fixed pointing in some desired direction.

3. Strips with square roots

However, rational numbers are not the only ones that can be obtained. *We can construct effective strips of length* $a+\sqrt{b}$, *where* a *and* b *are any integers!* To prove this, we first take the case that b is either odd or a multiple of 4.

In this case, we can write

$$b = s\,t \ , \tag{3.1}$$

where s and t are either both even or both odd (the number 1 is also allowed here). Write

$$b = x^2 - y^2 \ , \qquad x = \frac{s+t}{2} \ , \quad y = \frac{s-t}{2} \ . \tag{3.2}$$

Here then, x and y are both integers. Our construction is then the one depicted in Fig. 3: x, y, and $\sqrt{b}$ form a rectangular triangle. Here, we also make use of the well-known fact that the triangle with sides of lengths 3, 4 and 5 is a rectangular one.

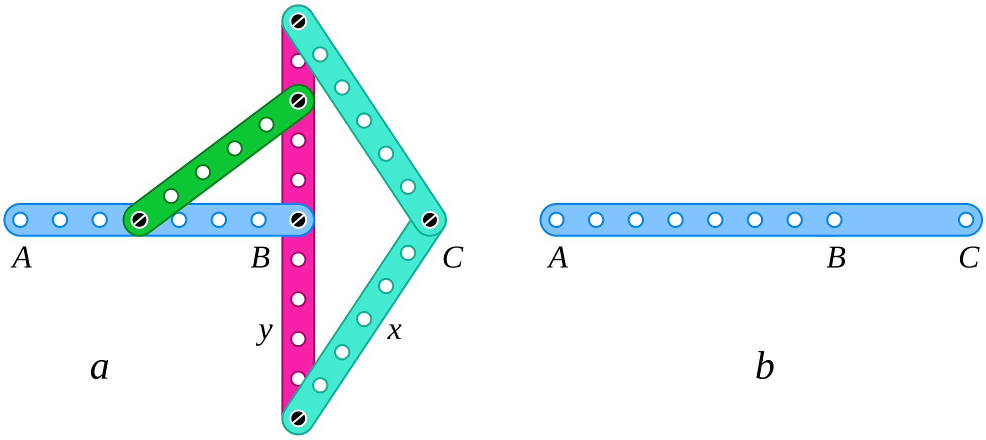

Figure 3: Construction of a generalized strip that effectively has holes on the positions A, $A+1$, $\cdots$ B, and C, where the distance BC is the square root of the number b of Equation (3.2). In this example, $a = 7$, $x = 6$ and $y = 5$. At the right is the strip that one effectively obtains, with length $a + \sqrt{b} = 7 + \sqrt{11}$.

Note that the procedure described here works for square roots of all numbers b of the form $b = 4n$, $b = 4n + 1$, or $b = 4n + 3$, where n is an integer. For $b = 4n + 2$ it does not work. However, now we have the technique to replace the integer strip of length x by one of length $\sqrt{4n+3}$; the strip of length y can be given, for instance, the length $y = 1$. Thus, by repeating the procedure, we see that the roots of the numbers $b = 2n+2$ can also be taken. Furthermore, combining the construction of this section with that of the previous one, it is easy to convince oneself that also the square root of any rational number can be obtained. In many cases, the procedure can be streamlined further to yield more elegant solutions.

4. The straight line

The constructions of the previous chapters were all rigid; that is, with all bolts in place all points are fixed, apart from an overall displacement or rotation of the work piece. It is of interest, however, to consider objects where one degree of freedom is left. If one strip is fixed at one point only, its other points can move around in perfect circles. When other strips are attached, remaining edges will have the freedom to move in more complicated orbits. A very important question is: can we attach strips in such a way that one of the end points can only move along a perfect straight line segment? Of course, a strip with infinite length would have an end point moving on a straight line; however, we will exclude strips of length ∞.

Finding approximate solutions is not hard. In fact, one can easily make gadgets where the deviations from a perfect straight line are extremely small. However, in this study, I am not interested in approximate solutions, only in exact ones. Remarkably, an exact solution to the straight line problem also exists. First, consider the construction in Figure 4. It has one degree of freedom: the distance x between the points A and B can be chosen to be anything between the sum and the difference of the lengths p and q.

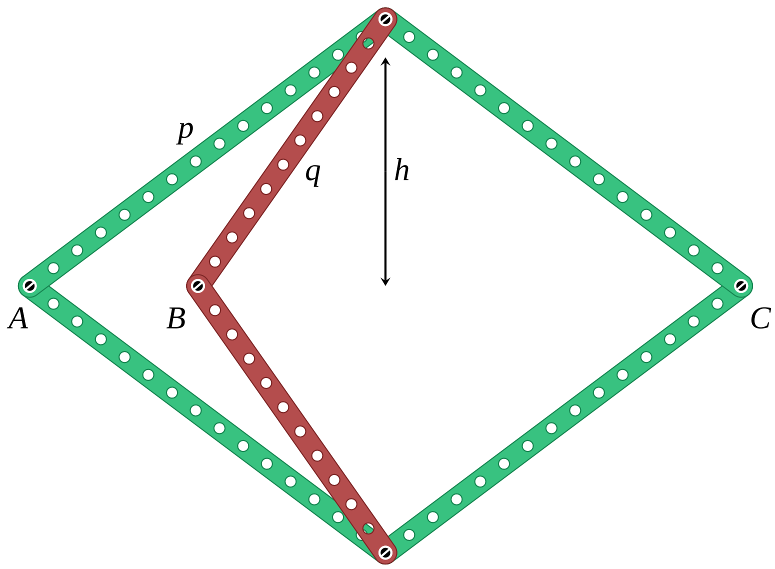

Figure 4: A mechanical device that keeps the points A, B and C on one line, while the product of the segments (AB) and (BC) is kept fixed at the value $p^2 - q^2$, see text.

Once x is chosen, the length y of the adjacent segment (BC) is fixed. Its value can easily be calculated. Let h be half of the vertical separation between the upper and lower joints (See Fig. 4). Defining $a^2 = p^2 - h^2$ and $b^2 = q^2 - h^2$, we see readily that

$$x = a - b\ , \qquad y = a + b\ ,$$
$$x \cdot y \;=\; a^2 - b^2 \;=\; p^2 - q^2\ . \tag{4.1}$$

Write $p^2 - q^2 = r^2$. The product of $x \cdot y = r^2$ is kept fixed! This is a very useful property.

Using this device, we can rigorously connect points $\vec{x}$ to points $-r^2 \cdot \vec{x}/x^2$, assuming that the point B is fixed to the origin O of our coordinate grid. We call this an *inversion*. A feature of inversions is that if the point A describes a circle or a straight line, then C also goes along a circle or a straight line. It is easy to confine A to a circle: we just fix it with a strip of length ℓ to some other fixed point M. If now the circle of A is arranged to go through the origin O, then C is tied to a straight line. See Fig. 5, where the math of the construction is visualized.

What has effectively been constructed here, is a strip with a gliding point.

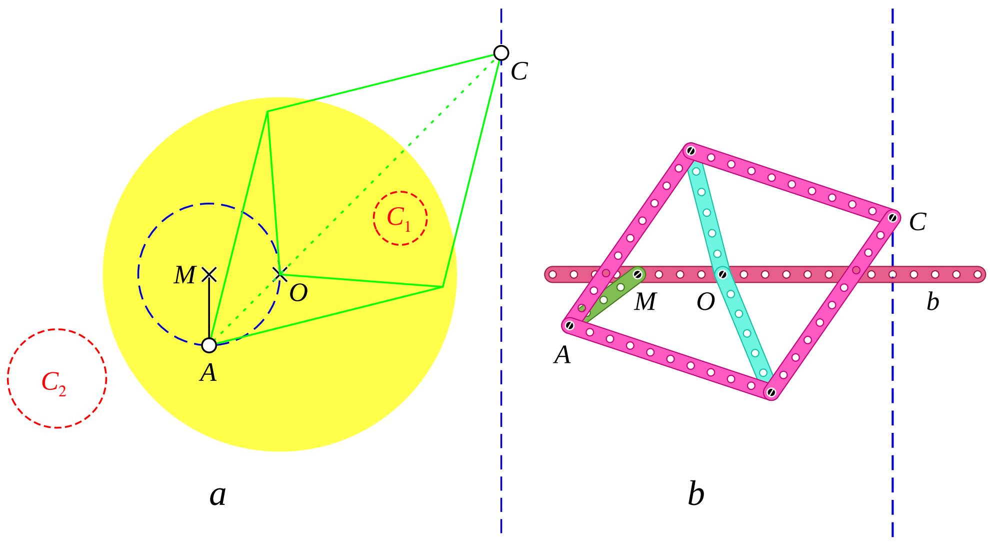

Figure 5: Construction where one point keeps the freedom to move along a fixed line. The device of Fig. 4 is used. a) The inversion transforms circles (or straight lines) such as C_1 into circles (or straight lines), here C_2 . b) Using strips. The point C can only move along the dotted straight line, which is orthogonal to, and fixed with respect to, the base strip b .

5. Bisecting angles

The device of the previous section can now be used to bisect an angle. Consider two strips, attached to one another at some arbitrary angle α . How do we fix a strip, through the same point, forming angles $\alpha/2$ with both these strips, *i.e.*, construct the bisectrix?

This may seem to be easy. Two equal strips will hold the bisectrix in place. However, these two strips cannot be fixed to the bisectrix at a fixed position; together, they have to glide. So, here is where we use the device of the previous section. See Fig. 6.

And now a little surprise: we can also trisect an angle. And divide an angle by 103 or any other integral number. This is because we can use Fig. 6 to force two adjacent angles to be equal, and so any number of adjacent angles can be forced to be equal. Thus, we see that the strips can do *more* than compass and straightedge can do in Euclidean geometry! Trisecting an angle is not possible using just compasses and straightedges in a 'legal' way.

Dividing an angle by a large number may be rather costly in terms of numbers of strips and bolts. Even the bisection of an angle, as described above, appears to be a bit complicated. Indeed, a simpler solution does exist, see *Meccano Math II* (October 2008).

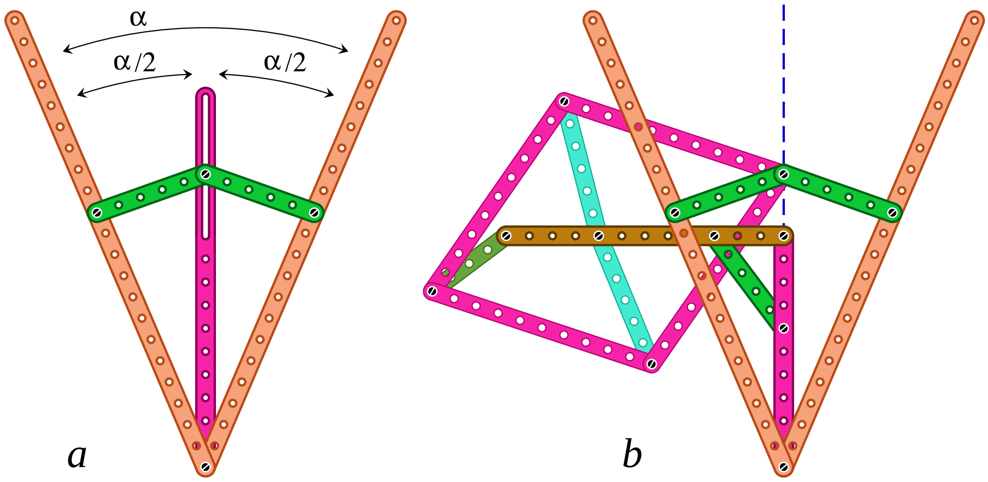

Figure 6: Bisecting an angle. a) What would be needed is a strip with a slit, through which the two auxiliary strips could glide. Such a strip however was not in our set, but can be mimicked (b) using the device of Fig. 5.

6. The regular pentagon

This whole investigation started when I wanted to construct a rigid regular pentagon in the easiest possible way. We start with five strips of the same length, but to fix them rigidly of course more strips are needed. The simplest procedure is to fix the distance between two next-to-nearest-neighbor edge-points. For this, a strip is required whose length is related to that of an edge strip by the Golden Ratio, *i.e.*, we need this length to be multiplied by $\frac{1}{2}(1+\sqrt{5})$. It was shown in Section 3 how to make an "effective strip" with that length. One more strip is then needed to make the whole thing rigid, and this turns out to be very easy to do. Thus, Figure 7 a is found. The pentagon in there is rigid. The other pentagons, Fig. 7 b — f , are variations on the theme, with increasing degrees of sophistication, as explained in the caption. All pentagons are exactly regular.

What about the regular heptagon and higher multigon figures? Fixing the two angles adjacent to one edge to be equal, requires at least 9 strips. In a heptagon (see Fig. 8), four such constraints must be imposed. If we include the seven edges, this leads to $7+4\cdot 9=43$ strips. For a multigon of n edges, $10n-27$ strips are required, using this method. With a bit more sophistication, I found heptagons made out of 35 pieces, 27 pieces, and any n-gon out of $7n-20$ integral strips (assuming n to be odd; if n is even, a slightly different algorithm is needed). A bit of serendipity led to a heptagon of only 15 pieces (see Fig. 9).

7.

Contributions and suggestions by readers are welcome.[2] In the mean time, the author produced a sequel to this study, see *Meccano Math II*. It shows a smarter way to bisect (or multisect) an angle, and gives some solutions to the exercises given below.

8. Exercises

1. Use the mathematical information of Fig. 7 e only to find another solution for the rigid regular pentagon without crossings (Fig. 7 f used something new).

2. All constructions done with straightedge and compass can also be produced using meccano strips. To prove this, construct a universal straightedge and a universal compass out of integer strips. It is not very easy — tip: use the straight line theorem of Section 4, fig. 6. The results will look rather complex, but that does not matter.

3. Show that, given a strip with length x, where x is integral, rational or otherwise, one can also construct a strip with length $\sqrt[3]{x}$.

4 a. Find a way to bisect an arbitrary angle, using just 9 auxiliary pieces, besides the 3 that define the angles (The text in *Meccano Math I* only gives the solution for at least 11 extra parts, see Fig. 6, but two of these can be avoided).

4 b. However, one can also bisect an angle using only 7 extra parts. Find such a solution, without consulting *Meccano Math II.*

[2] A few enthusiastic responses have been received until now (October 2008), including intelligent solutions to the problems.

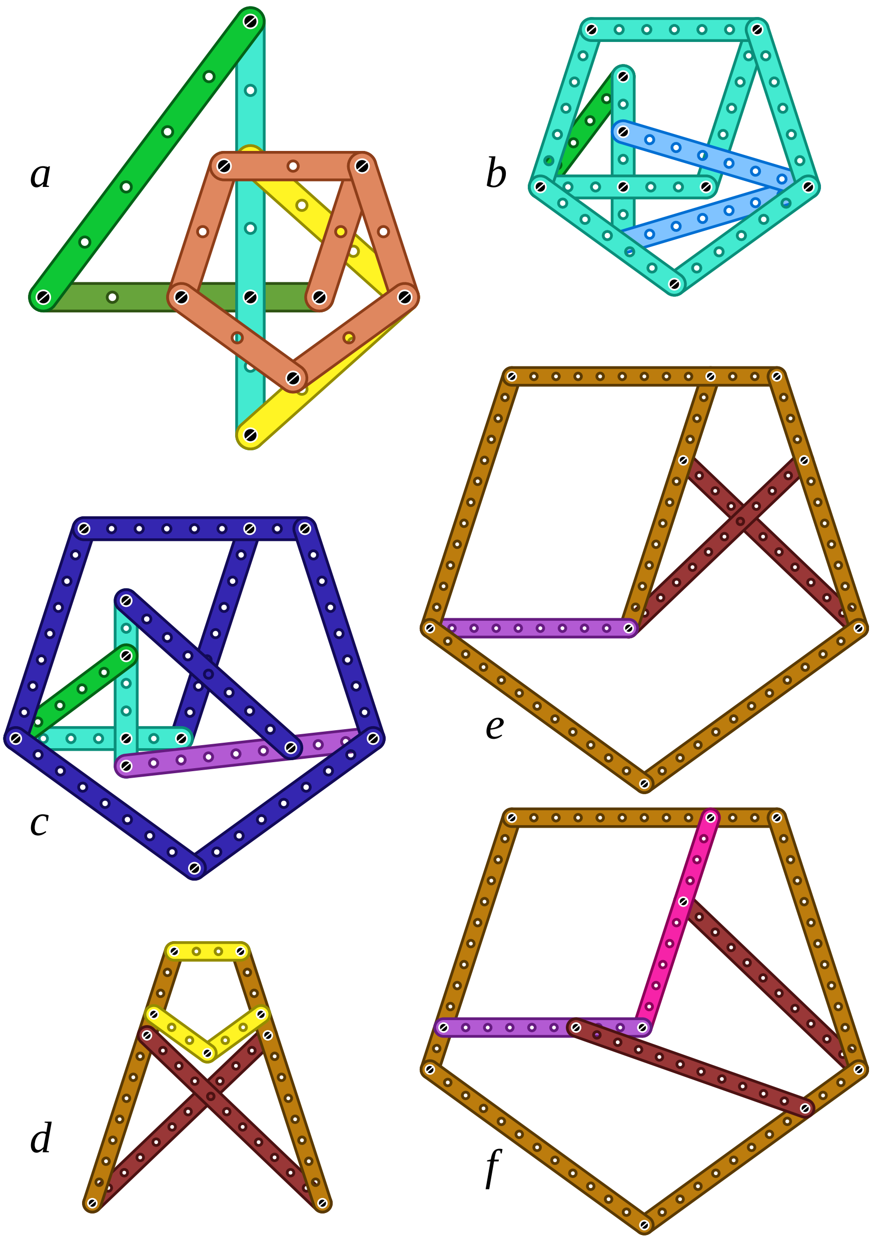

Figure 7: Rigid regular pentagons. *a*) The most straightforward construction. *b*) Variation on this theme: I wanted to have all auxiliary strips inside the pentagon. *c*) Slightly more delicate: I wanted the bolts not to overlap with other strips. *d*) Using only two auxiliary strips. I had to lengthen two of the side elements. One can prove that *i*) single strips cannot connect two adjacent sides, and *ii*) Single strips cannot be connected to sides inside the pentagon at both ends. *e*) With 4 auxiliary strips inside the pentagon. *f*) 4 strips without crossings.

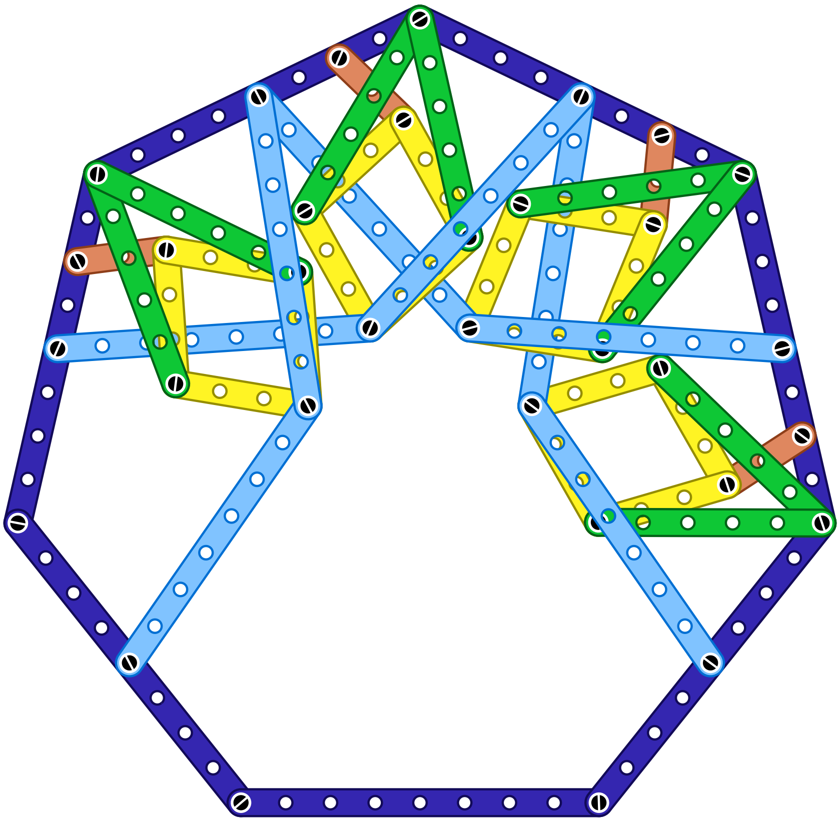

Figure 8: The rigid regular heptagon, using the method of equalizing angles as explained in Section 5. It is built out of 43 integral strips. This algorithm can be extended to any muligon.

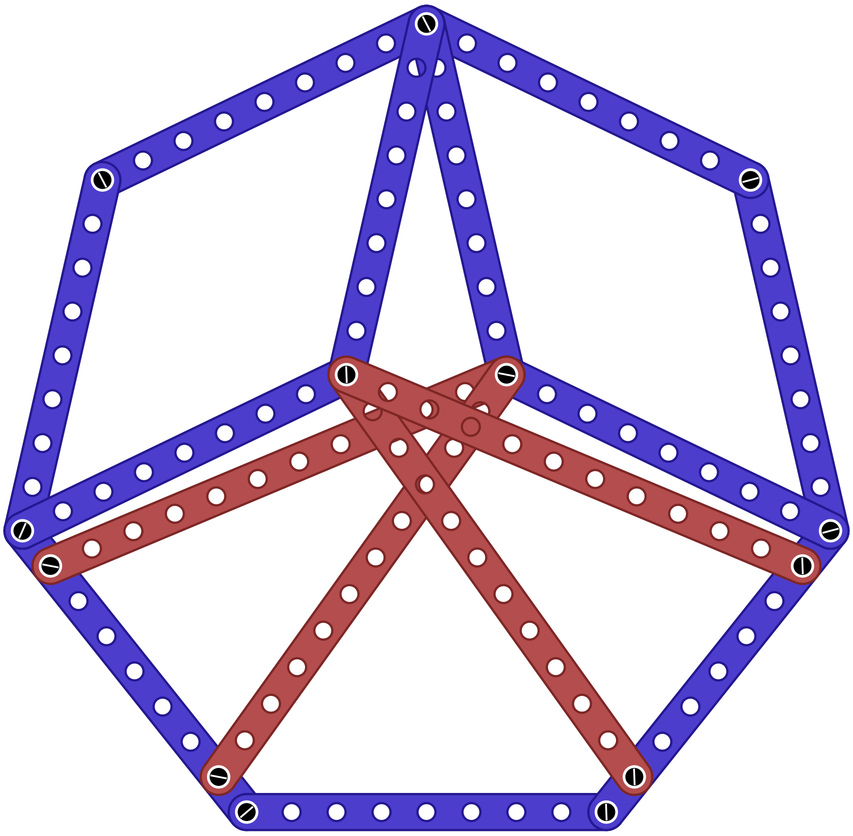

Figure 9: Heptagon made of only 15 strips.