The Dual of Duopoly Is Complementary Monopoly: or, Two of Cournot's Theories Are One

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The observation that two theories share the same formal structure, that is, differ only in the interpretation placed on symbols, can almost always be used to simplify a body of knowledge. The purpose of this communication is to demonstrate that Cournot's theories of duopoly and complementary monopoly are formally identical; furthermore, a precise statement of the correspondence which identifies them serves to extend a famous criticism of the duopoly theory.

Cournot's duopoly theory (Cournot, 1963, chap. vii) applies to a market situation in which two producers sell identical products, and his complementary monopoly theory (Cournot, 1963, chap. ix) to a market situation in which two producers sell products which are of no use unless combined in a fixed ratio (say 1:1) to form a composite commodity. Edgeworth observed that, in the former case, "there cannot well be supposed two prices; and [in the latter case] . . . there cannot be supposed two (independent variations of the) quantities" (Edgeworth, 1925, p. 122).

For two producers (*A* and *B*) this suggests a formal definition of duopoly as a situation in which the sum of the producers' outputs (denoted by $q^a + q^b$) determines a price for the output of each producer (denoted by $p = G[q^a + q^b]$), and complementary monopoly as a situation in which the sum of the prices charged by the producers (denoted by $p^a + p^b$) determines a demand for the output of each producer (denoted by $q = F[p^a + p^b]$).

Prices and quantities are determined in the Cournot solutions to the duopoly and complementary monopoly problems according to the analysis shown on the following page. This presentation establishes the formal equivalence of the two theories; it is immediately clear how one can be obtained from the other by a simple reinterpretation of symbols. A consequence of the equivalence is that a theorem for one theory is a theorem for the other; for example, the well-known result that the quantity

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Duopoly	Complementary Monopoly
Given the quantity demanded, $q = q^a + q^b$, $G(q)$ is the price at which this quantity is demanded (called the demand price at q)	Given price, $p = p^a + p^b$, $F(p)$ is the quantity demanded at this price
$G(q)$ is a decreasing function of q	$F(p)$ is a decreasing function of p
The reaction of A to q^b is the q^a that maximizes $q^a \times G(q^a + q^b)$	The reaction of A to p^b is the p^a that maximizes $p^a \times F(p^a + p^b)$
The reaction curve for B is defined symmetrically	The reaction curve for B is defined symmetrically
q^a and q^b (note $q^a = q^b$) are determined by the intersection of the reaction curves of A and B	p^a and p^b (note $p^a = p^b$) are determined by the intersection of the reaction curves of A and B
$p = G(q^a + q^b)$	$q = F(p^a + p^b)$

supplied under duopoly is greater than the quantity supplied under pure monopoly may be translated into the proposition that the price charged under complementary monopoly is higher than the price charged under pure monopoly.

The focus of Edgeworth's criticism of the Cournot duopoly solution is the observation:

> (1) At a positive profit equilibrium, each duopolist can obtain a greater revenue by reducing his price a little and selling the quantity that clears the market (provided, of course, the other duopolist does not change his price).[1]

Edgeworth was apparently unaware of the possibility that this criticism could be reinterpreted so as to be applicable to the case of complementary monopoly. He believed that the Cournot solution was more plausible for complementary monopoly (Edgeworth, 1925, pp. 136–37) and observed, as his only objection to the Cournot solution, that it requires each monopolist to act as if the other monopolist will not change his price. This is of course a very important criticism, which in fact applies equally well to Cournot's duopoly solution; however, it is not the counterpart of (1). (1′) is (1) reinterpreted for the case of complementary monopoly; it appears never to have been used to criticize Cournot's solution to the complementary monopoly problem.

> (1′) At a positive profit equilibrium, each monopolist can obtain a greater revenue by reducing his quantity a little and selling at the price that clears the market (provided, of course, the other monopolist does not change his quantity).[2]

[1] A quantity q clears the market if either (a) demand price at q = the price at which q is supplied (called the supply price at q), or (b) demand price at $q <$ supply price at q, and $q = 0$.

[2] A price p clears the market if either (a) quantity demanded at p = quantity supplied at p, or (b) quantity demanded at $p <$ quantity supplied at p, and $p = 0$.

By now the reader may not be surprised to learn that propositions (1) and (1′) share the same formal proof. Here is a proof of (1); a proof of (1′) is obtained by replacing p by q, q by p, and G by F.[3]

Proof.—Let q^a, q^b, G, and $p = G(q^a + q^b)$ characterize a Cournot equilibrium. If p^b is constant at p, and p^a is set at $p - \delta$, then the market for B's product can only be cleared if $q^b = 0$.[4] In this case, A's revenue will change from $p \cdot q^a$ to $(p - \delta) \cdot G^{-1}(p - \delta)$. Since $G^{-1}(p - \delta) \rightarrow q^a + q^b = 2q^a$ as $\delta \rightarrow 0$, it follows that there exists a reduction in p^a (that is a δ) that increases A's revenue.

References

Cournot, Augustin. *Researches into the Mathematical Principles of the Theory of Wealth.* Homewood, Ill.: Richard D. Irwin, Inc., 1963.

Edgeworth, F. Y. "The Pure Theory of Monopoly," in F. Y. Edgeworth (ed.). *Papers Relating to Political Economy.* Vol. 1. London: Macmillan Co., for the Royal Econ. Soc., 1925.

[3] We assume that G is continuous and that A can satisfy the entire demand at price $p - \delta$. In the latter assumption, we differ from Edgeworth. If we restrict A to an output of less than $G^{-1}(p - \delta)$, then an asymmetry with the complementary monopoly theory obtains—unless we admit price ceilings on the output of A.

[4] The justification for this assertion may be somewhat obscure for the case of complementary monopoly. It is the following: Since the products of A and B have no use unless combined in a 1:1 ratio, there must be an excess supply of B at every price p^b; and, in this case, the market for B can only be cleared if $p^b = 0$.