Fictionalist Nominalism and Applied Mathematics

For at least a century, mathematics has been a fertile source of arguments against nominalism. Among such arguments, arguments that are based on applied mathematics have at least one important advantage over arguments based on pure mathematics. Nominalists, upon being convinced that the Banach-Tarski Theorem—which has no conceivable application to the physical world—was inconsistent with nominalism, might well be willing to say, "So much the worse for the Banach-Tarski Theorem." But few nominalists would be willing simply to toss *Non-Linear Partial Differential Equations for Scientists and Engineers* (by Professor Lokenath Debnath; "a very hard act to follow," according to one reviewer) into the fire. It would be a heroic nominalist indeed who was willing to dismiss that book—or any of thousands of other books of a similar nature—with the words, "*Does it contain any abstract reasoning concerning quantity or number*? Yes. Commit it then to the flames: for it can contain nothing but sophistry and illusion."

In my view, arguments from applied mathematics are particularly difficult for *fictionalist* nominalists to deal with.[1] In section 1, I will present an argument based on applied mathematics against nominalism *simpliciter*. (The argument is coextensive with the section. Although I in fact accept the argument, Section 1 is no more than a presentation or statement of the argument, and none of the declarative sentences it contains should be taken to be an assertion of the author's.) In section 2, I will present a modification of the argument of section 1 that is directed specifically against fictionalist nominalism. In section 3, I will consider a reply to the argument of section 2 that is based on the common fictionalist contention that "the mathematical fiction" can be shown to be a "conservative extension" of nominalistically acceptable discourse.

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1. An Argument for Rejecting Nominalism

Let us consider a simple problem whose solution requires an application of mathematics to the physical world.[2] We suppose that Alice wishes to find the number of cans (cylindrical cans, each with the dimensions of a "sample" can she has in her possession) of Sherwin-Williams Porch and Floor Enamel (her favorite floor paint) needed to paint the floor of the Mary Lou Beasley Reading Room—a rectangular room, the length of whose sides she is in a position to measure—given that one needs 0.02 cubic units of this paint to cover a surface whose area is 100 square units. ("Units" are units of linear measure: meters, centimeters, nanometers, inches, leagues, parsecs.) This simple problem, as befits a simple problem, is easily solved. To solve it, Alice need only apply the following general principle to the floor of the Beasley Room:

1.1 The rectangular-surface/paint-can numerical requirement principle

NRP For any rectangular surface R and any numbers[3], x, y, z, and w: if x and y measure the lengths of two adjacent sides of R in Ls, and if z measures the height of a can of paint[4] in Ls and w measures its diameter in Ls, then the number of cans of paint that are needed to paint R is[5] 0.00025 (xy/w^2z).

Note that **NRP** has been presented in the form of a schema. *Instances* of this schema are obtained by replacing each occurrence therein of the dummy symbol 'Ls' with (the same) plural "expression signifying a unit of linear measure" ("ULM"). Examples of ULMs are: 'meter', 'inch', 'league', 'nanometer', 'terrestrial polar diameter', 'cubit', and 'astronomical unit'. Thus, one instance of **NRP** will contain the phrase '. . . if x and y measure the lengths of two adjacent sides of R in meters, and if z measures the height of a can of paint in meters . . .'.[6]

To *accept* or *assent to* **NRP** (or any schema) is to accept or assent to ('to commit oneself to accepting'? 'to be prepared to assent to'?) all its instances.

Alice need only apply **NRP** to the case at hand—yes. But in order to apply **NRP** to the case at hand, she will first have to know of two numbers that they measure the lengths of two adjacent sides of the floor of the Beasley Room in meters (or in nanometers or in parsecs . . .) . And she will have to acquire an analogous piece of knowledge for two numbers and the height of a can of paint in meters (or whatever) and the diameter of a can of paint in meters (or whatever). We may take it for granted that Alice has mastered the art of employing a measuring rod calibrated in meters (or whatever) to measure such things as the lengths and heights and diameters of physical objects whose dimensions are within a few orders of magnitude of her own. Let us suppose that she has applied this art to the floor of the Beasley Room (using a measuring rod that is indeed calibrated in meters) and has discovered that the following proposition:

(Adjec) Two adjacent sides of the floor of the Beasley Room are 10 meters in length and 6 meters in length

is true. But if she is to apply **NRP** to the case at hand, she will need somehow to "move" from the truth of (Adjec) to the truth of

(Nom) The numbers 10 and 6 measure the lengths of two adjacent sides of the floor of the Beasley Room in meters.

That (Adjec) entails (Nom) might be thought to be obvious, and from the point of view of the applied mathematician there is certainly a sense in which it is obvious—*so* obvious, in fact, that, in all probability, no physicist or engineer or architect or surveyor or carpenter or painter has ever explicitly raised the following question or any question remotely resembling it: What legitimates inferences like the following: "One side of the floor of the Beasley Room (a rectangular surface) is 10 meters in length, and an adjacent side of that floor is 6 meters in length; the area in square meters of a rectangular surface is equal to the product of the lengths in meters of any one of its sides and an adjacent side; $10 \times 6 = 60$; therefore, 10 meters × 6 meters = 60 meters^2 (or, more idiomatically, 60 square meters); therefore, the area of the floor of the Beasley Room is 60 square meters"? But the legitimacy (i.e., soundness) of this inference—or any inference of its kind—cannot be taken for granted in a philosophical discussion in which the existence of numbers is in question. It cannot be taken to be obvious that (Adject) entails (Nom) if the latter proposition has the sense it must have in a serious philosophical debate about the existence of numbers. In any such debate, it must be recognized that numerals like '10' and '6' are ambiguous: all numerals have both an *adjectival* sense and a *nominal* sense.

Since the adjectival/nominal distinction applies to integers as well as to rational and real numbers, let us first examine that case—the simplest case. Consider the sentence:

> It is possible for a jury's deliberations to end in a tie vote, since a jury has twelve members and twelve is an even number.

The first occurrence of 'twelve' in this sentence is an adjective (modifying the plural noun 'members')[7]. Its second occurrence, however, is a noun: the subject-term of the sentence 'Twelve is an even number'. Note that the obviously nominal phrase 'the number twelve' can be substituted for the second occurrence of 'twelve' in this sentence *salva grammatica*, but not for the first. Suppose we recognize this ambiguity of numerals by writing adjectival numerals in ordinary text and nominal numerals (but let us avoid the distractingly comic effect of the juxtaposition of two dactyls with almost the same pattern of consonants by speaking simply of 'adjectivals' and 'nominals') in boldface. (If you like this device better, you may read, e.g., '**10**' in the sequel as 'the number 10'.)

Can we deduce:

(Nom) **10** and **6** measure the lengths of two adjacent sides of the floor of the Beasley Room in meters

from:

(Adjec) Two adjacent sides of the floor of the Beasley Room are 10 meters in length and 6 meters in length?

As we (in effect) noted above, everyone who uses mathematics to draw useful conclusions from data that were obtained by counting things or measuring things makes such inferences freely and, as it were, unconsciously. For example (I quote from a pamphlet on probability and statistics—obviously from a very low-numbered page of that work):

> *Problem*: What is the probability of drawing a spade in a single draw from a standard deck of cards?
>
> *Solution*: There are 13 spades—favorable outcomes—among the 52 cards. Hence, the probability of drawing a spade is **13/52** or **0.25**.

(Of course the nominals in the original text were not in boldface; that's my work.) Nevertheless, (Nom) cannot be deduced from (Adjec) as a matter of formal logic (and the 'Hence' in the "spades problem" is not an abbreviation of 'It is a logical consequence of this statement that').

Every universal instantiation of an instance of **NRP** on the variables '*R*', '*x*', '*y*', '*z*', and '*w*' will be a sentence in which each of these variables is replaced by a true denoting phrase.[8]

If Alice is to deduce (Nom) from (Adjec), she will need an additional premise. I propose the following:

1.2 The first adjectival-nominal correspondence principle

CP1 If two adjacent sides of a rectangular are surface are A Ls in length and B Ls in length, then **A** and **B** measure the lengths of two adjacent sides of that surface in Ls.

CP1, like **NRP**, is a schema. Instances of **CP1** are obtained by replacing 'A' and 'B' with any adjectivals, '**A**' with the nominal "corresponding to"[9] the adjectival that replaced 'A', '**B**' with the nominal corresponding to the adjectival that replaced 'B', and (once more) replacing the occurrences of 'Ls' uniformly with any plural ULM.[10] And, of course, Alice will need a second correspondence principle to apply **NRP** to the floor of the Beasley Room:[11]

1.3 The second adjectival-nominal correspondence principle

CP2 If a can of paint is A Ls in height, and B Ls in diameter, then **A** measures the height of a can of paint in Ls and **B** measures its diameter in Ls.[12]

Now let us return to Alice and her problem. Let us suppose that she accepts **NRP**, **CP1**, and **CP2**. She has made her measurements of the floor of the Beasley Room, and has discovered that (Adjec) is true. Applying the appropriate universal instantiation of the appropriate instance of **CP1** to these data, she deduces the truth of (Nom). She proceeds to measure her "sample" paint can, and discovers that it is 20 centimeters (0.2 meters) in height and 17 centimeters (0.17 meters) in diameter. Obtaining the proposition that is the appropriate universal instantiation of the appropriate instance of **CP2**, she deduces from that proposition and her data the truth of:

(Nom2) **0.2** measures the height of a can of paint in meters and **0.17** measures its diameter in meters.

From (Nom), (Nom2), the following universal instantiation of an instance of **NRP**:

> If **10** and **6** measure the lengths of two adjacent sides of the floor of the Beasley Room in meters, and if **0.2** measures the height of a can of paint in meters and **0.17** measures its diameter in meters, then the number of cans of paint that are needed to paint the floor of the Beasley Room is:
>
> $$(\mathbf{0.00025} \times \mathbf{6} \times \mathbf{10}) \div (\mathbf{0.17} \times \mathbf{0.17} \times \mathbf{0.2}),$$

and the truth of arithmetic:

$$(\mathbf{0.00025} \times \mathbf{6} \times \mathbf{10}) \div (\mathbf{0.17} \times \mathbf{0.17} \times \mathbf{0.2}) = \mathbf{2.6},$$

(Alice has a firm grasp of the concept "significant digits"[13]), she deduces that she will need 2.6 cans of paint to paint the floor of the Beasley Room.[14] She proceeds to the nearest Sherwin-Williams outlet and buys three[15] cans of Porch and Floor Enamel (Antiquarian Brown, if you want to know). And in the event she is not disappointed: when the floor has been painted, she finds that she has used all the paint in two of the cans and that the third is four-tenths full. (The last stage of Alice's reasoning shows clearly why she needed universal instantiations of instances of **CP1** and **CP2** as premises: to obtain a sentence that is a universal instantiation of a sentence whose variables range over numbers—the range of the variables '*x*', '*y*', '*z*', and '*w*' in **NRP**—one must replace those variables with nominals).

It was no accident that Alice was not disappointed. Paint a thousand rectangular floors of wildly varying dimensions with Sherwin-Williams Porch and Floor Enamel; let the areas of the floors be both great and small; let the lengths of two adjacent sides of some of the floors be in some cases equal, in other cases very nearly the same, and, in other cases still, very different. You will find that applying **CP1**, **CP2**, and **NRP** to all those floors always yields the right result—for each of those thousand floors, applying them will yield a number that turns out to be the number of cans of paint needed to paint it.[16] And *why* does applying our principles always yield the right result? Why is it that these principles are applicable to the physical world? Well, if the principles are *true*, the fact that their application always yields the right result is no more puzzling than the fact that a valid argument with true premises has a true conclusion. But this explanation of the applicability of our principles to the physical world is not available to nominalists. Consider, for example, **CP1**. If this principle has only true instances, any formally valid argument that has an instance of **CP1** as one of its premises and all of whose other premises are true has a true conclusion. And the argument whose premises are:

> The floor of the Beasley Room is a rectangular surface.
>
> Two adjacent sides of the floor of the Beasley Room are 10 meters in length and 6 meters in length.
>
> If two adjacent sides of a rectangular surface are 10 meters in length and 6 meters in length, then **10** and **6** measure the lengths of two adjacent sides of that surface in meters.

and whose conclusion is:

> $\exists x\ \exists y$ x and y measure the lengths of two adjacent sides of the floor of the Beasley Room in meters

is formally valid.[17] This conclusion of this formally valid argument is inconsistent with nominalism, owing to the fact that only numbers—abstract, mathematical objects—satisfy the condition 'x and y measure the lengths of two adjacent sides of the floor of the Beasley Room in meters'.

If there are no numbers (and if there are rectangular surfaces that are 10 meters on one side and 6 meters on an adjacent side—or 11 cubits on one side and 3.1516×10^{-16} parsecs on an adjacent side, or what have you), then every instance of **CP1** is false, and the instances of **NRP**, while true, are only vacuously so. (And, of course, if all instances of **CP1** are false for that reason, all instances of **CP2** will be false for that reason.) Anyone who denies the existence of numbers—nominalists and anyone else, who is, like Calvin,[18] a "math atheist"—must therefore regard the empirically verifiable fact that applying these principles to the physical world always yields the right result as a *mystery*. If **CP1** and **CP2** are false (and if **NRP** is only vacuously true), God knows why applying these principles "works."

If we're *willing* to leave the fact that applying **CP1** and **CP2** and **NRP** to the physical world always gives the right results a mystery, of course, we can always treat the person who applies them—or treat *any* applied mathematician—and his or her pencil and paper as a black box, as a calculator made of flesh and blood and cellulose and graphite (its "programming" will be a set of procedures for making sequences of pencil marks on paper that the mathematician has somehow internalized).[19] But a black box is a black box: if it works and if no one knows how it works, then how it works is a mystery—and *why* it works is a mystery.

Nominalism is therefore to be rejected because it renders the applicability of mathematics to the physical world a mystery.

2. Against Fictionalist Nominalism

Now let us examine a version of the argument presented in section 1 that is directed specifically at *fictionalist* nominalism. To state this version of the argument, we first set out, in its full pedantic glory,[20] the reasoning by which Alice reached the conclusion that she would need 2.6 cans of paint to paint the floor of the Beasley Room:[21]

(1) The floor of the Beasley Room is a rectangular surface.

Premise

(2) If two adjacent sides of a rectangular surface are 10 meters in length and 6 meters in length, then **10** and **6** measure the lengths of two adjacent sides of that surface in meters.

Premise (instance of **CP1**)

(3) If two adjacent sides of the floor of the Beasley Room are 10 meters in length and 6 meters in length, then **10** and **6** measure the lengths of two adjacent sides of the floor of the Beasley Room in meters.

MP (1), *UI* (2)

(4) Two adjacent sides of the floor of the Beasley Room are 10 meters in length and 6 meters in length.

Premise

(5) **10** and **6** measure the lengths of two adjacent sides of the floor of the Beasley Room in meters.

MP (3), (4)

(6) If a can of paint is 0.2 meters in height and 0.17 meters in diameter, then **0.2** measures the height of a can of paint in meters and **0.17** measures its diameter in meters.

Premise (instance of **CP2**)

(7) A can of paint is 0.2 meters in height and 0.17 meters in diameter.

Premise

(8) **0.2** measures the height of a can of paint in meters and **0.17** measures its diameter in meters.

MP (6), (7)

(9) For any rectangular surface *R* and any numbers, *x*, *y*, *z*, and *w*: if *x* and *y* measure the lengths of two adjacent sides of *R* in meters, and if *z* measures the height of a can of pain in meters and *w* measures the diameter of a can of paint in meters, then the number of cans of paint that are needed to paint *R* is:

0.00025 (xy/w^2z).

Premise (instance of **NRP**)

(10) For any numbers, *x*, *y*, *z*, and *w*: if *x* and *y* measure the lengths of two adjacent sides of the floor of the Beasley Room in meters, and if *z* measures the height of a can of paint in meters and *w* measures the diameter of a can of paint in meters, then the

number of cans of paint that are needed to paint the floor of the Beasley Room is:

0.00025 (xy/w^2z).

MP (1), *UI* (9)

(11) If **10** and **6** measure the lengths of two adjacent sides of the floor of the Beasley Room in meters, and if **0.2** measures the height of a can of paint in meters and **0.17** measures its diameter in meters, then the number of cans of paint that are needed to paint the floor of the Beasley Room is:

(0.00025 × 6 × 10) ÷ (0.17 × 0.17 × 0.2).

UI (10)

(12) The number of cans of paint that are needed to paint the floor of the Beasley Room is:

(0.00025 × 6 × 10) ÷ (0.17 × 0.17 × 0.2).

MP (5), (8), (11)

(13) **(0.00025 × 6 × 10) ÷ (0.17 × 0.17 × 0.2) = 2.6**

Premise (truth of arithmetic)

(14) The number of cans of paint that are needed to paint the floor of the Beasley Room is **2.6**.

Euclid's Law (substitution of identicals), (12), (13)

(15) If the number of cans of paint that are needed to paint the floor of the Beasley Room is **2.6**, then 2.6 cans of paint are needed to paint the floor of the Beasley Room.

Premise (instance of the schema discussed in note 14)

(16) 2.6 cans of paint are needed to paint the floor of the Beasley Room.

MP (14), (15)

To return to a point made in part 1, if all the premises of this argument are true, it is no mystery why its conclusion is true. The premises are propositions (1), (2), (4), (6), (7), (9), (13), and (15). Nominalists will have no quarrel with (1), (4), and (7). At any rate, I will assume that they will not.[22]

But nominalists who have accepted these three premises of the argument must reject premises (2) and (6), since they would not accept all the logical consequences of (1), (2), and (4) or all the logical consequences of (6) and (7). Premises (6) and (7), for example, logically imply:

> $\exists x$ *x* measures the height of a can of paint in meters,

and objects that satisfy '*x* measures the height of a can of paint in meters' are just exactly those objects that are called "numbers." Nominalists will, of course, also want to reject premise (13), which entails a proposition that could be expressed in "logicians' English" in these words:

> There are numbers *x*, *y*, *z*, *u*, *v*, and *w* such that the result of dividing the product of *x*, *y*, and *z* by the product of *u*, *u*, and *v* is identical with *w*,

but our antinominalistic argument will appeal only to the fact that nominalists must reject premises (2) and (6).[23]

Fictionalist nominalists will, of course, regard (2) and (6) as false but nevertheless "true in the applied mathematical fiction." Or, to simplify matters, let us suppose that they are speaking of the applied *arithmetical* fiction —a fiction whose vocabulary comprises no specifically mathematical language but numerals (both adjectivals and nominals) and various operator-expressions like 'the product of . . . and . . .' and 'the result of dividing . . . by . . .'. Before we consider the implications of the thesis that the statements (2) and (6) are false but true in the applied arithmetical fiction, let us ask what it means to speak of truth in the *applied* arithmetical fiction—at least to the extent of asking which propositions enjoy this status.

We specify these propositions as follows. First, all truths of pure arithmetic ("Alleged truths!" the nominalist interjects[24]) are true in the applied arithmetical fiction. (For example: 'The sum of **7** and **5** is identical with **12**'; '$\exists x$ the sum of **7** and **5** is identical with *x*'; 'Every number has a successor'; 'The product of **2** and any number *x* is identical with the sum of *x* and *x*'.) Secondly, certain adjectival-nominal correspondence principles will be true in the applied arithmetical fiction. (Roughly speaking, those that everyone who accepts the existence of numbers would regard as trivial or analytic. For example: 'If there are five sheep in a field, then **five** is the number of sheep in that field'; 'If a Philistine champion is

9.512 cubits in height, then **9.512** measures the height of that Philistine champion in cubits'.) Thirdly, the class of propositions that are true in the applied arithmetical fiction is closed under logical deduction (that is, deduction that is valid according to the rules of standard, textbook nominal-variable quantifier logic with identity—"horseshoe pushing and quantifier dropping"); finally, the class of propositions that are true in the applied arithmetical fiction is the smallest class of propositions that satisfies these three conditions. And we say that a proposition is false in the applied arithmetical fiction if and only if its negation is true in the applied mathematical fiction. As is generally the case with truth and falsity in a fiction, there will be many statements that are neither true nor false in the applied arithmetical fiction. For example, 'John Wilkes Booth assassinated Lincoln', 'Lee Harvey Oswald assassinated Lincoln', 'There were four Stuart kings of England', '**Four** is the number of the Stuart Kings of England', and 'If there were four Stuart kings of England, then **four** is the number of Martian moons' are neither true nor false in the applied arithmetical fiction.

Now let 'F' represent the operator 'it is true in the applied arithmetical fiction that'. Fictionalist nominalists, although they regard (2) and (6) as false, regard the two statements:

(2f) F (If two adjacent sides of a rectangular surface are 10 meters in length and 6 meters in length, then **10** and **6** measure the lengths of two adjacent sides of that surface in meters).

(6f) F (If a can of paint is 0.2 meters in height and 0.17 meters in diameter, then **0.2** measures the height of a can of paint in meters and **0.17** measures its diameter in meters),

as true, true without qualification. But they cannot "demythologize" Alice's reasoning by the simple expedient of replacing her premise (2) with (2f) and her premise (6) with (6f)—for, of course, the resulting argument would not be valid (or at any rate, not formally, demonstrably valid). The only conclusion remotely resembling '2.6 cans of paint are needed to paint the floor of the Beasley Room' that can be validly deduced from the premises of the revised argument is:

F 2.6 cans of paint are needed to paint the floor of the Beasley Room.[25]

But this proposition is of no use to Alice, who needs to know how many cans of paint to buy. That is to say, she needs to know how many cans of paint are needed to paint the floor of the Beasley Room, and not how many cans of paint are needed to paint the floor of the Beasley Room according to some fiction—in some *story*. If it is true in the romance novel *My Wicked Pirate*[26] that there is a treasure chest containing millions in Spanish doubloons buried at latitude 14.547032°, longitude -82.65358°, the fact that *in the fiction* there is a chest of that description at that precise location, fact though it indeed is, is not the sort of fact that treasure hunters like E. Lee Spence and Robert F. Marx can exploit in their professional capacity. And Alice's position in respect of the arithmetical fiction and its side-measuring numbers and diameter-measuring numbers is precisely the same as the position of Spence and Marx in respect of the romance fiction and its treasure-chest-depositing pirates (never mind its feisty Yorkshire heiresses). If a treasure-chest-depositing pirate exists only according to a fiction, no appeal to the properties he has in the fiction will be of use to treasure hunters; and, for precisely the same reason, if a physical-object-measuring number exists only according to a fiction, no appeal to the properties it has in the fiction will be of use to prospective paint-can buyers.

"That's all very well," the fictionalist nominalist replies,

> but the force of your analogy rests on the assumption that we know nothing more about the applied arithmetical fiction than that it is, as its name implies, a fiction. Your analogy invites us to ignore the fact that there are such things as reliable fictions—or at least fictions that are reliable in certain respects. It is reasonable to believe of certain fictions, for example, that certain classes of propositions are true if they are true in those fictions. (Consider a novel about Shakespeare and Elizabeth I, written by a well-regarded professional historian whose specialty is late 16th-century England. It is reasonable to believe that the 'large scale' or 'background' statements about the condition of England in that period that are true in the fiction are true *simpliciter*.) I maintain that the applied arithmetical fiction is in a certain sense reliable—although in a sense different from the sense in which an historical novel might be said to be reliable about certain matters. I maintain that the applied arithmetical fiction is a *conservative extension of nominalistically acceptable discourse*.

3. The Applied Arithmetical Fiction As a Conservative Extension of Nominalistically Acceptable Discourse

What does it mean to say that the applied arithmetical fiction is conservative (for so I will abbreviate 'is a conservative extension of nominalistically acceptable discourse')?

Let us say, first, that a sentence (a declarative sentence, a sentence that can be used to make an assertion), is *nominalistically acceptable* if it contains no numerals but adjectivals[27] and contains no quantifiers that bind variables that range over numbers. An *application of arithmetic* is a formally valid argument all of whose premises are either nominalistically acceptable or true in the applied arithmetical fiction, and whose conclusion is nominalistically acceptable. The following argument is thus an application of arithmetic:

(1) The Lincoln Memorial is a rectangular structure that rests on the surface of the earth.

(2) One side of the Lincoln Memorial is 57.8 meters in length, and an adjacent side of the Lincoln Memorial is 36.1 meters in length.

(3) If one side of a rectangular structure is 57.8 meters in length, and an adjacent side of that rectangular structure is 36.1 meters in length, then **57.8** measures the length of one side of that structure in meters, and **36.1** measures the length of an adjacent side of that structure in meters.

(4) For any numbers *x* and *y* and any rectangular structure *z* that rests on the surface of the earth, if *x* measures one side of *z* in meters, and *y* measures an adjacent side of *z* in meters, then the product of *x* and *y* measures the area of the portion of the surface of the earth that is occupied by *z* in square meters.

(5) The product of **57.8** and **36.1** is **2086.58**.

(6) If **2086.58** measures the area of the portion of the surface of the earth that is occupied by the Lincoln Memorial in square meters, then the area of the portion of the surface of the earth that is occupied by the Lincoln Memorial is 2086.58 square meters.

hence,

> The area of the portion of the surface of the earth that is occupied by the Lincoln Memorial is 2086.58 square meters.

And, of course, the argument set out in Section 2 (the "Beasley Room" argument) is an application of arithmetic.[28]

Now, finally:

> The applied arithmetical fiction is conservative $=_{df}$
>
> Every application of arithmetic has the following property: if its nominalistically acceptable premises are true, its conclusion is true.

As regards the premises of the above application of arithmetic (the "Lincoln Memorial" argument), the fictionalist nominalist will say that:

- premises (1) and (2) are nominalistically acceptable. Whether they are true is an empirical question, a question to be answered by taking a look at the Lincoln Memorial and measuring its sides.
- premise (3) is not nominalistically acceptable. It is false if its antecedent is true, which it is if both (1) and (2) are true. If it is true, it is vacuously true.
- premise (4) is not nominalistically acceptable. It is, however, vacuously true, owing to the fact that there are no numbers.
- premise (5) is not nominalistically acceptable. And it is false, since there are no such things as **57.8**, **36.1**, their product, or **2086.58**.
- premise (6) is not nominalistically acceptable. It is vacuously true owing to the fact that its antecedent implies the existence of **2086.58**, which does not exist.

It is a consequence of these theses that if premises (1) and (2) of this argument are true, two of its other premises, (3) and (5), are false. Nevertheless, the fictionalist nominalist is confident that if premises (1) and (2) of the argument are true, its conclusion is true—owing, of course, to the fact that the applied arithmetical fiction is conservative.

We must now raise the question: Why does the fictionalist nominalist think that the applied arithmetical fiction is conservative?

It is commonly supposed that this has been proved.[29] Let us, as lawyers say, stipulate that it has been proved. It is clear, however, that any proof of the thesis that the applied arithmetical fiction is conservative will depend on nominalistically unacceptable premises. The proof will be a piece of pure, not applied, mathematics, and many of its premises will be, according to the nominalist, simply *untrue*.[30]

How then can the nominalist suppose that it has been proved that the applied arithmetical fiction is conservative? A proof (or "proof") with false premises can, of course, have a true conclusion, but why does the nominalist suppose that the proof in question, the proof of 'The applied arithmetical fiction is conservative' is one of those proofs with false premises that is fortunate enough to have a true conclusion? (Of course, it follows from nominalism itself that all applications of arithmetic are such that if their nominalistically acceptable premises are true, their conclusion is also true—simply because, as we saw in note 30, it follows from nominalism that although arithmetic is certainly frequently applied, there are no applications of arithmetic. But then it also follows from nominalism that all applications of arithmetic with true nominalistically acceptable premises have false conclusions.)

No doubt all the false premises of the proof are true in the fiction called, let's say, "the theory of the real numbers" (and no doubt all the vacuously true premises of the proof are nonvacuously true in that fiction), but that's only to say that (although false in reality) they're true according to a certain story. Every false premise of the proof that the applied arithmetical fiction is conservative can, of course, be "turned into" a truth by prefixing it with 'F_R' ('it is true in the fiction called "the theory of the real numbers" that'), but of what benefit is that to the nominalists? At best, that fact will allow them to prove the proposition:

F_R the applied arithmetical fiction is conservative.

And that proposition is of no use to them. To contend that the applied arithmetical fiction is conservative on the ground that it is conservative according to a fiction called 'the theory of the real numbers' would be like contending that the Holy Spirit proceeds from the Father and the Son on the ground that the Holy Spirit proceeds from the Father and the Son according to a fiction called "the Nicene Creed."

Or think of matters this way. Suppose that instead of the applied arithmetical fiction, we have a computer, Arithmos, that will answer any question put to it, provided that the answer to the question can be calculated, using the principles of arithmetic, from the information contained in the question. If, for example, you ask Arithmos, "If two sides of rectangular structure resting on the surface of the earth are 57.8 meters in length and 36.1 meters in length, what is the area of the portion of the surface of the earth that is occupied by that structure?", Arithmos will reply, "2086.58 square meters." One reason to trust the answers that Arithmos gives us would be that we have asked it many, many such questions—questions varying widely in subject-matter and involving a very diverse range of numbers—and it has given the right answer every time. But, if we are nominalists, what *other* reason could we have for trusting Arithmos? And what answer could we give to the question, "*Why* does Arithmos always give the right answer?" than something like, "It just *does*. That, apparently, is its nature: Arithmos has a numerically veridical virtue."

A Platonist could do better. Platonists—those with the requisite mathematical skills—could examine Arithmos's program and *prove* that it will always give the right answer to a question if the answer to that question can be arithmetically deduced from the numerical information contained in the question. (That is, they could prove that statement if it were true—which it certainly could be.) Or so those Platonists would *claim*. Nominalists, however, will—or at least should—say that the Platonists' "proofs" fail to be proofs because they have many false premises and many other premises that are only vacuously true. (The Platonist's "proof"—the nominalists say—may contain the true statement 'Every number has a successor', but it might just as well have contained the true statement 'Every number is even'.)

To return to a point I made earlier, it seems that the nominalist must regard those who apply arithmetic to the physical world much as they regard Arithmos: an applied arithmetician (taken together with his or her pencil and paper) is a black box, "a calculator made of flesh and blood and cellulose and graphite." (The programming of this calculator is a set of procedures for making sequences of pencil marks on paper that one of its components, the human arithmetician, has somehow internalized). We trust their (nominalistically acceptable) numerical answers to our nominalistically acceptable questions because those answers are always right—or at any rate, only very rarely wrong. But to the question, "And *why* do those procedures they have internalized work?" the consistent nominalist has no answer but "They just *do*."[31]

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NOTES

1. Fictionalist nominalism may be informally characterized as a form of nominalism that does not forbid reference to or quantification over abstract objects, provided that such reference and quantification is treated as a fiction—provided that it is recognized by those who make statements that involve such reference and quantification that, while many of these statements are true (and nonvacuously true) *in the fiction*, they are in fact false (or, at best, vacuously true) in reality. The argument of the paper will apply to any precise statement of "fictionalist nominalism" of which this is an apt informal characterization.

2. To solve this problem, one need grasp no mathematical concepts more recondite than "area" and "volume," and need have mastered no mathematical technique more elaborate than multiplication and division. But if I (or some other writer: a writer who shared my philosophical views but not my mathematical limitations) had chosen an illustrative problem concerning the angular velocity of the precession of the perihelion of the orbit of a planet or the dynamics of vortex lines in superconducting transistors or heat flow in a cylinder with constant surface temperature, the philosophical lesson of the problem would have been the same—although perhaps obscured by the complexity of the example.

3. The "numbers" that we speak of may be real numbers or they may be only rational numbers. Presumably, applied mathematics could "get by" with only rational numbers.

4. Here, and in the sequel, a "can of paint" is a can of Sherwin-Williams Porch and Floor Enamel whose dimensions are those of the sample can mentioned in the statement of the problem.

5. The word 'is' is to be taken as meaning 'is identical with', since the expressions that flank this word are terms. The expression that follows 'is' is equivalent to:

$$(xy/100 \times 0.02) \div 3.14\,(w/2)^2 z.$$

Here 0.02 is the number of cubic units of liquid paint needed to paint 100 square units of surface area, and 3.14 is an approximation of π. (The volume of a right cylinder in cubic units is equal to π times the square of the radius of its base in those units times its height in those units.) The expression in essence says this: the number of cans needed to paint a rectangular surface is the number of "100 square unit" portions needed to make up that surface times the number of cubic units of liquid paint needed to paint a single 100-square-unit portion of surface (that is, the number of cubic units of liquid paint needed to paint the surface) divided by the number of cubic units of liquid paint in one can of paint. The expression in the text comes from the above expression by a transformation that lecturers in the mathematical sciences call "Simplifying, we obtain . . .".

6. Suppose one believes that there is such a thing as the number 20 and believes that a certain tower is 20 meters high. One will then, surely, grant that the number 20, the height of the tower, and the unit of linear measure "the meter" stand in certain salient relation. I express this relation by saying that the number 20 measures the height of the tower in meters (and I use the same expression, *mutatis mutandis* to describe the relation between numbers and the various dimensions of physical objects of all kinds and all units of linear measure). Anyone who prefers some other way of expressing this relation should feel free to make the appropriate substitutions in the remainder of this paper.

7. That numerals are sometimes adjectives is immediately evident in languages in which adjectives agree in, e.g., gender with the noun they modify; in Latin, for example, the three kings (Caspar and Balthasar and Melchior) are '*tres* reges', but their three gifts (gold and frankincense and myrrh) are '*tria* dona'—owing to the fact that 'rex' ('king') is masculine and 'donum' ('gift') is neuter.

8. "A true denoting phrase" does not mean "A phrase that denotes something." 'The mad elephant that trampled the peonies in my garden this morning' and '**1/0**' are true denoting phrases, or at least fail to be true denoting phrases only when they occur in certain intensional contexts. The intuitive idea behind the words 'true denoting phrase' is something like this: a phrase that must be understood as *purporting to denote something* or as *representing itself as denoting something*—although it may well fail to do what it purports to do or represents itself as doing. Can this intuitive idea be made more precise? Quine has a well-known test, and a reasonably precise one, for determining whether an expression is a true denoting phrase (which is not in fact a term he uses): a true denoting phrase is a nominal expression that occupies a syntactic position that renders it subject to existential generalization. So, for example, the term that is the object of the preposition in the sentence 'She did it for my sake' is a noun-phrase but not a true denoting phrase, owing to the fact that '$\exists x$ she did it for x' does not follow from that sentence by existential generalization. And the ordinary-text occurrence of the word 'twelve' in the sentence 'A jury has twelve members and **twelve** is an even number' is not a true denoting phrase because it is not a noun-phrase to start with: '$\exists x$ a jury has x members' is not even grammatical. But both occurrences of '**twelve**' in '**Twelve** numbers the members of a jury and **twelve** is an even number' are true denoting phrases (even if, as nominalists suppose, they fail to denote anything): the sentence '$\exists x$ x numbers the members of a jury and x is an even number' is grammatical, follows from '**Twelve** numbers the members of a jury and **twelve** is an even number' by existential generalization, and is even true—or will be regarded as true by anyone who accepts the existence of numbers. It must be conceded, however, that the "intuitive idea behind the words 'true denoting phrase'" and the "existential-generalization test" are not entirely happy bedfellows. '**1/0**', I have said—guided by that intuitive idea—is, although it denotes nothing, a true denoting phrase. But does '$\exists x$ x does not exist' follow from '**1/0** does not exist' by existential generalization? One would hope not—at least if the latter sentence is true (as many would suppose) and the former is false (as many would suppose—Quine and I among them). There are various ways in which the intuitive idea and the formal test might be reconciled. Any such attempt at reconciliation, however, would be of no relevance to the argument against nominalism that is considered in the text. The intuitive idea and the formal test are in apparent conflict only in cases in which the

nouns and noun-phrases to which they apply occur in certain special contexts. (Of which '. . . does not exist' is representative'. Normally a subject-predicate sentence is said to be true just in the case that its subject denotes an object and that object belongs to the extension of its predicate. Suppose, however, that certain philosophers contend that various natural-language subject-predicate sentences do not conform to this rule, among them sentences whose predicate is 'does not exist'; such sentences, these philosophers maintain, are true just in the case that their subject is a true denoting phrase that lacks a denotation; and their predicate is therefore "a logical [i.e., grammatical] but not a real predicate." These philosophers will therefore regard '**1/0** does not exist' as true. If they also regard '$\exists x\, x$ does not exist' as false, they will have to reject or modify the existential-generalization test for true denoting phrases.) And no such contexts occur in the argument. All occurrences of boldface numerals in this paper are thus true denoting phrases by the existential generalization test.

9. It is easy to say what it means to say that a nominal and an adjectival "correspond" if the nominal denotes (or, if you like, purports to denote) an integer. The definition of correspondence can be given in the form of a schema: 'Q**A** corresponds to QA if and only if **A** is the cardinal number of any set that has A members'. Instances of this schema are obtained by replacing '**A**' with a nominal, 'A' with an adjectival, 'Q**A**' with the quotation-name of that nominal, and 'QA' with the quotation-name of that adjectival. (At any rate, this definition should be acceptable to anyone who thinks that there are numbers. Some nominalists may protest that they regard all the right-hand constituents of the instances of this schema false, and others that they regard them as all vacuously true. I'll leave it to the nominalists to provide their own account of adjectival-nominal correspondence.) It is much more difficult to formulate a schematic definition for the case of nominals that denote rational or real numbers (because of the great variety of contexts in which such nominals and their corresponding adjectivals can occur). I will not even try.

10. Unless 'A' or 'B' is replaced by '1' or 'one'. In that case, the ULM following (the adjectival) '1' or 'one' should be singular.

11. It would not be *too* difficult to formulate a much more general adjectival-nominal correspondence principle, a principle that pertained not to only to such limited classes of things as rectangular surfaces and paint cans, but to all "physical objects," or all "objects with spatial dimensions," or some such general count-phrase, and to display our two "special-purpose" principles as special cases of the general principle. The general principle would be a schema containing (in addition to dummy letters representing the positions of adjectivals, nominals, and plural ULMs) dummy letters representing the positions of occurrences of two other kinds of expression. Members of one of these two classes of dummy letters would indicate the positions of—well, call them "adverbs of linear dimension"; for example: 'in height', 'high', 'long' 'in diameter' 'thick' 'across' 'tall', 'wide', and '. . . in length on one side and . . . in length on an adjacent side'. The members of the other would indicate the positions of "dimensional operators": 'the height of', 'the width of', 'the lengths of two adjacent sides of . . .'. To formulate such a principle, however, is not an entirely straightforward task—as the final items in each of the two lists of examples in the previous two sentences perhaps suggest—and to do so would not contribute anything of substance to our argument, so we shall not attempt to state a fully general adjectival-nominal correspondence principle.

12. Instances of **CP2** are not universal statements; they are rather conditionals whose antecedents and consequents are universal statements. (Remember that all "cans of paint"

are of the same dimensions.) An example may make this point clear. The instance of **CP2** that Alice will use could be read this way: If every can of paint has this property: it is 0.2 meters in height and 0.17 meters in diameter, then every can of paint has *this* property: **0.2** measures its height in meters and **0.17** measures its diameter in meters.

13. Throughout this paper, I play fast and loose with "significant digits," and I often use '$x = y$' to mean something like 'x and y are either the same number or can be thought of as the same number for purposes of practical calculation'. For example, the result of dividing **0.00025 × 6 × 10** by **0.17 × 0.17 × 0.2** is not **2.6** but something in the vicinity of **2.595155709342561**. (That's as many digits as the calculator app in my iPhone will supply me with. The digits in the decimal part of this expression must, of course, start to "repeat" at some point.) In principle, I could have avoided this fastness and looseness and represented Alice as carrying all the physically meaningless digits that turn up at any point in her calculations through to the end, and then simply noting that the repeating decimal expression that occurs at the end of her calculation represents a number between **2** and **3**. The fastness and looseness is therefore irrelevant to the argument of the paper.

14. Strictly speaking, she must use yet another adjectival-nominal correspondence schema to reach this conclusion (a schema that "goes in the other direction"—from nominal to adjectival): 'If **A** is the number of cans of paint that are needed to paint a floor, then A cans of paint are needed to paint that floor'—or, more generally, 'If the number of *X*s that *F* is **A**, then A *Xs F*'. Instances of the schema are obtained by replacing '*Xs*' with a plural count-noun or "count-phrase," and '*F*' with a plural verb or verb-phrase.

15. She wasn't able to use the sample can as a source of paint. It was empty or contained paint of the wrong color or something.

16. Given only that 0.02, a number I picked out of the air, is indeed the number of cubic units of liquid Sherwin-Williams Porch and Floor Enamel needed to paint 100 square units of surface. (The volume of liquid paint of a given kind that is needed to paint a surface of a given area is of course an empirical question: it is a quantity whose value cannot be deduced from first principles but, to borrow an expression that has some currency among physicists, "needs to be filled in by hand.")

17. At any rate, it would be if I had stated **CP1** a little more pedantically ('If *R* is a rectangular surface, and two adjacent sides of *R* are A Ls in length and B Ls in length, then **A** and **B** measure the lengths of two adjacent sides of *R* in Ls').

18. The cartoon character created by Bill Watterson, not the author of *Institutio Christianae religionis*.

19. To anticipate somewhat: This strikes me as a not inaccurate description of the implications of fictionalist nominalism for our understanding of applied mathematics.

20. At least to the extent of stating explicitly all the premises required for the reasoning to be logically valid. I will not, however, introduce as much of the formal apparatus of quantification into the premises as would be required to demonstrate the validity of the reasoning by logic-textbook horseshoe-pushing and quantifier-dropping.

21. I expect she employed this argument only "tacitly." Being an admirably practical woman, she would no doubt have reacted to a request to see her reasoning spelled out in full logic-text detail much as another Alice reacted to Humpty-Dumpty's request for a fully explicit proof of the proposition "If there are 365 days in a year, and if exactly one day of the year is one's birthday, then exactly 364 days of the year are one's unbirthdays." (Humpty-Dumpty, you will remember, reserved passing judgment on the validity of Alice's supposedly fully explicit proof till he had had time to study it carefully. One

wonders whether, in the course of his later meticulous study of the proof, he discovered that her reasoning, while correct, was in fact not fully explicit—depending as it did, on certain unstated premises, of which 'If there are 365 days in a year, then **365** numbers the days in a year' may serve as an example.)

22. I will assume that nominalists will not object to statements simply because they contain adjectivals. Nominalistically acceptable paraphrases of sentences containing "integral adjectivals"—'There are 16 cats in the barn', and so on—into sentences that do not contain integral adjectivals are always available in principle (consult almost any logic textbook), however unwieldy they might be in practice. Nominalistically acceptable paraphrases of sentences containing nonintegral adjectivals ('1.5 meters', 'four and three-quarters hours' . . .) as sentences that contain only integral adjectivals are available to nominalists (nominalists who are willing to adopt a very rich ontology of the physical world). For example, the sentence 'This rod is 1.5 meters long' can be paraphrased as a sentence that contains only integral adjectivals. Here is one such paraphrase (the paraphrase assumes that a "rod" is a right cylinder):

> This rod is the fusion of 3 nonoverlapping objects *x*, *y*, and *z*; *x* and *y* are of the same size and shape; *x* and *z* are of the same size and shape; *y* and *z* are of the same size and shape; if the fusion of *x* and *y* is a connected object, it is 1 meter long; if the fusion of *x* and *z* is a connected object, it is 1 meter long; if the fusion of *y* and *z* is a connected object, it is 1 meter long.

23. As for the other two premises of the argument, (9) and (15), the nominalist will regard these as true, but true for reasons that imply that they have no application to anything in the real world: (9) is a vacuously true universal statement, and (15) is true because—and only because—it is a conditional with a false antecedent. (In the sequel, I will sometimes use the phrase 'vacuously true' to describe statements that are true because and only because they are conditionals with false antecedents.)

24. If we wish to specify this class of statements without saying anything that might offend the scrupulous fictionalist, we could always identify them with the statements that are formally derivable from some suitable set of statements, such as Peano's Axioms. These days, I suppose, every schoolchild knows that for any finite and consistent set of statements in the language of pure arithmetic, there will be true statements in that language that are not derivable from that set (or if the fictionalist prefers: there will be a statement *p* in that language such that neither *p* nor its negation is so derivable). We may ignore this difficulty in practice, however, for no arithmetical statement such that neither it nor its denial is deducible from a set of statements as rich as Peano's Axioms will ever figure in any application of arithmetic to the physical world that a human being could actually carry out.

25. This proposition does not, of course, follow from the revised premise-set by the rules of ordinary nominal-variable logic alone; it does follow, however if we supplement those rules with a rule that obviously should be a rule of logical inference if the arithmetical fiction operator is taken to be a logical symbol. Call the result of prefixing that operator to a sentence the "fictionalization" of that sentence; then the fictionalization rule (so to name it) is:

> If *q* is deducible in ordinary nominal-variable logic from *p*, then if the fictionalization of *p* occurs as a line in a deduction, the fictionalization of *q* may occur as a subsequent line.

The fictionalization rule is valid owing to the clause in our specification of the class of propositions that are true in the applied arithmetical fiction that states that that class is closed under logical deduction (i.e., deduction in "ordinary nominal-variable logic").

26. Rona Sharon, *My Wicked Pirate* (Cape Town: Zebra Press, 2006). "Azure-eyed Alanis was by far the most exquisite treasure ever claimed by the black pirate known as the Viper, but his motives went deeper than his silken promise to ravish the feisty Yorkshire heiress."

27. In order to avoid unnecessary complication, I will simply ignore those sentences that are or might be unacceptable to fictionalist nominalists (*qua* nominalists), but unacceptable for reasons unrelated to the application of arithmetic to the physical world—'Spiders and insects share many important anatomical characteristics', for example, or 'There are structures that do not satisfy the axiom of multiplicative identity, but satisfy all the other ring axioms', or 'Nominalism is false'. We should note that nominalists will regard some sentences that are not "nominalistically acceptable" as true (or as expressing propositions that are true)—albeit only vacuously true. Consider, for example Paul Erdös's first great theorem: Between every integer greater than **1** and its double, there is a prime. The nominalist will regard this sentence as true because it is a logical consequence of 'There are no integers'.

28. By the strict terms of our definition, 'All Greeks are mortal; Socrates is a Greek; *hence* Socrates is mortal' and Anselm's ontological argument and Wittgenstein's private-language argument are (provided the latter two are presented in some formally valid form) applications of arithmetic. This consequence of the definition is harmless. But feel free to consign these arguments to the class of "degenerate" or "improper" or "null" applications of arithmetic.

29. It is not clear to me that it *has* been proved that all nominalistically acceptable statements that can be deduced by applications of arithmetic from true nominalistically acceptable statements are true if the class of nominalistically acceptable statements includes statements that are the results of physical measurements—statements like 'One side of the Lincoln Memorial is 57.8 meters in length'—statements, that is, that contain numerical information in a form that must be "filtered through" appropriate adjectival-nominal correspondence principles before that information can be manipulated arithmetically. And if this is true in the case of arithmetic, it will certainly be true in cases involving more advanced mathematics. Consider, for example, the nominalistic reconstruction of classical gravitational mechanics that is the centerpiece of Hartry Field's *Science without Numbers* (Princeton, N.J.: Princeton University Press, 1980). Suppose a working astronomer had only Field's version of gravitational mechanics to work with. This astronomer, who wishes to determine by calculation (using, to be sure, as much pure mathematics as she likes) various things like the mass and the perihelion distance of Venus (such things can be stated in nominalistically acceptable sentences) has made various measurements, measurements that can be stated in nominalistically acceptable sentences like 'The maximum angular separation of Venus and the sun is 46.5°'. (The numerical information that these measurements provide is supposed to constitute the starting point of her calculations.) I do not understand how these numerical measurements are to be "plugged into" Field's version of mechanics. But I am entirely out of my mathematical depth when I consider questions like this, and it may well be that they have simple and straightforward answers.

30. It is worth noting that even the nonvacuous truth of the statement to be proved ('Every application of arithmetic has the following property: if its nominalistically accept-

able premises are true, its conclusion is true') can hardly be supposed to be consistent with nominalism. If this statement is nonvacuously true, there must be such objects as "applications of arithmetic." And applications of arithmetic are arguments of a certain sort, and are thus presumably much the same sort of thing as the well-defined objects called "proofs" that are quantified over in mathematical logic. (E.g., 'A *proof* is any finite, nonempty sequence of formulas each of which is either an axiom or follows from earlier elements in the sequence by the rules of deduction'; 'Every proof is a *proof of* its final element'; 'It is evident from Definition 6 that every proof has a Gödel number'.)

31. I thank Dr. Kenny Boyce for stimulating criticisms of the arguments of this essay—criticisms which I expect he will think I have been insufficiently attentive to.