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Project

RAND

ON THE HAMILTONIAN GAME  
(A Traveling Salesmen Problem)

Julia Robinson

RM-303

5 December 1949

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# PROJECT RAND

RESEARCH MEMORANDUM

ON THE HAMILTONIAN GAME  
(A Traveling Salesmen Problem)

Julia Robinson

5 December 1949

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## 1. Introduction

The purpose of this note is to give a method for solving a problem related to the traveling salesman problem. It seems worthwhile to give a description of the original problem.\* One formulation is to find the shortest route for a salesman starting from Washington, visiting all the state capitals and then returning to Washington.\*\* More generally, to find the shortest closed curve containing  $n$  given points in the plane.

Clearly, it is sufficient to consider curves made up of line segments joining pairs of the given points. Also, unless all the points lie on a straight line, the optimal path will not pass through any point twice. Hence the problem can be stated as follows:

Arrange the  $n$  points in a cyclic order so that the sum of the distance between consecutive points is a minimum.

In this statement of the problem, arbitrary real numbers can be assigned as the "distances" between ordered pairs of distinct points. Thus, the "distance" from A to B need not be the same as from B to A. We shall sometimes refer to the "length" of AB instead of the "distance" from A to B.

Since there are only a finite number of paths to consider, the problem consists in finding a method for picking out the optimal path when  $n$  is moderately large, say  $n = 50$ . In this case, there are more than  $10^{62}$  possible paths, so we can not simply try them all. Even for as few as 10 points, some short cuts are desirable.

\* Actually, the problem may go back to W. R. Hamilton. See R. W. Ball: Mathematical Recreations on the Hamiltonian game.

\*\* In this paper I shall not be concerned with the various possible applications of the problem solved here.

## 2. Statement of the problem

An unsuccessful attempt to solve the above problem led to a solution of the following:

Given  $n$  points and all the "distances" between ordered pairs of distinct points. The problem is to find a system of ordered circuits such that:

- i. Each point lies on exactly one circuit.
- ii. Each circuit contains at least 2 points.
- iii. No circuit passes through the same point more than once.
- iv. The total "length" of the circuits is a minimum.

However at first glance, it looks more difficult than the traveling salesman problem, for there are obviously many more systems of circuits than circuits. Actually the topological characterization of a system of circuits is much simpler than that of a single circuit and can be used to solve this problem.

The method presented here of handling this problem will enable us to check whether a given system of circuits is optimal or, if not, to find a better one. I believe it would be feasible to apply it to as many as 50 points provided suitable calculating equipment is available.

## 3. Description of the method.

Number the points  $1, 2, \dots, n$ . Put  $D = \parallel d_{ij} \parallel$ , where  $d_{ij}$  is the distance from  $i$  to  $j$ ,  $d_{ii} = 0$ . Let  $\mathcal{S}$  be the set of directed segments comprising the proposed system of circuits. We wish to determine if this system is optimal or, if not, to find a better system.

Construct the auxiliary matrix  $S = \parallel s_{ij} \parallel$  as follows:

For each  $i$ , determine  $i'$  so that  $ii' \in \mathcal{S}$ . Then put,

$$s_{11'} = + \infty$$

and

$$s_{1j} = d_{j1'} - d_{11'} \quad \text{for } j \neq 1'.$$

Now think of the S-matrix as giving new "distances" between the given points and look for a closed circuit of negative S-length. If there is such a circuit, it will have from 2 to n points. Suppose  $C = i_0 i_1 \dots i_k$  is a circuit of negative S-length. Then make up a new system of circuits  $S'$  by modifying  $S$  in the following way:

Remove	Add
$i_0 i_0'$	$i_1 i_0'$
$i_1 i_1'$	$i_2 i_1'$
⋮	⋮
⋮	⋮
$i_k i_k'$	$i_0 i_k'$

The new system of circuits  $S'$ , thus obtained, has a shorter total D-length than  $S$ . In fact, if we let  $L_A(a)$  be the length of  $a$  measured by the matrix A, then

$$L_D(S') = L_D(S) + S(C).$$

We then apply the same procedure to  $S'$ .

Suppose we can not find a circuit of negative S-length. Then we attempt to show that  $S$  is optimal. To do this, enforce the triangle inequality,

$s_{ij} \leq s_{ik} + s_{kj}$ \* that is, if  $s_{ij} > s_{ik} + s_{kj}$  replace  $s_{ij}$  by  $s_{ik} + s_{kj}$ . These replacements can be carried out in any order.

If a matrix is eventually obtained for which the triangle inequality holds, then  $\checkmark$  is the best system of circuits. If not, there must be some circuit of negative S-length. To find one, we must keep track of the changes made in the S-matrix. For example, under the  $i, j^{\text{th}}$  entry in the S-matrix, write (ij). Then if  $s_{ij}$  is replaced by  $s_{ik} + s_{kj}$ , replace the (ij) by (ikj). Similarly, if  $s_{iKj}$  is replaced by  $s_{iHk} + s_{kMj}$ , then replace (iKj) by (iHkMj). (Here K, H and M are finite sequences of numbers from 1 to n.) Thus, the entry in the  $i, j^{\text{th}}$  place will always be the length of the path indicated from i to j. If there is a negative circuit in the S-matrix, then at some stage a negative number can be put on the main diagonal of the modified S-matrix. We can then easily obtain the corresponding circuit in the S-matrix.

#### 4. A numerical example

As an example, we take a set of six points with the following distance matrix:

0	1	4	2	8	7
6	0	5	2	1	9
4	8	0	7	2	6
5	5	5	0	4	8
6	1	5	7	0	4
3	9	1	2	6	0

D

\*  $i, j$  and  $k$  need not be distinct.

As a first trial system of circuits  $\mathcal{S}$  take the two circuits 12531 and 464. Then  $\mathcal{S} = \{12, 25, 53, 31, 46, 64\}$ . Hence  $1' = 2, 2' = 5, 3' = 1, 4' = 6, 5' = 3$  and  $3' = 1$ . Next construct the S-matrix:

0	+∞	+7	+4	0	+8
+7	0	+1	+3	+∞	+5
+∞	+2	0	+1	+2	-1
-1	+1	-2	0	-4	+∞
-1	0	+∞	0	0	-4
0	0	+5	+∞	+5	0

S

We now look for a closed circuit of negative S-length. After a few trials, we find the circuit  $C = 456234$  with S-length = -6. We obtain  $\mathcal{S}'$  from  $\mathcal{S}$  by removing 46, 53, 64, 25 and 31 from  $\mathcal{S}$  and adjoining 56, 63, 24, 35 and 41. We then obtain the  $\mathcal{S}'$  matrix:

0	+∞	+7	+4	0	+8
0	0	+5	+∞	+5	0
+6	-1	0	+2	+∞	+4
+∞	+1	-1	0	+1	-2
+3	+5	+2	+4	0	+∞
+3	+4	+∞	+4	+4	0

S'

Since we do not find a negative circuit in  $S'$ , we try to enforce the triangle inequality, keeping track of the changes we make in case there is a negative circuit. We give one intermediate matrix as an example and the final one in which the triangle inequality holds.

0 (11)	+5 (142)	+2 (153)	+4 (14)	0 (15)	+2 (146)
0 (21)	0 (22)	+2 (2153)	+4 (214)	0 (215)	0 (26)
-1 (321)	-1 (32)	0 (33)	+2 (34)	-1 (3215)	-1 (326)
-2 (4321)	-2 (432)	-1 (43)	0 (44)	-2 (43215)	-2 (46)
+1 (5321)	+1 (532)	+2 (53)	+4 (54)	0 (55)	+1 (5326)
+2 (64321)	+2 (6432)	+3 (643)	+4 (64)	+2 (643215)	0 (66)

Intermediate modified matrix

0 (11)	+1 (1532)	+2 (155)	+4 (14)	0 (15)	+1 (15326)
0 (21)	0 (22)	+2 (2155)	+4 (214)	0 (215)	0 (26)
-1 (321)	-1 (32)	0 (33)	+2 (34)	-1 (3215)	1 (326)
-2 (4321)	-2 (432)	-1 (43)	0 (44)	-2 (43215)	-2 (46)
+1 (5321)	+1 (532)	+2 (53)	+4 (54)	0 (55)	+1 (5326)
+2 (64321)	+2 (6432)	+3 (643)	+4 (64)	+2 (643215)	0 (66)

Final S'-matrix with  $\Delta$ -inequality holding

Hence  $\mathcal{S}'$  is the optimal system of circuits. It consists of the two circuits 1241 and 3563 and has D-length = 15.

### 5. Justification of the method.

First, notice that a set of  $n$  directed segments satisfies 1 - iii of Section 1, if and only if

1. Each of the  $n$  points is an initial point of one of the segments;
2. Each of the  $n$  points is a terminal point of one of the segments;
3. Each segment is between distinct points.

To see this, think of the terminal points as a permutation of the initial points. This permutation can be expressed as a product of cyclic permutations. These are the circuits.

This insures that, if there is a circuit  $C$  of negative  $S$ -length and if  $\mathcal{L}'$  is obtained from  $\mathcal{L}$  by the rule given in Section 3, then  $\mathcal{L}'$  will also be an admissible system of circuits. This is clear since, if a segment with initial point  $a$  is removed, one is also added and conversely. Similarly, for the terminal points. Hence 1 and 2 remain satisfied. Furthermore, if  $C$  is of negative  $S$ -length, it can not contain any segments in common with  $\mathcal{L}$ , for these have  $S$ -length  $+\infty$ ; therefore, the segments added to  $\mathcal{L}$  are between distinct points.

Let  $C = i_0 i_1 \dots i_k$ . Then

$$\begin{aligned} \ell_S(C) &= s_{i_0 i_1} + s_{i_1 i_2} + \dots + s_{i_k i_0} \\ &= (d_{i_1 i_0'} - d_{i_0 i_0'}) + (d_{i_2 i_1'} - d_{i_1 i_1'}) + \dots + (d_{i_0 i_k'} - d_{i_k i_k'}) \\ &= (d_{i_1 i_0'} + d_{i_2 i_1'} + \dots + d_{i_0 i_k'}) - (d_{i_0 i_0'} + d_{i_1 i_1'} + \dots + d_{i_k i_k'}) \end{aligned}$$

Hence  $\ell_D(\mathcal{L}') - \ell_D(\mathcal{L}) = \ell_S(C)$ . Thus, we see that if there is a circuit  $C$  of negative  $S$ -length, then  $\mathcal{L}$  is not an optimal system and we can construct a system  $\mathcal{L}'$  of shorter total length.

Conversely, if  $\mathcal{L}$  is not optimal, then we will show that there is a circuit

$\mathcal{C}$  of negative S-length. Let  $\mathcal{S}'$  be a system of circuits of shorter total D-length than  $\mathcal{S}$ . Let  $\mathcal{A}$  be the set of segments in  $\mathcal{S}$  but not in  $\mathcal{S}'$  and  $\mathcal{B}$  be the set of segments in  $\mathcal{S}'$  but not in  $\mathcal{S}$ . Let

$\mathcal{A} = \{i_0 i'_0, i_1 i'_1, \dots, i_k i'_k\}$ . Then  $\mathcal{B}$  must consist of a set of segments with the same initial points as in  $\mathcal{A}$ , with the same terminal points and the same number of segments. Hence let  $\mathcal{B} = \{j_0 i'_0, \dots, j_k i'_k\}$ , where  $j_0, j_1, \dots, j_k$  is a permutation of  $i_0, i_1, \dots, i_k$ . Then

$$\begin{aligned} \ell_D(\mathcal{S}') - \ell_D(\mathcal{S}) &= (d_{j_0 i'_0} - d_{i_0 i'_0}) + (d_{j_1 i'_1} - d_{i_1 i'_1}) + (d_{j_k i'_k} - d_{i_k i'_k}) \\ &= s_{i_0 j_0} + s_{i_1 j_1} + \dots + s_{i_k j_k} \end{aligned}$$

Express the permutation  $\begin{pmatrix} i_0 & i_1 & \dots & i_k \\ j_0 & j_1 & \dots & j_k \end{pmatrix}$  as the product of cycles, say

$\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_t$ . Then by rearranging and collecting terms of

$$s_{i_0 j_0} + s_{i_1 j_1} + \dots + s_{i_k j_k}$$

we see that this sum is just

$$\ell_S(\mathcal{C}_1) + \dots + \ell_S(\mathcal{C}_t)$$

where  $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_t$  are the circuits corresponding to the cycles of the permutation. Hence

$$\ell_D(\mathcal{S}') - \ell_D(\mathcal{S}) = \ell_S(\mathcal{C}_1) + \dots + \ell_S(\mathcal{C}_t).$$

Since  $\mathcal{L}'$  is shorter than  $\mathcal{L}$ , one of the circuits  $C_1, \dots, C_t$  must have negative S-length. Therefore  $\mathcal{L}$  is optimal if and only if there is no closed circuit of negative S-length.

It remains to show that the non-existence of a circuit of negative S-length is equivalent to the existence of a modified S-matrix for which the triangle inequality holds. Assume first that A is a modified S-matrix and that the triangle inequality holds in A. Let C be a circuit of negative S-length. It corresponds to a circuit C' of negative A-length. Let  $C' = i_0 i_1 i_2 \dots i_k$ . Then  $C'' = i_0 i_2 \dots i_k$  is also of negative A-length since  $a_{i_0 i_2} \leq a_{i_0 i_1} + a_{i_1 i_2}$ . Hence, if there is any circuit of negative A-length we can find a one-point circuit of negative length i.e. for some i,  $a_{ii} < 0$ . But this is impossible since then  $a_{ii} + a_{ii} < a_{ii}$  contrary to the assumption that the triangle inequality holds.

On the other hand, if there is no circuit of negative S-length we can enforce the triangle inequality. The resulting matrix will give in the i, j<sup>th</sup> place the S-length of the shortest path from i to j. If there is no circuit of negative length, there clearly is a shortest path between any two points.

JR:je