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Numerical expressions for precession formulae and mean elements for the Moon and the planets

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Abstract. We present, in this paper, a coherent set of formulae giving numerical expressions for precession quantities and mean elements of the Moon and the planets. First, using the notations of Lieske et al. (1977), we construct expressions for the precession quantities based upon the use of the secular variations of the ecliptic pole from the planetary theories built at the Bureau des Longitudes and taking into account recent determinations of the precession constant and of the obliquity in J2000. Also we give the derivatives of these expressions with respect to the masses of the planets, to the precession constant and to the obliquity. So, this set of formulae is applicable whenever the values of the planetary masses and of the constants are improved. Afterwards, we give the mean elements of the Moon and the planets connected to the fixed J2000 ecliptic and connected to the ecliptic of date. At last, we give formulae which enable one to compute approximate ephemerides of the Moon and the planets from mean elements.

Key words: celestial mechanics – planets and satellites: general – Moon

1. Introduction

For the last ten years or so, the Bureau des Longitudes has built new analytical theories for the motion of the Moon, the Sun and the planets from Mercury to Neptune: ELP 2000-82 (Chapront-Touzé & Chapront 1983) and ELP 2000-85 (Chapront-Touzé & Chapront 1988) for the Moon, VSOP82 (Bretagnon 1982) and VSOP87 (Bretagnon & Francou 1988) for the Sun and all planets except Pluto, TOP82 (Simon 1983) for the four outer planets, JASON84 (Simon & Bretagnon 1984) for Jupiter and Saturn. These theories are much more precise than the former theories of Brown, Le Verrier and Gaillot or Newcomb. They have been adjusted to the numerical integration DE200-LE200 of the Jet Propulsion Laboratory (Standish & Williams 1981; Standish 1982; Standish 1990), which has used most of the available observations.

In the same time many works have been made to determine new formulae for the precession. The precessional formulae derived by (Lieske et al. 1977) are based upon the use of the secular variations of the ecliptic pole from Newcomb's theory of the Sun. The use of the secular expressions of the motion of the ecliptic from VSOP82 in the computation of the precession has been proposed by Bretagnon & Chapront (1981). Formulae extending to 10000 years were computed by Laskar (1986), based on VSOP82 and on his general theory (Laskar 1985). Also, the theory of the rotation of the rigid earth has been improved by Kinoshita (1977), Zhu & Groten (1989) and Kinoshita & Souchay (1990). On the other hand, new determinations of the precession constant have been obtained by several authors putting into evidence an error about $3''$/thousand of years in the IAU 1976 values. Also a new value of the obliquity in J2000 has been determined by analysis of DE200/LE200 (Standish 1982) by fitting theories to that numerical integration.

According to these various improvements we propose in this paper a coherent set of formulae giving numerical expressions for the precession quantities and their derivatives with respect to the planetary masses, to the precession constant and to the obliquity and numerical expressions for the mean elements of the Moon and the planets. These expressions are consistent with the value $50\overset{''}{.}2882$/yr of the precession constant determined by Williams et al. (1991), the value $23°26'21\overset{''}{.}412$ of the obliquity in J2000 from DE200/LE200, and, for the Moon, the value $-24\overset{''}{.}9/(\text{century})^2$ of the tidal secular acceleration recommended in the IERS Standards 1992. Also we give in Sects. 4 and 6 formulae which enable one to compute approximate ephemerides of the Moon and the planets from mean elements.

2. Expressions for precession quantities

In this section we give expressions for precession quantities based upon the use of modern expressions of the motion of the ecliptic and consistent with the most recent values of the precession constant and of the obliquity. Furthermore we give the derivatives of these expressions with respect to the

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masses of the planets, to the precession constant and to the obliquity. So, this set of formulae is applicable whenever the values of the planetary masses and of the constants are improved.

2.1. Precession quantities

In 1976, the XVI Assembly of the IAU in Grenoble adopted a system of astronomical constants containing new values of the precession and obliquity constants. Lieske et al. (1977) then developed a set of expressions for the precession quantities based on these new constants and using the Newcomb theory to describe the motion of the ecliptic pole relative to a fixed ecliptic. We here construct a new set of expressions for the precession quantities using a motion of the ecliptic defined by the planetary theories built at the Bureau des Longitudes and taking into account recent determinations of the constant of general precession in longitude.

2.1.1. The constants $\mathcal{P}_1$ and ε_0

We note $\mathcal{P}_1$ the constant of the general precession in longitude and ε_0 the obliquity in J2000 (JD 2 451 545.0).

The IAU 1976 value of $\mathcal{P}_1$ is

$$\mathcal{P}_1 = 50\,290''.966 \text{ by thousand of Julian years}. \quad (1)$$

But it is a well known fact that the error in $\mathcal{P}_1$ is about 3″/thousand of years. So our formulae incorporate the recent value of $\mathcal{P}_1$ determined by Williams et al. (1991).

ε_0 has the rotational sense as defined by Standish (1981); its value is derived from the JPL numerical integration DE200-LE200 through the different determinations of Standish (1982), Bretagnon (1982), Chapront-Touzé & Chapront (1983).

The quantities used are therefore:

$$\mathcal{P}_1 = 50\,288''.200 \text{ by thousand of Julian years}$$
$$\varepsilon_0 = 23°26'21''.412. \quad (2)$$

Note that the so obtained expressions for the precession will be consistent with the nutation theory ZMOA 1990 (Herring 1991) which involves this value of $\mathcal{P}_1$.

We use too the secular variations of the variables q and p describing the motion of the mean ecliptic of date on the fixed ecliptic J2000. These variables are defined under the following form:

$$q = \sin \tfrac{1}{2}\pi \cos \Pi ,$$
$$p = \sin \tfrac{1}{2}\pi \sin \Pi ,$$

where π is the inclination of the ecliptic and Π the longitude of the ascending node.

From the expressions of q and p given in Sect. 5.8, we can express the secular variations of the variables $\sin\pi \sin\Pi$ and $\sin\pi\cos\Pi$:

$$\sin\pi\sin\Pi = 41''.9971t + 19''.3971t^2 - 0''.2235t^3 - 0''.0104t^4 + 0''.0002t^5 ,$$
$$\sin\pi\cos\Pi = -468''.0927t + 5''.1043t^2 + 0''.5223t^3 - 0''.0057t^4 - 0''.0001t^5 , \quad (3)$$

where t is counted in thousands of Julian years from J2000.

These expressions are sensibly different from those of Lieske et al. (1977) which were issued from the Newcomb theory:

$$\sin\pi\sin\Pi = 41''.976t + 19''.447t^2 - 0''.179t^3 ,$$
$$\sin\pi\cos\Pi = -468''.150t + 5''.059t^2 + 0''.344t^3 . \quad (4)$$

With the expressions (2) and (3) and the equations of the precession motion of the mean equator (Kinoshita 1977; Kinoshita & Souchay 1990), we can determine the variations of the precession $\mathcal{P}$ and the obliquity ε, following Laskar (1986):

$$\mathcal{P} = 50\,288''.200t + 111''.2022t^2 + 0''.0773t^3 - 0''.2353t^4 - 0''.0018t^5 + 0''.0002t^6 ,$$
$$\varepsilon = 23°26'21''.412 - 468''.0927t - 0''.0152t^2 + 1''.9989t^3 - 0''.0051t^4 - 0''.0025t^5 . \quad (5)$$

This result has been obtained using the following values of the geodesic precession $\mathcal{P}_g$ and of the dynamical ellipticity H_d:

$$\mathcal{P}_g = 19''.199 \text{ by thousand of Julian years}$$
$$H_d = 0.003\,273\,7752 ,$$

where $\mathcal{P}_g$ is the value computed by Brumberg et al. (1991) and H_d is obtained by adjustment to the initial conditions (2).

The discrepancies [Lieske et al. (1977) – formulae (5)] are given hereafter:

$$\Delta\mathcal{P} = 2''.766t - 0''.0892t^2 - 0''.0833t^3 + 0''.2353t^4 + 0''.0018t^5 - 0''.0002t^6 ,$$
$$\Delta\varepsilon = 0''.036 - 0''.0573t - 0''.0438t^2 - 0''.1859t^3 + 0''.0051t^4 + 0''.0025t^5 ,$$

where t is counted in thousands of Julian years from J2000. Except for $\mathcal{P}_1$ and for the coefficient in t^2 of $\mathcal{P}$ which presents a change of 0″.0051, this solution is the same as the one of Laskar (1986).

Formulae (3) and (5) enable one to compute all the quantities of precession. For that, let us consider a basic epoch σ_0 (here J2000), an arbitrary fixed epoch σ_F and the epoch of date σ_D. Let T represent the time from σ_0 to σ_F and t represent the time from σ_F to σ_D. We use the notations of Lieske et al. (1977): for whatever angular variable α, α_A represents the accumulated precessional displacement from σ_F to σ_D. T and t are measured in thousands of Julian years of 365 250 d:

$$T = [\text{Julian date } (\sigma_F) - \text{Julian date } (\sigma_0)]/365\,250 ,$$
$$t = [\text{Julian date } (\sigma_D) - \text{Julian date } (\sigma_F)]/365\,250 .$$

The precession quantities are then given hereunder:

$$\sin\pi_A \sin\Pi_A = (41''9971 - 75''3286T + 0''3179T^2 + 0''3178T^3 + 0''0007T^4 - 0''0004T^5)t + (19''3971 + 0''5740T - 0''2541T^2 - 0''0005T^3 + 0''0006T^4)t^2 + (-0''2235 + 0''0859T + 0''0033T^2 - 0''0003T^3)t^3 + (-0''0104 - 0''0004T + 0''0002T^2)t^4 + 0''0002t^5\,,$$

$$\sin\pi_A \cos\Pi_A = (-468''0927 - 0''0305T + 5''9967T^2 - 0''0205T^3 - 0''0125T^4 - 0''0002T^5)t + (5''1043 - 3''1633T - 0''0326T^2 + 0''0138T^3 - 0''0002T^4)t^2 + (0''5223 + 0''0318T - 0''0066T^2 - 0''0004T^3)t^3 + (-0''0057 + 0''0019T - 0''0001T^2)t^4 - 0''0001t^5\,,$$

$$\pi_A = (469''9729 - 6''7011T + 0''0448T^2 - 0''0019T^3 - 0''0001T^4)t + (-3''3505 + 0''0448T - 0''0028T^2 - 0''0002T^3 + 0''0001T^4)t^2 + (-0''1237 - 0''0004T - 0''0002T^2 + 0''0001T^3)t^3 + (0''0003 - 0''0001T + 0''0001T^2)t^4\,,$$

$$\Pi_A = 174°52'23''433 + 32929''659T + 95''352T^2 - 0''005T^3 - 0''459T^4 - 0''010T^5 + (-8679''270 - 15''851T - 0''113T^2 - 0''448T^3 - 0''019T^4)t + (15''342 - 0''019T - 0''432T^2 - 0''023T^3)t^2 + (0''005 - 0''208T - 0''015T^2)t^3 + (-0''037 - 0''005T)t^4 - 0''001t^5\,,$$

$$\mathcal{P}_A = (50288''200 + 222''4045T + 0''2095T^2 - 0''9408T^3 - 0''0090T^4 + 0''0010T^5)t + (111''2022 + 0''2095T - 1''4111T^2 - 0''0180T^3 + 0''0026T^4)t^2 + (0''0773 - 0''9410T - 0''0180T^2 + 0''0035T^3)t^3 + (-0''2353 - 0''0090T + 0''0026T^2)t^4 + (-0''0018 + 0''0010T)t^5 + 0''0002t^6\,,$$

$$\theta_A = (20042''0207 - 85''3131T - 0''2111T^2 + 0''3642T^3 + 0''0008T^4 - 0''0005T^5)t + (-42''6566 - 0''2111T + 0''5463T^2 + 0''0017T^3 - 0''0012T^4)t^2 + (-41''8238 + 0''0359T + 0''0027T^2 - 0''0001T^3)t^3 + (-0''0731 + 0''0019T + 0''0009T^2)t^4 + (-0''0127 + 0''0011T)t^5 + 0''0004t^6\,,$$

$$\zeta_A = (23060''9097 + 139''7495T - 0''0038T^2 - 0''5918T^3 - 0''0037T^4 + 0''0007T^5)t + (30''2226 - 0''2523T - 0''3840T^2 - 0''0014T^3 + 0''0007T^4)t^2 + (18''0183 - 0''1326T + 0''0006T^2 + 0''0005T^3)t^3 + (-0''0583 - 0''0001T + 0''0007T^2)t^4 - 0''0285t^5 - 0''0002t^6\,,$$

$$z_A = (23060''9097 + 139''7495T - 0''0038T^2 - 0''5918T^3 - 0''0037T^4 + 0''0007T^5)t + (109''5270 + 0''2446T - 1''3913T^2 - 0''0134T^3 + 0''0026T^4)t^2 + (18''2667 - 1''1400T - 0''0173T^2 + 0''0044T^3)t^3 + (-0''2821 - 0''0093T + 0''0032T^2)t^4 + (-0''0301 + 0''0006T)t^5 - 0''0001t^6\,,$$

$$\varepsilon_A = 23°26'21''412 - 468''0927T - 0''0152T^2 + 1''9989T^3 - 0''0051T^4 - 0''0025T^5 + (-468''0927 - 0''0305T + 5''9967T^2 - 0''0205T^3 - 0''0125T^4 - 0''0002T^5)t + (-0''0152 + 5''9967T - 0''0308T^2 - 0''0250T^3 - 0''0006T^4)t^2 + (1''9989 - 0''0205T - 0''0250T^2 - 0''0008T^3)t^3 + (-0''0051 - 0''0125T - 0''0006T^2)t^4 + (-0''0025 - 0''0002T)t^5\,,$$

$$\omega_A = 23°26'21''412 - 468''0927T - 0''0152T^2 + 1''9989T^3 - 0''0051T^4 - 0''0025T^5 + (5''1294 - 9''1954T + 0''0298T^2 + 0''0389T^3 + 0''0002T^4)t^2 + (-7''7276 + 0''0235T + 0''0987T^2 - 0''0001T^3)t^3$$

$$+(-0''.0048+0''.0954T-0''.0007T^2)t^4$$
$$+(0''.0333-0''.0009T)t^5-0''.0003t^6\,,$$
$$\psi_A=(50385''.0672+49''.2595T-0''.1344T^2$$
$$-0''.2115T^3+0''.0017T^4+0''.0003T^5)t$$
$$+(-107''.2374-1''.0919T+1''.3673T^2$$
$$+0''.0137T^3-0''.0028T^4)t^2+(-1''.1424$$
$$+2''.6425T+0''.0087T^2-0''.0111T^3)t^3$$
$$+(1''.3279-0''.0110T-0''.0170T^2)t^4$$
$$+(-0''.0094-0''.0123T)t^5-0''.0035t^6\,,$$
$$\chi_A=(105''.5794-188''.8214T-0''.1888T^2$$
$$+0''.7950T^3+0''.0101T^4-0''.0009T^5)t$$
$$+(-238''.1379-1''.0910T+3''.0291T^2$$
$$+0''.0290T^3-0''.0059T^4)t^2$$
$$+(-1''.2117+3''.9055T+0''.0229T^2-0''.0159T^3)t^3$$
$$+(1''.7024-0''.0038T-0''.0214T^2)t^4$$
$$+(-0''.0077-0''.0145T)t^5-0''.0040t^6\,. \qquad (6)$$

2.2. Derivatives of the precession quantities with respect to the masses of the planets and the constants $\mathcal{P}_1$ and ε_0

2.2.1. Derivatives with respect to the masses of the planets

The planetary elements used in this paper are coming from VSOP87 and JASON84 solutions which have been computed using the values of the masses of IAU 1976. More accurate values of the masses are now available. The IAU 1976 and the IERS standards 1992 masses (McCarthy 1992) are given in Table 1. Also, we give in this table the maximal relative errors on the masses $\Delta m/m$ which represent the maximum value between the difference IERS 1992 mass – IAU 1976 mass and an estimation of the error on the IERS value (for this estimation see, for example, Chapront-Touzé et al. 1993).

The use of different values of the planetary masses only slightly modifies the variations of the plane of the ecliptic and consequently the expressions of the precession. Note X one of the precession quantities given by Eq. (6), the corrections ΔX given by relative variations $\delta m_p/m_p$ of the planetary masses m_p have the form

$$\Delta X=\left[m_p\frac{\partial X}{\partial m_p}\right]\frac{\delta m_p}{m_p}\,.$$

We note M, V, Ma, J, S, U, N the relative variations $\delta m_p/m_p$ corresponding to Mercury, Venus, Mars, Jupiter, Saturn, Uranus and Neptune, respectively (the error on the Earth + Moon mass is too small for having influence on the precession quantities). Then, for all the precession quantities, the corrections ΔX coherent with the estimation of

Table 1. Ratios of mass of Sun to masses of the planets for the IAU 1976 and the IERS 1992 systems and maximal relative errors on the masses $\Delta m/m$

	IAU 1976	IERS 1992	$\Delta m/m$ ($\times 10^{-6}$)
Mercury	6 023 600	6 023 600	100
Venus	408 523.5	408 523.71	1.25
Earth + Moon	328 900.5	328 900.56	0.2
Mars	3 098 710	3 098 708	3
Jupiter	1047.355	1047.3486	7
Saturn	3498.5	3497.90	200
Uranus	22 869	22 902.94	1500
Neptune	19 314	19 412.24	6000

$\Delta m/m$ given in Table 1 are given by:

$$\Delta(\sin\pi_A\sin\Pi_A)=[(3.0\mathrm{M}-29\mathrm{J}-5.7\mathrm{S}-0.04\mathrm{N})$$
$$+(-39\mathrm{J}-3.1\mathrm{S})T]t\,,$$
$$\Delta(\sin\pi_A\cos\Pi_A)=[(-2.7\mathrm{M}-286\mathrm{V}-159\mathrm{J}-13.0\mathrm{S}$$
$$-0.04\mathrm{N})+(1.4\mathrm{S})T]t\,,$$
$$\Delta\pi_A=[(3.0\mathrm{M}+291\mathrm{V}+156\mathrm{J}+12.4\mathrm{S}+0.04\mathrm{N})]t\,,$$
$$\Delta\Pi_A=[(-1228\mathrm{M}-18371\mathrm{V}-2410\mathrm{Ma}+18994\mathrm{J}$$
$$+2995\mathrm{S}-7\mathrm{U}+18.3\mathrm{N})+(92\mathrm{M}+10490\mathrm{V}$$
$$+6018\mathrm{J}+502\mathrm{S}+3\mathrm{U}+1.8\mathrm{N})T]$$
$$+[(46\mathrm{M}+5245\mathrm{V}+3009\mathrm{J}+251\mathrm{S}$$
$$+1.5\mathrm{U}+0.9\mathrm{N})]t\,,$$
$$\Delta\mathcal{P}_A=[(1.8\mathrm{M}+190\mathrm{V}+106\mathrm{J}+8.7\mathrm{S}$$
$$+0.03\mathrm{N})T]t+[(53\mathrm{J}+4.4\mathrm{S})]t^2\,,$$
$$\Delta\theta_A=[(2.8\mathrm{M}-27\mathrm{J}-5.2\mathrm{S}-0.03\mathrm{N})$$
$$+(-29\mathrm{J}-2.3\mathrm{S})T]t+[(-1.2\mathrm{S})]t^2\,,$$
$$\Delta\zeta_A=[(1.1\mathrm{S})+(64\mathrm{J}+5.2\mathrm{S})T]t+[(1.3\mathrm{S})]t^2\,,$$
$$\Delta z_A=[(1.1\mathrm{S})+(64\mathrm{J}+5.2\mathrm{S})T]t+[(49\mathrm{J}+4.0\mathrm{S})]t^2\,,$$
$$\Delta\varepsilon_A=[(-2.7\mathrm{M}-286\mathrm{V}-159\mathrm{J}-13.0\mathrm{S}-0.04\mathrm{N})T]$$
$$+[(-2.7\mathrm{M}-286\mathrm{V}-159\mathrm{J}-13.0\mathrm{S}$$
$$-0.04\mathrm{N})+(1.4\mathrm{S})T]t\,,$$
$$\Delta\omega_A=[(-2.7\mathrm{M}-286\mathrm{V}-159\mathrm{J}-13.0\mathrm{S}-0.04\mathrm{N})T]\,,$$
$$\Delta\psi_A=[(7.0\mathrm{M}+156\mathrm{V}-67\mathrm{J}-13.1\mathrm{S}-0.09\mathrm{N})$$
$$+(1.3\mathrm{S})T]t+[(-36\mathrm{J}-3.0\mathrm{S})]t^2\,,$$
$$\Delta\chi_A=[(7.7\mathrm{M}+170\mathrm{V}-73\mathrm{J}-14.3\mathrm{S}-0.10\mathrm{N})$$
$$+(-1.6\mathrm{M}-175\mathrm{V}-97\mathrm{J}-8.0\mathrm{S}-0.03\mathrm{N})T]t$$
$$+[(-1.6\mathrm{M}-175\mathrm{V}-97\mathrm{J}-8.0\mathrm{S}-0.03\mathrm{N})]t^2\,. \qquad (7)$$

As an illustration of formulae (7), we give hereunder the values which have to be added to the expressions (6) to obtain the precession quantities computed with the IERS 1992 values of the planetary masses:

$$\Delta(\sin\pi_A \sin\Pi_A) = (-0''.0010 - 0''.0008T)t\,,$$
$$\Delta(\sin\pi_A \cos\Pi_A) = (-0''.0029 + 0''.0002T)t\,,$$
$$\Delta\pi_A = 0''.0027t\,,$$
$$\Delta\Pi_A = 0''.555 + 0''.104T + 0''.052t\,,$$
$$\Delta P_A = 0''.0019Tt + 0''.0011t^2\,,$$
$$\Delta\theta_A = (-0''.0009 - 0''.0006T)t - 0''.0002t^2\,,$$
$$\Delta\zeta_A = (0''.0002 + 0''.0013T)t + 0''.0002t^2\,,$$
$$\Delta z_A = (0''.0002 + 0''.0013T)t + 0''.0010t^2\,,$$
$$\Delta\varepsilon_A = -0''.0029T + (-0''.0029 + 0''.0002T)t\,,$$
$$\Delta\omega_A = -0''.0029T\,,$$
$$\Delta\psi_A = (-0''.0023 + 0''.0002T)t - 0''.0007t^2\,,$$
$$\Delta\chi_A = (-0''.0025 - 0''.0017T)t - 0''.0017t^2\,. \quad (8)$$

2.2.2. Derivatives with respect to the constants $\mathcal{P}_1$ and ε_0

The corrections ΔX of the precession quantities X given by variations $\Delta\mathcal{P}_1$ and $\Delta\varepsilon_0$ of $\mathcal{P}_1$ and ε_0 have the form

$$\Delta X = \frac{\partial X}{\partial\mathcal{P}_1}\Delta\mathcal{P}_1 + \frac{\partial X}{\partial\varepsilon_0}\Delta\varepsilon_0\,.$$

For all the precession quantities, these corrections to be added to the formulae (6) are given by:

$$\Delta(\sin\pi_A \sin\Pi_A) = [(-0.0023\Delta\mathcal{P}_1)T]t\,,$$
$$\Delta(\sin\pi_A \cos\Pi_A) = [(0.0004\Delta\mathcal{P}_1)T^2]t\,,$$
$$\Delta\pi_A = 0\,,$$
$$\Delta\Pi_A = [(\Delta\mathcal{P}_1)T + (0.003\Delta\mathcal{P}_1)T^2]\,,$$
$$\Delta P_A = [\Delta\mathcal{P}_1 + (0.0062\Delta\mathcal{P}_1 - 0.0016\Delta\varepsilon_0)T]t + [(0.0031\Delta\mathcal{P}_1)]t^2\,,$$
$$\Delta\theta_A = [(0.3978\Delta\mathcal{P}_1 + 0.2236\Delta\varepsilon_0) + (-0.0017\Delta\mathcal{P}_1)T]t + [(-0.0008\Delta\mathcal{P}_1)]t^2 + [(-0.0025\Delta\mathcal{P}_1)]t^3\,,$$
$$\Delta\zeta_A = [(0.4588\Delta\mathcal{P}_1 - 0.0485\Delta\varepsilon_0) + (0.0038\Delta\mathcal{P}_1)T]t + [(0.0009\Delta\mathcal{P}_1)]t^2 + [(0.0011\Delta\mathcal{P}_1)]t^3\,,$$
$$\Delta z_A = [(0.4588\Delta\mathcal{P}_1 - 0.0485\Delta\varepsilon_0) + (0.0038\Delta\mathcal{P}_1)T]t + [(0.0028\Delta\mathcal{P}_1)]t^2 + [(0.0011\Delta\mathcal{P}_1)]t^3\,,$$
$$\Delta\varepsilon_A = [\Delta\varepsilon_0] + [(0.0004\Delta\mathcal{P}_1)T^2]t + [(0.0004\Delta\mathcal{P}_1)T]t^2\,,$$
$$\Delta\omega_A = [\Delta\varepsilon_0] + [(-0.0005\Delta\mathcal{P}_1)T]t^2 + [(-0.0003\Delta\mathcal{P}_1)]t^3\,,$$
$$\Delta\psi_A = [(\Delta\mathcal{P}_1 - 0.0013\Delta\varepsilon_0) + (0.0010\Delta\mathcal{P}_1)T]t + [(-0.0021\Delta\mathcal{P}_1 + 0.0021\Delta\varepsilon_0)]t^2\,,$$
$$\Delta\chi_A = [(-0.0012\Delta\varepsilon_0) + (-0.0057\Delta\mathcal{P}_1 + 0.0021\Delta\varepsilon_0)T]t + [(-0.0057\Delta\mathcal{P}_1 + 0.0027\Delta\varepsilon_0) + (0.0003\Delta\mathcal{P}_1)T^2]t^2 + [(0.0003\Delta\mathcal{P}_1)T]t^3\,. \quad (9)$$

In expressions (9), $\Delta\mathcal{P}_1$ is measured in arcseconds by thousand of years and $\Delta\varepsilon_0$ is measured in arcseconds. The precision of the corrections (9) is better than $0''.001$ for Π_A and better than $0''.0001$ for the other precession quantities for

$|\Delta\mathcal{P}_1| \leq 0''.5$ by thousand of years,

$|\Delta\varepsilon_0| \leq 0''.1$.

The precession formulae (6) and the derivatives (7) and (9) can be used over the interval −4000, 8000. Over this range of time, the internal uncertainties are exceeded by the uncertainty on the precession constant which is now about $0''.3$ by thousand of years. With a more precise value of the precession constant, the formulae (6), (7) and (9) as a whole can ensure a precision of $0''.001$ over the interval 1000, 3000, of $0''.1$ over −1000, 5000 and of $1''$ over −4000, 8000.

3. Lunar mean elements

3.1. Origin and formulation

Geocentric osculating elliptic elements of the Moon, i.e. semi-major axis a, eccentricity e, inclination i, mean longitude λ, longitude of the perigee ϖ, and longitude of the node Ω, have been derived from the periodic terms of the semi-analytical theory ELP 2000-82 and the secular terms of the theory ELP 2000-85. The general formulation of these osculating elements is

$$s = s_0 + s_1 t + s_2 t^2 + s_3 t^3 + s_4 t^4 + S_0 + S'_0 + tS'_1 + t^2 S'_2\,, \quad (10)$$

t is the barycentric time (TDB) in Julian centuries reckoned from J2000 (Julian TDB date 2451 545.0); s_0, s_1, s_2, s_3, s_4 are constants; S_0 is a Fourier series with four arguments, polynomial functions of time, i.e. Delaunay arguments D, F, l, l'. The Fourier series S'_0, S'_1, S'_2 have 13 arguments, linear functions of t, which are $\bar{D}$, $\bar{F}$, $\bar{l}$, $\bar{l}'$ the linear parts of Delaunay arguments, the mean longitude of the Moon referred to the mean equinox of date, and the mean longitudes of planets given in Sect. 5.8.

The secular part of s:

$$\langle s\rangle = s_0 + s_1 t + s_2 t^2 + s_3 t^3 + s_4 t^4$$

is the mean element corresponding to s. Delaunay arguments D, F, l, l' are derived from the mean elements $\langle\lambda\rangle$, $\langle\varpi\rangle$, $\langle\Omega\rangle$ for the Moon and the quantities $\langle\lambda_T\rangle$, $\langle\varpi_T\rangle$ for the Earth–Moon barycenter by:

$$D = \langle\lambda\rangle - \langle\lambda_T\rangle + 180°\,,$$
$$F = \langle\lambda\rangle - \langle\Omega\rangle\,,$$
$$l = \langle\lambda\rangle - \langle\varpi\rangle\,,$$
$$l' = \langle\lambda_T\rangle - \langle\varpi_T\rangle\,, \quad (11)$$

$\langle\lambda_T\rangle$ is the heliocentric mean mean longitude of the Earth–Moon barycenter; in ELP theories, $\langle\varpi_T\rangle$ is close to the mean longitude of the Earth–Moon barycenter perihelion.

3.2. Reference frames

The basic geocentric osculating elements and mean elements (immediately derived from the theory) are referred to the mean ecliptic of date and to the "departure point" γ'_{2000} of this plane defined by

$$N\gamma'_{2000} = N\gamma_{2000}\,,$$

N being the node of the mean ecliptic of date on the mean ecliptic of J2000 and γ_{2000} the mean equinox of J2000.

The osculating elements and the mean elements referred to the mean ecliptic and mean equinox of date are obtained by adding to the basic expressions of λ, ϖ, Ω and $\langle\lambda\rangle$, $\langle\varpi\rangle$, $\langle\Omega\rangle$ the accumulated precession $\mathcal{P}_A$ from J2000 to the date.

By using other precession formulae, the basic osculating elements can be converted to osculating elements referred to the mean ecliptic and equinox of J2000. The secular parts of the latter are the mean elements referred to the mean ecliptic and equinox of J2000. Note that, for ϖ, Ω, and λ, the mean elements referred to J2000 are identical to the mean elements of the theory referred to the mean ecliptic of date and to the departure point. a and e do not depend on the reference frame.

We emphasize that, in all this Sect. 3, ecliptics and equinoxes are dynamical. They are also inertial as defined by Standish (1981).

3.3. Effects of the constants

The analytical solution of the differential equations for the orbital motion of the Moon depends mainly on the following constants:

(a) Six integration constants i.e. coefficient s_1 for λ referred to the mean ecliptic of date and departure point and denoted as ν (*sidereal mean motion in* J2000), coefficients s_0 for λ, ϖ, Ω and two constants E (*eccentricity constant*) and Γ (*inclination constant*) respectively close to coefficients s_0 for e and $\sin(s_0/2)$ for i;

(b) $G' = Gm_T$, m_T being the Earth's mass;

(c) $\sigma_1 = m_L/(m_T+m_L)$ derived from the constant $\mu = m_L/m_T$, m_L being the lunar mass;

(d) $\sigma_2 = (m_T+m_L)/(m_S+m_T+m_L)$ derived from the constant $\mu' = m_S/(m_T+m_L)$, m_S being the solar mass;

(e) The tidal part of s_2 for λ, $s^T_{2,\lambda}$, replaced by the constant $k_2\delta$ proportional to $s^T_{2,\lambda}$ with a factor derived from the integration of the tidal forces following (Williams et al. 1978);

(f) The precession constant: $\mathcal{P}_1$;

(g) Solar constants: eccentricity, mean motion and constant term of $\langle\lambda_T\rangle$ and constant term of $\langle\varpi_T\rangle$.

In ELP 2000 and ELP 2000-85, the values of G' and of the masses are equal or very close to the values of the IAU 1976 system of constants. The values of the integration constants and of the solar constants are derived from a fit of ELP 2000-82 to the JPL numerical integration DE200/LE200 (Standish & Williams 1981) over the time span [1990; 2000], but for the fit, the values of G' and μ in ELP 2000-82 have been replaced by the DE200/LE200 values. The value of $s^T_{2,\lambda}$ is close to the DE200/LE200 value. The values of the constants involved in ELP 2000, except integration constants, and the corresponding values involved in DE200/LE200 are quoted in Table 2; the value of the precession constant in both solutions is the IAU 1976 value quoted in formula (1).

Since the construction of DE200/LE200 and ELP 2000, more precise values of the constants G', μ, μ', $s^T_{2,\lambda}$ and $\mathcal{P}_1$ have been proposed. We have quoted in the column "1992 values" of Table 2 the values of μ, μ' and $s^T_{2,\lambda}$ adopted in the IERS standards 1992 (McCarthy 1992). The value of G' is the value compatible with the use of TDB from (Seidelmann & Fukushima 1992). For $\mathcal{P}_1$, we adopt the value issued from (Williams et al. 1991) and given in formula (2).

The values of the integration constants depend on the values of constants G', μ, μ', $s^T_{2,\lambda}$. An estimate of the corrections to the integration constants induced by the new values of G', μ, μ', and $s^T_{2,\lambda}$ may be obtained by the following process:

(a) The corrections to the coefficients of series S_0, S_1, S'_1 of formula (10) induced by the new values of G', μ, μ', and $s^T_{2,\lambda}$ are negligible. It will be checked later that the corrections induced by the new values of the integration constants are negligible also. Then only mean elements are affected by the change of the constants.

(b) We suppose that the values of the solar constants and of E and Γ do not change. These values are given in Table 3.

Table 2. Physical constants

	ELP 2000	DE200/LE200	1992 values	Unit
G'	3.986005	3.98600448	3.98600435[a]	$10^{14}\ \mathrm{m^3\,s^{-2}}$
μ	0.0123000202	0.0123000342	0.012300034[b]	Dimensionless
μ'	328900.5	328 900.55	328 900.56[b]	Dimensionless
$s^T_{2,\lambda}$	−11.94731	−11.9	−12.45[b]	$''\mathrm{cy}^{-2}$

[a] From (Seidelmann & Fukushima 1992). [b] From IERS Standards 1992.

Table 3. Sidereal mean motion in J2000, eccentricity and inclination constants

	ELP 2000	1992 values	Unit
ν	1 732 559 343.73604	1 732 559 343.4847	$''\ \mathrm{cy}^{-1}$
E	0.054 879 9046	0.0548799046	Dimensionless
Γ	0.044 751 3054	0.0447513054	Dimensionless

(c) We suppose that, at the mean epoch of observations, the values of the sidereal mean motion, of the mean longitude referred to the mean equinox of J2000 and of Delaunay arguments D, F, l are well determined by DE200/LE200 through the LLR observations.

(d) The expressions of these quantities in function of time only involve their values in J2000, the constants ν and $s^{\mathrm{T}}_{2,\lambda}$ and to a smaller amount μ and μ', since constants E and Γ and solar constants are left unchanged.

(e) Taking the two remarks above into account, it is easy to deduce, from the corrections "1992 values − ELP 2000 values" to $s^{\mathrm{T}}_{2,\lambda}$ and to other constants of Table 2, corrections to ν and to the values in J2000 of D, F, l and $\langle\lambda\rangle$ referred to the mean equinox of J2000. The corrected value of ν is given in Table 3.

(f) From these corrections and from the "1992 values" of Table 2 we deduce new expressions of mean elements $\langle\varpi\rangle$, $\langle\Omega\rangle$, $\langle\lambda\rangle$ referred to the mean ecliptic and equinox of J2000 and of $\langle a\rangle$. The corrections to $\langle e\rangle$ and $\langle i\rangle$ are found to be negligible. We note that these new expressions do not depend on a change to the precession constant.

(g) The correction to the precession constant, "formula (2) value − formula (1) value", is introduced only in the expressions of $\langle\varpi\rangle$, $\langle\Omega\rangle$, $\langle\lambda\rangle$ referred to the mean ecliptic and equinox of date. Following (Williams & Melbourne 1982) and (Zhu & Mueller 1983), using these new expressions referred to the mean ecliptic and equinox of date implies adopting a correction $\Delta m\,t$ to the sidereal time of (Aoki et al. 1982) in order not to change the lunar hour angle and the universal time. m is the precession in right ascension.

(h) The mean epoch of observations used in DE200/LE200 has been taken as J1975.0 (Julian TDB date 2442413.75).

3.4. Expressions of lunar mean elements

(a) The following expressions of the lunar mean elements are computed for the values "ELP 2000" of the constants of Table 2 and the integration constants fitted to DE200/LE200, except for μ which has been given the exact IAU 1976 value: $\mu = 0.01230002$. For $\langle i\rangle$, $\langle\varpi\rangle$, $\langle\Omega\rangle$, and $\langle\lambda\rangle$, both expressions referred to the mean ecliptic and equinox of J2000 and expressions referred to the mean ecliptic and equinox of date are given. The latter involve the IAU 1976 value of the precession constant:

$$\langle a\rangle = 383397.7916 + 0.0038t\,,$$

$$\langle e\rangle = 0.055545526 - 0.000000016t\,.$$

(a.1.) Mean elements referred to the mean ecliptic and equinox of J2000

$$\langle i\rangle = 5^\circ.15668983 - 0''.00008t + 0''.02966t^2 - 0''.000042t^3 - 0''.00000013t^4\,,$$

$$\langle\varpi\rangle = 83^\circ.35324299 + 14643420''.2632t - 38''.2776t^2 - 0''.045047t^3 + 0''.00021301t^4\,,$$

$$\langle\Omega\rangle = 125^\circ.04455504 - 6967919''.3622t + 6''.3622t^2 + 0''.007625t^3 - 0''.00003586t^4\,,$$

$$\langle\lambda\rangle = 218^\circ.31665436 + 1732559343''.73604t - 5''.8883t^2 + 0''.006604t^3 - 0''.00003169t^4\,.$$

(a.2) Mean elements referred to the mean ecliptic and equinox of date ($\mathcal{P}_1 = 5029''.0966\ \mathrm{cy}^{-1}$)

$$\langle i\rangle = 5^\circ.15668983 - 0''.00008t\,,$$

$$\langle\varpi\rangle = 83^\circ.35324299 + 14648449''.3598t - 37''.1656t^2 - 0''.044970t^3 + 0''.00018948t^4\,,$$

$$\langle\Omega\rangle = 125^\circ.04455504 - 6962890''.2656t + 7''.4742t^2 + 0''.007702t^3 - 0''.00005939t^4\,,$$

$$\langle\lambda\rangle = 218^\circ.31665436 + 1732564372''.83263t - 4''.7763t^2 + 0''.006681t^3 - 0''.00005522t^4\,.$$

(b) The following expressions of the lunar mean elements are computed for the "1992 values" of the constants of Table 2, by the method described in Sect. 3.3. For $\langle i\rangle$, $\langle\varpi\rangle$, $\langle\Omega\rangle$, $\langle\lambda\rangle$, both expressions referred to the mean ecliptic and equinox of J2000 and expressions referred to the mean ecliptic and equinox of date are given. In the latter case, we give both the expressions involving the IAU 1976 value of the precession constant and those involving the precession constant from (Williams et al. 1991).

$$\langle a\rangle = 383397.7725 + 0.0040t\,,$$

$$\langle e\rangle = 0.055545526 - 0.000000016t\,.$$

(b.1) Mean elements referred to the mean ecliptic and equinox of J2000

$$\langle i\rangle = 5^\circ.15668983 - 0''.00008t + 0''.02966t^2 - 0''.000042t^3 - 0''.00000013t^4\,,$$

$$\langle\varpi\rangle = 83^\circ.35324312 + 14643420''.2669t - 38''.2702t^2 - 0''.045047t^3 + 0''.00021301t^4\,,$$

$$\langle\Omega\rangle = 125^\circ.04455501 - 6967919''.3631t + 6''.3602t^2 + 0''.007625t^3 - 0''.00003586t^4\,,$$

$$\langle\lambda\rangle = 218^\circ.31664563 + 1732559343''.48470t - 6''.3910t^2 + 0''.006588t^3 - 0''.00003169t^4\,.$$

(b.2.) Mean elements referred to the mean ecliptic and equinox of date ($\mathcal{P}_1 = 5029''.0966\,\mathrm{cy}^{-1}$)

$$\langle i \rangle = 5^\circ.15668983 - 0''.00008t\,,$$

$$\langle \varpi \rangle = 83^\circ.35324312 + 14648449''.3635t - 37''.1582t^2 - 0''.044970t^3 + 0''.00018948t^4\,,$$

$$\langle \Omega \rangle = 125^\circ.04455501 - 6962890''.2665t + 7''.4722t^2 + 0''.007702t^3 - 0''.00005939t^4\,,$$

$$\langle \lambda \rangle = 218^\circ.31664563 + 1732564372''.58130t - 5''.2790t^2 + 0''.006665t^3 - 0''.00005522t^4\,.$$

(b.3) Mean elements referred to the mean ecliptic and equinox of date ($\mathcal{P}_1 = 5028''.8200\,\mathrm{cy}^{-1}$)

$$\langle i \rangle = 5^\circ.15668983 - 0''.00008t\,,$$

$$\langle \varpi \rangle = 83^\circ.35324312 + 14648449''.0869t - 37''.1582t^2 - 0''.044970t^3 + 0''.00018948t^4\,,$$

$$\langle \Omega \rangle = 125^\circ.04455501 - 6962890''.5431t + 7''.4722t^2 + 0''.007702t^3 - 0''.00005939t^4\,,$$

$$\langle \lambda \rangle = 218^\circ.31664563 + 1732564372''.30470t - 5''.2790t^2 + 0''.006665t^3 - 0''.00005522t^4\,.$$

In all these expressions, t is TDB measured in Julian centuries from J2000 (Julian TDB date 2451545.0), $\langle a \rangle$ is measured in km, and $\langle e \rangle$ is dimensionless. In $\langle i \rangle$, $\langle \varpi \rangle$, $\langle \Omega \rangle$, and $\langle \lambda \rangle$, the constant term is given in degrees and the coefficient of $t^n (n > 0)$ in $''\,\mathrm{cy}^{-n}$.

3.5. Delaunay arguments

(a) The expressions of Delaunay arguments D, F, l, l' for the values "ELP 2000" of the constants of Table 2 are:

$$D = 297^\circ.85020420 + 1602961601''.4603t - 5''.8679t^2 + 0''.006609t^3 - 0''.00003169t^4\,,$$

$$F = 93^\circ.27209932 + 1739527263''.0983t - 12''.2505t^2 - 0''.001021t^3 + 0''.00000417t^4\,,$$

$$l = 134^\circ.96341138 + 1717915923''.4728t + 32''.3893t^2 + 0''.051651t^3 - 0''.00024470t^4\,,$$

$$l' = 357^\circ.52910918 + 129596581''.0481t - 0''.5532t^2 + 0''.000136t^3 - 0''.00001149t^4\,.$$

(b) The expressions of Delaunay arguments for the "1992 values" of the constants of Table 2 are:

$$D = 297^\circ.85019547 + 1602961601''.2090t - 6''.3706t^2 + 0''.006593t^3 - 0''.00003169t^4\,,$$

$$F = 93^\circ.27209062 + 1739527262''.8478t - 12''.7512t^2 - 0''.001037t^3 + 0''.00000417t^4\,,$$

$$l = 134^\circ.96340251 + 1717915923''.2178t + 31''.8792t^2 + 0''.051635t^3 - 0''.00024470t^4\,,$$

$$l' = 357^\circ.52910918 + 129596581''.0481t - 0''.5532t^2 + 0''.000136t^3 - 0''.00001149t^4\,.$$

Delaunay arguments do not depend on the reference frame and consequently do not depend on the precession constant. The expressions above are computed by formulae (11), coefficients s_2, s_3, s_4 for $\langle \lambda_\mathrm{T} \rangle$ and s_1, s_2, s_3, s_4 for $\langle \varpi_\mathrm{T} \rangle$ being derived from Sect. 5, and coefficients s_0, s_1 for $\langle \lambda_\mathrm{T} \rangle$ and s_0 for $\langle \varpi_\mathrm{T} \rangle$ being derived from the fit of ELP 2000-82 to DE200/LE200.

Besides lunar theory, the expressions of Delaunay arguments and $\langle \Omega \rangle$ referred to the mean ecliptic and equinox of date are necessary for the computation of the nutation. Using the expressions of this paper instead of the expressions given in (Seidelmann 1982) induces corrections of a few 10^{-5} arcsecond to the nutation. But even if these corrections are much smaller than the actual precision of the nutation, the expressions of the lunar arguments from this paper should be used with the new theories of nutation, e.g. (Herring 1991) and (Zhu et al. 1990), since the rigid Earth part of these theories uses ELP 2000-82 series. Furthermore, the expression of $\langle \Omega \rangle$ computed with the value of the precession constant from (Williams et al. 1991) is perfectly consistent with the nutation theory ZMOA 1990 (Herring 1991) which involves the same value of the precession constant.

The expression of argument ψ involved in Table 5 below, computed for the "1992 values" of the constants of Table 2 and the precession constant from (Williams et al. 1991), is

$$\psi = 310^\circ.17137918 - 6967051''.4360t + 6''.2068t^2 + 0''.007618t^3 - 0''.00003219t^4\,.$$

The corrections on ψ due to the change of constants are negligible towards the precision of Table 5.

In the expressions above, t is TDB measured in Julian centuries from J2000. The constant terms are given in degrees and the coefficients of t^n $(n > 0)$ in $''\,\mathrm{cy}^{-n}$.

3.6. Validity of lunar elements

The time span of validity of the lunar elements is restricted mainly by the bound to the maximum power of time involved in the corresponding mean elements. The optimal use takes place within [4000 B.C.; A.D. 8000]. Within this time span, the effects of the powers of time greater than 1 in $\langle a \rangle$, $\langle e \rangle$ and $\langle i \rangle$ referred to the mean ecliptic of date, are negligible and the corresponding terms have been disregarded here. On the other hand, independently of the theories, the accuracies of λ and, to a much smaller amount, of ϖ and Ω are restricted by the uncertainty on the tidal part of coefficients s_2 for λ, $s^\mathrm{T}_{2,\lambda}$ which is about $0''.75\,\mathrm{cy}^{-2}$

for the constant of ELP 2000 and 0″.50 cy^{-2} for the "1992 constant" of Table 2 (Williams & Standish 1989).

4. Approximate ephemerides of the Moon in elliptic elements

Usually, for the Moon, the mean elements are not sufficient for computing positions, even to a low precision. In most cases, it is necessary to add to mean elements at least the leading periodic terms, and for a more precise computation, further periodic terms and t^n. periodic terms. The latter become necessary for distant times.

We give in Table 4 the leading terms of the series S_0 of formula (10) for a, e and for i, ϖ, Ω, λ referred to the mean ecliptic of date and departure point (genuine series) or to the mean ecliptic and equinox of date. The coefficients of the series S'_0, S'_1, S'_2 are smaller. By adding the periodic terms of Table 4 to the expressions of mean elements $\langle a \rangle$, $\langle e \rangle$ and $\langle i \rangle$, $\langle \varpi \rangle$, $\langle \Omega \rangle$, $\langle \lambda \rangle$ referred to the mean ecliptic and equinox of date, given in Sect. 3.4, one can obtain a, e and i, ϖ, Ω, λ in the same reference frame, with a precision about 200 km for a, 0.001 for e, 50″ for i, 3° for ϖ, 0°.3 for Ω and 0°.1 for λ. For a better precision, more complete expressions of series S_0, S'_0, S'_1, S'_2 are given in (Chapront-Touzé & Chapront 1991).

Converting the genuine series for i, ϖ, Ω, λ, referred to the mean ecliptic of date and departure point, to series referred to the mean ecliptic and equinox of J2000 gives rise to large complementary terms in series tS'_1 and $t^2S'_2$ of formula (10) and to a complementary series $t^3S'_3$. These terms depend on Delaunay arguments and on argument:

$$\psi = \langle \Omega \rangle - \Pi_A ,$$

where $\langle \Omega \rangle$ is referred to the mean equinox of date and Π_A is computed from the precession formulae (6) by substituting the values of T and t obtained for σ_F = date and σ_D = J2000 (the expression of T to be substituted in formulae (6) is then equal to the current argument t of this section divided by ten and the expression of t to be substituted in formulae (6) is the opposite of the expression of T).

This effect is specific to the lunar theory. From a mathematical point of view, it is due to the fact that the period of the lunar node being short with respect to the time span of validity of the solution, sine or cosine of such an argument cannot be developed with respect to time as it is the case for planets. More intuitively, the geocentric motion of the Moon is tied to the heliocentric motion of the Earth and for this reason, the formulation of lunar theory is simpler in the mean ecliptic of date than in the mean ecliptic of J2000.

The leading complementary terms are given in Table 5. For i, ϖ and λ, we have restricted ourselves to the terms with coefficients greater than 0″.1 cy^{-1} in S'_1, 0″.001 cy^{-2} in S'_2, and 0″.00001 cy^{-3} in S'_3, and for Ω to the terms with coefficients greater than 1″ cy^{-1} in S'_1, 0″.01 cy^{-2} in S'_2, and 0″.0001 cy^{-3} in S'_3. To sum up, elements of the

Table 4. Leading periodic terms of lunar osculating elements referred to the mean ecliptic and equinox of date

a (km)	i (″)	Ω (″)
$3400.4 \cos 2D$	$486.26 \cos(2D-2F)$	$-5392 \sin(2D-2F)$
$-635.6 \cos(2D-l)$	$-40.13 \cos 2D$	$-540 \sin l'$
$-235.6 \cos l$	$37.51 \cos 2F$	$-441 \sin 2D$
$218.1 \cos(2D-l')$	$25.73 \cos(2l-2F)$	$423 \sin 2F$
$181.0 \cos(2D+l)$	$19.97 \cos(2D-l'-2F)$	$-288 \sin(2l-2F)$
e (dimensionless)	**ϖ (″)**	**λ (″)**
$0.014216 \cos(2D-l)$	$-55609 \sin(2D-l)$	$-3332.9 \sin 2D$
$0.008551 \cos(2D-2l)$	$-34711 \sin(2D-2l)$	$1197.4 \sin(2D-l)$
$-0.001383 \cos l$	$-9792 \sin l$	$-662.5 \sin l'$
$0.001356 \cos(2D+l)$	$9385 \sin(4D-3l)$	$396.3 \sin l$
$-0.001147 \cos(4D-3l)$	$7505 \sin(4D-2l)$	$-218.0 \sin(2D-l')$
$-0.000914 \cos(4D-2l)$	$5318 \sin(2D+l)$	
$0.000869 \cos(2D-l'-l)$	$3484 \sin(4D-4l)$	
$-0.000627 \cos 2D$	$-3417 \sin(2D-l'-l)$	
$-0.000394 \cos(4D-4l)$	$-2530 \sin(6D-4l)$	
$0.000282 \cos(2D-l'-2l)$	$-2376 \sin 2D$	
$-0.000279 \cos(D-l)$	$-2075 \sin(2D-3l)$	
$-0.000236 \cos 2l$	$-1883 \sin 2l$	
$0.000231 \cos 4D$	$-1736 \sin(6D-5l)$	
$0.000229 \cos(6D-4l)$	$1626 \sin l'$	
$-0.000201 \cos(2l-2F)$	$-1370 \sin(6D-3l)$	

Table 5. Complementary terms for lunar osculating elements referred to the mean ecliptic and equinox of J2000

Variable	S'_1 t-périodic	S'_2 t^2. périodic	S'_3 t^3. périodic
i ($''$)	$46.997\cos\psi$	$-0.0297\cos 2\psi$	$-0.00016\cos\psi$
	$-0.614\cos(2D-2F+\psi)$	$-0.0335\cos\psi$	$0.00004\cos 3\psi$
	$0.614\cos(2D-2F-\psi)$	$0.0012\cos(2D-2F+2\psi)$	$0.00004\cos 2\psi$
ϖ, λ ($''$)	$2.116\sin\psi$	$-0.0015\sin\psi$	
	$-0.111\sin(2D-2F-\psi)$		
Ω($''$)	$-520.77\sin\psi$	$0.660\sin 2\psi$	$0.0014\sin\psi$
	$13.66\sin(2D-2F+\psi)$	$0.371\sin\psi$	$-0.0011\sin 3\psi$
	$1.12\sin(2D-\psi)$	$-0.035\sin(2D-2F+2\psi)$	$-0.0009\sin 2\psi$
	$-1.06\sin(2F-\psi)$	$-0.015\sin(2D-2F+\psi)$	

Moon referred to the mean ecliptic and equinox of J2000 are obtained by adding both terms of Table 4 (and subsequent terms if necessary) and terms of Table 5, multiplied by the convenient power of t, to the expressions of mean elements referred to the mean ecliptic and equinox of J2000 given in Sect. 3.4.

An approximate ephemeris of the Moon in polar coordinates can be computed from the tables given in Chapront-Touzé & Chapront (1988) and to a better precision in Chapront-Touzé & Chapront (1991).

5. Mean elements of the planets

5.1. Notations

We use classical notations for the elliptic elements of a planet:

a : semi − major axis, λ : mean longitude

e : eccentricity, ϖ : longitude of perihelion

i : inclination, Ω : longitude of the ascening node

We use also the variables:

$k : e\cos\varpi$ $\quad h : e\sin\varpi$

$q : \sin\frac{1}{2}i\cos\Omega$ $\quad p : \sin\frac{1}{2}i\sin\Omega$

At last we note $\bar\lambda$ the mean mean longitude: $\bar\lambda = \lambda^\circ + Nt$, where λ° is the integration constant for λ and where N is the mean mean motion of the planet.

The elements which are tabulated in this paper are: a, λ, e, ϖ, i, Ω, k, h, q, p. Among these variables we note σ elements other than mean longitudes λ.

5.2. Definition of mean elements of the planets

To define the mean elements it is necessary to briefly present the relations between general planetary theories and classical planetary theories.

5.2.1. General planetary theories

In general planetary theories the solutions are searched on the form of Fourier series in the argument Φ, linear combination of 24 components if we consider 8 planets:

$$\Phi = \sum_{j=1}^{8} r_j\bar\lambda_j + \sum_{j=1}^{8} l_j\psi_j + \sum_{j=1}^{8} m_j\theta_j\,, \tag{12}$$

where $\bar\lambda_j$, ψ_j, θ_j are the mean mean longitude, the argument of Lagrange–Laplace solution in eccentricity, the argument of Lagrange–Laplace solution in inclination, for the planet j; r_j, l_j, m_j are integers.

It must be noted that the period of arguments $\bar\lambda_j$ is included between 0.25 yr (Mercury) and 165 yr (Neptune) when period of arguments ψ_j and θ_j are of the order of several 10 000 years. We call *short period part* of the argument Φ the expression $\sum_{j=1}^{8} r_j\bar\lambda_j$ and *long period part* of Φ the expression $\sum_{j=1}^{8} l_j\psi_j + \sum_{j=1}^{8} m_j\theta_j$. In the same way we call *long period arguments* the arguments (12) for which $r_j = 0$ and we note them Φ^*:

$$\Phi^* = \sum_{j=1}^{8} l_j\psi_j + \sum_{j=1}^{8} m_j\theta_j\,, \tag{13}$$

For the planet j, solutions will have the form of Fourier series:

$$\sigma_j = \sigma_j^0 + \sum_{\Phi^*} A_{\Phi^*,j}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\Phi^* + \sum_{\Phi} A_{\Phi,j}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\Phi\,,$$

$$\lambda_j = \bar\lambda_j + \sum_{\Phi^*} B_{\Phi^*,j}\sin\Phi^* + \sum_{\Phi} B_{\Phi,j}\sin\Phi. \tag{14}$$

In the expressions (14), σ_j are cosine series for the variables a, e, i, k, q and sine series for the variables ϖ, Ω, h, p.

5.2.2. Classical planetary theories

Starting from the expressions (14) we can develop with respect to time the long period arguments Φ^* and the long period part of the agruments Φ. So we find the developments of the classical planetary theories under the form of Poisson series in the mean mean longitudes $\bar\lambda_j$. For the planet j, we have

$$\sigma_j = \sigma_j^0 + \sigma_j^1 t + \sigma_j^2 t^2 + \ldots + \sigma_j^p t^p + S_j^0 + tS_j^1 + \ldots + t^p S_j^p ,$$

$$\lambda_j = \lambda_j^0 + N_j t + l_j^2 t^2 + \ldots + l_j^p t^p + L_j^0 + tL_j^1 + \ldots + t^p L_j^p , \quad (15)$$

where t is the barycentric time (TDB), $\sigma_j^0, \ldots \sigma_j^p$, λ_j^0, N_j, $l_j^2, \ldots, l_j^p$ are numerical coefficients and $S_j^0, \ldots, S_j^p$, $L_j^0, \ldots, L_j^p$ are Fourier series in the mean mean longitudes $\bar{\lambda}_j$. The value of p is depending of the problem resolved by the theory. For example, $p = 5$ for VSOP82 (theory for the motion of all planets), $p = 6$ for TOP82 (motion of the four outer planets), $p = 20$ for JASON84 (mutual perturbations of Jupiter and Saturn).

5.2.3. Mean elements

Mean elements of the variables σ_j and λ_j are the secular part of the expressions (15), corresponding to the development with respect to time of the long period perturbations $\sum_{\Phi^*} A_{\Phi^*,j} \left\{ {\cos \atop \sin} \right\} \Phi^*$ and $\sum_{\Phi^*} B_{\Phi^*,j} \sin \Phi^*$ of the expressions (14). We note them $\langle\sigma_j\rangle$ and $\langle\lambda_j\rangle$:

$$\langle\sigma_j\rangle = \sigma_j^0 + \sigma_j^1 t + \sigma_j^2 t^2 + \ldots + \sigma_j^p t^p ,$$

$$\langle\lambda_j\rangle = \lambda_j^0 + N_j t + l_j^2 t^2 + \ldots + l_j^p t^p . \quad (16)$$

We must note that:

— In the expressions (16), numerical coefficients σ_j^1, $\sigma_j^2, \ldots, \sigma_j^p$, $l_j^2, \ldots, l_j^p$ are issued from the theory. The mean mean motions N_j and the integration constants σ_j^0 and λ_j^0 are obtained by fitting the solutions in form (15) to the observations or to numerical integration fitted with the observations, such as DE200 (Standish 1982). Thus, mean elements are strictly dependent on the theory from which they are issued.

— In expressions (15) and (16), the time t is measured from a fixed origin t_0. In our theories this origin is J2000 (JD 2 451 545.0) and the mean elements (16) are referred to the mean dynamical ecliptic and equinox J2000. Ecliptics and equinoxes are also inertial as defined by Standish (1981).

— From these elements, mean elements referred to the mean dynamical ecliptic and equinox of date can be computed, using precession formulae. In the adjustment of DE200 to the observations the optical observations have been reduced using the IAU 1976 values of $\mathcal{P}_1$ given by expression (1). Since the observations of the outer planets are essentially optical, the transformation to the ecliptic and equinox of date has been done with formulae using that value of the precession constant. On the other hand, the observations related to Mercury, Venus, the Earth and Mars are mainly distance measures which do not depend on the precession constant and the transformation has been done with the formulae of Sect. 2.

— In the expressions given in Sects. 5.8 and 5.9, the value of p of the formulae (16) has been chosen in order to have [4000 B.C.; A.D. 8000] as time span of validity.

5.3. Secular terms of the semi-major axes

In classical planetary theories, semi-major axes contain secular terms of the third order with respect to the masses (Simon & Bretagnon 1978; Duriez 1978). Numerically these terms are small but they give, by integration, important secular terms in the perturbations of the mean longitudes specially for Jupiter and Saturn. For these two planets they were computed with great precision in the JASON84 theory.

5.4. Use of the mean elements

Mean elements are mainly useful for:

(a) the determination of the starting integration constants in building classical planetary theories. For example, our theories have been built with starting integration constants issued from the mean elements of Le Verrier and Le Verrier & Gaillot.

(b) the determination of the integration constants of general planetary theories (see, for example, Laskar 1988).

(c) the amelioration of general planetary theories by fitting the long period terms of general theories to the mean elements of classical theories. For example, in the general theory of the couple Jupiter–Saturn of Bretagnon & Simon (1990), some long period terms of the semi-major axes have been determined by fitting to the secular terms of the semi-major axes computed at a very good precision by the Simon & Joutel's method (1988)

(d) the check of the accuracy of general planetary theories. For example, Laskar (1990), in his study of the chaotic motion of the solar system, has evaluated the accuracy of his secular system by comparison with the mean elements of VSOP82.

We must note that, for the uses (c) and (d), it is necessary to compute mean elements with a good precision, which means that expressions (16) must be developed to a high order of the time.

5.5. Why two sets of elements e, ϖ, i, Ω and k, h, q, p?

It is important to emphasize that the two sets of mean elements $\langle e\rangle$, $\langle\varpi\rangle$, $\langle i\rangle$, $\langle\Omega\rangle$ and $\langle k\rangle$, $\langle h\rangle$, $\langle q\rangle$, $\langle p\rangle$ are not redundant. A mean element $\langle\sigma_j\rangle$ in form (16) is indeed computed from the complete solution in form (15). So, a change of variables for computing the mean elements $\langle e\rangle$, $\langle\varpi\rangle$, $\langle i\rangle$, $\langle\Omega\rangle$ from the elements k, h, q, p must be performed on the complete solution (15). A change of variables only performed on mean elements $\langle k\rangle$, $\langle h\rangle$, $\langle q\rangle$, $\langle p\rangle$ in form (16) would give unaccurate results. The error is particularly important for the outer planets. As an example, the mean value of the eccentricity of Neptune given by the formulae of Sect. 5.8 is $e_0 = 0.0094557470$, the value given by $\sqrt{k_0^2 + h_0^2}$ (where k_0 and h_0 are the mean values of k and h of Neptune) would be 0.0089880948 and the relative error would be of $5\,10^{-2}$.

General and classical planetary theories may be constructed either using the set of variables a, λ, e, ϖ, i, Ω, or using the set of a, λ, k, h, q, p. The mean elements $\langle e\rangle$, $\langle\varpi\rangle$, $\langle i\rangle$, $\langle\Omega\rangle$ will be useful only in the first case and the mean elements k, h, q, p, only in the second case.

5.6. Is it possible to compute approximate ephemerides with mean elements?

Expressions (16) do not contain periodic perturbations of the complete solutions (15) and it must be clear that the purpose of the mean elements is *not* to compute ephemerides. However it is natural to ask the question: can expressions (16) give approximate ephemerides of the planets if we substitute numerical values of time t in them and what is the precision? The answer is different for the inner planets (including Mars) and for the outer planets:

— For the inner planets, periodic perturbations of classical planetary theories are small. Thus, for instance, the amplitudes of the most important periodic terms of the mean longitudes are 7″ for Mercury (term of period 5.5 yr), 4″ (period 0.8 yr) for Venus, 7″ (period 1783 yr) and 4″ (period 0.55 yr) for the Earth, 57″ (period 1783 yr) and 17″ (period 1.1 yr) for Mars. So, mean elements can provide approximate ephemerides for these planets with a precision of a few tens of arcseconds.

— For the outer planets, periodic perturbations of classical planetary theories are very large. The amplitudes of the most important periodic terms of the mean longitudes are 1183″ (great inequality $2\bar\lambda_J - 5\bar\lambda_S$, with a period of 883 yr) and 129″ (period 61 yr) for Jupiter, 2912″ (period 883 yr) and 536″ (period 19.9 yr) for Saturn, 3102″ (argument $\bar\lambda_U - 2\bar\lambda_N$, with a period of 4233 yr) and 703″ (period 13.8 yr) for Uranus, 2099″ (period 4233 yr) and 911″ (period 12.8 yr) for Neptune. So, we see that mean elements cannot provide approximate ephemeris for these planets.

5.7. Approximate ephemerides by adding trigonometric terms

We have seen that mean elements do not intend to give approximate ephemerides. However it is easy to add a few trigonometric terms to their formulation to obtain ephemerides with very satisfying precision as well over the interval 1800–2050 as over the interval 1000–3000. We give these trigonometric terms in Sect. 6.

5.8. Mean elements of the planets referred to the mean dynamical ecliptic and equinox J2000

The mean elements of the planets referred to the mean dynamical ecliptic and equinox J2000 are given for the variables a, λ, e, ϖ, i, Ω, k, h, q, p. They were obtained from VSOP87 for Mercury, Venus, the Earth, Mars, Uranus and Neptune and from JASON84 for Jupiter and Saturn. As explained in Bretagnon & Francou (1988), for the variables k, h, q, p of Mercury, Venus, the Earth and Mars, VSOP87 contains the mean elements of VSOP82 improved for the terms of high degree with respect to time by the polynomials taken out of the general theory of Laskar (1986).

In our formulae, t is TDB measured in thousands of Julian years from J2000 (JD 2451545.0):

$$t = (\mathrm{JD} - 2\,451545.0)/365250\,,$$

a is measured in au, e, k, h, q, p are dimensionless. For the angles (λ, ϖ, i, Ω), the constant terms are in degrees and fraction of degrees, the coefficients of time are in arcseconds.

5.8.1. Mean elements of Mercury

$$a = 0.3870983098\,,$$

$$\lambda = 252^\circ\!.25090552 + 5381016286''\!.88982t - 1''\!.92789t^2 + 0''\!.00639t^3\,,$$

$$e = 0.2056317526 + 0.0002040653t - 28349\,10^{-10}t^2 - 1805\,10^{-10}t^3 + 23\,10^{-10}t^4 - 2\,10^{-10}t^5\,,$$

$$\varpi = 77^\circ\!.45611904 + 5719''\!.11590t - 4''\!.83016t^2 - 0''\!.02464t^3 - 0''\!.00016t^4 + 0''\!.00004t^5\,,$$

$$i = 7^\circ\!.00498625 - 214''\!.25629t + 0''\!.28977t^2 + 0''\!.15421t^3 - 0''\!.00169t^4 - 0''\!.00002t^5\,,$$

$$\Omega = 48^\circ\!.33089304 - 4515''\!.21727t - 31''\!.79892t^2 - 0''\!.71933t^3 + 0''\!.01242t^4\,,$$

$$k = 0.0446605976 - 0.0055211462t - 0.0000186057t^2 + 7912\,10^{-10}t^3 + 59\,10^{-10}t^4 - 2\,10^{-10}t^5\,,$$

$$h = 0.2007233137 + 0.0014375012t - 0.0000797412t^2 - 3046\,10^{-10}t^3 + 81\,10^{-10}t^4 - 1\,10^{-10}t^5\,,$$

$$q = 0.0406156338 + 0.0006543312t - 0.0000107122t^2 + 2246\,10^{-10}t^3 - 38\,10^{-10}t^4\,,$$

$$p = 0.0456355046 - 0.0012763366t - 0.0000091335t^2 + 1899\,10^{-10}t^3 - 64\,10^{-10}t^4\,.$$

5.8.2. Mean elements of Venus

$a = 0.7233298200\,,$

$\lambda = 181^\circ.97980085 + 2106641364''.33548t + 0''.59381t^2 - 0''.00627t^3\,,$

$e = 0.0067719164 - 0.0004776521t + 98127\,10^{-10}t^2 + 4639\,10^{-10}t^3 + 123\,10^{-10}t^4 - 3\,10^{-10}t^5\,,$

$\varpi = 131^\circ.56370300 + 175''.48640t - 498''.48184t^2 - 20''.50042t^3 - 0''.72432t^4 + 0''.00224t^5\,,$

$i = 3^\circ.39466189 - 30''.84437t - 11''.67836t^2 + 0''.03338t^3 + 0''.00269t^4 + 0''.00004t^5\,,$

$\Omega = 76^\circ.67992019 - 10008''.48154t - 51''.32614t^2 - 0''.58910t^3 - 0''.04665t^4\,,$

$k = -0.0044928213 + 0.0003125902t + 0.0000060406t^2 - 6835\,10^{-10}t^3 + 49\,10^{-10}t^4 + 6\,10^{-10}t^5\,,$

$h = 0.0050668473 - 0.0003612124t + 0.0000184676t^2 + 328\,10^{-10}t^3 - 61\,10^{-10}t^4 - 2\,10^{-10}t^5\,,$

$q = 0.0068241014 + 0.0013813383t - 0.0000109094t^2 - 18642\,10^{-10}t^3 + 60\,10^{-10}t^4 + 7\,10^{-10}t^5\,,$

$p = 0.0288228577 - 0.0004038479t - 0.0000623289t^2 + 2473\,10^{-10}t^3 + 423\,10^{-10}t^4 - 1\,10^{-10}t^5\,.$

5.8.3. Mean elements of the Earth

$a = 1.0000010178\,,$

$\lambda = 100^\circ.46645683 + 1295977422''.83429t - 2''.04411t^2 - 0''.00523t^3\,,$

$e = 0.0167086342 - 0.0004203654t - 0.0000126734\,t^2 + 1444\,10^{-10}t^3 - 2\,10^{-10}t^4 + 3\,10^{-10}t^5\,,$

$\varpi = 102^\circ.93734808 + 11612''.35290t + 53''.27577t^2 - 0''.14095t^3 + 0''.11440t^4 + 0''.00478t^5\,,$

$i = 469''.97289t - 3''.35053t^2 - 0''.12374t^3 + 0''.00027t^4 - 0''.00001t^5 + 0''.00001t^6\,,$

$\Omega = 174^\circ.87317577 - 8679''.27034t + 15''.34191t^2 + 0''.00532t^3 - 0''.03734t^4 - 0''.00073t^5 + 0''.00004t^6\,,$

$k = -0.0037408165 - 0.0008226742t + 0.0000276246t^2 + 11696\,10^{-10}t^3 - 270\,10^{-10}t^4 - 7\,10^{-10}t^5\,,$

$h = 0.0162844766 - 0.0006202965t - 0.0000338263t^2 + 8510\,10^{-10}t^3 + 277\,10^{-10}t^4 - 5\,10^{-10}t^5\,,$

$q = -0.0011346887t + 0.0000123731t^2 + 12654\,10^{-10}t^3 - 137\,10^{-10}t^4 - 3\,10^{-10}t^5\,,$

$p = 0.0001018038t + 0.0000470200t^2 - 5417\,10^{-10}t^3 - 251\,10^{-10}t^4 + 5\,10^{-10}t^5\,.$

5.8.4. Mean elements of Mars

$a = 1.5236793419 + 3\,10^{-10}t\,,$

$\lambda = 355^\circ.43299958 + 689050774''.93988t + 0''.94264t^2 - 0''.01043t^3\,,$

$e = 0.0934006477 + 0.0009048438t - 80641\,10^{-10}t^2 - 2519\,10^{-10}t^3 + 124\,10^{-10}t^4 - 10\,10^{-10}t^5\,,$

$\varpi = 336^\circ.06023395 + 15980''.45908t - 62''.32800t^2 + 1''.86464t^3 - 0''.04603t^4 - 0''.00164t^5\,,$

$i = 1^\circ.84972648 - 293''.31722t - 8''.11830t^2 - 0''.10326t^3 - 0''.00153t^4 + 0''.00048t^5\,,$

$\Omega = 49^\circ.55809321 - 10620''.90088t - 230''.57416t^2 - 7''.06942t^3 - 0''.68920t^4 - 0''.05829t^5\,,$

$k = 0.0853656025 + 0.0037633015t - 0.0002465778t^2 - 36731\,10^{-10}t^3 + 1111\,10^{-10}t^4 + 3\,10^{-10}t^5\,,$

$h = -0.0378997324 + 0.0062465746t + 0.0001552948t^2 - 63488\,10^{-10}t^3 - 659\,10^{-10}t^4 + 7\,10^{-10}t^5\,,$

$q = 0.0104704257 + 0.0001713853t - 0.0000407749t^2 - 13883\,10^{-10}t^3 + 92\,10^{-10}t^4 + 18\,10^{-10}t^5\,,$

$p = 0.0122844931 - 0.0010802008t - 0.0000192222t^2 + 8719\,10^{-10}t^3 + 309\,10^{-10}t^4\,.$

5.8.5. Mean elements of Jupiter

$$a = 5.2026032092 + 19132\,10^{-10}t - 39\,10^{-10}t^2 - 60\,10^{-10}t^3 - 10\,10^{-10}t^4 + 1\,10^{-10}t^5\,,$$

$$\lambda = 34^\circ.35151874 + 109256603''.77991t - 30''.60378t^2 + 0''.05706t^3 + 0''.04667t^4 + 0''.00591t^5 - 0''.00034t^6\,,$$

$$e = 0.0484979255 + 0.0016322542t - 0.0000471366t^2 - 20063\,10^{-10}t^3 + 1018\,10^{-10}t^4 - 21\,10^{-10}t^5 + 1\,10^{-10}t^6\,,$$

$$\varpi = 14^\circ.33120687 + 7758''.75163t + 259''.95938t^2 - 16''.14731t^3 + 0''.74704t^4 - 0''.02087t^5 - 0''.00016t^6\,,$$

$$i = 1^\circ.30326698 - 71''.55890t + 11''.95297t^2 + 0''.34909t^3 - 0''.02710t^4 - 0''.00124t^5 + 0''.00003t^6\,,$$

$$\Omega = 100^\circ.46440702 + 6362''.03561t + 326''.52178t^2 - 26''.18091t^3 - 2''.10322t^4 + 0''.04453t^5 + 0''.01154t^6\,,$$

$$k = 0.0469857457 + 0.0011300656t - 0.0001092396t^2 - 43089\,10^{-10}t^3 + 1963\,10^{-10}t^4 + 24\,10^{-10}t^5 - 2\,10^{-10}t^6\,,$$

$$h = 0.0120038766 + 0.0021714660t + 0.0000985396t^2 - 51635\,10^{-10}t^3 - 990\,10^{-10}t^4 + 69\,10^{-10}t^5\,,$$

$$q = -0.0020656001 - 0.0003134485t - 0.0000167052t^2 + 7975\,10^{-10}t^3 + 365\,10^{-10}t^4 - 2\,10^{-10}t^5 - 1\,10^{-10}t^6\,,$$

$$p = 0.0111837479 - 0.0002342791t + 0.0000208686t^2 + 5272\,10^{-10}t^3 - 342\,10^{-10}t^4 + 5\,10^{-10}t^5\,.$$

5.8.6. Mean elements of Saturn

$$a = 9.5549091915 - 0.0000213896t + 444\,10^{-10}t^2 + 670\,10^{-10}t^3 + 110\,10^{-10}t^4 - 7\,10^{-10}t^5 - 1\,10^{-10}t^6\,,$$

$$\lambda = 50^\circ.07744430 + 43996098''.55732t + 75''.61614t^2 - 0''.16618t^3 - 0''.11484t^4 - 0''01452t^5 + 0''.00083t^6\,,$$

$$e = 0.0555481426 - 0.0034664062t - 0.0000643639t^2 + 33956\,10^{-10}t^3 - 219\,10^{-10}t^4 - 3\,10^{-10}t^5 + 6\,10^{-10}t^6\,,$$

$$\varpi = 93^\circ.05723748 + 20395''.49439t + 190''.25952t^2 + 17''.68303t^3 + 1''.23148t^4 + 0''.10310t^5 + 0''.00702t^6\,,$$

$$i = 2^\circ.48887878 + 91''.85195t - 17''.66225t^2 + 0''.06105t^3 + 0''.02638t^4 - 0''.00152t^5 - 0''.00012t^6\,,$$

$$\Omega = 113^\circ.66550252 - 9240''.19942t - 66''.23743t^2 + 1''.72778t^3 + 0''.26990t^4 + 0''.03610t^5 - 0''.00248t^6\,,$$

$$k = -0.0029599926 - 0.0052959042t + 0.0003092222t^2 + 0.0000129279t^3 - 6347\,10^{-10}t^4 - 54\,10^{-10}t^5 + 8\,10^{-10}t^6\,,$$

$$h = 0.0554296096 - 0.0037559081t - 0.0003198421t^2 + 0.0000159875t^3 + 3022\,10^{-10}t^4 - 231\,10^{-10}t^5 + 2\,10^{-10}t^6\,,$$

$$q = -0.0087174677 + 0.0008017413t + 0.0000414442t^2 - 19997\,10^{-10}t^3 - 896\,10^{-10}t^4 + 6\,10^{-10}t^5 + 2\,10^{-10}t^6\,,$$

$$p = 0.0198914760 + 0.0005944060t - 0.0000523589t^2 - 12993\,10^{-10}t^3 + 856\,10^{-10}t^4 - 16\,10^{-10}t^5 - 1\,10^{-10}t^6\,.$$

5.8.7. Mean elements of Uranus

$$a = 19.2184460618 - 3716\,10^{-10}t + 979\,10^{-10}t^2\,,$$

$$\lambda = 314^\circ.05500511 + 15424811''.93933t - 1''.75083t^2 + 0''.02156t^3\,,$$

$$e = 0.0463812221 - 0.0002729293t + 0.0000078913t^2 + 2447\,10^{-10}t^3 - 171\,10^{-10}t^4\,,$$

$$\varpi = 173^\circ.00529106 + 3215''.56238t - 34''.09288t^2 + 1''.48909t^3 + 0''.06600t^4\,,$$

$$i = 0^\circ.77319689 - 60''.72723t + 1''.25759t^2 + 0''.05808t^3 + 0''.00031t^4\,,$$

$$\Omega = 74^\circ.00595701 + 2669''.15033t + 145''.93964t^2 + 0''.42917t^3 - 0''.09120t^4\,,$$

$$k = -0.0459513238 + 0.0001834412t - 0.0000008085t^2 - 4540\,10^{-10}t^3 + 218\,10^{-10}t^4\,,$$

$$h = 0.0056379131 - 0.0007496435t + 0.0000121020t^2 - 4209\,10^{-10}t^3 - 171\,10^{-10}t^4\,,$$

$$q = 0.0018591507 - 0.0001244938t - 0.0000020737t^2 + 762\,10^{-10}t^3\,,$$

$$p = 0.0064861701 - 0.0001174473t + 0.0000031780\,t^2 + 732\,10^{-10}\,t^3\,.$$

5.8.8. Mean elements of Neptune

$$a = 30.1103868694 - 16635\,10^{-10}\,t + 686\,10^{-10}\,t^2\,,$$

$$\lambda = 304^{\circ}.34866548 + 7865503''.20744t + 0''.21103t^2 - 0''.00895t^3\,,$$

$$e = 0.0094557470 + 0.0000603263t + 0t^2 - 483\,10^{-10}\,t^3\,,$$

$$\varpi = 48^{\circ}.12027554 + 1050''.71912t + 27''.39717t^2\,,$$

$$i = 1^{\circ}.76995259 + 8''.12333t + 0''.08135t^2 - 0''.00046t^3\,,$$

$$\Omega = 131^{\circ}.78405702 - 221''.94322t - 0''.78728t^2 - 0''.28070t^3 + 0''.00049t^4\,,$$

$$k = 0.0059997757 + 0.0000087130t - 0.0000011990t^2 - 403\,10^{-10}\,t^3\,,$$

$$h = 0.0066924241 + 0.0000782434t + 0.0000008080t^2 - 395\,10^{-10}\,t^3\,,$$

$$q = -0.0102914782 - 0.0000007273t - 0.0000000657t^2 + 167\,10^{-10}\,t^3\,,$$

$$p = 0.0115168398 + 0.0000257554t + 0.0000001938t^2 + 133\,10^{-10}t^3\,.$$

5.9. Mean elements of the planets referred to the mean dynamical ecliptic and equinox of date

Mean elements referred to the mean dynamical ecliptic and equinox of date are computed from mean elements referred to ecliptic and equinox J2000 using formulae of precession as explained in Sect. 5.2. They are given for the same variables and are issued from VSOP87 for Mercury, Venus, the Earth, Mars, Uranus and Neptune and from JASON84 for Jupiter and Saturn.

In our formulae t is TDB measured in thousands of Julian years from J2000 (JD 2 451 545.0):

$$t = (JD - 2\,451545.0)/365250\,,$$

a is measured in au, e, k, h, q, p are dimensionless. For the angles (λ, ϖ, i, Ω), the constant terms are in degrees and fraction of degrees, the coefficients of time are in arcseconds.

5.9.1. Mean elements of Mercury

$$a = 0.3870983098\,,$$

$$\lambda = 252^{\circ}.25090552 + 5381066598''.20037t + 109''.25943t^2 + 0''.06522t^3 - 0''.23500t^4 - 0''.00179t^5 + 0''.00020t^6\,,$$

$$e = 0.2056317526 + 0.0002040653t - 28349\,10^{-10}t^2 - 1805\,10^{-10}t^3 + 23\,10^{-10}t^4 - 2\,10^{-10}t^5\,,$$

$$\varpi = 77^{\circ}.45611904 + 56030''.42645t + 106''.35716t^2 + 0''.03418t^3 - 0''.23516t^4 - 0''.00176t^5 + 0''.00020t^6\,,$$

$$i = 7^{\circ}.00498625 + 65''.57301t - 6''.51516t^2 + 0''.20113t^3 + 0''.00019t^4 - 0''.00019t^5\,,$$

$$\Omega = 48^{\circ}.33089304 + 42700''.01444t + 63''.14994t^2 + 0''.77259t^3 - 0''.20893t^4 - 0''.00219t^5 + 0''.00016t^6\,,$$

$$k = 0.0446605976 - 0.0544807963t - 0.0018059782t^2 + 0.0006632523t^3 + 0.0000149034t^4 - 23668\,10^{-10}t^5 - 597\,10^{-10}t^6\,,$$

$$h = 0.2007233137 + 0.0123309371t - 0.0073733874t^2 - 0.0001849726t^3 + 0.0000445200t^4 + 10075\,10^{-10}t^5 - 1028\,10^{-10}t^6\,,$$

$$q = 0.0406156338 - 0.0093417782t - 0.0009192871t^2 + 0.0000651977t^3 + 37416\,10^{-10}t^4 - 1284\,10^{-10}t^5 - 67\,10^{-10}t^6\,,$$

$$p = 0.0456355046 + 0.0085265821t - 0.0009553697t^2 - 0.0000671085t^3 + 33005\,10^{-10}t^4 + 1711\,10^{-10}t^5 - 37\,10^{-10}t^6\,.$$

5.9.2. Mean elements of Venus

$$a = 0.7233298200\,,$$

$$\lambda = 181^{\circ}.97980085 + 2106691666''.31989t + 111''.65021t^2 + 0''.05368t^3 - 0''.23516t^4 - 0''.00179t^5 + 0''.00020t^6\,,$$

$$e = 0.0067719164 - 0.0004776521t + 98127\,10^{-10}t^2 + 4639\,10^{-10}t^3 + 123\,10^{-10}t^4 - 3\,10^{-10}t^5\,,$$

$$\varpi = 131^{\circ}.56370300 + 50477''.47081t - 387''.42545t^2 - 20''.44048t^3 - 0''.95948t^4 + 0''.00044t^5 + 0''.00020t^6,$$

$$i = 3^{\circ}.39466189 + 36''.13261t - 0''.31523t^2 - 0''.02525t^3 + 0''.00085t^4 - 0''.00008t^5,$$

$$\Omega = 76^{\circ}.67992019 + 32437''.57636t + 146''.22586t^2 - 0''.33446t^3 - 0''.23007t^4 - 0''.00088t^5 + 0''.00009t^6,$$

$$k = -0.0044928213 - 0.0009230666t + 0.0002250026t^2 - 0.0000014513t^3 - 16810\ 10^{-10}t^4 + 627\ 10^{-10}t^5 + 50\ 10^{-10}t^6,$$

$$h = 0.0050668473 - 0.0014568806t - 0.0000583901t^2 + 0.0000226090t^3 - 6041\ 10^{-10}t^4 - 998\ 10^{-10}t^5 + 43\ 10^{-10}t^6,$$

$$q = 0.0068241014 - 0.0045125642t - 0.0001183914t^2 + 0.0000177623t^3 + 5244\ 10^{-10}t^4 - 173\ 10^{-10}t^5 - 11\ 10^{-10}t^6,$$

$$p = 0.0288228577 + 0.0011583648t - 0.0003491466t^2 - 0.0000087743t^3 + 6535\ 10^{-10}t^4 + 264\ 10^{-10}t^5 - 2\ 10^{-10}t^6.$$

5.9.3. Mean elements of the Earth

$$a = 1.0000010178,$$

$$\lambda = 100^{\circ}.46645683 + 1296027711''.03429t + 109''.15809t^2 + 0''.07207t^3 - 0''.23530t^4 - 0''.00180t^5 + 0''.00020t^6,$$

$$e = 0.0167086342 - 0.0004203654t - 0.000012\,6734t^2 + 1444\ 10^{-10}t^3 - 2\ 10^{-10}t^4 + 3\ 10^{-10}t^5,$$

$$\varpi = 102^{\circ}.93734808 + 61900''.55290t + 164''.47797t^2 - 0''.06365t^3 - 0''.12090t^4 + 0''.00298t^5 + 0''.00020t^6,$$

$$k = -0.0037408165 - 0.0047928949t + 0.0002812540t^2 + 0.0000740171t^3 - 26974\ 10^{-10}t^4 - 3810\ 10^{-10}t^5 + 86\ 10^{-10}t^6,$$

$$h = 0.0162844766 - 0.0015323228t - 0.0007203925t^2 + 0.0000324712t^3 + 58589\ 10^{-10}t^4 - 1719\ 10^{-10}t^5 - 213\ 10^{-10}t^6.$$

5.9.4. Mean elements of Mars

$$a = 1.5236793419 + 3\ 10^{-10}\,t,$$

$$\lambda = 355^{\circ}.43299958 + 689101069''.33069t + 111''.78674t^2 + 0''.05624t^3 - 0''.23516t^4 - 0''.00180t^5 + 0''.00020t^6,$$

$$e = 0.0934006477 + 0.0009048438t - 80641\ 10^{-10}t^2 - 2519\ 10^{-10}t^3 + 124\ 10^{-10}t^4 - 10\ 10^{-10}t^5,$$

$$\varpi = 336^{\circ}.06023395 + 66274''.84990t + 48''.51610t^2 + 1''.93131t^3 - 0''.28118t^4 - 0''.00344t^5 + 0''.00020t^6,$$

$$i = 1^{\circ}.84972648 - 21''.63885t + 4''.59350t^2 - 0''.02376t^3 - 0''.01708t^4 + 0''.00065t^5 + 0''.00005t^6,$$

$$\Omega = 49^{\circ}.55809321 + 27792''.68736t + 5''.60611t^2 + 8''.16222t^3 - 0''.45709t^4 - 0''.04722t^5 + 0''.00435t^6,$$

$$k = 0.0853656025 + 0.0130045425t - 0.0042870473t^2 - 0.0002595083t^3 + 0.0000354092t^4 + 15988\ 10^{-10}t^5 - 1104\ 10^{-10}t^6,$$

$$h = -0.0378997324 + 0.0270616164t + 0.0022454557t^2 - 0.0004514091t^3 - 0.0000226552t^4 + 21921\ 10^{-10}t^5 + 959\ 10^{-10}t^6,$$

$$q = 0.0104704257 - 0.0016892678t - 0.0000827820t^2 + 0.0000036153t^3 + 169\ 10^{-10}t^4 + 142\ 10^{-10}t^5 + 3\ 10^{-10}t^6,$$

$$p = 0.0122844931 + 0.0013708983t - 0.0001073425t^2 - 0.0000026091t^3 - 231\ 10^{-10}t^4 - 34\ 10^{-10}t^5 + 14\ 10^{-10}t^6.$$

5.9.5. Mean elements of Jupiter

$$a = 5.2026032092 + 19132\ 10^{-10}\,t - 39\ 10^{-10}t^2 - 60\ 10^{-10}t^3 - 10\ 10^{-10}t^4 + 1\ 10^{-10}t^5,$$

$$\lambda = 34^{\circ}.35151874 + 109306899''.89453t + 80''.38700t^2 + 0''.13327t^3 - 0''.18850t^4 + 0''.00411t^5 - 0''.00014t^6,$$

$$e = 0.0484979255 + 0.0016322542t - 0.0000471366t^2 - 20063\ 10^{-10}t^3 + 1018\ 10^{-10}t^4 - 21\ 10^{-10}t^5 + 1\ 10^{-10}t^6,$$

$$\varpi = 14^{\circ}.33120687 + 58054''.86625t + 370''.95016t^2 - 16''.07110t^3 + 0''.51186t^4 - 0''.02268t^5 + 0''.00004t^6,$$

$$i = 1^{\circ}.30326698 - 197''.87442t + 1''.67744t^2 - 0''.00838t^3 - 0''.00735t^4 + 0''.00085t^5 + 0''.00004t^6,$$

$$\Omega = 100^{\circ}.46440702 + 36755''.18747t + 145''.13295t^2 + 1''.45556t^3 - 0''.59609t^4 - 0''.04324t^5 + 0''.00175t^6,$$

$$k = 0.0469857457 - 0.0017969926t - 0.0020420604t^2 - 0.0000402595t^3 + 0.0000168641t^4 + 6000\,10^{-10}t^5 - 623\,10^{-10}t^6$$

$$h = 0.0120038766 + 0.0136285825t + 0.0000425103t^2 - 0.0002108419t^3 - 0.0000061928t^4 + 11097\,10^{-10}t^5 + 444\,10^{-10}t^6\,,$$

$$q = -0.0020656001 - 0.0019057660t + 0.0001082507t^2 + 0.0000089680t^3 - 3638\,10^{-10}t^4 - 117\,10^{-10}t^5 - 7\,10^{-10}t^6\,,$$

$$p = 0.0111837479 - 0.0008397312t - 0.0001594973t^2 + 0.0000079342t^3 + 3790\,10^{-10}t^4 - 67\,10^{-10}t^5 - 1\,10^{-10}t^6\,.$$

5.9.6. Mean of elements of Saturn

$$a = 9.5549091915 - 0.0000213896\,t + 444\,10^{-10}t^2 + 670\,10^{-10}t^3 + 110\,10^{-10}t^4 - 7\,10^{-10}t^5 - 1\,10^{-10}t^6\,,$$

$$\lambda = 50^{\circ}.07744430 + 44046398''.47038t + 186''.86817t^2 - 0''.10748t^3 - 0''.35004t^4 - 0''.01630t^5 + 0''.00103t^6\,,$$

$$e = 0.0555481426 - 0.0034664062t - 0.0000643639t^2 + 33956\,10^{-10}t^3 - 219\,10^{-10}t^4 - 3\,10^{-10}t^5 + 6\,10^{-10}t^6\,,$$

$$\varpi = 93^{\circ}.05723748 + 70695''.40745t + 301''.51155t^2 + 17''.74174t^3 + 0''.99628t^4 + 0''.10132t^5 + 0''.00722t^6\,,$$

$$i = 2^{\circ}.48887878 - 134''.50388t - 5''.46800t^2 + 0''.31168t^3 + 0''.03207t^4 - 0''.00237t^5 - 0''.00023t^6$$

$$\Omega = 113^{\circ}.66550252 + 31575''.16875t - 43''.83321t^2 - 8''.09520t^3 + 0''.18433t^4 + 0''.06867t^5 - 0''.00276t^6\,,$$

$$k = -0.0029599926 - 0.0188130068t + 0.0012832568t^2 + 0.0003847521t^3 - 0.0000214188t^4 - 25250\,10^{-10}t^5 + 1149\,10^{-10}t^6\,,$$

$$h = 0.0554296096 - 0.0044777281t - 0.0032610492t^2 + 0.0002000704t^3 + 0.0000346305t^4 - 17436\,10^{-10}t^5 - 1558\,10^{-10}t^6\,,$$

$$q = -0.0087174677 - 0.0029141582t + 0.0001573853t^2 + 0.0000123470t^3 - 7068\,10^{-10}t^4 - 347\,10^{-10}t^5 + 38\,10^{-10}t^6\,,$$

$$p = 0.0198914760 - 0.0016330327t - 0.0002233181t^2 + 0.0000111755t^3 + 6174\,10^{-10}t^4 - 482\,10^{-10}t^5 - 24\,10^{-10}t^6\,.$$

5.9.7. Mean elements of Uranus

$$a = 19.2184460618 - 3716\,10^{-10}t + 979\,10^{-10}t^2\,,$$

$$\lambda = 314^{\circ}.05500511 + 15475106''.01961t + 109''.40272t^2 + 0''.09474t^3 - 0''.23521t^4 - 0''.00180t^5 + 0''.00020t^6\,,$$

$$e = 0.0463812221 - 0.0002729293t + 0.0000078913t^2 + 2447\,10^{-10}t^3 - 171\,10^{-10}t^4\,,$$

$$\varpi = 173^{\circ}.00529106 + 53509''.64266t + 77''.06068t^2 + 1''.56227t^3 - 0''.16921t^4 - 0''.00180t^5 + 0''.00020t^6\,,$$

$$i = 0^{\circ}.77319689 + 27''.87845t + 13''.49529t^2 - 0''.33095t^3 - 0''.03444t^4 + 0''.00171t^5 + 0''.00012t^6\,,$$

$$\Omega = 74^{\circ}.00595701 + 18760''.59902t + 482''.21068t^2 + 66''.54269t^3 - 3''.52490t^4 - 0''.32819t^5 + 0''.03056t^6\,,$$

$$k = -0.0459513238 - 0.0011912655t + 0.0015449434t^2 + 0.0000112035t^3 - 83536\,10^{-10}t^4 - 513\,10^{-10}t^5 + 165\,10^{-10}t^6\,,$$

$$h = 0.0056379131 - 0.0119540733t - 0.0001355308t^2 + 0.0001320336t^3 + 7849\,10^{-10}t^4 - 4140\,10^{-10}t^5 - 33\,10^{-10}t^6\,,$$

$$q = 0.0018591508 - 0.0005713216t - 0.0000197484t^2 - 49846\,10^{-10}t^3 + 391\,10^{-10}t^4 + 267\,10^{-10}t^5 + 3\,10^{-10}t^6\,,$$

$$p = 0.0064861701 + 0.0002340588t + 0.0000106579t^2 - 11892\,10^{-10}t^3 - 4589\,10^{-10}t^4 - 14\,10^{-10}t^5 + 12\,10^{-10}t^6\,.$$

5.9.8. Mean elements of Neptune

$$a = 30.1103868694 - 16635\,10^{-10}t + 686\,10^{-10}t^2\,,$$

$$\lambda = 304^{\circ}.34866548 + 7915799''.13277t + 111''.17536t^2 + 0''.06468t^3 - 0''.23514t^4 - 0''.00180t^5 + 0''.00020t^6\,,$$

$$e = 0.0094557470 + 0.0000603263t + 0t^2 - 483\,10^{-10}t^3\,,$$

$$\varpi = 48^{\circ}.12027554 + 51346''.64445t + 138''.36149t^2 + 0''.07363t^3 - 0''.23514t^4 - 0''.00180t^5 + 0''.00020t^6\,,$$

$$i = 1^{\circ}.76995259 - 335''.09412t - 2''.54991t^2 + 0''.09845t^3 + 0''.00101t^4 - 0''.00005t^5 - 0''.00001t^6\,,$$

$$\Omega = 131^\circ\!.78405702 + 39679''\!.34159t + 93''\!.42773t^2 - 2''\!.29323t^3 - 0''\!.33948t^4 - 0''\!.00479t^5 - 0''\!.00006t^6\,,$$

$$k = 0.0059997757 - 0.0016231779t - 0.0002022477t^2 + 0.0000148438t^3 + 12298\,10^{-10}t^4 - 323\,10^{-10}t^5 - 33\,10^{-10}t^6\,,$$

$$h = 0.0066924241 + 0.0015412377t - 0.0001928011t^2 - 0.0000180270t^3 + 8157\,10^{-10}t^4 + 686\,10^{-10}t^5 - 8\,10^{-10}t^6\,,$$

$$q = -0.0102914782 - 0.0016743192t + 0.0003058350t^2 + 56782\,10^{-10}t^3 - 13752\,10^{-10}t^4 - 133\,10^{-10}t^5 + 25\,10^{-10}t^6\,,$$

$$p = 0.0115168399 - 0.0025854022t - 0.0001182648t^2 + 237436\,10^{-10}t^3 + 2469\,10^{-10}t^4 - 639\,10^{-10}t^5 - 9\,10^{-10}t^6\,.$$

6. Approximate ephemerides of the planets

6.1. Construction

Approximate ephemerides may be constructed either from the set of variables a, λ, e, ϖ, i, Ω, or from the set a, λ, k, h, q, p. In this section we choose to consider the set of variables a, λ, e, ϖ, i, Ω. Among these variables, we note ρ variables other than the semi-major axes a and the mean longitudes λ.

We can obtain approximate ephemerides of the planets by using the mean elements of these variables referred to the ecliptic and equinox J2000 and by adding a few trigonometric terms to the formulation of the mean elements of the variables a and λ.

We must note that:

— The so-obtained expressions are useful only for computing heliocentric coordinates of the planets and are not simplified series of the osculating elements.

— As said before, the heliocentric coordinates must be computed with the set of variables a, λ, e, ϖ, i, Ω, and not with the set a, λ, k, h, q, p.

6.2. The added trigonometric terms

The added trigonometric terms are represented under the form of Poisson series of the argument μ related to the mean mean motions of Jupiter and Saturn N_5 and N_6 by

$$\mu = (N_5 - N_6)t/880 = 0.359\,536\,20\,t\,, \qquad (17)$$

where t is measured in thousands of years from J2000.

We choose this representation because perturbations are more convergent under this form than under the classical form of Poisson series of the mean longitudes, for Jupiter and Saturn (Simon et al. 1992). For the other planets the convergence of perturbations is the same than under the classical form.

6.3. Approximate ephemerides over the interval 1800–2050

We can compute approximate ephemerides of the planets over the interval 1800–2050 by using developments of the variables ρ, a and λ under the form

$$\rho = \rho^0 + \rho^1 t\,,$$

$$a = a^0 + a^1 t + \sum_{i=1}^{8}[C_i^a\cos(p_i\mu) + S_i^a\sin(p_i\mu)]\,,$$

$$\lambda = \lambda^0 + N\,t + \sum_{i=1}^{8}[C_i^l\cos(q_i\mu) + S_i^l\sin(q_i\mu)]\,. \qquad (18)$$

In expression (18):

— ρ represents the variables e, ϖ, i, Ω;

— t is the time measured in thousands of years from J2000 (JD 2 451 545.0);

— μ is the argument defined by (17);

— p_i and q_i are integers;

— ρ^0, a^0, λ^0 are the constant parts of the mean elements of the variables ρ (e, ϖ, i, Ω), a and λ, referred to the ecliptic and equinox J2000 and given in Sect. 5.8;

— ρ^1, a^1, N are the coefficients of t of these mean elements;

— the arguments $p_i\mu, q_i\mu$ and the values of the coefficients $C_i^a, S_i^a, C_i^l, S_i^l$ for the eight major planets are given in Table 6.

6.4. Approximate ephemerides over the interval 1000–3000

We can compute approximate ephemerides of the planets over the interval 1000–3000 by adding to the developments (18) the complements:

$$\delta\rho = \rho^2 t^2\,,$$

$$\delta a = a^2 t^2 + t[C_9^a\cos(p_9\mu) + S_9^a\sin(p_9\mu)]\,,$$

$$\delta\lambda = \lambda^2 t^2 + \sum_{i=9}^{10} t[C_i^l\cos(q_i\mu) + S_i^l\sin(q_i\mu)]\,. \qquad (19)$$

In expressions (19) ρ^2, a^2, λ^2 are the coefficients of t^2 of the mean elements of the variables ρ, a and λ, given in Sect. 5.8.

The arguments $p_i\mu$, $q_i\mu$ and the values of the coefficients C_9^a, S_9^a, C_i^l, S_i^l non equal to zero are given in Table 6.

6.5. Computer routines

The subroutines which enable one to compute these approximate ephemerides of the planets are available at the Bureau des Longitudes. They give the heliocentric longitude, latitude and radius vector and the cartesian coordinates from the elliptic elements.

Table 6. Trigonometric terms to be added to the mean elements for computing approximate ephemerides using formulae (17), (18) and (19). Units are 10^{-7} au for C_i^a and S_i^a, 10^{-7} radian for C_i^l and S_i^l

i	$p_i\mu$	C_i^a	S_i^a	$q_i\mu$	C_i^l	S_i^l	$p_i\mu$	C_i^a	S_i^a	$q_i\mu$	C_i^l	S_i^l
Mercury							*Venus*					
1	69613μ	4	−29	3086μ	21	−342	21863μ	−156	−48	21863μ	−160	524
2	75645μ	−13	−1	15746μ	−95	136	32794μ	59	−125	32794μ	−313	−149
3	88306μ	11	9	69613μ	−157	−23	26934μ	−42	−26	10931μ	−235	−35
4	59899μ	−9	6	59899μ	41	62	10931μ	6	−37	73μ	60	117
5	15746μ	−9	−6	75645μ	−5	66	26250μ	19	18	4387μ	−74	151
6	71087μ	−3	5	88306μ	42	−52	43725μ	−20	−13	26934μ	−76	122
7	142173μ	−1	4	12661μ	23	−33	53867μ	−10	−20	1473μ	−27	−71
8	3086μ	4	0	2658μ	30	17	28939μ	−12	−2	2157μ	34	−62
Earth–Moon barycenter							*Mars*					
1	16002μ	64	−150	10μ	−325	−105	6345μ	124	−621	10μ	2268	854
2	21863μ	−152	−46	16002μ	−322	−137	7818μ	621	532	6345μ	−979	−205
3	32004μ	62	68	21863μ	−79	258	15636μ	−145	−694	7818μ	802	−936
4	10931μ	−8	54	10931μ	232	35	7077μ	208	−20	1107μ	602	−240
5	14529μ	32	14	1473μ	−52	−116	8184μ	54	192	15636μ	−668	140
6	16368μ	−41	24	32004μ	97	−88	14163μ	−57	−94	7077μ	−33	−341
7	15318μ	19	−28	4387μ	55	−112	1107μ	30	71	8184μ	345	−97
8	32794μ	−11	22	73μ	−41	−80	4872μ	15	−73	532μ	201	−232
9										10μ	−55	536
Jupiter							*Saturn*					
1	1760μ	−23437	−14614	19μ	7610	−56980	574μ	62911	139737	19μ	−18549	138606
2	1454μ	−2634	−19828	1760μ	−4997	8016	0μ	−119919	0	574μ	30125	−13478
3	1167μ	6601	−5869	1454μ	−7689	1012	880μ	79336	24667	287μ	20012	−4964
4	880μ	6259	1881	287μ	−5841	1448	287μ	17814	51123	306μ	−730	1441
5	287μ	−1507	−4372	1167μ	−2617	−3024	19μ	−24241	−5102	1760μ	824	−1319
6	2640μ	−1821	−2255	880μ	1115	−3710	1760μ	12068	7429	12μ	23	−1482
7	19μ	2620	782	574μ	−748	318	1167μ	8306	−4095	31μ	1289	427
8	2047μ	−2115	930	2640μ	−607	503	306μ	−4893	−1976	38μ	−352	1236
9	1454μ	−1489	913	19μ	6074	3767	574μ	8902	−9566	19μ	−14767	−9167
10				1454μ	354	577				574μ	−2062	−1918

Table 6 (continued)

i	$p_i\mu$	C_i^a	S_i^a	$q_i\mu$	C_i^l	S_i^l	$p_i\mu$	C_i^a	S_i^a	$q_i\mu$	C_i^l	S_i^l
Uranus							*Neptune*					
1	204μ	389061	−138081	4μ	−135245	71234	0μ	−412235	0	4μ	89948	−47645
2	0μ	−262125	0	204μ	−14594	−41116	102μ	−157046	28492	102μ	2103	11647
3	177μ	−44088	37205	177μ	4197	5334	106μ	−31430	133236	106μ	8963	2166
4	1265μ	8387	−49039	8μ	−4030	−4935	4μ	37817	69654	8μ	2695	3194
5	4μ	−22976	−41901	31μ	−5630	−1848	98μ	−9740	52322	98μ	3682	679
6	385μ	−2093	−33872	200μ	−2898	66	1367μ	−13	−49577	1367μ	1648	0
7	200μ	−615	−27037	1265μ	2540	434	487μ	−7449	−26430	487μ	866	−244
8	208μ	−9720	−12474	102μ	−306	−1748	204μ	9644	−3593	204μ	−154	−419
9	204μ	6633	18797	4μ	2939	3780				4μ	−1963	−2531
10				204μ	1986	−701				102μ	−283	48

6.6. Precision of the approximate ephemerides

6.6.1. Precision over the interval 1800–2050

For 225 dates over the interval 1800–2050, we have compared the heliocentric spherical variables (longitude L, latitude B and distance R) issued from DE200 both with the same variables computed from our developments (18) and those computed from the keplerian elements given in Sect. 5.8 of the Explanatory Supplement to the Astronomical Almanach (1992). Keplerian elements of the Explanatory Supplement are not mean elements but approximate formulae under a very simple form (a constant term and a term in t) for each keplerian element of the eight major planets and Pluto.

Results of these comparisons are given in Table 7. We see that, for the heliocentric longitudes, the precision of our approximate ephemerides is about a few arcseconds for Mercury, Venus, the Earth and Neptune and about a few ten of arcseconds for the other planets. The gain in precision of our formulae, in connection with the formulae of the Explanatory Supplement, lies between 4 and 10 for the longitudes and the distances except for the longitude of Uranus for which the gain is only 1.5. Note that, due to the more exhaustive nature of our comparison, the errors of the formulae of the Explanatory Supplement are a little greater than indicated in Table 5.8.2 of this book.

6.6.2. Precision over the interval 1000–3000

We have compared the heliocentric variables computed from our developments (18) and (19) with the same variables issued from the theories VSOP82 and JASON84 for 366 dates over the interval 1000–3000. The precision is better than 1.5 times the precision over the interval 1800–2050. Note that the formulae of the Explanatory Supplement cannot give significant approximate ephemerides over a so large interval.

Table 7. Maximum errors over the interval 1800–2050 of our approximate ephemerides computed from formulae (18) and Table 6, and of the Keplerian elements of the Explanatory Supplement (1992). Errors are given in heliocentric longitude (L), latitude (B), and distance (R)

	Approximate ephemerides			Explanatory Supplement		
	L (″)	B (″)	R (1000 km)	L (″)	B (″)	R (1000 km)
Mercury	4	1	0.3	25	4	2
Venus	5	1	0.8	25	2	5
E–M barycenter	6	1	1.0	24	5	7
Mars	17	1	7.7	129	5	57
Jupiter	71	5	76	528	23	664
Saturn	81	13	267	790	78	2498
Uranus	86	7	712	132	37	3605
Neptune	11	1	253	62	8	1705

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