Application of Dispersion Relations to Low-Energy Meson-Nucleon Scattering*

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Relativistic dispersion relations are used to derive equations for low-energy S-, P-, and D-wave meson-nucleon scattering under the assumption that the (3,3) resonance dominates the dispersion integrals. The P-wave equations so obtained differ only slightly from those of the static fixed-source theory. The conclusions of the static theory are re-examined in the light of their new derivation.

1. INTRODUCTION

DISPERSION relations for the scattering of π mesons by nucleons have been given by many authors.[1] Proofs[2] based on the field theory formalism have been given for the special case of forward scattering by Symanzik, Lehmann and Jost, and Bogoliubov, for angles infinitesimally near forward by Symanzik, and for arbitrary angles by Bogoliubov.

In the neighborhood of the forward direction, the dispersion relations express real parts of scattering amplitudes as integrals over sums of partial-wave cross sections. Since these quantities are at least in principle measurable, one has at hand a multiple infinity of sum rules which may be compared directly with experiment. This is the procedure which has been followed by Anderson, Davidon and Kruse,[3] by Haber-Schaim,[4] and by Davidon and Goldberger.[5]

Another use to which the dispersion relations may be put has been pointed out by Oehme[6] who showed that in the static limit, provided higher waves than $l=1$ are neglected, the equations of the static P-wave theory are obtained together with a similar set of static S-wave equations.

We will here consider the problem in an intermediate way, one which is motivated at the same time by the success of the static P-wave theory in correlating meson scattering and photoproduction experiments and by the observed dominance of the (3,3) resonance in dispersion integrals. In effect we shall assume that the (3,3) resonance not only dominates dispersion integrals but exhausts them. Once this assumption is made, the equations of the static theory follow naturally, since in the energy range of the resonance the nucleon recoil velocity is small, $v/c\lesssim\frac{1}{3}$. Including effects of order v/c gives us the contributions to the dispersion integrals of the resonance region accurate to about 10%. Our most important conclusion will be that these assumptions lead to effective range relations for the P-wave phase shifts; they do not make possible a determination of the actual location of the (3,3) resonance, which must be taken from experiment. Once the (3,3) phase shift is known the S-, D-, and small P-wave phase shifts may be directly calculated in our approximation. The validity of the approximation is hard to estimate without knowing the partial wave decomposition of cross sections in the Bev region; order of magnitude estimates based on known total cross sections indicate that the resonant state is correctly given to about 10%, but that the contribution of high-energy cross sections to a typical small amplitude, although less than 10% of the (3,3) amplitude, is still comparable to the small amplitude itself. More detailed studies of this question are now being made.

In Sec. II we introduce the necessary variables and describe the transformation from relativistically invariant scattering amplitudes to a partial wave decomposition in the center-of-mass system. This material is not new, nor is it related to meson theory; we include it only for convenient reference. In Sec. III we give dispersion relations and use them to derive equations for S-, P-, and D-wave scattering. The algebra in this section is quite complicated. We advise the interested reader (as opposed to the dedicated one) to read up to and including Eq. (3.18), by which time the method of calculation should be clear. The essential results of the remaining algebra are contained in Eq. (3.20), (3.30), (3.33), (3.34), and (3.35). In Sec. IV we discuss the conditions imposed on the P-wave scattering amplitude by the P-wave equations, and re-examine the solutions of the static theory in the light of their new derivation.

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[1] For a complete list of references see R. H. Capps and G. Takeda, Phys. Rev. **103**, 1877 (1956).
[2] Reported at the International Conference on Theoretical Physics, Seattle, Washington, September, 1956.
[3] Anderson, Davidon, and Kruse, Phys. Rev. **100**, 339 (1955).
[4] U. Haber-Schaim, Phys. Rev. **104**, 1113 (1956).
[5] W. Davidon and M. L. Goldberger, Phys. Rev. **104**, 1119 (1956).
[6] R. Oehme, Phys. Rev. **100**, 1503 (1955); **102**, 1174 (1956).

2. KINEMATICAL CONSIDERATIONS

Let the four-vector momenta of the incident and outgoing pion be q_1 and q_2, respectively, while those of the initial and final nucleon are p_1 and p_2. Momentum-energy conservation,

$$p_1+q_1=p_2+q_2, \tag{2.1}$$

means that only three of the four vectors are independent.[7] We choose to consider the combinations

$$P=\tfrac{1}{2}(p_1+p_2),\quad Q=\tfrac{1}{2}(q_1+q_2),\quad \kappa=\tfrac{1}{2}(q_1-q_2), \tag{2.2}$$

as the three independent four-vectors.

The mass shell restrictions, $p_1^2=p_2^2=-M^2$, $q_1^2=q_2^2=-1$, mean that

$$\begin{gathered} P\cdot\kappa=Q\cdot\kappa=0,\\ P^2+\kappa^2=-M^2,\quad Q^2+\kappa^2=-1, \end{gathered} \tag{2.3}$$

so that there are only two independent scalars, which we may take as

$$\nu=-P\cdot Q/M \quad\text{and}\quad \kappa^2. \tag{2.4}$$

The second of these variables is one-quarter of the invariant momentum-transfer squared:

$$\kappa^2=\tfrac{1}{4}(q_1-q_2)^2=\tfrac{1}{2}\mathbf{q}^2(1-\cos\theta), \tag{2.5}$$

where $\mathbf{q}$ is the three-vector momentum and θ the scattering angle in the center-of-mass system. Also

$$\nu=\nu_L-(\kappa^2/M), \tag{2.6}$$

where ν_L is the incident meson energy in the lab system. Thus ν is almost equal to the lab energy for moderate momentum transfer.

To form further invariants, we must use the Dirac operators. By virtue of the Dirac equation,

$$\begin{gathered} (i\gamma\cdot p_1+\mathrm{M})u_1=0,\\ (i\gamma\cdot p_2+\mathrm{M})u_2=0. \end{gathered} \tag{2.7}$$

Thus the invariants $i\gamma\cdot p_1$ and $i\gamma\cdot p_2$ may be anticommuted through the matrix element until they act on the initial or final spinor, respectively, where by (2.7) they give a constant. The same is true for $i\gamma\cdot\kappa=i\gamma\cdot(p_2-p_1)/2$, so that only $i\gamma\cdot Q$ remains as an independent scalar.

The S matrix can be written

$$S=\delta_{fi}-(2\pi)^4 i\delta^4(p_2+q_2-p_1-q_1)\times\left(\frac{M^2}{4E_1E_2\omega_1\omega_2}\right)^{\frac{1}{2}}\bar{u}_2Tu_1, \tag{2.8}$$

where E_1 and E_2 are initial and final nucleon energies and ω_1 and ω_2 the initial and final meson energies. The spinor normalization is

$$\bar{u}_2u_2=\bar{u}_1u_1=1. \tag{2.9}$$

Our considerations have shown that T may be written

$$T=-A+i\gamma\cdot QB, \tag{2.10}$$

where A and B are functions of ν and κ^2, and are also matrices in charge space. Charge independence limits this last complication to a doubling of the number of functions. Let β be the state of the final meson ($\beta=1, 2, 3$) and α that of the initial. Then

$$A_{\beta\alpha}=\delta_{\beta\alpha}A^{(+)}+\tfrac{1}{2}[\tau_\beta,\tau_\alpha]A^{(-)}, \tag{2.11}$$

and

$$B_{\beta\alpha}=\delta_{\beta\alpha}B^{(+)}+\tfrac{1}{2}[\tau_\beta,\tau_\alpha]B^{(-)}, \tag{2.12}$$

where $A^{(\pm)}$ and $B^{(\pm)}$ are simply functions of ν and κ^2.

It is frequently useful to express the $(\pm)$ amplitudes in terms of the total isotopic spin. One finds easily

$$A^{(+)}=\tfrac{1}{3}(A^{(\frac{1}{2})}+2A^{(\frac{3}{2})}),\qquad A^{(-)}=\tfrac{1}{3}(A^{(\frac{1}{2})}-A^{(\frac{3}{2})}),\ \text{etc.} \tag{2.13}$$

Finally we state without proof the relation between the A's and B's and the conventional scattering amplitudes in states of definite parity and angular momentum. For this relation it is convenient to introduce the center-of-mass variables

$$\begin{aligned} W&=\text{total energy},\\ E&=\text{total nucleon energy},\\ x&=\cos\theta. \end{aligned} \tag{2.14}$$

In terms of these variables,

$$\frac{1}{4\pi}A^{(\pm)}=\frac{W+M}{E+M}f_1^{(\pm)}-\frac{W-M}{E-M}f_2^{(\pm)}, \tag{2.15}$$

$$\frac{1}{4\pi}B_4^{(\pm)\frac{3}{2}}=\frac{1}{E+M}f_1^{(\pm)}+\frac{1}{E-M}f_2^{(\pm)}. \tag{2.16}$$

Here f_1 and f_2 are simply related to the scattering cross sections in the center-of-mass system:

$$\frac{d\sigma}{d\Omega}=\sum\left|\left\langle f\left|f_1+\frac{\boldsymbol{\sigma}\cdot\mathbf{q}_2\boldsymbol{\sigma}\cdot\mathbf{q}_1}{q_2q_1}f_2\right|i\right\rangle\right|^2 \tag{2.17}$$

where the symbol $\sum$ represents a sum over final and average over initial spin states. In (2.17) we have suppressed the superscripts referring to charge states.

Finally

$$f_1=\sum_{l=0}^{\infty}f_{l+}P_{l+1}'(x)-\sum_{l=2}^{\infty}f_{l-}P_{l-1}'(x), \tag{2.18}$$

$$f_2=\sum_{l=1}^{\infty}(f_{l-}-f_{l+})P_l'(x), \tag{2.19}$$

where $f_{l\pm}$ is the scattering amplitude in the state of

[7] In this paper four-vectors are represented by italicized symbols, thus: p. Three-dimensional vectors are represented by bold-face symbols, thus: $\mathbf{p}$. The four-dimensional inner product is $p\cdot q=\Sigma p_\lambda q_\lambda=\mathbf{p}\cdot\mathbf{q}-p_0q_0$. We also set $\hbar=c=\mu=1$, where μ is the pi-meson mass. The nucleon mass is M.

parity $-(-1)^l$ and total angular momentum $j=l\pm\frac{1}{2}$. $P_l'(x)$ is the first derivative of the conventionally normalized Legendre polynomial.

The $f_\pm$ are normalized so that

$$(j+\tfrac{1}{2})\,\mathrm{Im}f_{l\pm}=\frac{q}{4\pi}\sigma_{l\pm}, \tag{2.20}$$

where $\sigma_{l\pm}$ is the total cross section of the partial wave involved. Thus, for energies below the two-meson threshold,

$$f_{l\pm}=e^{i\delta_{l\pm}}\frac{\sin\delta_{l\pm}}{q}, \tag{2.21}$$

where $\delta_{l\pm}$ is the real phase shift in the state $l_\pm$; above this threshold a representation of the form (2.21) still holds, but with complex $\delta_{l\pm}$.

3. DISPERSION RELATIONS

The form that the dispersion relations take depends on the behavior of $A(\nu,\kappa^2)$ and $B(\nu,\kappa^2)$ for very large values of ν. In what follows we shall make a drastic (and probably incorrect) assumption about this high-frequency behavior: we shall assume that A and B approach zero sufficiently rapidly so that all the integrals we introduce converge uniformly (as functions of κ^2) in the neighborhood of $\kappa^2=0$. With this assumption, we have

$$\mathrm{Re}A^{(\pm)}(\nu,\kappa^2)=\frac{P}{\pi}\int_{1-\kappa^2/M}^{\infty}d\nu'\,\mathrm{Im}A^{(\pm)}(\nu',\kappa^2)\left(\frac{1}{\nu'-\nu}\pm\frac{1}{\nu'+\nu}\right), \tag{3.1}$$

and

$$\mathrm{Re}B^{(\pm)}(\nu,\kappa^2)=\frac{g_r^2}{2M}\left(\frac{1}{\nu_B-\nu}\mp\frac{1}{\nu_B+\nu}\right)+\frac{P}{\pi}\int_{1-\kappa^2/M}^{\infty}d\nu'\,\mathrm{Im}B^{(\pm)}(\nu',\kappa^2)\left(\frac{1}{\nu'-\nu}\mp\frac{1}{\nu'+\nu}\right). \tag{3.2}$$

Here $\nu_B=-(1/2M)-(\kappa^2/M)$ and g_r^2 is the rationalized, renormalized (according to the Lepore-Watson renormalization convention[8]) pseudoscalar coupling constant. Experimentally, $g_r^2/4\pi\approx 14$. As usual the symbol P stands for principal value.

In this paper we wish to exhibit the consequences of augmenting Eq. (3.1) and Eq. (3.2) with the assumption of the dominance of the (3,3) state. Put differently, we shall investigate the effects of (3,3) contributions to the integrals in Eq. (3.1) and (3.2). Now if the integrals in question do not converge sufficiently rapidly to make (3.1) and (3.2) meaningful equations, it is still possible to obtain valid equations by subtraction at some fixed value of ν. The new equations so obtained will contain arbitrary functions of κ^2 which do not arise from resonant integrals, and which at present cannot be predicted by dispersion theory. We shall therefore take the point of view that the existence of these arbitrary functions is one among many high-energy effects which we shall not attempt to evaluate; we shall therefore use Eq. (3.1) and Eq. (3.2). Finally, in order to simplify the results, we shall make the valid approximation that the nucleon velocity is small in the resonance region; we shall, however, carry our results to include first order terms in this velocity (as opposed to the static limit, which keeps only zero-order terms). It should be understood, then, in the following that although we shall continue to write integrals with an infinite upper limit we really have in mind as an upper limit the energy at which the (3,3) state fails to dominate, say three or four hundred Mev lab energy.

We first change the variables in (3.1) and (3.2) to ν_L and κ^2 by means of (2.6). We find

$$\mathrm{Re}A^{(\pm)}(\nu_L,\kappa^2)=\frac{P}{\pi}\int_1^{\infty}d\nu_L'\,\mathrm{Im}A^{(\pm)}(\nu_L',\kappa^2)\times\left(\frac{1}{\nu_L'-\nu_L}\pm\frac{1}{\nu_L'+\nu_L-2\kappa^2/M}\right), \tag{3.3}$$

and

$$\mathrm{Re}B^{(\pm)}(\nu_L,\kappa^2)=\frac{g_r^2}{2M}\left(\frac{1}{\nu_0-\nu_L}\mp\frac{1}{\nu_0+\nu_L-2\kappa^2/M}\right)+\frac{P}{\pi}\int_1^{\infty}d\nu_L'\,\mathrm{Im}B^{(\pm)}(\nu_L',\kappa^2)\times\left(\frac{1}{\nu_L'-\nu_L}\mp\frac{1}{\nu_L'+\nu_L-2\kappa^2/M}\right), \tag{3.4}$$

where $\nu_0=-1/2M$.

Next we solve (2.15) and (2.16) for f_1 and f_2. We find easily that

$$f_1=\left(\frac{E+M}{2W}\right)\left(\frac{A+(W-M)B}{4\pi}\right), \tag{3.5}$$

and

$$f_2=\left(\frac{E-M}{2W}\right)\left(\frac{-A+(W+M)B}{4\pi}\right). \tag{3.6}$$

As long as we intend to carry our calculation only to a finite order in v/c (or equivalently $1/M$), we save ourselves a great deal of trouble by expanding in powers of κ^2. It will become obvious that S-wave amplitudes may be expected to be of order $g_r^2/M\approx Mf_r^2$ (although in fact they are greatly reduced from this value for reasons that are still obscure), P waves of order f_r^2 (although the enhanced state here is of order unity, of course) and D waves of order f_r^2/M, as long as high-energy contributions to (3.3) and (3.4) are

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[8] K. M. Watson and J. V. Lepore, Phys. Rev. 76, 1157 (1949).

negligible. In any case, these orders of magnitude should be kept in mind as a basis for comparison of different terms. Here f_r^2 is, of course, the rationalized pseudovector coupling constant, $f_r^2=g_r^2/4M^2$.

Let us write (2.18) and (2.19) in terms of κ^2 rather than the c.m. angle θ. Including D waves (but no higher), we have

$$f_1=f_S-f_{D\frac{3}{2}}+3f_{P\frac{3}{2}}\left(1-\frac{2\kappa^2}{q^2}\right)+\tfrac{1}{2}f_{D\frac{5}{2}}\left[15\left(1-\frac{2\kappa^2}{q^2}\right)^2-3\right]+\cdots, \tag{3.7}$$

and

$$f_2=f_{P\frac{1}{2}}-f_{P\frac{3}{2}}+3\left(1-\frac{2\kappa^2}{q^2}\right)(f_{D\frac{3}{2}}-f_{D\frac{5}{2}})+\cdots, \tag{3.8}$$

where q^2 is the c.m. meson momentum.

If we set $\kappa^2=0$ in (3.7), we obtain

$$f_1(0)\cong f_S+3f_{P\frac{3}{2}}, \tag{3.9}$$

since $f_D\ll f_S$ in the region of interest. Differentiating (3.7) with respect to κ^2 and then setting $\kappa^2=0$, we find

$$f_1'(0)\cong-6f_{P\frac{3}{2}}/q^2, \tag{3.10}$$

where again D waves have been neglected. Combining (3.9) and (3.10), we find

$$f_S=f_1(0)+\tfrac{1}{2}q^2f_1'(0)+\sim D\text{ waves}. \tag{3.11}$$

The prime here stands for differentiation with respect to κ^2. The argument zero also stands for κ^2. The amplitudes $f(0)$, $f'(0)$, etc., are of course still functions of energy. Similarly,

$$-\frac{6}{q^2}f_{P\frac{3}{2}}=f_1'(0)+\tfrac{1}{2}q^2f_1''(0)+\sim F\text{ waves}, \tag{3.12}$$

$$f_{P\frac{1}{2}}-f_{P\frac{3}{2}}=f_2(0)+\tfrac{1}{2}q^2f_2'(0)+\sim F\text{ waves}, \tag{3.13}$$

$$\frac{60}{q^4}f_{D\frac{5}{2}}=f_1''(0)+\sim F\text{ waves}, \tag{3.14}$$

$$-\frac{6}{q^2}(f_{D\frac{3}{2}}-f_{D\frac{5}{2}})=f_2'(0)+\sim F\text{ waves}. \tag{3.15}$$

Finally, on the right-hand side of Eq. (3,3) we shall, at least for the moment, keep only the (3.3) state—that is, we write

$$\frac{1}{4\pi}\operatorname{Im}A^{(\pm)}(\nu_L',\kappa^2)=\left[\frac{3(W'+M)(1-2\kappa^2/q^2)}{E'+M}+\frac{W'-M}{E'-M}\right]\operatorname{Im}f_3^{(\pm)}, \tag{3.16}$$

$$\frac{1}{4\pi}\operatorname{Im}B^{(\pm)}(\nu_L',\kappa^2)=\left[\frac{3(1-2\kappa^2/q^2)}{E'+M}-\frac{1}{E'-M}\right]\operatorname{Im}f_3^{(\pm)}, \tag{3.17}$$

where

$$f_3^{(+)}=\tfrac{2}{3}f_{33},\quad f_3^{(-)}=-\tfrac{1}{3}f_{33}. \tag{3.18}$$

We first calculate the resonance contribution to the S-wave amplitudes:

$$\begin{aligned}\operatorname{Re}f_S^{(\pm)}&\cong f_1^{(\pm)}(0)+\tfrac{1}{2}q^2f_1^{(\pm)\prime}(0)=\frac{1}{4\pi}\cdot\frac{E+M}{2W}\left[A^{(\pm)}(0)+\tfrac{1}{2}q^2A^{(\pm)\prime}(0)+(W-M)(B^{(\pm)}(0)+\tfrac{1}{2}q^2B^{(\pm)\prime}(0))\right]\\
&=\frac{E+M}{2W}\Bigg\{\frac{(W-M)g^2}{2M}\left(\frac{1}{\nu_0-\nu_L}\mp\frac{1}{\nu_0+\nu_L}\mp\frac{q^2/M}{(\nu_0+\nu_L)^2}\right)+\frac{P}{\pi}\int_1^\infty d\nu_L'\operatorname{Im}f_3^{(\pm)}(\nu_L')\Bigg[\frac{1}{\nu_L'-\nu_L}\Bigg(\frac{3(W'+M)(1-q^2/q'^2)}{E'+M}\\
&\quad+\frac{W'-M}{E'-M}+(W-M)\left(\frac{3(1-q^2/q'^2)}{E'+M}-\frac{1}{E'-M}\right)\Bigg)\pm\frac{1}{\nu_L'+\nu_L}\Bigg(\frac{3(W'+M)(1-q^2/q'^2)}{E'+M}+\frac{W'-M}{E'-M}\\
&\quad-(W-M)\left(\frac{3(1-q^2/q'^2)}{E'+M}-\frac{1}{E'-M}\right)\Bigg)\pm\frac{q^2/M}{(\nu_L'+\nu_L)^2}\Bigg(\frac{3(W'+M)}{E'+M}+\frac{W'-M}{E'-M}\\
&\quad-(W-M)\left(\frac{3}{E'+M}-\frac{1}{E'-M}\right)\Bigg)\Bigg]\Bigg\},\end{aligned} \tag{3.19}$$

where g^2 is the nonrationalized coupling constant,

$$g^2=g_r^2/4\pi.$$

We next express everything in terms of center-of-mass energies, using the relation $\nu_L=(W^2-M^2-1)/2M$. There results, to the desired order in $1/M$,

$$f_S^{(\pm)}\cong\frac{M}{W}\left\{-\frac{g^2}{2M}\left[\left(1-\frac{\omega}{2M}\right)\pm\left(1+\frac{\omega}{2M}\right)\right]+\frac{2M}{\pi}\int_1^\infty\frac{d\omega'}{q'^2}\left[\left(1+\frac{2\omega'}{M}+\frac{\omega}{M}\right)\pm\left(1+\frac{2\omega'}{M}-\frac{\omega}{M}\right)\right]\operatorname{Im}f_3^{(\pm)}(\omega')\right\}, \tag{3.20}$$

where $\omega=W-M$, $\omega'=W'-M$, and terms of order $1/M$ relative to those kept are left out.

We see that the strong energy dependence of f_3 produces no reflection on the S-wave energy dependence, which to a high degree of approximation is given (except for the trivial phase space factor M/W) by a constant for $f^{(+)}$ and a constant times ω for $f^{(-)}$.

It is a simple matter to add the low-energy contribution of the S-wave amplitude under the integrals. Here we may be very careless in taking nonrelativistic limits, since the term $f_2/(E-M)$ in Eq. (2.15) and (2.16) receives no contribution from S waves. Thus, to lowest order in $1/M$,

$$f_S\cong A(\kappa^2=0)/4\pi, \tag{3.21}$$

so that Eq. (3.20) becomes

$$\mathrm{Re}f_S{}^{(\pm)}\cong-\left[\left(\lambda^{(+)}-\frac{\omega}{2M}\lambda^{(-)}\right)\pm\left(\lambda^{(+)}+\frac{\omega}{2M}\lambda^{(-)}\right)\right]+\frac{P}{\pi}\int d\omega'\,\mathrm{Im}f_S{}^{(\pm)}(\omega')\left[\frac{1}{\omega'-\omega}\pm\frac{1}{\omega'+\omega}\right], \tag{3.22}$$

where explicit expressions for $\lambda^{(\pm)}$ are provided by Eq. (3.20):

$$\lambda^{(+)}=\frac{g^2}{2M}-\frac{2M}{\pi}\int\frac{d\omega'}{q'^2}\left(1+\frac{2\omega'}{M}\right)\mathrm{Im}f_3{}^{(+)}(\omega'), \tag{3.23}$$

$$\lambda^{(-)}=\frac{g^2}{2M}+\frac{4M}{\pi}\int\frac{d\omega'}{q'^2}\mathrm{Im}f_3{}^{(-)}(\omega'). \tag{3.24}$$

These are the Oehme[6] equations. Now the contribution of the S-wave integrals in (3.22) at threshold is small, so that $\lambda^{(\pm)}$ may be determined by experiment.

According to Orear,[9]

$$f^{(\frac{3}{2})}(q=0)=-0.11,\qquad f^{(\frac{1}{2})}(q=0)=+0.16. \tag{3.25}$$

thus

$$f^{(+)}(0)=-0.02,\quad f^{(-)}(0)=+0.09,$$

and

$$\lambda^{(+)}=0.01,\quad \lambda^{(-)}=0.6. \tag{3.26}$$

These numbers for $\lambda^{(\pm)}$ are in fact not inconsistent with (3.23) and (3.24) integrated over the (3.3) resonance. We prefer not to take such an agreement seriously since the approximate constancy of high-energy cross sections (as observed in cosmic rays) argues strongly against the validity of Eq. (3.1) as it stands, so that at least one subtraction must presumably be made in Eq. (3.1).

Of course, if one is primarily interested in calculating threshold scattering the partial wave reduction we have performed is unnecessary since only S waves contribute at zero kinetic energy. The constants $\lambda^{(\pm)}$ should therefore be calculated directly from (3.1) and (3.2). The result of such a calculation for $\lambda^{(+)}$ is inconclusive, whereas for $\lambda^{(-)}$, as carried out by Haber-Schaim,[4] it yields a number in surprisingly close agreement with (3.25). This presumably means that in practice no subtraction need be carried out to make $(3.1)^{(-)}$ a correct equation.

We turn next to the derivation of P-wave equations. We calculate first

$$f_{P\frac{1}{2}}{}^{(\pm)}-f_{P\frac{3}{2}}{}^{(\pm)}\cong f_2{}^{(\pm)}(0)+\tfrac{1}{2}q^2f_2{}^{(\pm)\prime}(0)=\frac{1}{4\pi}\left(\frac{E-M}{2W}\right)\{-A^{(\pm)}(0)-\tfrac{1}{2}q^2A^{(\pm)\prime}(0)+(W+M)[B^{(\pm)}(0)+\tfrac{1}{2}q^2B^{(\pm)\prime}(0)]\}. \tag{3.27}$$

The Born approximation to (3.27) is easily found, since (3.19) and (3.20) inform us that

$$\frac{1}{4\pi}\left(\frac{E+M}{2W}\right)(W-M)[B^{(\pm)}(0)+\tfrac{1}{2}q^2B^{(\pm)\prime}(0)]|_{\mathrm{Born}}\cong\frac{M}{W}\left(-\frac{g^2}{2M}\right)\left[\left(1-\frac{\omega}{2M}\right)\pm\left(1+\frac{\omega}{2M}\right)\right].$$

Thus

$$\left(\frac{E-M}{2W}\right)(W+M)[B^{(\pm)}(0)+\tfrac{1}{2}q^2B^{(\pm)\prime}(0)]|_{\mathrm{Born}}\cong-\frac{q^2}{\omega}f^2\left(1-\frac{\omega}{2M}\right)\left[\left(1-\frac{\omega}{2M}\right)\pm\left(1+\frac{\omega}{2M}\right)\right], \tag{3.28}$$

where $f^2=g^2/4M^2\cong0.08$.

The contribution of the (3.3) integrals is, however, different from the S-wave case:

$$\begin{aligned}\mathrm{Re}(f_{P\frac{1}{2}}{}^{(\pm)}-f_{P\frac{3}{2}}{}^{(\pm)})\cong&-\frac{q^2f^2}{\omega}\left(1-\frac{\omega}{2M}\right)\left[\left(1-\frac{\omega}{2M}\right)\pm\left(1+\frac{\omega}{2M}\right)\right]+\frac{q^2}{4M}\cdot\frac{P}{\pi}\int_1^\infty\frac{d\nu_L'}{W}\\&\times\mathrm{Im}f_3{}^{(\pm)}(\nu_L')\left\{\frac{1}{\nu_L'-\nu_L}\left[-\frac{3(W'+M)(1-q^2/q'^2)}{M+E'}-\frac{W'-M}{E'-M}+(W+M)\left(\frac{3(1-q^2/q'^2)}{M+E'}-\frac{1}{E'-M}\right)\right]\right.\\&\pm\frac{1}{\nu_L'+\nu_L}\left[-\frac{3(W'+M)}{E'+M}(1-q^2/q'^2)-\frac{W'-M}{E'-M}-(W+M)\left(\frac{3(1-q^2/q'^2)}{E'+M}-\frac{1}{E'-M}\right)\right]\\&\left.\pm\frac{q^2/M}{(\nu_L'+\nu_L)^2}\left[-\frac{3(W'+M)}{E'+M}-\frac{W'-M}{E'-M}-(W+M)\left(\frac{3}{E'+M}-\frac{1}{E'-M}\right)\right]\right\}.\end{aligned} \tag{3.29}$$

[9] J. Orear, Phys. Rev. 100, 288 (1955).

Again we express ν_L and ν_L' as functions of ω and ω' and expand in powers of ω/M and ω'/M. The result is

$$\mathrm{Re}(f_{P\frac{1}{2}}{}^{(\pm)}-f_{P\frac{3}{2}}{}^{(\pm)})=-\frac{q^2f^2}{\omega}\left[\left(1-\frac{\omega}{2M}\right)\pm\left(1+\frac{\omega}{2M}\right)\right]\left(1-\frac{\omega}{2M}\right)$$
$$+q^2\frac{P}{\pi}\int_1^\infty\frac{d\omega'}{q'^2}(-\mathrm{Im}f_3{}^{(\pm)}(\omega'))\left(\frac{1}{\omega'-\omega}+\frac{1}{M}\mp\frac{1}{\omega'+\omega}\right).\quad(3.30)$$

Notice that except for the added constant $1/M$, all $1/M$ corrections under the integral have gone into the energy variable $\omega=W-M$. That is, the form of (3.30), except for the added constant, is the same as in the static theory[10]; the difference is in the meaning of the variable ω, which is now obviously the appropriate one for describing low-energy P-wave scattering.

Next we calculate $f_{P\frac{3}{2}}$:

$$-\frac{6}{q^2}\,\mathrm{Re}f_{P\frac{3}{2}}\cong f_1{}^{(\pm)\prime}(0)+\tfrac{1}{2}q^2f_1{}^{(\pm)\prime\prime}(0)$$
$$=\left(\frac{E+M}{2W}\right)[A^{(\pm)\prime}(0)+\tfrac{1}{2}q^2A^{(\pm)\prime\prime}(0)+(W-M)(B^{(\pm)\prime}(0)+\tfrac{1}{2}q^2B^{(\pm)\prime\prime}(0))],\quad(3.31)$$

or

$$-\frac{6}{q^2}\,\mathrm{Re}f_{P\frac{3}{2}}{}^{(\pm)}=\left(\frac{E+M}{2W}\right)\left\{\mp\frac{g^2}{2M}(W-M)\left(\frac{2/M}{(\nu_0+\nu_L)^2}+\frac{4q^2/M^2}{(\nu_0+\nu_L)^3}\right)+\frac{P}{\pi}\int_1^\infty d\nu_L'\,\mathrm{Im}f_3{}^{(\pm)}(\nu_L')\right.$$
$$\times\left[\frac{1}{\nu_L'-\nu_L}\left(-\frac{6(W'+M)}{E'+M}\cdot\frac{1}{q'^2}-\frac{6(W-M)}{E'+M}\cdot\frac{1}{q'^2}\right)\pm\frac{1}{\nu_L'+\nu_L}\left(-\frac{6(W'+M)}{E'+M}\cdot\frac{1}{q'^2}+\frac{6(W-M)}{E'+M}\cdot\frac{1}{q'^2}\right)\right.$$
$$\pm\frac{2/M}{(\nu_L'+\nu_L)^2}\left(\frac{3(W'+M)}{E'+M}+\frac{W'-M}{E'-M}-(W-M)\left(\frac{3}{E'+M}-\frac{1}{E'-M}\right)\right)\pm\frac{\frac{1}{2}q^2}{(\nu_L'+\nu_L)^2}\left(-\frac{24}{Mq'^2}\right)$$
$$\left.\left.\times\left(\frac{W'+M}{E'+M}-\frac{W-M}{E'+M}\right)\pm\frac{4q^2/M^2}{(\nu_L'+\nu_L)^3}\left(\frac{3(W'+M)}{E'+M}+\frac{W'-M}{E'-M}-(W-M)\left(\frac{3}{E'+M}-\frac{1}{E'-M}\right)\right)\right]\right\}.\quad(3.32)$$

An entirely analogous reduction of this equation gives

$$-6\,\mathrm{Re}\frac{f_{P\frac{3}{2}}{}^{(\pm)}}{q^2}=\mp\frac{4f^2}{\omega}+\frac{P}{\pi}\int_1^\infty\frac{d\omega'}{q'^2}$$
$$\times\mathrm{Im}f_{P\frac{3}{2}}{}^{(\pm)}(\omega')\left[-\frac{6}{\omega'-\omega}-\frac{6}{M}\mp\frac{2}{\omega'+\omega}\right].\quad(3.33)$$

Before going on to a discussion of these results, let us find expressions for the D waves:

$$\frac{60}{q^4}f_{D\frac{5}{2}}{}^{(\pm)}\cong f_1{}^{(\pm)\prime}(0),$$

or

$$f_{D\frac{5}{2}}{}^{(\pm)}\cong-\frac{q^4}{15M}\left[\frac{4f^2}{\omega^2}+\frac{2}{\pi}\int\frac{d\omega'}{q'^2}\frac{\mathrm{Im}f_3{}^{(\pm)}(\omega')}{(\omega'+\omega)^2}\right],\quad(3.34)$$

and

$$-\frac{6}{q^2}(f_{D\frac{3}{2}}{}^{(\pm)}-f_{D\frac{5}{2}}{}^{(\pm)})\cong f_2{}^{(\pm)\prime}(0),$$

or

$$-(f_{D\frac{3}{2}}{}^{(\pm)}-f_{D\frac{5}{2}}{}^{(\pm)})$$
$$=\pm\frac{1}{3}\frac{q^4}{m}\left[\frac{f^2}{\omega^2}-\frac{1}{\pi}\int\frac{d\omega'}{q'^2}\frac{\mathrm{Im}f_3{}^{(\pm)}(\omega')}{(\omega'+\omega)^2}\right].\quad(3.35)$$

4. DISCUSSION OF RESULTS

The P-wave equations may be rewritten (using (2.13))

$$\mathrm{Re}f_{11}=-\frac{8}{3}\frac{f^2q^2}{\omega}+\frac{3}{M}f^2q^2+\frac{16}{9}\frac{q^2}{\pi}\int\frac{d\omega'}{q'^2}\frac{\mathrm{Im}f_{33}(\omega')}{\omega'+\omega},$$

$$\mathrm{Re}f_{13}=\mathrm{Re}f_{31}=\tfrac{1}{4}\,\mathrm{Re}f_{11}-\frac{3}{4M}f^2q^2,$$

$$\mathrm{Re}f_{33}=\frac{4}{3}\frac{f^2q^2}{\omega}+\frac{q^2}{\pi}P\int\frac{d\omega'}{q'^2}$$
$$\times\mathrm{Im}f_{33}(\omega')\left[\frac{1}{\omega'-\omega}+\frac{1}{M}+\frac{1}{9}\left(\frac{1}{\omega'+\omega}\right)\right].\quad(4.1)$$

[10] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

These equations are almost the same as those of the static theory[10] (with the same assumption, of course, that the resonance region dominates so that the contributions of the "small phase shift" states may be neglected). To obtain the static equations from (4.1), one simply replaces q^2 by $v^2(q)q^2$ where $v^2(q)$ is the cutoff function, ω by $(1+q^2)^{\frac{1}{2}}$ [the reader will recall that here ω is the total center-of-mass energy: $\omega=q^2/2M+(1+q^2)^{\frac{1}{2}}$], and finally one drops the explicit $1/M$ terms in (4.1). Since all of these changes are small ones in the resonance region, we have in (4.1) a "derivation" of the static theory and a method of partially understanding its agreement with experiment.

Let us repeat this last point, since it is an important one. The reason the static theory agrees with experiment is that in the integrals on the right-hand side of the dispersion relations the resonance region dominates, so that Eq. (4.1) holds; Eq. (4.1) is in turn a consequence of static meson theory, provided that there also one assumes the dominance of the resonance integral for low-energy phenomena. What we have achieved, therefore, is not really a complete derivation of the static meson theory, but a set of instructions on how that theory must be used and which of its results are believable.

Let us replace Eq. (4.1) by the static equations [see reference 10, Eq. (40)]:

$$\mathrm{Re}f_\alpha=\lambda_\alpha\frac{q^2v^2(q)}{\omega}+\frac{q^2v^2(q)}{\pi}P\int\frac{d\omega'}{q'^2v^2(q')}\times\left[\frac{\mathrm{Im}f_\alpha(\omega')}{\omega'-\omega}+\sum_\beta\frac{A_{\alpha\beta}\,\mathrm{Im}f_\beta(\omega')}{\omega'+\omega}\right],\quad(4.2)$$

and look for solutions subject to the one-meson approximation and to the condition that the (3.3) resonance be properly located. These may or may not be such that the (3.3) resonance integral dominates for low values of ω, depending on $v^2(q)$ and, in the event of the existence of several solutions,[11] on which one is chosen. All those solutions which have substantial non(3-3) resonance contributions to the right-hand side of (4.2) for small ω we throw out. This will include all of those solutions having zeros in the low-energy region since these will necessarily also have extra resonances which will contribute to the integrals.

As shown in reference 10, the solutions of (4.2) for ω not too large are roughly of the form

$$f_\alpha\cong\frac{\lambda_\alpha q^2/\omega}{1-r_\alpha\omega-i\lambda_\alpha q^3/\omega},\quad(4.3)$$

independent of the shape of the cutoff function (provided it is singular enough to produce the observed resonance), and provided there are no zeros of f_α. As shown in reference (11), however, any zeros whose corresponding resonances do not contribute to the right-hand side of (4.2) must be on the real axis, but quite far from $\omega=0$, so that they result only in new effective ranges for the three states.

If one actually tries to solve Eq. (4.2) for the location of the resonance, one finds

$$1/r_{33}=\omega_r\approx1/f^2\omega_{\max},\quad(4.4)$$

where $\omega_{\max}$ is the cutoff energy. The location of the resonance at a moderate energy with a small coupling constant $f^2=0.08$, therefore, necessitates a high cutoff, $\omega_{\max}\gg1$. This circumstance in turn insures the approximate constancy of the effective ranges r_α.

This set of statements may be approximately deduced directly from Eq. (4.1). In particular, if we consider $1/9\ll1$ and $1/M\gg1$, then the equation for f_{33} becomes

$$f_{33}=\frac{\lambda q^2}{\omega}+\frac{q^2}{\pi}\int\frac{d\omega'}{q'^2}\frac{\mathrm{Im}f_{33}(\omega')}{\omega'-\omega-i\epsilon},\quad(4.5)$$

with $\lambda=(4/3)f^2$ or, for a narrow resonance,

$$f_{33}\cong\frac{\lambda q^2}{\omega}+\frac{q^2}{\omega_r-\omega}\cdot\frac{1}{\pi}\int\frac{d\omega'}{q'^2}\mathrm{Im}f_{33}(\omega')\quad(4.6)$$

$$\cong\frac{\lambda q^2}{\omega}+\frac{\lambda q^2}{\omega_r-\omega}=\frac{\lambda q^2/\omega}{1-\omega/\omega_r},\quad(4.7)$$

where (4.6) follows from (4.5) provided $\omega_r-\omega\gg\Gamma$ where Γ is the width of the resonance; in going from (4.6) to (4.7) we have actually integrated (4.3) over the resonance, again assuming the width to be small. One may also derive (4.7) directly from (4.5) by noting that for $\omega\gg\omega_r$, $f_{33}\approx0$; thus the integral in (4.6) must equal λ.

Thus the effective-range formula is approximately consistent with (4.1) for any resonance energy; the equation does not, therefore, determine the resonance, since the left and right sides are approximately equal for any ω_r. Thus small terms in the equation, such as $1/9$, or $1/M$, or f^2, or the high-energy contribution, will actually determine the precise location of the resonance. In this way we can reconcile the cutoff dependence of the resonance energy predicted by Eq. (4.4) and the dominance of the resonance integrals in Eq. (4.2).

We conclude that the shape of the resonance curve is determined by our considerations once the position of the resonance has a given value; this position however, we have not been able to determine from first principles.

For the small phase shifts, we have only to do the integrals over $d\omega'$. We find easily

$$f_{11}\cong-\frac{8}{3}\frac{f^2q^2/\omega}{1+\omega/\omega_r},\quad(4.8)$$

$$f_{13}\cong{}_{31}\cong\tfrac{1}{4}f_{11},$$

[11] Castillejo, Dalitz, and Dyson, Phys. Rev. **101**, 453 (1956).

where we have again set Γ, $1/M$, and 1/9 approximately equal to zero.

Let us summarize: the assumption of the dominance of the (3.3) contribution to low-energy dispersion integrals, together with the experimental location of the (3.3) resonance, leads to the following results:

(1) The P-wave phase-shifts approximately satisfy effective-range formulas

$$\frac{\lambda_\alpha q^3}{\omega}\cot\delta_\alpha \cong 1-\omega r_\alpha, \qquad (4.9)$$

with

$$\lambda_{11}=-(8/3)f^2,\quad \lambda_{33}=\tfrac{4}{3}f^2,\quad \lambda_{13}=\lambda_{31}=-\tfrac{2}{3}f^2,$$
$$r_{33}=1/\omega_r,\quad r_{11}\cong r_{13}=r_{31}\cong -r_{33}.$$

Also, to order $1/M$, $f_{13}=f_{31}$, as in the static theory.

(2) The S-wave amplitudes should be approximately given by the two zero energy scattering lengths, $\lambda^{(\pm)}$; since we have not considered the addition of any arbitrary constants, the only interesting fact that emerges is that even to order $1/M$ the strong energy dependence of the (3,3) state has no reflection in the S-wave energy dependence.

(3) The D-wave phase shifts are approximately given by[12]

$$\delta_{D_{\frac{3}{2}}}{}^{\frac{1}{2}}=-\lambda_D\left[1+\frac{112}{9}\left(\frac{\omega}{\omega+\omega_r}\right)^2\right], \qquad (4.10)$$

$$\delta_{D_{\frac{3}{2}}}{}^{\frac{3}{2}}=\lambda_D\left[2-\frac{28}{9}\left(\frac{\omega}{\omega+\omega_r}\right)^2\right], \qquad (4.11)$$

$$\delta_{D_{\frac{5}{2}}}{}^{\frac{1}{2}}=\lambda_D\left[4-\frac{32}{9}\left(\frac{\omega}{\omega+\omega_r}\right)^2\right], \qquad (4.12)$$

$$\delta_{D_{\frac{5}{2}}}{}^{\frac{3}{2}}=-\lambda_D\left[8+\frac{8}{9}\left(\frac{\omega}{\omega+\omega_r}\right)^2\right], \qquad (4.13)$$

[with $\lambda_D=(1/15)(f^2/M)q^5/\omega^2$], as one finds simply by carrying out the integrals in (3.34) and (3.35) in the zero-width approximation. Now $\lambda_D\approx 0.21°$ at $\omega=\omega_r$, so that the D wave phase shifts are all very small. Since, however, $\delta_{D_{\frac{5}{2}}}{}^{\frac{3}{2}}$ is of the order of magnitude two degrees at $\omega\cong\omega_r$, and since the weighting factor for a $j=5/2$ state is 3, the present analysis in terms of P and S states is unreliable as far as the small P phase shifts and S phase shifts are concerned in the resonance region.

Finally, which of these results will survive the addition of contributions from high-energy cross sections? It is our tentative opinion that only the (3.3) amplitude will stand this test, since the others, with the possible exception of f_{11}, are so small that very small corrections can change them by their own order of magnitude: the present theory predicts $\delta_{13}\approx\delta_{31}-4°$ at the resonance, which is just the order of magnitude of the high-energy contributions we have estimated from the known total cross sections at 1 Bev. The chances of the present theory adequately describing δ_{13}, δ_{31}, and the D waves are thus very small. A slight consolation is perhaps that the argument can be turned around, and eventual measurement of δ_{13} and δ_{31} used to provide information on the high-energy cross sections.

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[12] The D-wave phase shifts given here have been previously obtained by V. Wataghin (unpublished). We would like to thank Dr. Wataghin for informing us of his results.