MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

1. If $z \neq 0$ be a complex number such that $\left|z - \frac{1}{z}\right| = 2$, then the maximum value of $|z|$ is

(A) $\sqrt{2}$ (B) 1

(C) $\sqrt{2}-1$ (D) $\sqrt{2}+1$

Answer (D)

Sol. $\left|z - \frac{1}{z}\right| \geq \left||z| - \frac{1}{|z|}\right|$

$\Rightarrow \left||z| - \frac{1}{|z|}\right| \leq 2$

Let $|z| = r$

$\left|r - \frac{1}{r}\right| \leq 2$

$-2 \leq r - \frac{1}{r} \leq 2$

$r - \frac{1}{r} \geq -2$ and $r - \frac{1}{r} \leq 2$

$r^2 + 2r - 1 \geq 0$ and $r^2 - 2r - 1 \leq 0$

$r \in \left[-\infty, -1-\sqrt{2}\right] \cup \left[-1+\sqrt{2}, \infty\right]$ and

$r \in \left[1-\sqrt{2}, 1+\sqrt{2}\right]$

Taking intersection $r \in \left[\sqrt{2}-1, \sqrt{2}+1\right]$

2. Which of the following matrices can **NOT** be obtained from the matrix $\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$ by a single elementary row operation?

(A) $\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

(C) $\begin{bmatrix} -1 & 2 \\ -2 & 7 \end{bmatrix}$ (D) $\begin{bmatrix} -1 & 2 \\ -1 & 3 \end{bmatrix}$

Answer (C)

Sol. (1) By $R_1 \to R_1 + R_2$, $\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ is possible

(2) By $R_1 \leftrightarrow R_2$, $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ is possible

(3) This matrix can't be obtained

(4) By $R_2 \to R_2 + 2R_1$, $\begin{bmatrix} -1 & 2 \\ -1 & 3 \end{bmatrix}$ is possible

3. If the system of equations

$x + y + z = 6$

$2x + 5y + \alpha z = \beta$

$x + 2y + 3z = 14$

has infinitely many solutions, then $\alpha + \beta$ is equal to

(A) 8 (B) 36

(C) 44 (D) 48

Answer (C)

Sol. $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & \alpha \\ 1 & 2 & 3 \end{vmatrix} = 1(15-2\alpha) - 1(6-\alpha) + 1(-1)$

$= 15 - 2\alpha - 6 + \alpha - 1$

$= 8 - \alpha$

For infinite solutions, $\Delta = 0 \Rightarrow \alpha = 8$

$\Delta_x = \begin{vmatrix} 6 & 1 & 1 \\ \beta & 5 & 8 \\ 14 & 2 & 3 \end{vmatrix} = 6(-1) - 1(3\beta - 112) + 1(2\beta - 70)$

$= -6 - 3\beta + 112 + 2\beta - 70$

$= 36 - \beta$

$\Delta_x = 0 \Rightarrow$ for $\beta = 36$

$\alpha + \beta = 44$

4. Let the function $f(x) = \begin{cases} \frac{\log_e(1+5x) - \log_e(1+\alpha x)}{x}; & \text{if } x \neq 0 \\ 10 & ; \text{if } x = 0 \end{cases}$ be continuous at $x = 0$. Then α is equal to

(A) 10 (B) –10

(C) 5 (D) –5

Answer (D)

Sol. $\lim_{x\to 0} \frac{\ln(1+5x)-\ln(1+\alpha x)}{x}$

$= 5 - \alpha = 10$

$\Rightarrow \alpha = -5$

5. If $[t]$ denotes the greatest integer $\le t$, then the value of $\int_0^1 \left[2x - |3x^2 - 5x + 2| + 1\right] dx$ is

(A) $\frac{\sqrt{37}+\sqrt{13}-4}{6}$ (B) $\frac{\sqrt{37}-\sqrt{13}-4}{6}$

(C) $\frac{-\sqrt{37}-\sqrt{13}+4}{6}$ (D) $\frac{-\sqrt{37}+\sqrt{13}+4}{6}$

Answer (A)

Sol. $I = \int_0^1 \left[2x - |3x^2 - 5x + 2| + 1\right] dx$

$$I = \int_0^{2/3} \left[\underbrace{-3x^2 + 7x - 2}_{I_1}\right] dx + \int_{2/3}^{1} \left[\underbrace{3x^2 - 3x + 2}_{I_2}\right] dx + 1$$

$$I_1 = \int_0^{t_1} (-2)dx + \int_{t_1}^{1/3} (-1)\,dx + \int_{1/3}^{t_2} 0.dx + \int_{t_2}^{2/3} dx$$

$= -t_1 - t_2 + \frac{1}{3}$, where $t_1 = \frac{7-\sqrt{37}}{6}$, $t_2 = \frac{7-\sqrt{13}}{6}$

$I_2 = \int_{2/3}^{1} 1dx = \frac{1}{3}$

$\therefore \quad I = \frac{1}{3} - t_1 - t_2 + \frac{1}{3} + 1 = \frac{5}{3} - \left[\frac{7-\sqrt{37}}{6} + \frac{7-\sqrt{13}}{6}\right]$

$= \frac{\sqrt{37}+\sqrt{13}-4}{6}$

6. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence such that $a_0 = a_1 = 0$ and $a_{n+2} = 3a_{n+1} - 2a_n + 1$, $\forall\ n \ge 0$.

Then $a_{25}a_{23} - 2a_{25}a_{22} - 2a_{23}a_{24} + 4a_{22}a_{24}$ is equal to

(A) 483 (B) 528

(C) 575 (D) 624

Answer (B)

Sol. $a_{n+2} = 3a_{n+1} - 2a_n + 1$, $\forall\ n \ge 0 \quad (a_0 = a_1 = 0)$

$(a_{n+2} - a_{n+1}) - 2(a_{n+1} - a_n) - 1 = 0$

Put $n = 0$

$(a_2 - a_1) - 2(a_1 - a_0) - 1 = 0$

$n = 1$

$(a_3 - a_2) - 2(a_2 - a_1) - 1 = 0$

$n = 2$

$(a_4 - a_3) - 2(a_3 - a_2) - 1 = 0$

⋮

$n = n$

$(a_{n+2} - a_{n+1}) - 2(a_{n+1} - a_n) - 1 = 0$

Adding,

$(a_{n+2} - a_1) - 2(a_{n+1} - a_0) - (n+1) = 0$

$\therefore \quad a_{n+2} - 2a_{n+1} - (n+1) = 0$

$n \to n - 2$

$a_n - 2a_{n-1} - n + 1 = 0$

Now, $a_{25}a_{23} - 2a_{25}a_{22} - 2a_{23}a_{24} + 4a_{22}a_{24}$

$= a_{25}(a_{23} - 2a_{22}) - 2a_{24}(a_{23} - 2a_{22})$

$= (a_{25} - 2a_{24})(a_{23} - 2a_{22})$

$= 24 \cdot 22 = 528$

7. $\sum_{r=1}^{20} (r^2+1)(r!)$ is equal to

(A) 22! – 21! (B) 22! – 2(21!)

(C) 21! – 2(20!) (D) 21! – 20!

Answer (B)

Sol. $\sum_{r=1}^{20} (r^2+1+2r-2r)r! = \sum_{r=1}^{20} \left((r+1)^2 - 2r\right) r!$

$= \sum_{r=1}^{20} \left[(r+1)(r+1)! - rr!\right] - \sum_{r=1} (r+1)r! = r!$

$= (2\cdot 2! - 1!) + (3\cdot 3! - 2\cdot 2!) + ... + (21.21! - 20.20!)$

$-\left[(2! - 1!) + (3! - 2!) + ... + (21! - 20!)\right]$

$= (21\cdot 21! - 1) - (21! - 1)$

$= 20.21! = (22-2)21!$

$= 22! - 2(21!)$

8. For $I(x) = \int \frac{\sec^2 x - 2022}{\sin^{2022} x} dx$, if $I\left(\frac{\pi}{4}\right) = 2^{1011}$, then

(A) $3^{1010} I\left(\frac{\pi}{3}\right) - I\left(\frac{\pi}{6}\right) = 0$

(B) $3^{1010} I\left(\frac{\pi}{6}\right) - I\left(\frac{\pi}{3}\right) = 0$

(C) $3^{1011} I\left(\frac{\pi}{3}\right) - I\left(\frac{\pi}{6}\right) = 0$

(D) $3^{1011} I\left(\frac{\pi}{6}\right) - I\left(\frac{\pi}{3}\right) = 0$

Answer (A)

Sol. $I(x)=\int \frac{\sec^2 x - 2022}{\sin^{2022} x}dx$

$=\int\left(\sec^2 x\cdot\sin^{-2022}x - 2022\sin^{-2022}x\right)dx$

$=\sin^{-2022}x\tan x+\int 2022\sin^{-2023}x\cos x\cdot\tan x\,dx$

$-\int 2022\sin^{-2022}x\,dx+c$

$I(x)=\sin^{-2022}x\ \tan x+c$

$\because\quad I\left(\frac{\pi}{4}\right)=2^{1011}\quad\Rightarrow\ c=2^{1011}-2^{1011}=0$

$\therefore\quad I\left(\frac{\pi}{3}\right)=\left(\frac{2}{\sqrt{3}}\right)^{2022}\sqrt{3},\ I\left(\frac{\pi}{6}\right)=2^{2022}\frac{1}{\sqrt{3}}$

So, option (A) : $\frac{3^{1010}2^{2022}}{3^{1011}}\cdot\sqrt{3}-\frac{2^{2022}}{\sqrt{3}}=0$

$\therefore$ Option (A) is correct

9. if the solution curve of the differential equation $\frac{dy}{dx}=\frac{x+y-2}{x-y}$ passes through the points (2, 1) and (k + 1, 2), k > 0, then

(A) $2\tan^{-1}\left(\frac{1}{k}\right)=\log_e\left(k^2+1\right)$

(B) $\tan^{-1}\left(\frac{1}{k}\right)=\log_e\left(k^2+1\right)$

(C) $2\tan^{-1}\left(\frac{1}{k+1}\right)=\log_e\left(k^2+2k+2\right)$

(D) $2\tan^{-1}\left(\frac{1}{k}\right)=\log_e\left(\frac{k^2+1}{k^2}\right)$

Answer (A)

Sol. $\frac{dy}{dx}=\frac{x+y-2}{x-y}=\frac{(x-1)+(y-1)}{(x-1)-(y-1)}$

Let $x - 1 = X$, $y - 1 = Y$

$\frac{dY}{dX}=\frac{X+Y}{X-Y}$

Let $Y=tX\Rightarrow\frac{dY}{dX}=t+X\frac{dt}{dX}$

$t+X\frac{dt}{dX}=\frac{1+t}{1-t}$

$X\frac{dt}{dX}=\frac{1+t}{1-t}-t=\frac{1+t^2}{1-t}$

$\int\frac{1-t}{1+t^2}dt=\int\frac{dX}{X}$

$\tan^{-1}t-\frac{1}{2}\ln\left(1+t^2\right)=\ln|X|+c$

$\tan^{-1}\left(\frac{y-1}{x-1}\right)-\frac{1}{2}\ln\left(1+\left(\frac{y-1}{x-1}\right)^2\right)=\ln|x-1|+c$

Curve passes through (2, 1)

$0 - 0 = 0 + c\Rightarrow c = 0$

If (k + 1, 2) also satisfies the curve

$\tan^{-1}\left(\frac{1}{k}\right)-\frac{1}{2}\ln\left(\frac{1+k^2}{k^2}\right)=\ln k$

$2\tan^{-1}\left(\frac{1}{k}\right)=\ln\left(1+k^2\right)$

10. Let $y = y(x)$ be the solution curve of the differential equation $\frac{dy}{dx}+\left(\frac{2x^2+11x+13}{x^3+6x^2+11x+6}\right)y=\frac{(x+3)}{x+1},\ x>-1,$ which passes through the point (0, 1). Then $y(1)$ is equal to

(A) $\frac{1}{2}$

(B) $\frac{3}{2}$

(C) $\frac{5}{2}$

(D) $\frac{7}{2}$

Answer (B)

Sol. $\frac{dy}{dx}+\left(\frac{2x^2+11x+13}{x^3+6x^2+11x+6}\right)y=\frac{(x+3)}{x+1},\ x>-1,$

Integrating factor I.F. = $e^{\int\frac{2x^2+11x+13}{x^3+6x^2+11x+6}dx}$

Let $\frac{2x^2+11x+13}{(x+1)(x+2)(x+3)}=\frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{x+3}$

$A = 2$, $B = 1$, $C = -1$

I.F. = $e^{(2\ln|x+1|+\ln|x+2|-\ln|x+3|)}$

$=\frac{(x+1)^2(x+2)}{x+3}$

Solution of differential equation

$y\cdot\frac{(x+1)^2(x+2)}{x+3}=\int(x+1)(x+2)dx$

$y\frac{(x+1)^2(x+2)}{x+3}=\frac{x^3}{3}+\frac{3x^2}{2}+2x+c$

Curve passes through (0, 1)

$1\times\frac{1\times2}{3}=0+c\quad\Rightarrow\quad c=\frac{2}{3}$

So, $y(1)=\frac{\frac{1}{3}+\frac{3}{2}+2+\frac{2}{3}}{\frac{\left(2^2\times3\right)}{4}}=\frac{3}{2}$

11. Let m_1, m_2 be the slopes of two adjacent sides of a square of side a such that $a^2 + 11a + 3(m_1^2 + m_2^2)$ = 220. If one vertex of the square is (10(cosα – sinα),10(sinα + cosα)), where $\alpha \in \left(0, \frac{\pi}{2}\right)$ and the equation of one diagonal is $(\cos\alpha - \sin\alpha)x + (\sin\alpha + \cos\alpha)y = 10$, then $72(\sin^4\alpha + \cos^4\alpha) + a^2 - 3a + 13$ is equal to :

(A) 119 (B) 128

(C) 145 (D) 155

Answer (B)

Sol. One vertex of square is

(10 (cosα – sinα), 10(sinα + cosα))

and one of the diagonal is

(cosα – sinα) x + (sinα + cosα)y = 10

So the other diagonal can be obtained as

(cosα + sinα)x – (cosα – sinα)y = 0

So, point of intersection of diagonal will be

(5(cosα – sinα), 5(cosα + sinα)).

Therefore, the vertex opposite to the given vertex is (0, 0).

So, the diagonal length = $10\sqrt{2}$

Side length (a) = 10

It is given that

$$a^2 + 11a + 3\left(m_1^2 + m_2^2\right) = 220$$

$$m_1^2 + m_2^2 = \frac{220 - 100 - 110}{3} = \frac{10}{3}$$

and $m_1 m_2 = -1$

Slopes of the sides are tanα and – cotα

$$\tan^2\alpha = 3 \text{ or } \frac{1}{3}$$

$72(\sin^4\alpha + \cos^4\alpha) + a^2 - 3a + 13$

$$= 72 \cdot \frac{\tan^4\alpha + 1}{\left(1 + \tan^2\alpha\right)^2} + a^2 - 3a + 13 = 128$$

12. The number of elements in the set

$$S = \left\{x \in \mathbb{R} : 2\cos\left(\frac{x^2 + x}{6}\right) = 4^x + 4^{-x}\right\} \text{ is :}$$

(A) 1 (B) 3

(C) 0 (D) infinite

Answer (A)

Sol. $S = \left\{x \in \mathbb{R} : 2\cos\left(\frac{x^2 + x}{6}\right) = 4^x + 4^{-x}\right\}$

LHS is less than or equal to 2 and RHS is greater than or equal to 2.

So equality holds only if LHS = RHS = 2

RHS is 2 when x = 0

and at x = 0, LHS is also 2.

So, only one solution exist.

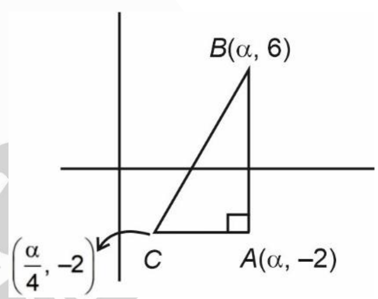
13. Let $A(\alpha, -2)$, $B(\alpha, 6)$ and $C\left(\frac{\alpha}{4}, -2\right)$ be vertices of a ΔABC. If $\left(5, \frac{\alpha}{4}\right)$ is the circumcentre of ΔABC, then which of the following is **NOT** correct about ΔABC.

(A) area is 24 (B) perimeter is 25

(C) circumradius is 5 (D) inradius is 2

Answer (B)

Sol.



Circumcentre of ΔABC

$$= \left(\frac{\alpha + \frac{\alpha}{4}}{2}, \frac{6 - 2}{2}\right)$$

$$= \left(\frac{5\alpha}{8}, 2\right)$$

$$= \left(5, \frac{\alpha}{4}\right)$$

⇒ α = 8

$$\text{area}(\Delta ABC) = \frac{1}{2} \cdot \frac{3\alpha}{4} \times 8 = 24 \text{ sq. units}$$

$$\text{Perimeter} = 8 + \frac{3\alpha}{4} + \sqrt{8^2 + \left(\frac{3\alpha}{4}\right)^2}$$

$$= 8 + 6 + 10 = 24$$

$$\text{Circumradius} = \frac{10}{2} = 5$$

$$r = \frac{\Delta}{s} = \frac{24}{12} = 2$$

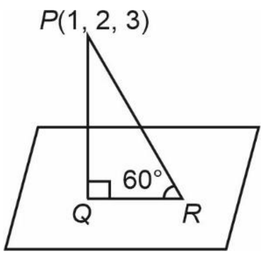
14. Let Q be the foot of perpendicular drawn from the point $P(1, 2, 3)$ to the plane $x + 2y + z = 14$. If R is a point on the plane such that $\angle PRQ = 60^\circ$, then the area of ΔPQR is equal to :

(A) $\frac{\sqrt{3}}{2}$ (B) $\sqrt{3}$

(C) $2\sqrt{3}$ (D) 3

Answer (B)

Sol.



$$PQ = \left|\frac{1+4+3-14}{\sqrt{6}}\right| = \sqrt{6}$$

$$QR = \frac{PQ}{\tan 60^\circ} = \frac{\sqrt{6}}{\sqrt{3}} = \sqrt{2}$$

Area $(\Delta PQR) = \frac{1}{2} \cdot PQ \cdot QR = \sqrt{3}$

15. If (2, 3, 9), (5, 2, 1), (1, λ, 8) and (λ, 2, 3) are coplanar, then the product of all possible values of λ is :

(A) $\frac{21}{2}$

(B) $\frac{59}{8}$

(C) $\frac{57}{8}$

(D) $\frac{95}{8}$

Answer (D)

Sol. $\because$ (2, 3, 9) (5, 2, 1), (1, λ, 8) and (λ, 2, 3) are coplanar.

$$\therefore \begin{vmatrix} \lambda-2 & -1 & -6 \\ -1 & \lambda-3 & -1 \\ 3 & -1 & -8 \end{vmatrix} = 0$$

$\therefore$ $8\lambda^2 - 67\lambda + 95 = 0$

$\therefore$ Product of all values of $\lambda = \frac{95}{8}$

16. Bag I contains 3 red, 4 black and 3 white balls and Bag II contains 2 red, 5 black and 2 white balls. One ball is transferred from Bag I to Bag II and then a ball is drawn from Bag II. The ball so drawn is found to be black in colour. Then the probability, that the transferred ball is red, is :

(A) $\frac{4}{9}$

(B) $\frac{5}{18}$

(C) $\frac{1}{6}$

(D) $\frac{3}{10}$

Answer (B)

Sol. Let $E \rightarrow$ Ball drawn from Bag II is black.

$E_R \rightarrow$ Bag I to Bag II red ball transferred.

$E_B \rightarrow$ Bag I to Bag II black ball transferred.

$E_W \rightarrow$ Bag I to Bag II white ball transferred.

$$P\left(\frac{E_R}{E}\right) = \frac{P\left(\frac{E}{E_R}\right) \cdot P(E_R)}{P\left(\frac{E}{E_R}\right)P(E_R) + P\left(\frac{E}{E_B}\right)P(E_B) + P\left(\frac{E}{E_W}\right)P(E_W)}$$

Here,

$P(E_R) = 3/10$, $P(E_B) = 4/10$, $P(E_W) = 3/10$

and

$P\left(\frac{E}{E_R}\right) = 5/10$, $P\left(\frac{E}{E_B}\right) = 6/10$, $P\left(\frac{E}{E_W}\right) = 5/10$

$$\therefore P\left(\frac{E_R}{E}\right) = \frac{15/100}{15/100 + 24/100 + 15/100}$$

$$= \frac{15}{54} = \frac{5}{18}$$

17. Let $S = \{z = x + iy : |z - 1 + i| \geq |z|, |z| < 2, |z + i| = |z - 1|\}$. Then the set of all values of x, for which $w = 2x + iy \in S$ for some $y \in R$, is

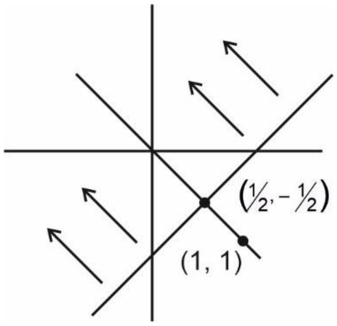
(A) $\left(-\sqrt{2}, \frac{1}{2\sqrt{2}}\right]$ (B) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{4}\right]$

(C) $\left(-\sqrt{2}, \frac{1}{2}\right]$ (D) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right]$

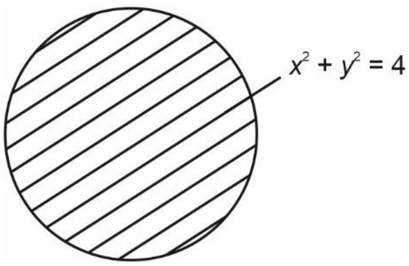
Answer (B)

Sol. $S : \{z = x + iy : |z - 1 + i| \geq |z|, |z| < 2, |z - i| = |z - 1|\}$

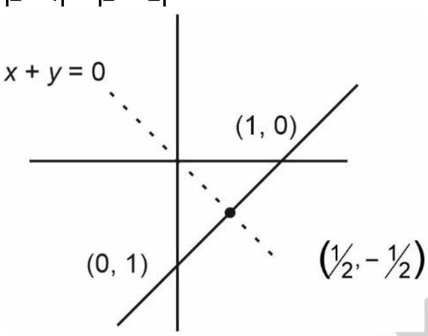
$|z - 1 + i| \geq |z|$

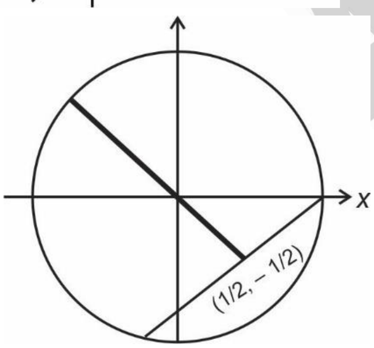


$|z| < 2$



$|z - i| = |z - 1|$





$\because$ $w \in S$ and $w = 2x + iy$

$2x < \dfrac{1}{2}$ $\qquad \therefore$ $x < \dfrac{1}{4}$

$(2x)^2 + (-2x)^2 < 4$

$4x^2 + 4x^2 < 4$

$x^2 < \dfrac{1}{2} \Rightarrow x \in \left(-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$

$\therefore$ $x \in \left(-\dfrac{1}{2}, \dfrac{1}{4}\right]$

18. Let $\vec{a}, \vec{b}, \vec{c}$ be three coplanar concurrent vectors such that angles between any two of them is same. If the product of their magnitudes is 14 and $(\vec{a} \times \vec{b}) \cdot (\vec{b} \times \vec{c}) + (\vec{b} \times \vec{c}) \cdot (\vec{c} \times \vec{a}) + (\vec{c} \times \vec{a}) \cdot (\vec{a} \times \vec{b}) = 168$, then $|\vec{a}| + |\vec{b}| + |\vec{c}|$ is equal to :

(A) 10 (B) 14

(C) 16 (D) 18

Answer (C)

Sol. $|\vec{a}||\vec{b}||\vec{c}| = 14$

$\vec{a} \wedge \vec{b} = \vec{b} \wedge \vec{c} = \vec{c} \wedge \vec{a} = \theta = \dfrac{2\pi}{3}$

$\vec{a} \cdot \vec{b} = -\dfrac{1}{2}|\vec{a}||\vec{b}|$

$\vec{b} \cdot \vec{c} = -\dfrac{1}{2}|\vec{b}||\vec{c}|$

$\vec{c} \cdot \vec{a} = -\dfrac{1}{2}|\vec{c}||\vec{a}|$

Now,

$(\vec{a} \times \vec{b}) \cdot (\vec{b} \times \vec{c}) + (\vec{b} \times \vec{c}) \cdot (\vec{c} \times \vec{a}) + (\vec{c} \times \vec{a}) \cdot (\vec{a} \times \vec{b})$

$= 168$ ...(i)

$(\vec{a} \times \vec{b}) \cdot (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{b})(\vec{b} \cdot \vec{c}) - (\vec{a} \cdot \vec{c})|\vec{b}|^2$

$= \dfrac{1}{4}|\vec{b}|^2 |\vec{a}||\vec{c}| + \dfrac{1}{2}|\vec{a}||\vec{b}|^2 |\vec{c}|$

$= \dfrac{3}{4}|\vec{a}||\vec{b}|^2 |\vec{c}|$ ...(ii)

Similarly $(\vec{b} \times \vec{c}) \cdot (\vec{c} \times \vec{a}) = \dfrac{3}{4}|\vec{a}||\vec{b}||\vec{c}|^2$ ...(iii)

$(\vec{c} \times \vec{a}) \cdot (\vec{a} \times \vec{b}) = \dfrac{3}{4}|\vec{a}|^2|\vec{b}||\vec{c}|$ ...(iv)

Substitute (ii), (iii), (iv) in (i)

$\dfrac{3}{4}|\vec{a}||\vec{b}||\vec{c}|\left[|\vec{a}| + |\vec{b}| + |\vec{c}|\right] = 168$

$\dfrac{3}{4} \times 14\left[|\vec{a}| + |\vec{b}| + |\vec{c}|\right] = 168$

$|\vec{a}| + |\vec{b}| + |\vec{c}| = 16$

19. The domain of the function $f(x) = \sin^{-1}\left(\dfrac{x^2 - 3x + 2}{x^2 + 2x + 7}\right)$ is :

(A) $[1, \infty)$ (B) $[-1, 2]$

(C) $[-1, \infty)$ (D) $(-\infty, 2]$

Answer (C)

Sol. $f(x)=\sin^{-1}\left(\frac{x^2-3x+2}{x^2+2x+7}\right)$

$-1\le\frac{x^2-3x+2}{x^2+2x+7}\le 1$

$\frac{x^2-3x+2}{x^2+2x+7}\le 1$

$x^2-3x+2\le x^2+2x+7$

$5x\ge -5$

$x\ge -1$ … (i)

$\frac{x^2-3x+2}{x^2+2x+7}\ge -1$

$x^2-3x+2\ge -x^2-2x-7$

$2x^2-x+9\ge 0$

$x\in R$ … (ii)

(i) $\cap$ (ii)

Domain $\in [-1, \infty)$

20. The statement $(p\Rightarrow q)\vee(p\Rightarrow r)$ is **NOT** equivalent to

(A) $(p\wedge(\sim r))\Rightarrow q$

(B) $(\sim q)\Rightarrow((\sim r)\vee p)$

(C) $p\Rightarrow(q\vee r)$

(D) $(p\wedge(\sim q))\Rightarrow r$

Answer (B)

Sol. (A) $(p\wedge(\sim r))\Rightarrow q$

$\sim(p\wedge\sim r)\vee q$

$\equiv(\sim p\vee r)\vee q$

$\equiv\sim p\vee(r\vee q)$

$\equiv p\rightarrow(q\vee r)$

$\equiv(p\Rightarrow q)\vee(p\Rightarrow r)$

(C) $p\Rightarrow(q\vee r)$

$\equiv\sim p\vee(q\vee r)$

$\equiv(\sim p\vee q)\vee(\sim p\vee r)$

$\equiv(p\rightarrow q)\vee(p\rightarrow r)$

(D) $(p\wedge\sim q)\Rightarrow r$

$\equiv p\Rightarrow(q\vee r)$

$\equiv(p\Rightarrow q)\vee(p\Rightarrow r)$

SECTION - B

Numerical Value Type Questions: This section contains 10 questions. In Section B, attempt any five questions out of 10. The answer to each question is a **NUMERICAL VALUE.** For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the second decimal place; e.g. 06.25, 07.00, –00.33, –00.30, 30.27, –27.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

1. The sum and product of the mean and variance of a binomial distribution are 82.5 and 1350 respectively. Then the number of trials in the binomial distribution is _______.

Answer (96)

Sol. Given $np + npq = 82.5$ … (1)

and $np\,(npq) = 1350$ … (2)

$\therefore\quad x^2-82.5x+1350=0$ ↗ Mean ↘ Variance

$\Rightarrow\quad x^2-22.5x-60x+1350=0$

$\Rightarrow\quad x-(x-22.5)-60(x-22.5)=0$

Mean = 60 and Variance = 22.5

$np = 60,\ npq = 22.5$

$\Rightarrow\quad q=\frac{9}{24}=\frac{3}{8},\ p=\frac{5}{8}$

$\therefore\quad n\frac{5}{8}=60 \qquad \Rightarrow n=96$

2. Let $\alpha, \beta(\alpha>\beta)$ be the roots of the quadratic equation $x^2-x-4=0$. If $P_n=\alpha^n-\beta^n$, $n\in\mathbb{N}$, then $\frac{P_{15}P_{16}-P_{14}P_{16}-P_{15}^2+P_{14}P_{15}}{P_{13}P_{14}}$ is equal to _______.

Answer (16)

Sol. $x^2-x-4=0$ ↗ α ↘ β and $P_n=\alpha^n-\beta^n$

$$\therefore\quad I=\frac{(P_{15}-P_{14})P_{16}-P_{15}(P_{15}-P_{14})}{P_{13}P_{14}}=\frac{(P_{16}-P_{15})(P_{15}-P_{14})}{P_{13}P_{14}}$$

$$\Rightarrow\quad I=\frac{(\alpha^{16}-\beta^{16}-\alpha^{15}+\beta^{15})(\alpha^{15}-\beta^{15}-\alpha^{14}+\beta^{14})}{\left(\alpha^{13}-\beta^{13}\right)\left(\alpha^{14}-\beta^{14}\right)}$$

$$\Rightarrow\quad I=\frac{(\alpha^{15}(\alpha-1)-\beta^{15}(\beta-1))(\alpha^{14}(\alpha-1)-\beta^{14}(\beta-1))}{\left(\alpha^{13}-\beta^{13}\right)\left(\alpha^{14}-\beta^{14}\right)}$$

As $\alpha^2 - \alpha = 4 \quad \Rightarrow \alpha - 1 = \frac{4}{\alpha}$ and $\beta - 1 = \frac{4}{\beta}$

$$\Rightarrow I = \frac{\left(\alpha^{15}\cdot\frac{4}{\alpha} - \beta^{15}\cdot\frac{4}{\beta}\right)\left(\alpha^{14}\cdot\frac{4}{\alpha} - \beta^{14}\cdot\frac{4}{\beta}\right)}{\left(\alpha^{13}-\beta^{13}\right)\left(\alpha^{14}-\beta^{14}\right)}$$

$$= \frac{16\left(\alpha^{14}-\beta^{14}\right)\left(\alpha^{13}-\beta^{13}\right)}{\left(\alpha^{14}-\beta^{14}\right)\left(\alpha^{13}-\beta^{13}\right)} = 16$$

3. Let $x = \begin{bmatrix}1\\1\\1\end{bmatrix}$ and $A = \begin{bmatrix}-1 & 2 & 3\\0 & 1 & 6\\0 & 0 & -1\end{bmatrix}$. For $k \in \mathbb{N}$, if $X'A^kX = 33$, then k is equal to _______.

Answer (10*)

Sol. Given $A = \begin{bmatrix}-1 & 2 & 3\\0 & 1 & 6\\0 & 0 & -1\end{bmatrix}$

$$A^2 = \begin{bmatrix}1 & 0 & 6\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}, \quad A^4 = \begin{bmatrix}1 & 0 & 12\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow A^k = \begin{bmatrix}1 & 0 & 3k\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$\therefore \quad X'A^kX = [111]\begin{bmatrix}1 & 0 & 3k\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = [3k+3]$$

$\Rightarrow$ [3k + 3] = 33 (here it shall be [33] as matrix can't be equal to a scalar)

i.e. [3k + 3] = 33

3k + 3 = [33] $\quad \Rightarrow k = 10$

If k is odd and apply above process, we don't get odd value of k

$\therefore$ k = 10

4. The number of natural numbers lying between 1012 and 23421 that can be formed using the digits 2, 3, 4, 5, 6 (repetition of digits is not allowed) and divisible by 55 is _______.

Answer (6)

Sol. Case-I When number is 4-digit number $\left(a\,b\,c\,d\right)$

here d is fixed as 5

So, (a, b, c) can be (6, 4, 3), (3, 4, 6), (2, 3, 6), (6, 3, 2), (3, 2, 4) or (4, 2, 3)

$\Rightarrow$ 6 numbers

Case-II No number possible

5. If $\sum_{k=1}^{10} K^2\left(10_{C_K}\right)^2 = 22000L$, then L is equal to _____.

Answer (221)

Sol. $\sum_{K=1}^{10} K^2\left({}^{10}C_K\right)^2 = 1^2\ {}^{10}C_1^2 + 2^2\ {}^{10}C_2^2 + ... + 10^2\ {}^{10}C_{10}$

Let $(1 + x)^{10} = {}^{10}C_0 + {}^{10}C_1 x + {}^{10}C_2 x^2 + \ldots + {}^{10}C_{10} x^{10}$

$\Rightarrow 10(1 + x)^9 = {}^{10}C_1 + 2\cdot{}^{10}C_2 x + \ldots + 10\cdot{}^{10}C_{10} x^9 \ldots(1)$

Similarly, $10(x + 1)^9 = 10\cdot{}^{10}C_0\, x^9 + 9\cdot{}^{10}C_1\, x^8 + \ldots + 1\cdot{}^{10}C_9$

$100(1+x)^{18}$ has required term with coefficient of x^9

i.e. ${}^{18}C_9\ 100 = 22000\ L$

$\Rightarrow$ L = 221

6. If [t] denotes the greatest integer $\le t$, then the number of points, at which the function $f(x) = 4|2x+3| + 9\left[x + \frac{1}{2}\right] - 12[x+20]$ is not differentiable in the open interval (–20, 20), is _______.

Answer (79)

Sol. $f(x) = 4|2x+3| + 9\left[x+\frac{1}{2}\right] - 12[x+20]$

$= 4|2x+3| + 9\left[x+\frac{1}{2}\right] - 12[x] - 240$

$f(x)$ is non differentiable at $x = -\frac{3}{2}$

and $f(x)$ is discontinuous at {–19, –18, ….., 18, 19}

as well as $\left\{-\frac{39}{2}, -\frac{37}{2}, \ldots, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \ldots, \frac{39}{2}\right\}$,

at same point they are also non differentiable

$\therefore$ Total number of points of non differentiability

= 39 + 40

= 79

7. If the tangent to the curve $y = x^3 - x^2 + x$ at the point (a, b) is also tangent to the curve $y = 5x^2 + 2x - 25$ at the point (2, –1), then |2a + 9b| is equal to _______.

Answer (195)

Sol. Slope of tangent to curve $y = 5x^2 + 2x - 25$

$$= m = \left(\frac{dy}{dx}\right)_{\text{at}(2,-1)} = 22$$

$\therefore$ Equation of tangent : $y + 1 = 22(x - 2)$

$\therefore$ $y = 22x - 45$.

Slope of tangent to $y = x^3 - x^2 + x$ at point (a, b)

$= 3a^2 - 2a + 1$

$3a^2 - 2a + 1 = 22$

$3a^2 - 2a - 21 = 0$

$\therefore$ $a = 3$ or $-\frac{7}{3}$

Also $b = a^3 - a^2 + a$

Then $(a, b) = (3, 21)$ or $\left(-\frac{7}{3}, -\frac{151}{9}\right)$.

$\left(-\frac{7}{3}, -\frac{151}{9}\right)$ does not satisfy the equation of tangent

$\therefore$ $a = 3, b = 21$

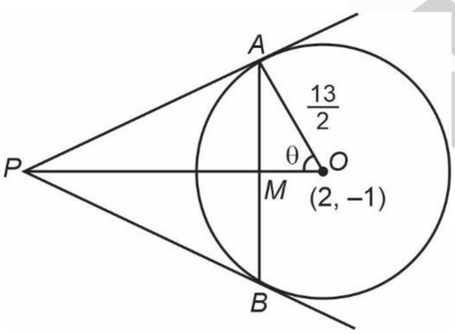
$\therefore$ $|2a + 9b| = 195$

8. Let AB be a chord of length 12 of the circle $(x-2)^2 + (y+1)^2 = \frac{169}{4}$. If tangents drawn to the circle at points A and B intersect at the point P, then five times the distance of point P from chord AB is equal to ________.

Answer (72)

Sol. Here $AM = BM = 6$

$$OM = \sqrt{\left(\frac{13}{2}\right)^2 - 6^2} = \frac{5}{2}$$



$$\sin\theta = \frac{12}{13}$$

In ΔPAO:

$$\frac{PO}{OA} = \sec\theta$$

$$PO = \frac{13}{2}\cdot\frac{13}{5} = \frac{169}{10}$$

$$\therefore \quad PM = \frac{169}{10} - \frac{5}{2} = \frac{144}{10} = \frac{72}{5}$$

$\therefore$ $5\,PM = 72$.

9. Let $\vec{a}$ and $\vec{b}$ be two vectors such that $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + 2|\vec{b}|^2$, $\vec{a}\cdot\vec{b} = 3$ and $|\vec{a}\times\vec{b}|^2 = 75$.

Then $|\vec{a}|^2$ is equal to _____.

Answer (14)

Sol. $\because$ $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + 2|\vec{b}|^2$

or $|\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b} = |\vec{a}|^2 + 2|\vec{b}|^2$

$\therefore$ $|\vec{b}|^2 = 6$ ...(i)

Now $|\vec{a}\times\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$

$75 = |\vec{a}|^2 \cdot 6 - 9$

$\therefore$ $|\vec{a}|^2 = 14$

10. Let

$S = \{(x, y) \in \mathbb{N}\times\mathbb{N} : 9(x-3)^2 + 16(y-4)^2 \le 144\}$

and $T = \{(x, y) \in \mathbb{R}\times\mathbb{R} : (x-7)^2 + (y-4)^2 \le 36\}$.

Then $n(S \cap T)$ is equal to ______.

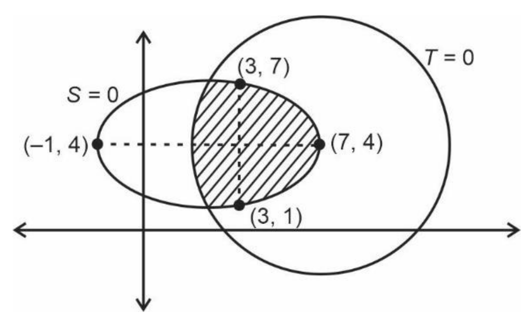
Answer (27)

Sol. $S = \left\{(x, y) \in \mathbb{N}\times\mathbb{N} : \frac{(x-3)^2}{16} + \frac{(y-4)^2}{9} \le 1\right\}$

represents all the integral points inside and on the ellipse $\frac{(x-3)^2}{16} + \frac{(y-4)^2}{9} = 1$, in first quadrant.

and $T = \{(x, y) \in \mathbb{R}\times\mathbb{R} : (x-7)^2 + (y-4)^2 \le 36\}$

represents all the points on and inside the circle $(x-7)^2 + (y-4)^2 = 36$.



$\therefore$ $n(S \cap T) = \{(3, 1), (2, 2), (3, 2), (4, 2), (5, 2), (2, 3), \ldots. (6, 5)\}$

Total number of points = 27

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