Computer Graphics
Ray Casting

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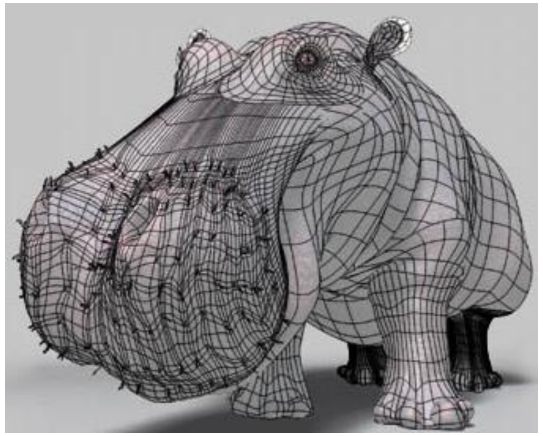
UNI FREIBURG

Outline

– Context
– Implicit surfaces
– Parametric surfaces
– Combined objects
– Triangles
– Axis-aligned boxes
– Iso-surfaces in grids
– Summary

Rendering

- Visibility / hidden surface problem
  - Object projection onto sensor plane
  - Ray-object intersections with ray casting
- Light transport / shading
  - Rendering equation
  - Phong illumination model



[Jeremy Birn]

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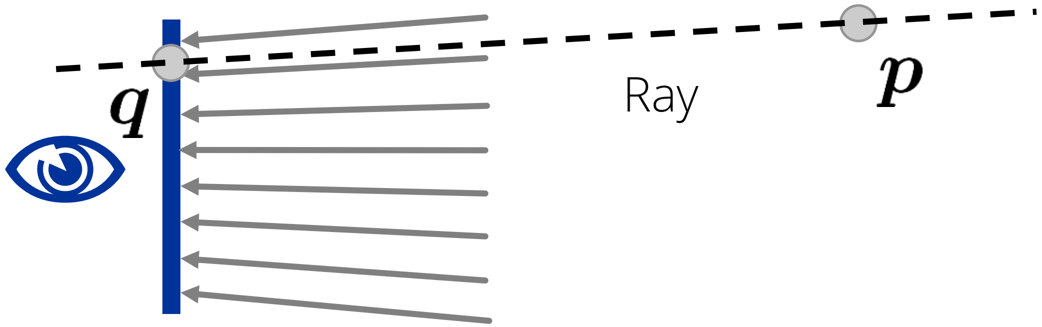
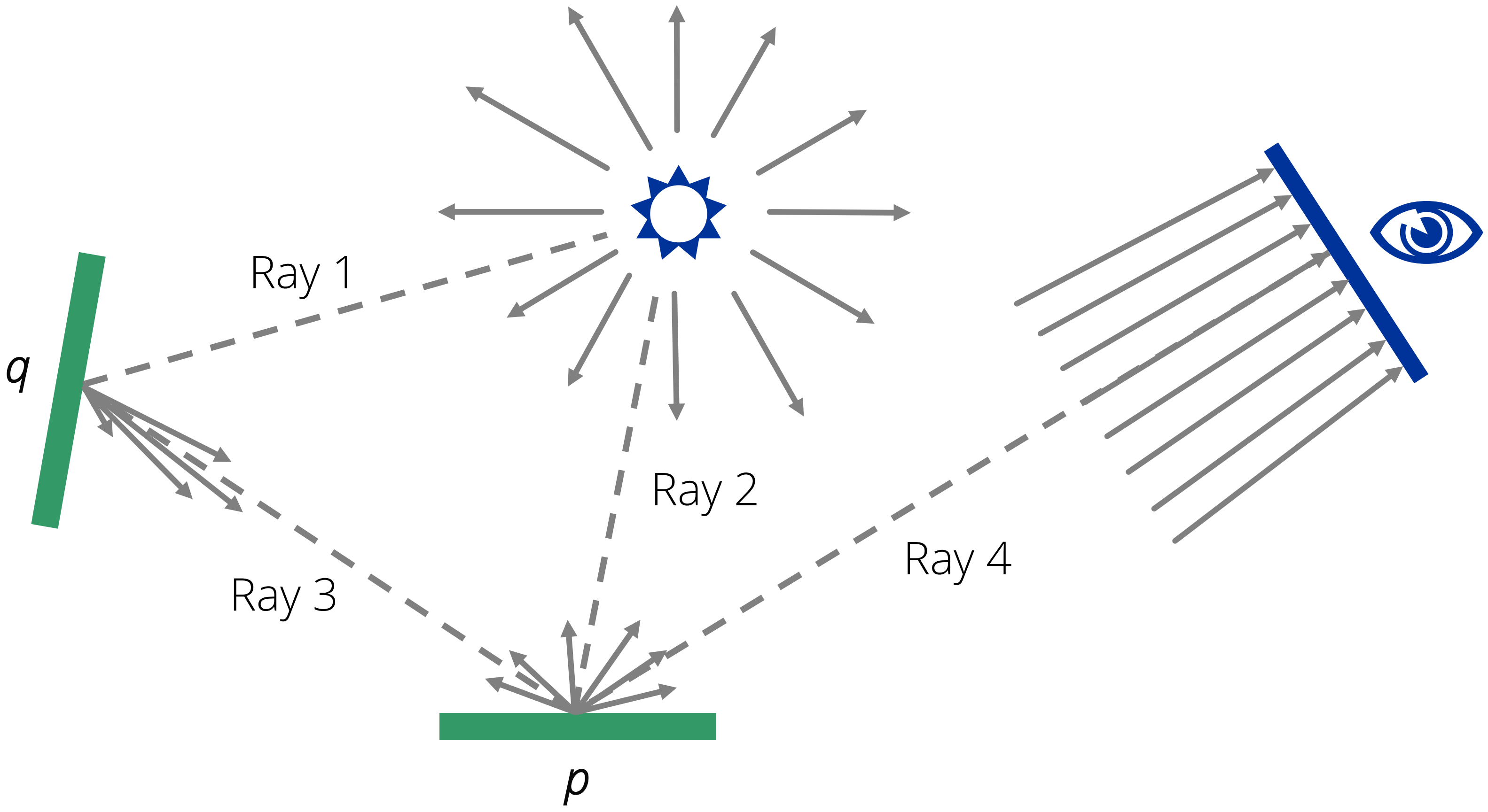
Ray Casting

- Computes ray intersections with the representation of a scene to estimate the projection of the scene onto the sensor



Ray Casting computes ray-scene intersections to estimate *q* from *p*.

Ray Tracing - Concept



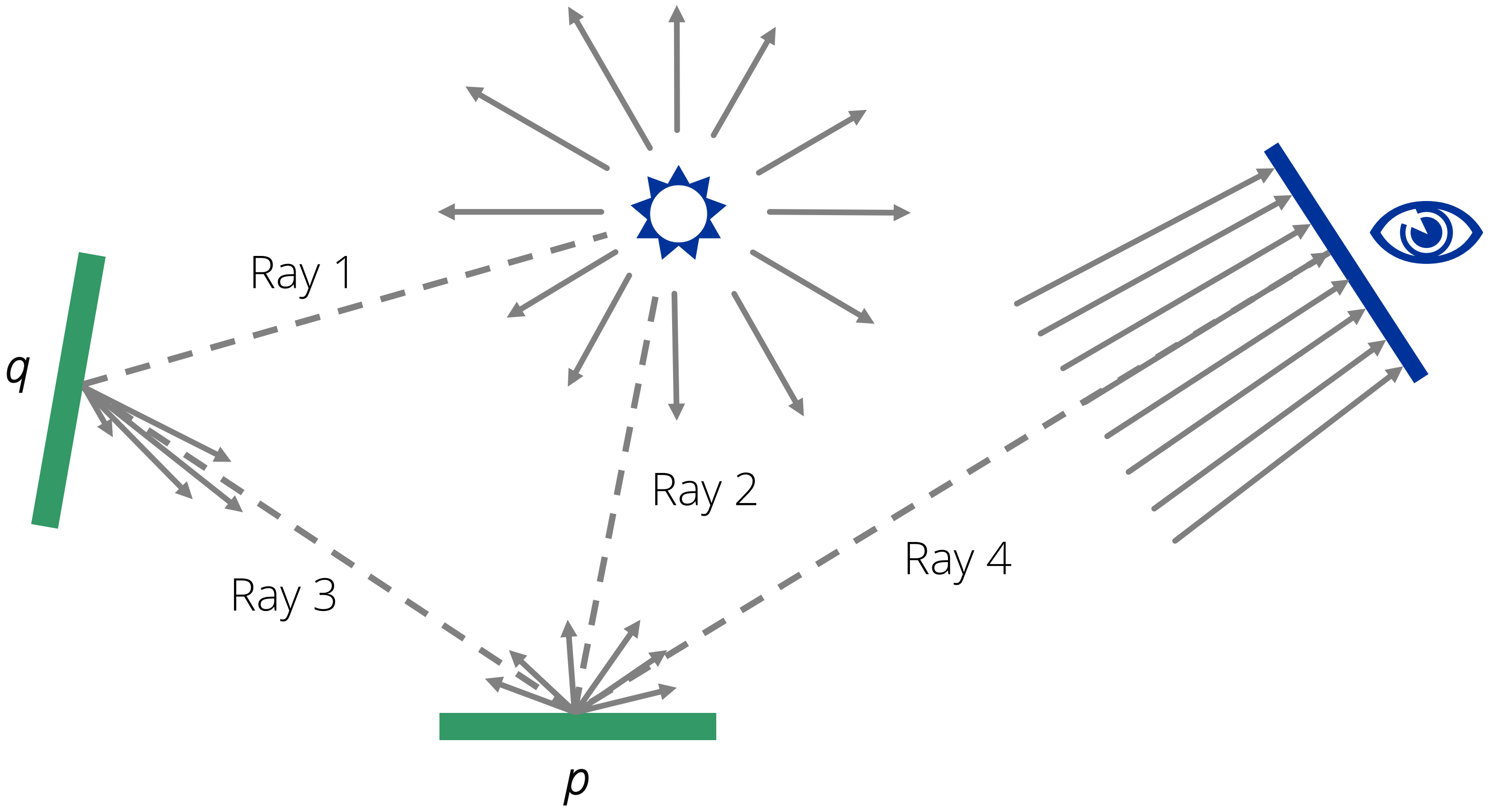
Ray 1
Outgoing light from source
Incoming light at surface
Direct illumination

Ray 2
Outgoing light from source
Incoming light at surface
Direct illumination

Ray 3
Outgoing light from surface
Incoming light at surface
Indirect illumination

Ray 4
Outgoing light from surface
Incoming light at sensor

Ray Tracing - Challenge

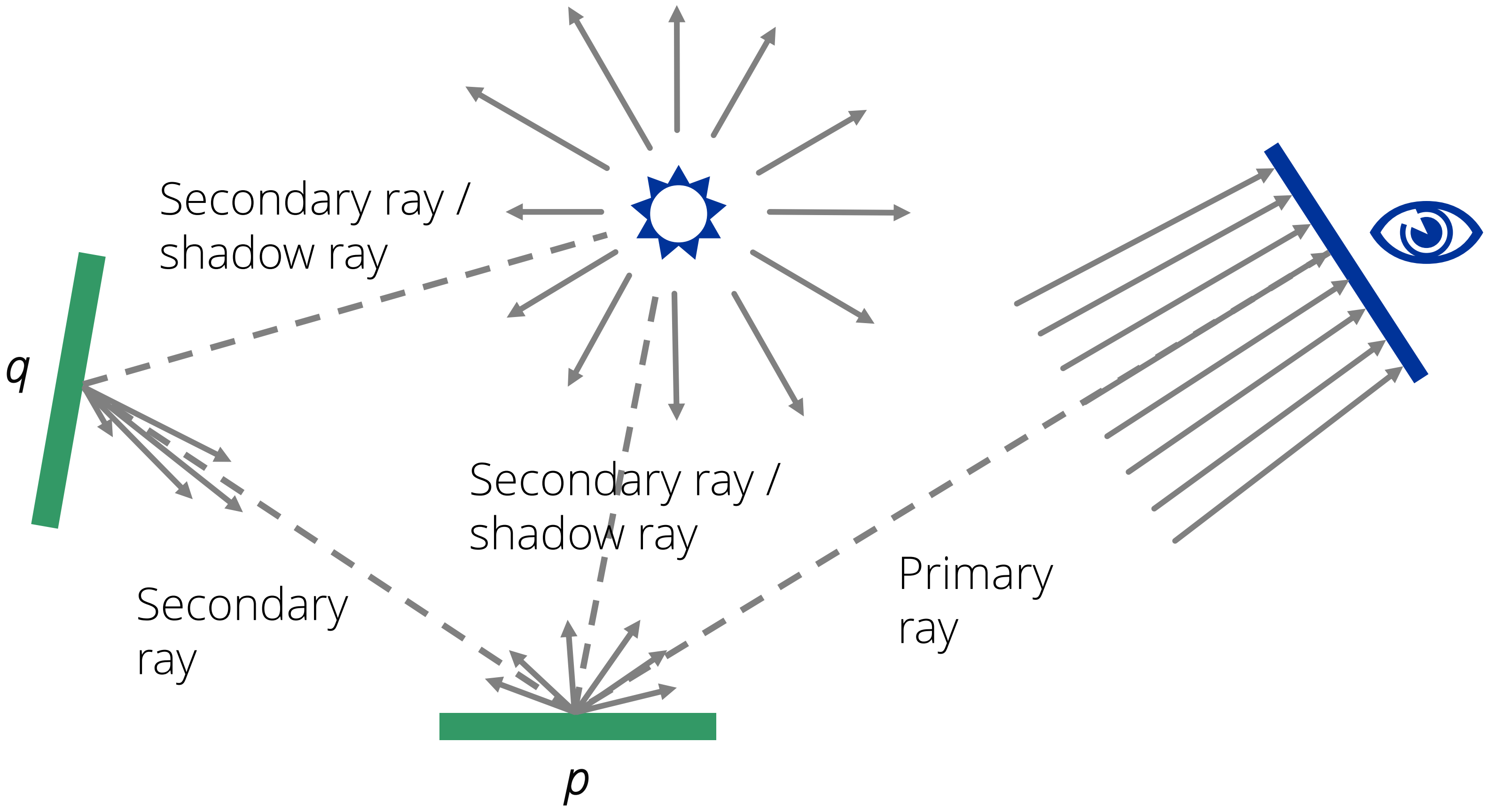


Ray 4
Incoming light at the sensor
Main goal of a ray tracer

Ray 1, 2, 3, …
Incoming / outgoing light
at all other paths is required
to compute light at ray 4

Ray 3
Two surfaces illuminate each other.
Outgoing light from *q* towards *p*
depends on outgoing light from
p towards *q* which depends on …

Ray Tracing - Terms



Primary rays
start / end at sensors

Secondary rays
do not start / end
at sensors

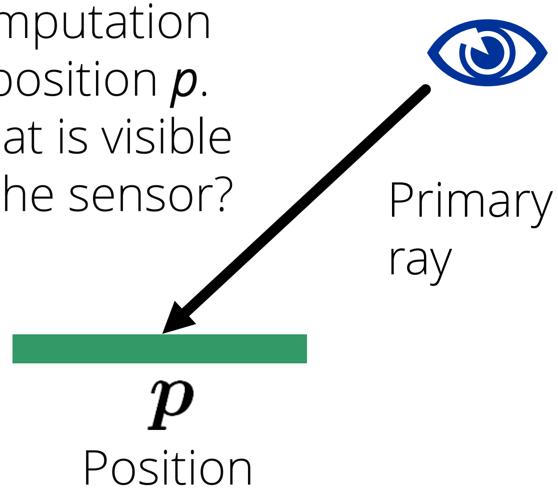
Shadow rays
start / end at light sources

Ray Casting and Ray Tracing

- Primary rays solve the visibility problem
  - What is visible at the sensor?
  - Ray casting
- Secondary rays are used to compute the light transport along a primary ray towards the viewer
  - Which color does it have?
  - Shading model / rendering equation
  - Ray tracing

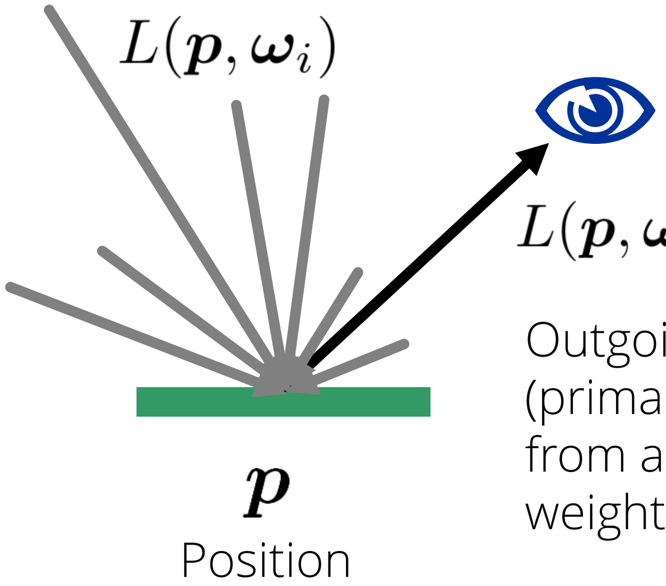
Ray Casting and Ray Tracing

Ray Casting
Computation of position $\boldsymbol{p}$. What is visible at the sensor?



Incoming light from direction $\boldsymbol{\omega}_i$ along a secondary ray

Ray Tracing
Computation of the light that is transported along primary rays. Which color does it have? Secondary rays are used.

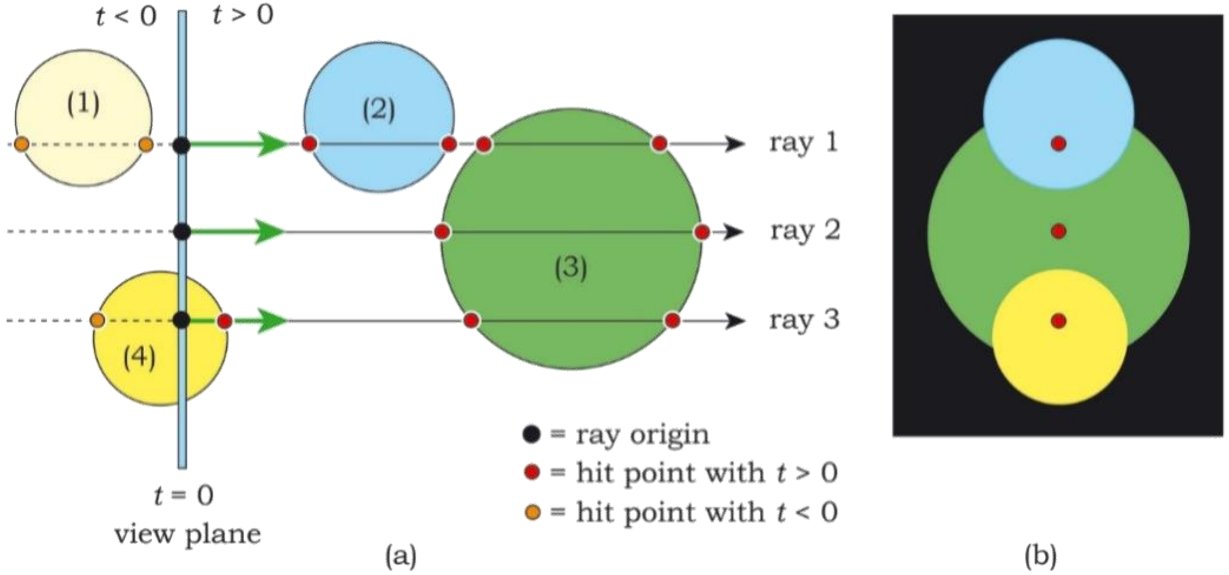


$$L(\boldsymbol{p}, \boldsymbol{\omega}_o) = \int_\Omega \ldots \mathrm{d}\omega_i$$

Outgoing light into direction $\boldsymbol{\omega}_o$ (primary ray) is a sum of incident light from all directions (secondary rays) weighted with material properties.

Ray Casting - Concept

- Ray
  - A half-line specified by an origin $\boldsymbol{o}$ and a direction $\boldsymbol{d}$
  - Parametric form $\boldsymbol{r}(t) = \boldsymbol{o} + t\boldsymbol{d}$ with $0 \leq t \leq \infty$
- Nearest intersection with all objects has to be computed, i.e. intersection with minimal $t \geq 0$



Orthographic camera with parallel rays [Suffern]

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Implicit Surfaces

- Function f implicitly defines a set of surface points
- For a surface point (x, y, z) : $\quad f(x,y,z) = 0$
- An intersection occurs, if a point on a ray satisfies the implicit equation $f(x,y,z) = f(\boldsymbol{r}(t)) = f(\boldsymbol{o} + t\boldsymbol{d}) = 0$
- E.g., all points $\boldsymbol{p} = (x,y,z)$ on a plane with surface normal $\boldsymbol{n}$ and offset $\boldsymbol{r}$ satisfy the equation $\boldsymbol{n} \cdot (\boldsymbol{p} - \boldsymbol{r}) = 0$
- The intersection with a ray can be computed based on t

$\boldsymbol{n} \cdot (\boldsymbol{o} + t\boldsymbol{d} - \boldsymbol{r}) = 0 \quad t = \frac{(\boldsymbol{r}-\boldsymbol{o})\cdot\boldsymbol{n}}{\boldsymbol{n}\cdot\boldsymbol{d}}$ if *d* is not orthogonal to *n*

Implicit Surfaces - Normal

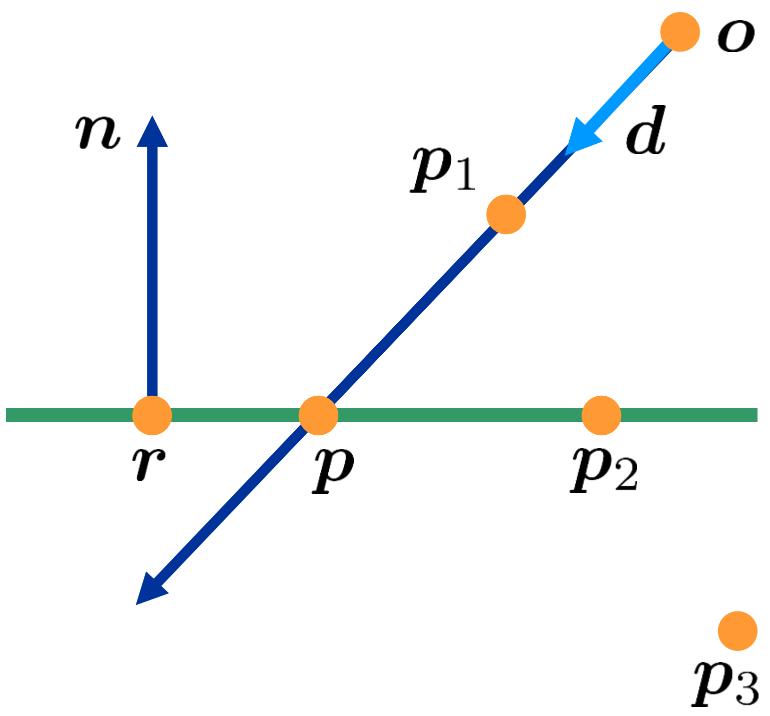
- Perpendicular to the surface
- Given by the gradient of the implicit function

$$\boldsymbol{n} = \nabla f(\boldsymbol{p}) = \left(\frac{\partial f(\boldsymbol{p})}{\partial x}, \frac{\partial f(\boldsymbol{p})}{\partial y}, \frac{\partial f(\boldsymbol{p})}{\partial z} \right)$$

- E.g., for a point $\boldsymbol{p} = (x, y, z)$ on a plane $f(\boldsymbol{p}) = \boldsymbol{n} \cdot (\boldsymbol{p} - \boldsymbol{r}) = 0$

$$\boldsymbol{n} = \nabla f(\boldsymbol{p}) = (\tfrac{\partial}{\partial x} n_x(x - r_x), \tfrac{\partial}{\partial y} n_y(y - r_y), \tfrac{\partial}{\partial z} n_z(z - r_z)) = (n_x, n_y, n_z)$$

Implicit Surfaces



Implicit surface

$$\boldsymbol{n} \cdot (\boldsymbol{p}_1 - \boldsymbol{r}) \neq 0$$
$$\boldsymbol{n} \cdot (\boldsymbol{p}_2 - \boldsymbol{r}) = 0$$
$$\boldsymbol{n} \cdot (\boldsymbol{p}_3 - \boldsymbol{r}) \neq 0$$
$$\boldsymbol{n} \cdot (\boldsymbol{p} - \boldsymbol{r}) = 0$$

Ray

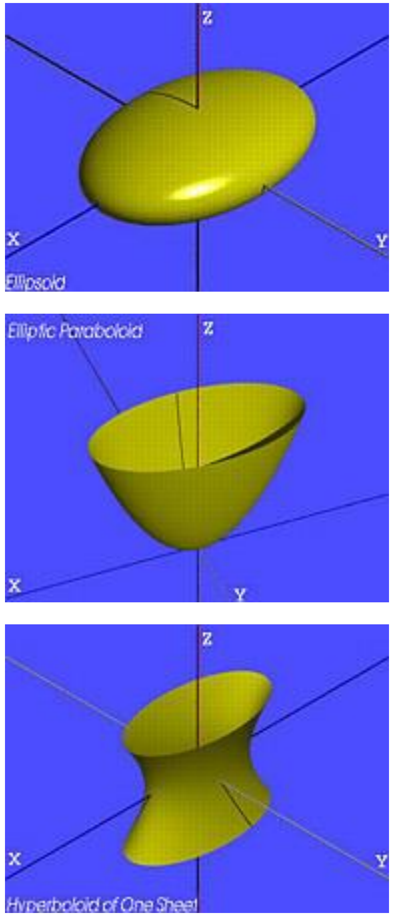
$$\boldsymbol{o} + t_1 \boldsymbol{d} = \boldsymbol{p}_1$$
$$\boldsymbol{o} + t \boldsymbol{d} = \boldsymbol{p}$$

Ray-surface intersection

$$\boldsymbol{n} \cdot (\boldsymbol{o} + t\boldsymbol{d} - \boldsymbol{r}) = 0$$

Quadrics

- E.g.
  - Sphere $$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{a^2}-1=0$$
  - Ellipsoid $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$$
  - Paraboloid $$\frac{x^2}{a^2}+\frac{y^2}{a^2}-z=0$$
  - Hyperboloid $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}-1=0$$
  - Cone $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$
  - Cylinder $$\frac{x^2}{a^2}+\frac{y^2}{a^2}-1=0$$
- Represented by quadratic equations, i.e. zero, one or two intersections with a ray



[Wikipedia: Quadric]

Quadrics - Sphere

– At the origin with radius one $f(\boldsymbol{p}) = x^2 + y^2 + z^2 - 1 = 0$

$$(o_x + td_x)^2 + (o_y + td_y)^2 + (o_z + td_z)^2 - 1 = 0$$

– Quadratic equation in *t*

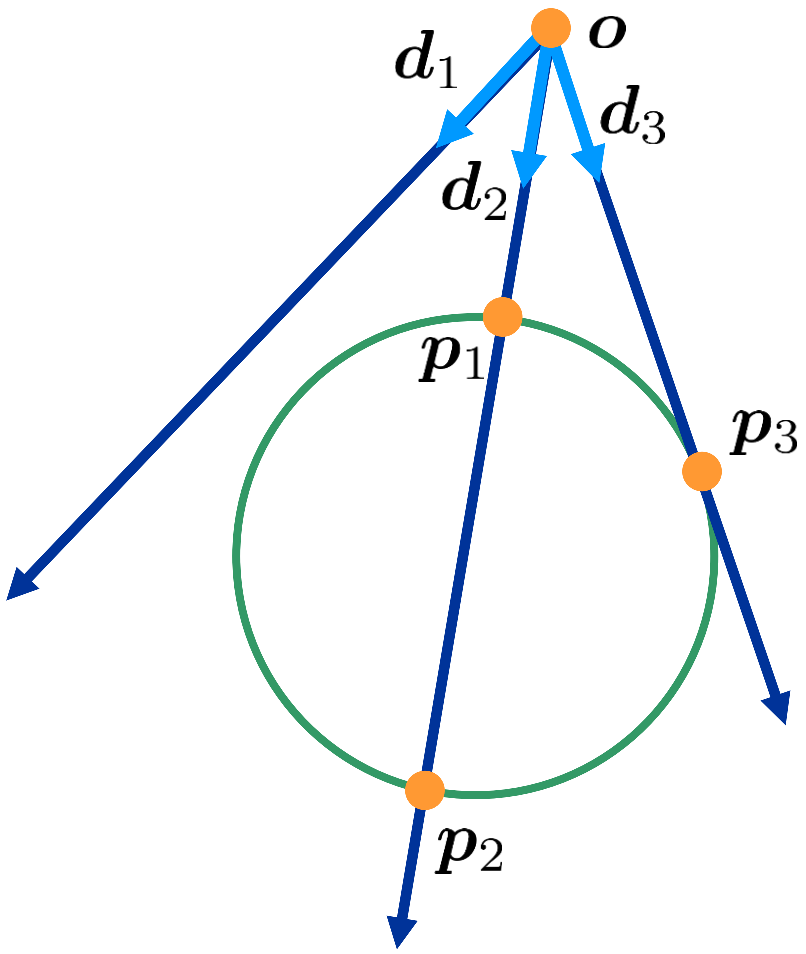
$$At^2 + Bt + C = 0 \qquad A = d_x^2 + d_y^2 + d_z^2 \qquad B = 2(d_x o_x + d_y o_y + d_z o_z)$$

$$t_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \qquad C = o_x^2 + o_y^2 + o_z^2 - 1$$

– Surface normal

$$\boldsymbol{n} = \nabla f(\boldsymbol{p}) = (2x, 2y, 2z)$$

Quadrics - Sphere



Ray 1: $\boldsymbol{r}(t) = \boldsymbol{o} + t\boldsymbol{d}_1$

$$B^2 - 4AC < 0$$

Ray 2: $\boldsymbol{r}(t) = \boldsymbol{o} + t\boldsymbol{d}_2$

$$t_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

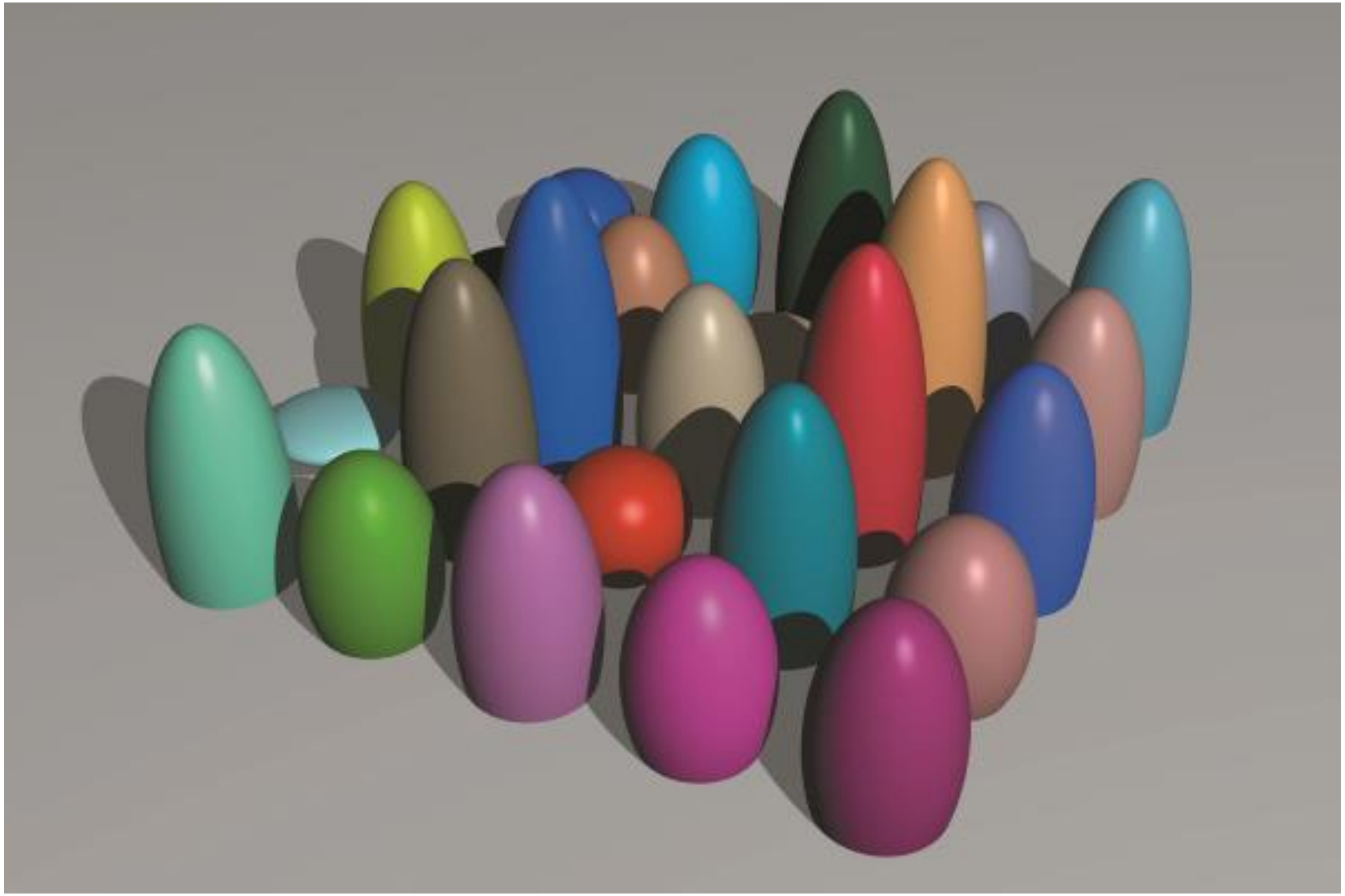
$$\boldsymbol{p}_{1,2} = \boldsymbol{o} + t_{1,2}\boldsymbol{d}_2$$

Ray 3: $\boldsymbol{r}(t) = \boldsymbol{o} + t\boldsymbol{d}_3$

$$t_3 = \frac{-B}{2A}$$

$$\boldsymbol{p}_3 = \boldsymbol{o} + t_3\boldsymbol{d}_3$$

Quadrics - Example



[Suffern]

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Parametric Surfaces

- Are represented by functions with 2D parameters

$$x = f(u,v) \quad y = g(u,v) \quad z = h(u,v)$$

- Intersection can be computed from a (non-linear) system with three equations and three unknowns

$$o_x + td_x = f(u,v) \quad o_y + td_y = g(u,v) \quad o_z + td_z = h(u,v)$$

- Normal vector

$$\boldsymbol{n}(u,v) = \underbrace{\left(\frac{\partial f}{\partial u}, \frac{\partial g}{\partial u}, \frac{\partial h}{\partial u}\right)}_{\text{Tangent}} \times \underbrace{\left(\frac{\partial f}{\partial v}, \frac{\partial g}{\partial v}, \frac{\partial h}{\partial v}\right)}_{\text{Tangent}}$$

Parametric Surfaces, e.g., Cylinder, Sphere

- Cylinder about *z*-axis with parameters ϕ and ν

$$
\begin{aligned}
x &= \cos\phi \qquad 0 \le \phi \le 2\pi \\
y &= \sin\phi \\
z &= z_{\min} + \nu(z_{\max} - z_{\min}) \quad 0 \le \nu \le 1
\end{aligned}
$$

- Sphere centered at the origin with parameters ϕ and θ

$$
\begin{aligned}
x &= \cos\phi\sin\theta \qquad 0 \le \phi \le 2\pi \\
y &= \sin\phi\sin\theta \qquad 0 < \theta \le \pi \\
z &= \cos\theta
\end{aligned}
$$

- Parametric representations are used to render partial objects, e.g. $\phi_{\min} \le \phi \le \phi_{\max}$

Parametric Surfaces, e.g., Disk, Cone

- Disk with radius *r* at height *h* along the *z*-axis with inner radius r_i with parameters *u* and ν

$$\phi = u\phi_{\max} \qquad 0 \le u \le 1$$
$$x = ((1-\nu)r_i + \nu r)\cos\phi \qquad 0 \le \nu \le 1$$
$$y = ((1-\nu)r_i + \nu r)\sin\phi$$
$$z = h$$

- Cone with radius *r* and height *h* and parameters *u* and ν

$$\phi = u\phi_{\max} \qquad 0 \le u \le 1$$
$$x = r(1-\nu)\cos\phi \qquad 0 \le \nu \le 1$$
$$y = r(1-\nu)\sin\phi$$
$$z = \nu h$$

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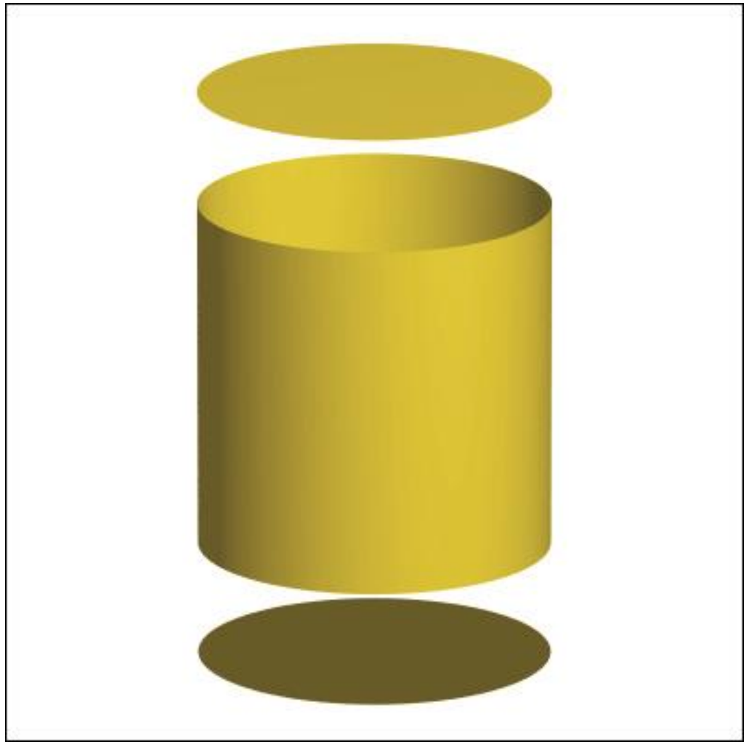
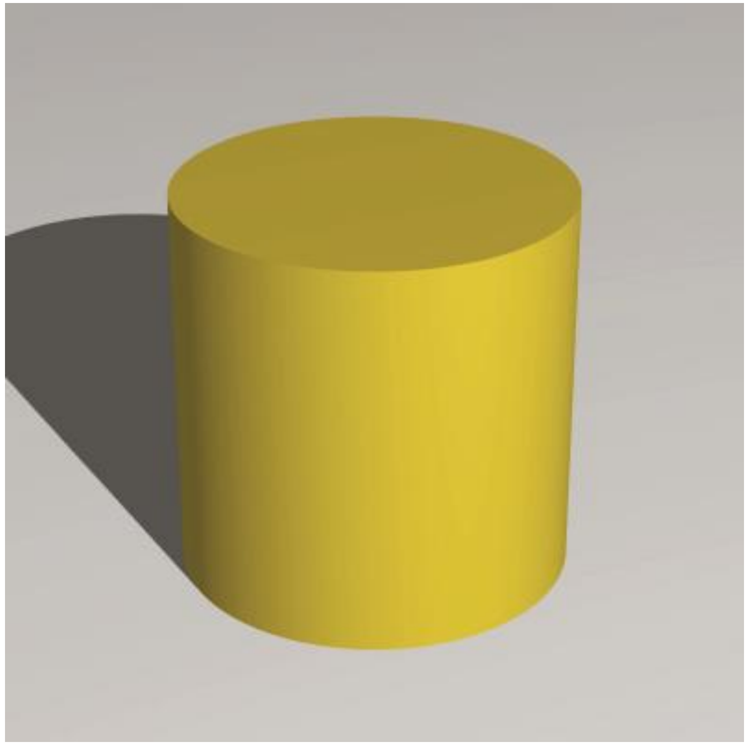
Compound Objects

– Consist of components

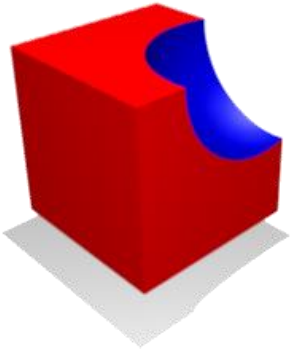




[Suffern]

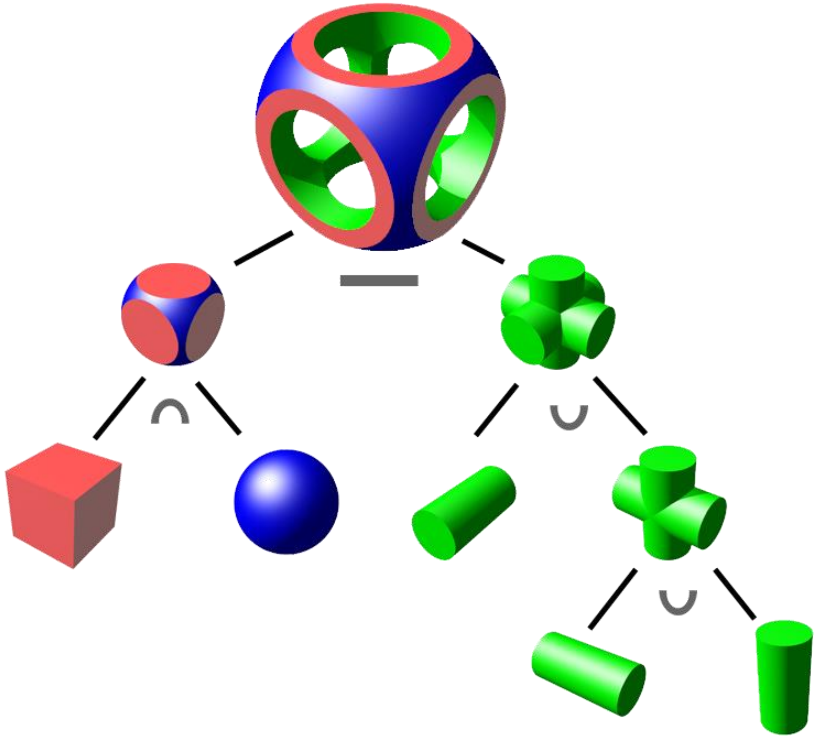
Constructive Solid Geometry CSG

– Combine simple objects to complex geometry using Boolean operators



Difference of a cube and a sphere. Sphere intersections are only considered inside the cube. Cube intersections are not considered inside the sphere.

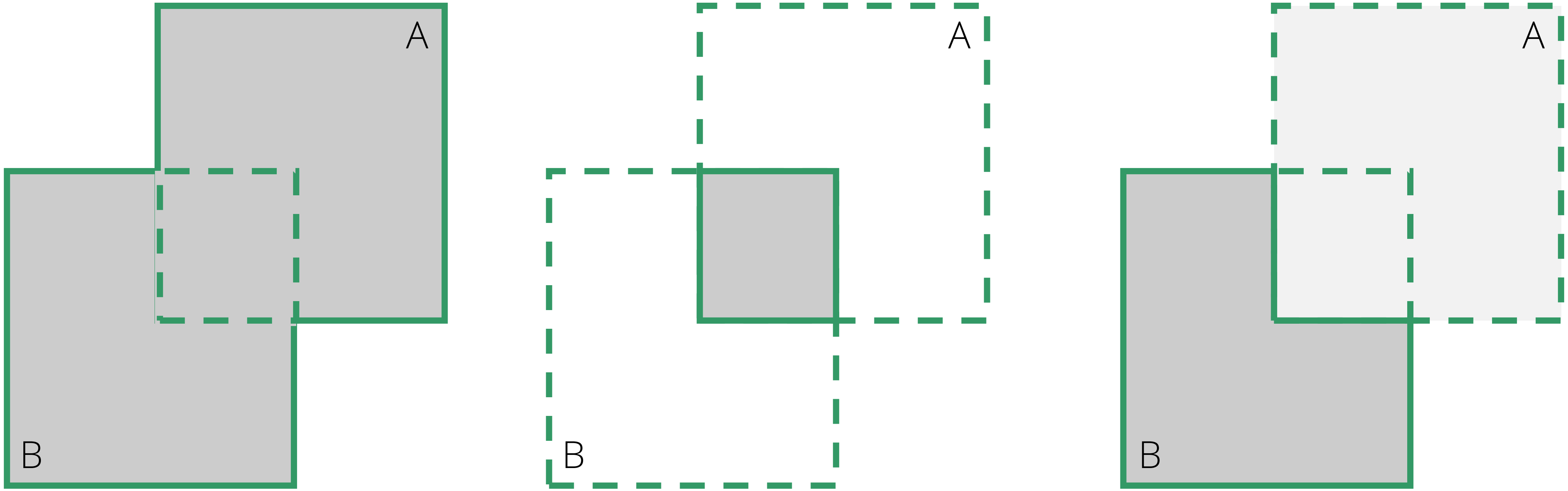
[Wikipedia: Constructive Solid Geometry]



[Wikipedia: Computergrafik]

Constructive Solid Geometry CSG

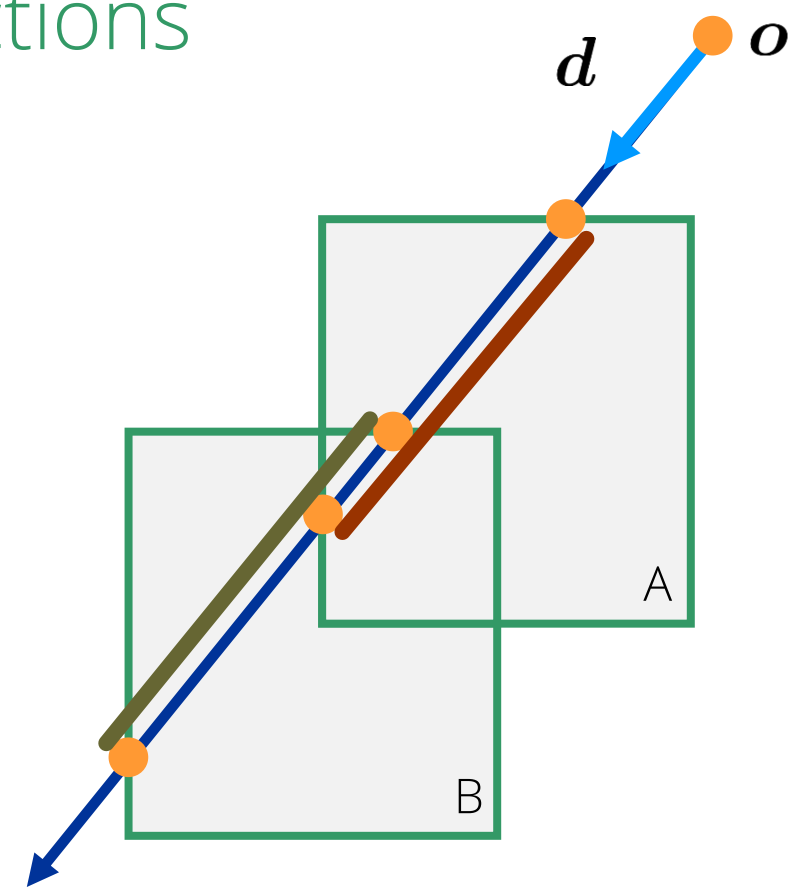
– Closed surfaces / defined volumes required



Union Intersection Difference B-A

Implementation

- Estimate and analyze all intersections
- Consider intervals inside objects
  - Works for closed surfaces
- Union
  - Closest intersection
- Intersection
  - Closest intersection with A inside B or closest intersection with B inside A
- Difference …



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Triangle Meshes

- Popular approximate surface representation
- Surface vertices connected to faces

[Wikipedia: Stanford bunny]

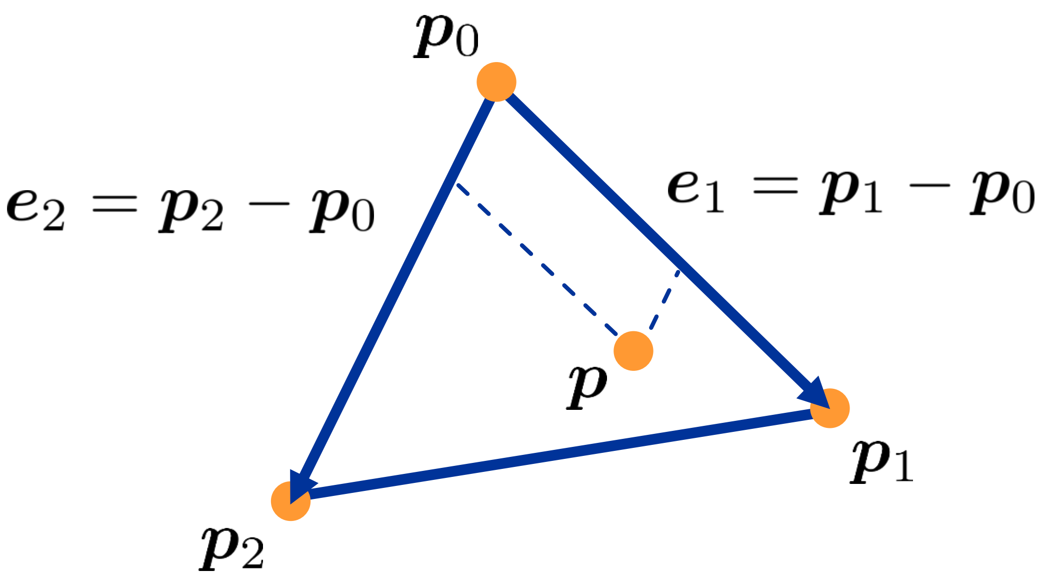
Triangles

– Parametric representation (Barycentric coordinates)

$$\boldsymbol{p}(b_1, b_2) = (1 - b_1 - b_2)\boldsymbol{p}_0 + b_1\boldsymbol{p}_1 + b_2\boldsymbol{p}_2$$

$$b_1 \geq 0 \qquad b_2 \geq 0 \qquad b_1 + b_2 \leq 1$$

Vertices $\boldsymbol{p}_0$, $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ form a triangle. $\boldsymbol{p}$ is an arbitrary point in the plane of the triangle.



$$\begin{aligned}\boldsymbol{p} &= \boldsymbol{p}_0 + b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2\\ &= \boldsymbol{p}_0 + b_1(\boldsymbol{p}_1 - \boldsymbol{p}_0) + b_2(\boldsymbol{p}_2 - \boldsymbol{p}_0)\\ &= \underbrace{(1 - b_1 - b_2)}_{b_0}\boldsymbol{p}_0 + b_1\boldsymbol{p}_1 + b_2\boldsymbol{p}_2\end{aligned}$$

Barycentric Coordinates - Properties

- $\boldsymbol{p}(b_0, b_1, b_2) = b_0\boldsymbol{p}_0 + b_1\boldsymbol{p}_1 + b_2\boldsymbol{p}_2$
- $b_0 + b_1 + b_2 = 1$
- $b_0 = b_1 = 0 \quad \Rightarrow \quad b_2 = 1 \quad \Rightarrow \quad \boldsymbol{p}(b_0, b_1, b_2) = \boldsymbol{p}_2$

  $\Rightarrow$ Point corresponds to a triangle vertex
- $b_0 = 0 \quad \Rightarrow \quad b_1 + b_2 = 1 \quad \Rightarrow \quad \boldsymbol{p}(b_0, b_1, b_2) = 0\boldsymbol{p}_0 + b_1\boldsymbol{p}_1 + (1 - b_1)\boldsymbol{p}_2$

  $\Rightarrow$ Point located on a triangle edge
- $b_0 \geq 0 \wedge b_1 \geq 0 \wedge b_2 \geq 0 \quad \Rightarrow$ Point located inside triangle
- $b_0 < 0 \vee b_1 < 0 \vee b_2 < 0 \quad \Rightarrow$ Point located outside triangle

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Triangles

– Potential intersection point is on the ray and in the triangle plane

$$\boldsymbol{o} + t\boldsymbol{d} = (1 - b_1 - b_2)\boldsymbol{p}_0 + b_1\boldsymbol{p}_1 + b_2\boldsymbol{p}_2$$

Point on the ray

Point in the triangle plane (not necessarily inside the triangle)

$$\boldsymbol{o} - \boldsymbol{p}_0 = -t\boldsymbol{d} + b_1(\boldsymbol{p}_1 - \boldsymbol{p}_0) + b_2(\boldsymbol{p}_2 - \boldsymbol{p}_0)$$

$$\boldsymbol{e}_1 = \boldsymbol{p}_1 - \boldsymbol{p}_0 \qquad \boldsymbol{e}_2 = \boldsymbol{p}_2 - \boldsymbol{p}_0 \qquad \boldsymbol{s} = \boldsymbol{o} - \boldsymbol{p}_0$$

$$\begin{pmatrix} -\boldsymbol{d} & \boldsymbol{e}_1 & \boldsymbol{e}_2 \end{pmatrix} \begin{pmatrix} t \\ b_1 \\ b_2 \end{pmatrix} = \boldsymbol{s}$$

Triangles - Intersection

– Solution

$$\begin{pmatrix} t \\ b_1 \\ b_2 \end{pmatrix} = \frac{1}{(\boldsymbol{d} \times \boldsymbol{e}_2) \cdot \boldsymbol{e}_1} \begin{pmatrix} (\boldsymbol{s} \times \boldsymbol{e}_1) \cdot \boldsymbol{e}_2 \\ (\boldsymbol{d} \times \boldsymbol{e}_2) \cdot \boldsymbol{s} \\ (\boldsymbol{s} \times \boldsymbol{e}_1) \cdot \boldsymbol{d} \end{pmatrix}$$

– Non-degenerated triangle and ray not parallel to triangle plane: $\frac{1}{(\boldsymbol{d} \times \boldsymbol{e}_2) \cdot \boldsymbol{e}_1}$

Triple product. Volume of a parallelepiped.

– Intersection inside triangle: $b_1 \geq 0 \quad b_2 \geq 0 \quad b_1 + b_2 \leq 1$

– Intersection in front of sensor: $t > 0$

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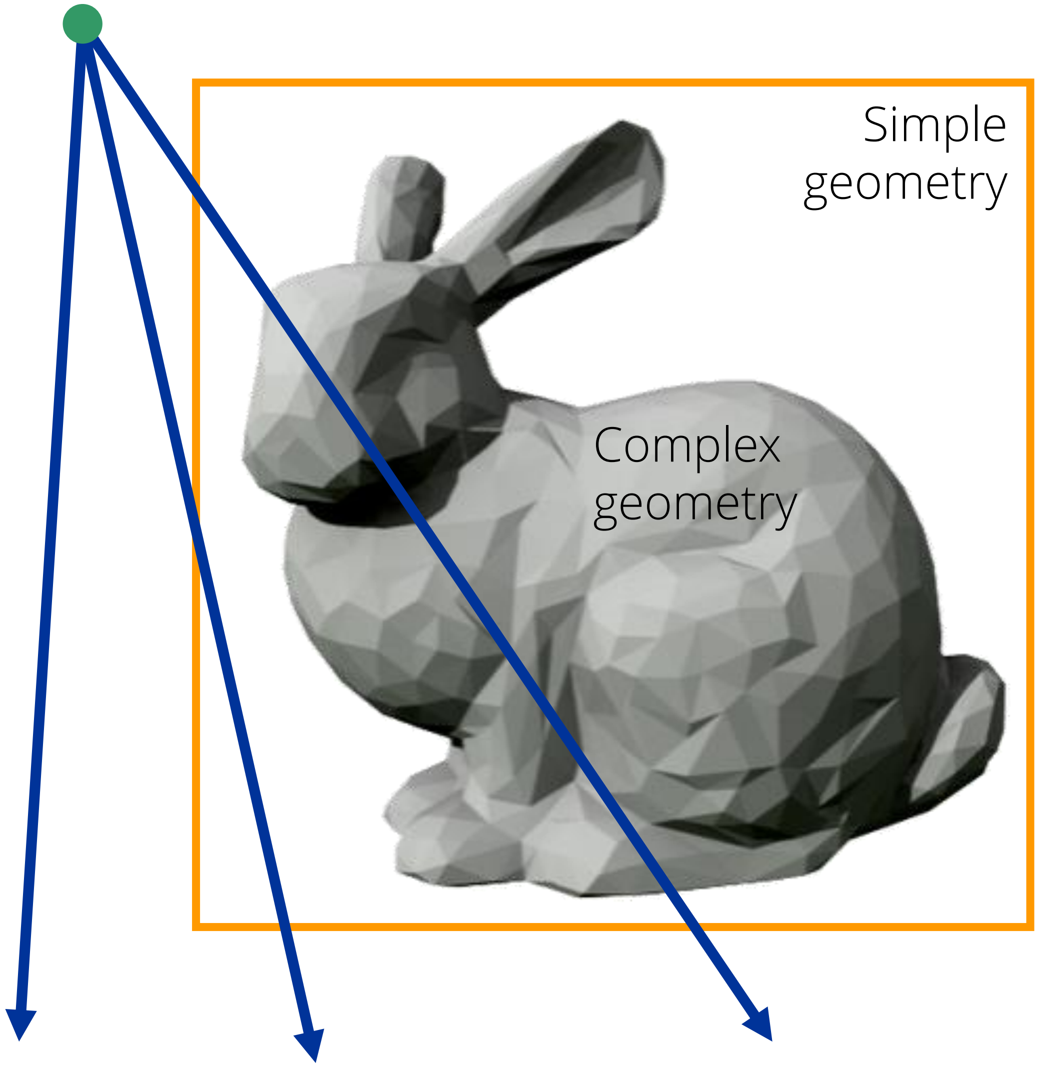
Motivation

- Simple geometry with an efficient intersection test encloses a complex geometry
- Rays that miss the simple geometry are not tested against the complex geometry

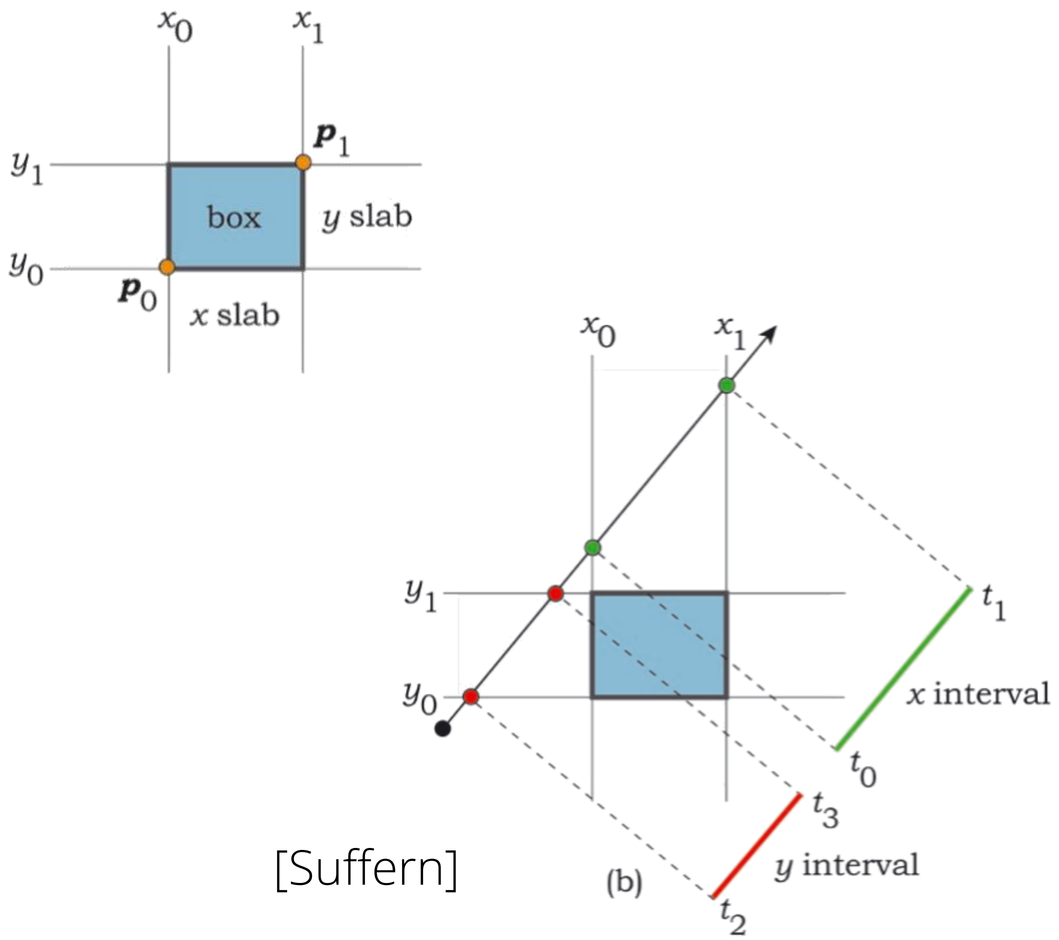


Axis-Aligned Bounding Box (AABB)

- Characteristics
  - Aligned with the principal coordinate axes
  - Simple representation (an interval per axis)
  - Efficient intersection test
  - Can be translated with object
  - Update required for other transformations
- Alternatives
  - Object-oriented boxes, k-DOPs, spheres

AABB

- Boxes are represented by slabs
- Intersections of rays with slabs are analyzed to check for ray-box intersection
  - E.g. non-overlapping ray intervals within different slabs indicate that the ray misses the box



[Suffern]

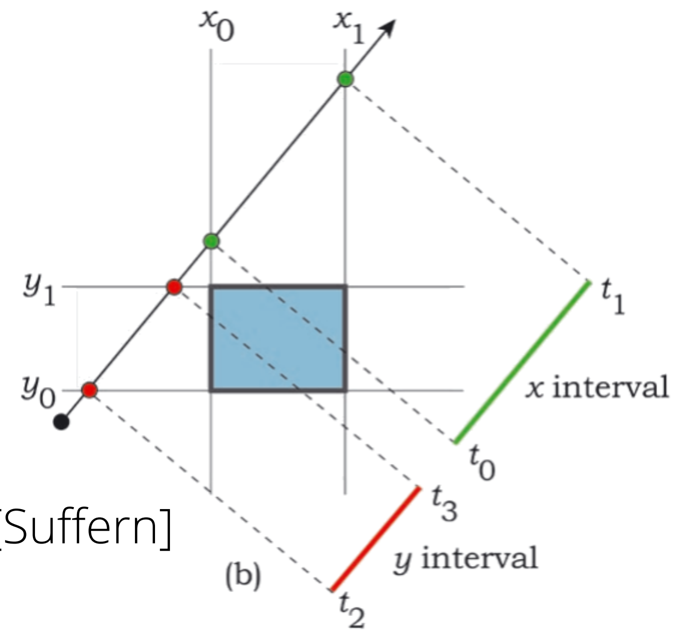
AABB – Intersection Test

- Ray-plane intersection
  - $\boldsymbol{n} \cdot (\boldsymbol{o} + t\boldsymbol{d} - \boldsymbol{r}) = 0$

    $t = \frac{(\boldsymbol{r}-\boldsymbol{o})\cdot\boldsymbol{n}}{\boldsymbol{n}\cdot\boldsymbol{d}}$
- Intersection with *x*-slab
  - $(1,0,0) \cdot (\boldsymbol{o} + t\boldsymbol{d} - (x_{0,1},0,0)) = 0$

    $t_{0,1} = \frac{x_{0,1}-o_x}{d_x}$
- Intersection with *y*-slab
  - $(0,1,0) \cdot (\boldsymbol{o} + t\boldsymbol{d} - (0,y_{0,1},0)) = 0$

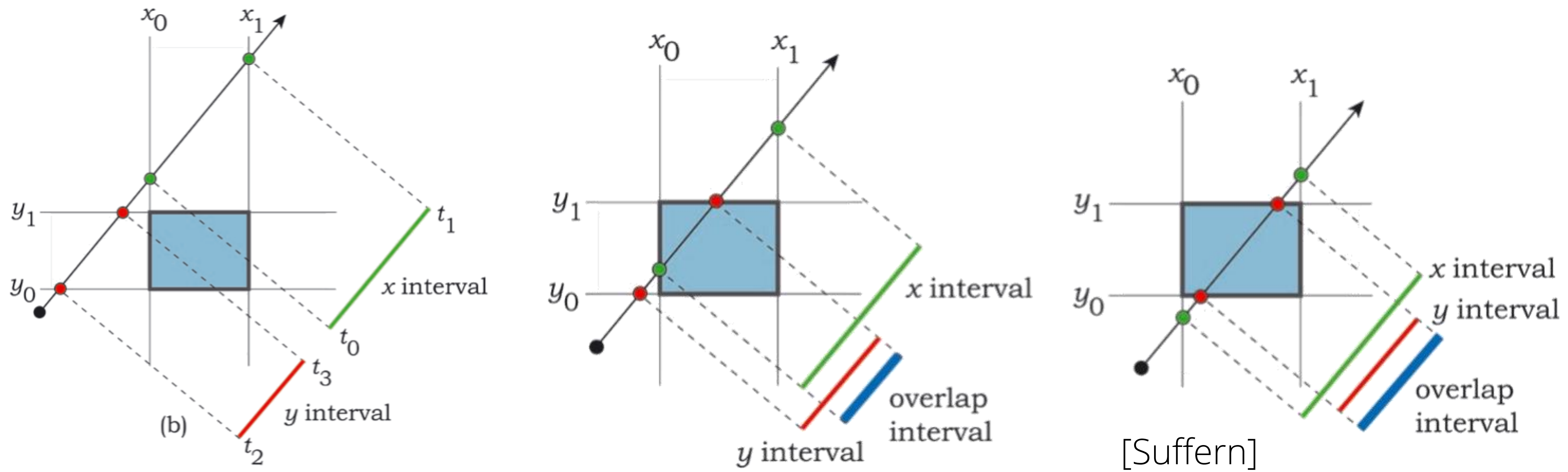
    $t_{2,3} = \frac{y_{0,1}-o_y}{d_y}$



[Suffern]

AABB – Intersection Test

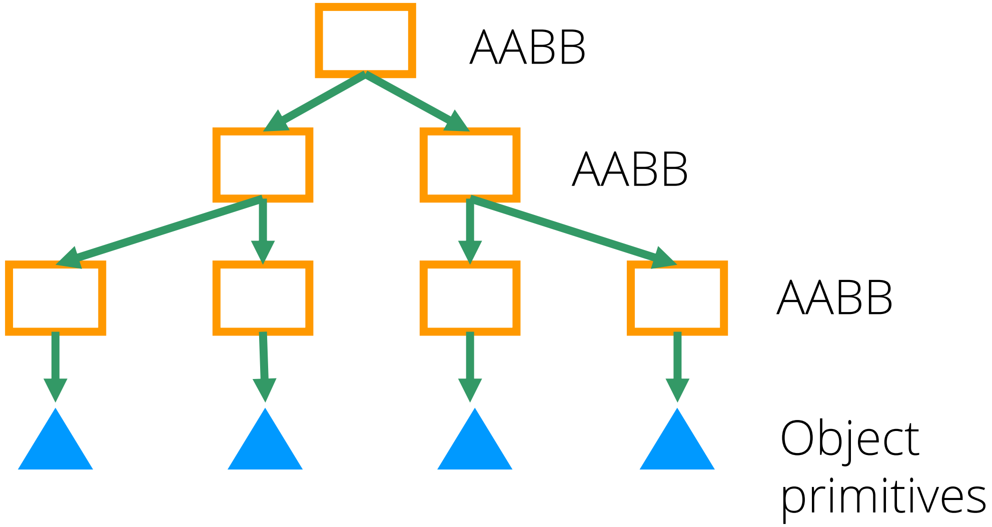
– Overlapping ray intervals inside an AABB indicate intersections

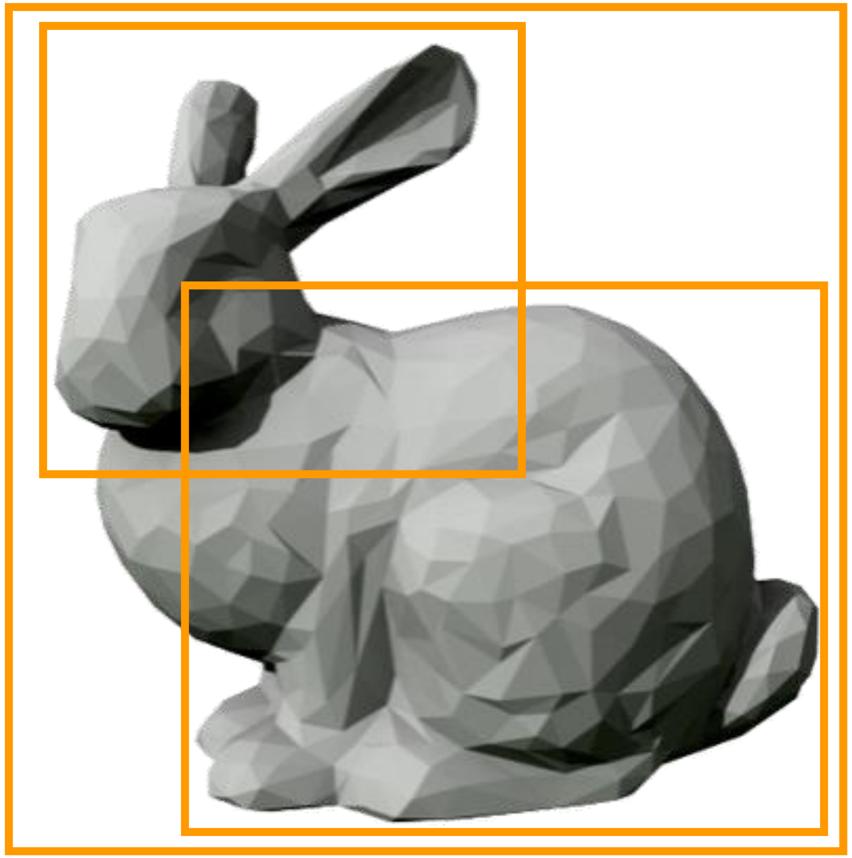


[Suffern]

Bounding Volume Hierarchies (BVH)

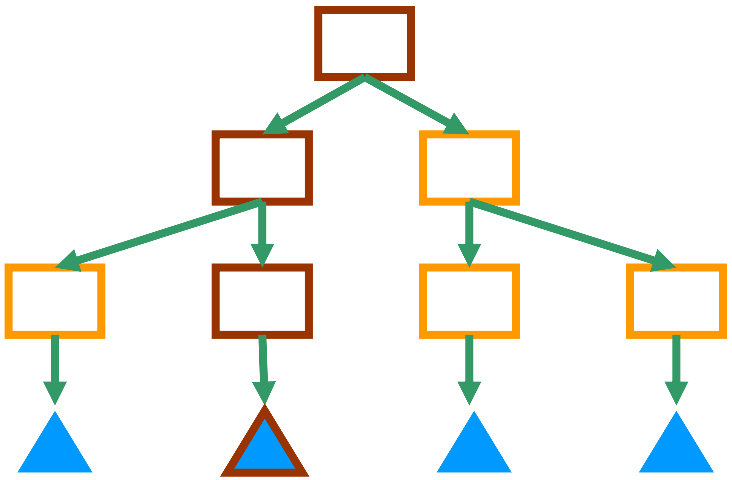
- AABBs can be combined to hierarchies





BVH – Intersection Test

- Traversing the BVH
  - If a box is intersected, test its children
- log *n* box tests for an object with *n* faces
- Efficient pruning of irrelevant regions
- Memory and pre-processing overhead



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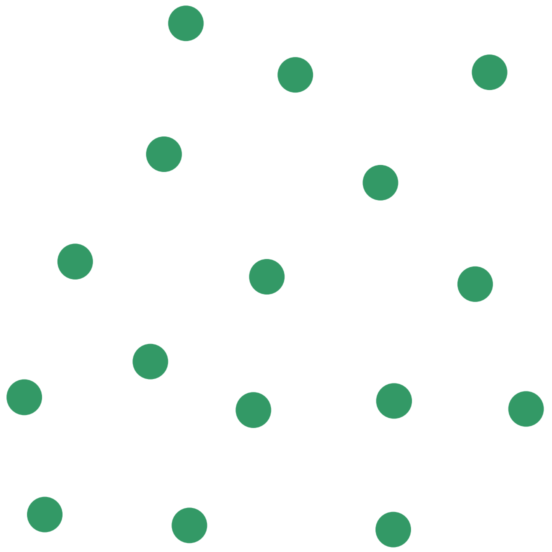
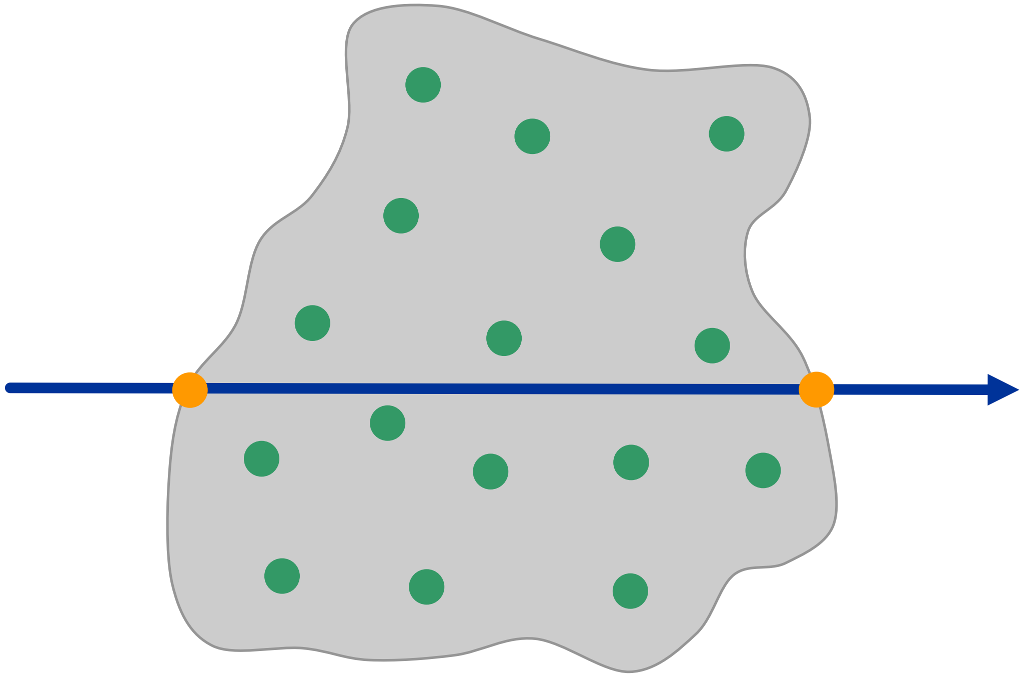
Motivation

– Ray casting of fluid surfaces

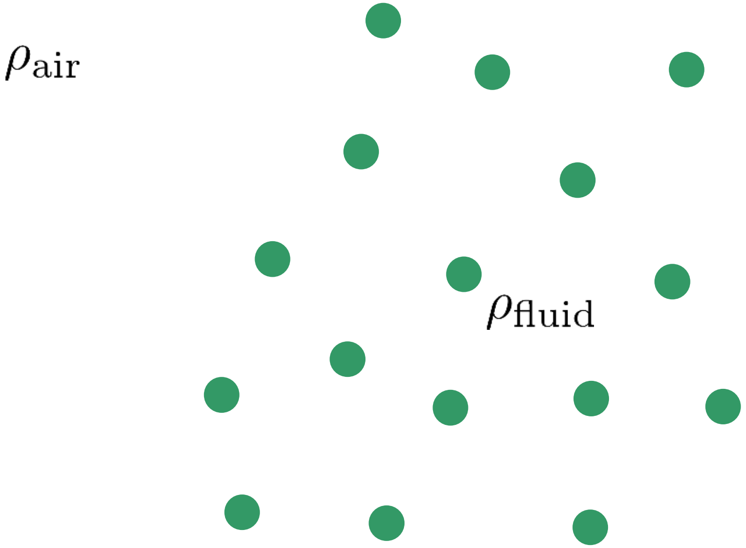


Fluid particles

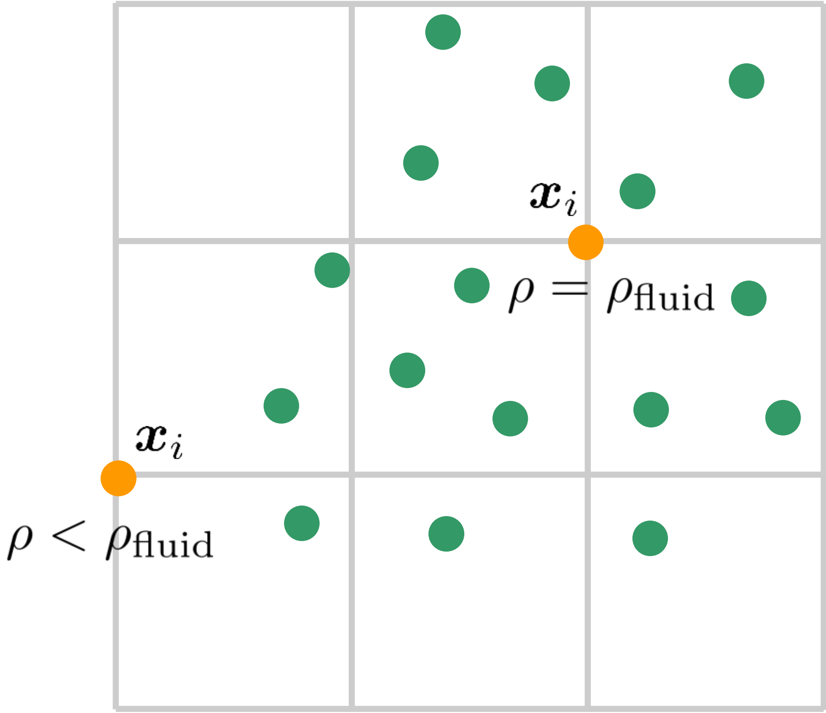


Ray-surface intersection
without explicit surface representation

Density Mapping onto Grid



Fluid particles
with densities

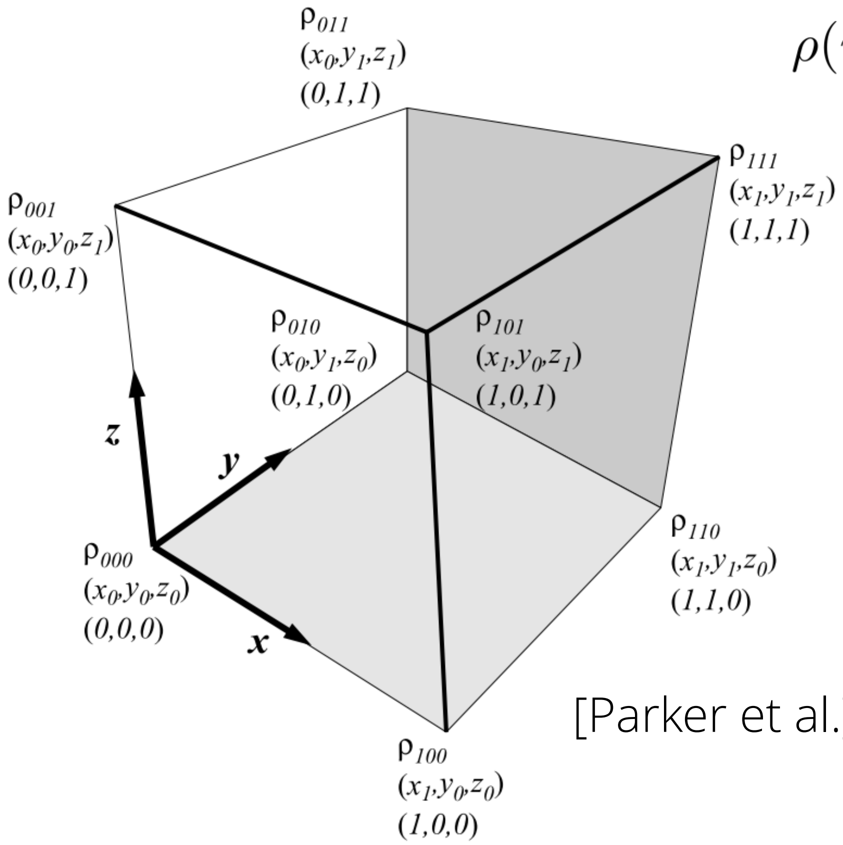


Density interpolation
at grid cells, e.g.

$$\rho(\boldsymbol{x}_i) = \sum_j V_j \rho_{\mathsf{fluid}} W(\|\boldsymbol{x}_j - \boldsymbol{x}_i\|)$$

Density Interpolation in a Grid Cell

– Trilinear interpolation of scalar values inside a grid cell



[Parker et al.]

$$
\begin{aligned}
\rho(u,v,w) = {} & (1-u)(1-v)(1-w)\rho_{000} + \\
& (1-u)(1-v)(w)\rho_{001} + \\
& (1-u)(v)(1-w)\rho_{010} + \\
& (u)(1-v)(1-w)\rho_{100} + \\
& (u)(1-v)(w)\rho_{101} + \\
& (1-u)(v)(w)\rho_{011} + \\
& (u)(v)(1-w)\rho_{110} + \\
& (u)(v)(w)\rho_{111}
\end{aligned}
$$

$$
u = \frac{x - x_0}{x_1 - x_0} \qquad v = \frac{y - y_0}{y_1 - y_0} \qquad w = \frac{z - z_0}{z_1 - z_0}
$$

Ray-Isosurface Intersection

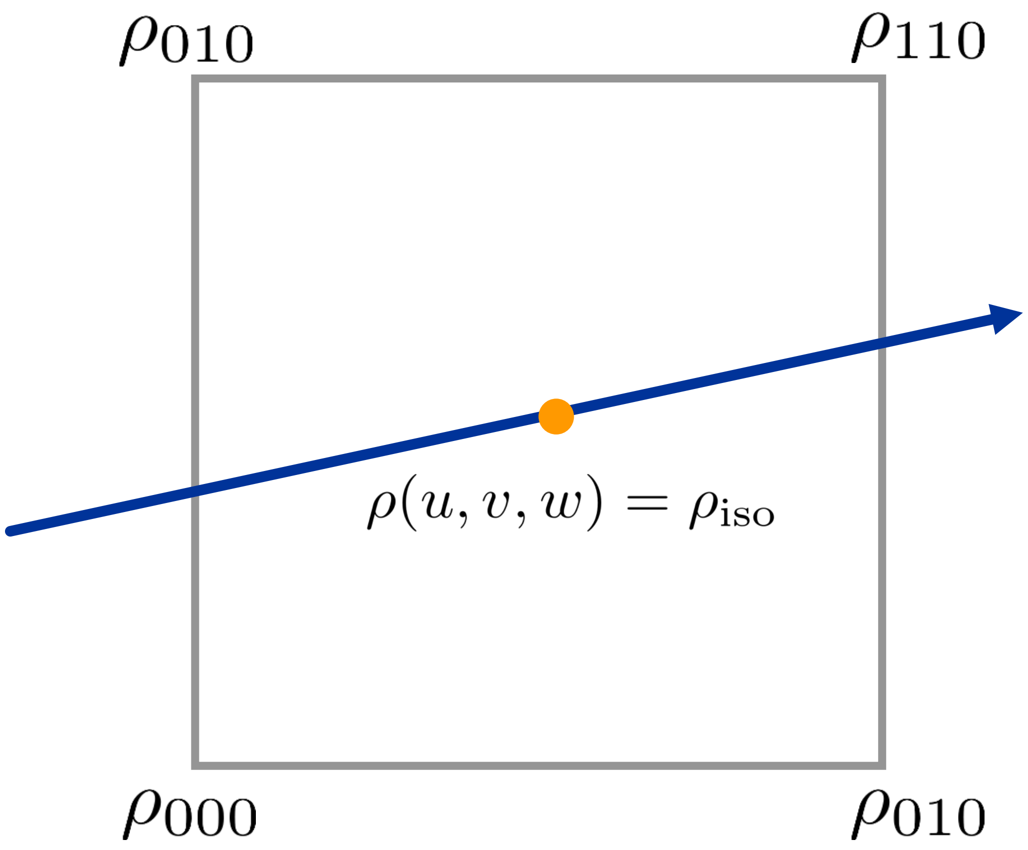
- Define an iso-value

  $\rho_{\text{air}} < \rho_{\text{iso}} < \rho_{\text{fluid}}$

- Ray $\boldsymbol{r}(t) = \boldsymbol{o} + t\boldsymbol{d}$
- Compute u, v, w, t with

  $\rho(u, v, w) = \rho_{\text{iso}}$ and

  $$\begin{pmatrix} x_0 + u(x_1 - x_0) \\ y_0 + v(y_1 - y_0) \\ z_0 + w(z_1 - z_0) \end{pmatrix} = \boldsymbol{o} + t\boldsymbol{d}$$



Intersection Normal

– Gradient of the density field

$$\boldsymbol{n} = \nabla\rho(x, y, z) = \left(\frac{\partial\rho(x,y,z)}{\partial x}, \frac{\partial\rho(x,y,z)}{\partial y}, \frac{\partial\rho(x,y,z)}{\partial z}\right)$$

– Approximated, e.g., with finite differences

$$n_x = \sum_{i,j,k=0,1} \frac{(-1)^{i+1} v_j w_k}{x_1 - x_0} \rho_{ijk}$$

$$n_y = \sum_{i,j,k=0,1} \frac{(-1)^{j+1} u_i w_k}{y_1 - y_0} \rho_{ijk}$$

$$n_z = \sum_{i,j,k=0,1} \frac{(-1)^{k+1} u_i v_j}{z_1 - z_0} \rho_{ijk}$$

Outline

- Context
- Implicit surfaces
- Parametric surfaces
- Combined objects
- Triangles
- Axis-aligned boxes
- Iso-surfaces in grids
- Summary

Ray Casting

- Very versatile concept to compute what is visible at a sensor
  - Implicit surfaces, parametric surfaces
- Expensive for complex geometries
  - Spatial data structures, e.g. bounding volume hierarchies
- Can be simple
  - Linear or quadratic formulations (plane, triangle, sphere)
- Can be involved
  - Implicit representation of iso-surfaces