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PREDICTING THE PRICE LEVEL IN A WORLD THAT CHANGES ALL THE TIME

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I. INTRODUCTION

The aim of this paper is to discuss ways of computing forecasts of the aggregate price level p that are consistent with given projections of the future levels of the money stock (M) and real output (y). The calculation of such price level forecasts becomes relevant now that many national authorities publish medium-term projections for the money supply. Since the ultimate aim of monetary policy is to create stable inflationary expectations, it is vital to know which future path of the price level is implied by any proposed path for the money stock. In this paper, no extraneous information will be used for the computation of the forecasts, apart from historic data for p, M, and y; the forecasts are based on a *limited quantity* of information. In a static world, therefore, it should not be hard to perform this particular problem in macroeconomic forecasting. One could use an "off-line" statistical method, such as ordinary least squares or vector autoregression, to estimate the laws of motion of p, M, and y and subsequently feed in projections for M and y to compute the corresponding values for p. The residual variance of the ex-post "forecast" errors would indicate the degree of precision with which genuine ex-ante forecasts could be made.

How relevant the static version of this forecasting exercise remains in a variable world depends upon the way in which the economic environment varies. To facilitate the distinctions between different types of changes, I shall first put forward an explicit model for p, M, and y in the next section of the paper. The model will consist of an inverted demand for money function that connects prices to money and output, a dynamic single-equation model for M, and a similar model for y. The model will then be used in section III to illustrate the analysis of three types of events, each of which implies a break with the past.

The first type of event is that studied by Brunner, Cukierman, and

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Meltzer (1980). It leads to serially correlated forecast errors for p but does not yet force agents to rethink the model they have been using to compute the forecasts. The second and third types of events to be discussed require agents to undertake a learning process for updating the forecasting formulas. The three classes of events that will be treated in this paper far from exhaust all possible ways in which a break with the past might occur. I shall mention briefly additional radical changes of regime and attempt to argue why their consequences cannot be analysed within the context of our simple model.

In sections IV and V of the paper I make an attempt to show how Kalman filtering methods can be used to model the learning processes that are associated with the types of events that can be analysed with the model. Quarterly data on money, real output, and prices from five Western European countries and the United States are used for the empirical work. Section IV shows how the Multi-State Kalman Filter (MSKF) can be used to simulate the way in which agents learn about a restricted class of changes in the law of motion of an exogenous variable. Section V is concerned with the Recursive Prediction Error (RPE) method of Anderson, Moore, and Ljung that can be used to model a learning process regarding transitory and permanent shifts in the demand for money. Both the MSKF method and the RPE algorithm are "on-line" methods, so that the forecasts and the parameter values generated are purely ex-ante and do not require any information from time periods that are yet to come. The paper terminates with a brief concluding section.

II. A THREE-VARIABLE MODEL FOR THE PRICE LEVEL

Exhibit 1 gives the three-equation model that will be analysed in this paper. The central equation of the model is equation (2) that uses an inverted demand for money function to connect the expected future price level in a closed economy with expected values of the money stock and the level of real output. $(1-\theta_5)$ is the income elasticity of the demand for money, and θ_7 is minus the elasticity of the demand for money with respect to the opportunity costs of holding money as measured by the expected rate of change of the price level. If the expected real rate of interest were constant, then $-\theta_7$ could also represent the interest elasticity of the demand for money. I do not deny that the fit of this inverted demand for money function over a historical period could be improved by including observed values for interest rates in addition to the term in expected inflation. However, such a procedure would only make it more difficult to use the results for assessing the feasibility of forming medium-term

Exhibit 1. - a three variable model for prices, money, and output

state equations (RPE method)

$$c^e_{t+1} = c^e_t + \theta_6 + \theta_5 \Delta y^e_{t+1} + \theta_4 \epsilon_t \qquad (1)$$

$$p^e_{t+1} = c^e_{t+1} + M^e_{t+1} - y^e_{t+1} + \theta_7(\hat{p}^e_{t+2}) \qquad (2)$$

observation equation

$$p_t = p^e_t + \theta_1(M_t - M^e_t) - \theta_2(y_t - y^e_t) + \epsilon_t \qquad (3)$$

auxiliary equations

$$M^e_{t+1} = M_t + \hat{M}^e_{t+1} \qquad (4)$$

$$y^e_{t+1} = y_t + \hat{y}^e_{t+1} \qquad (5)$$

expectations of exogenous variables (MSKF method)

$$\hat{M}^e_{t+1} = \hat{M}^e_t + (1-\psi_M)(\hat{M}_t - \hat{M}^e_t) \qquad (6)$$

$$\hat{y}^e_{t+1} = \hat{y}^e_t + (1-\psi_y)(\hat{y}_t - \hat{y}^e_t) \qquad (7)$$

meaning of symbols:

p - the natural logarithm of the aggregate price level

M - the natural logarithm of the money stock

y - the natural logarithm of the level of real output

A caret " ^" indicates the relative rate of growth of a variable.

θ_1, θ_2, θ_4, θ_5, θ_6, θ_7 and ψ_M, ψ_y are parameters; agents learn about their optimal values as time goes on. ϵ_t is a serially uncorrelated error-term.

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projections of the price level, because our ability to predict movements in real interest rates is still limited. Therefore I have limited the information set in this model to data on money and output, past data on the price level and internally generated rational expectations of future rates of inflation. This information set can be appropriate for a model of a closed economy or an economy on flexible exchange rates; in an open economy with fixed exchange rates, one would want to include the rate of change of some foreign price index in the information set that agents use to form their inflation forecasts.

Equation (2) is an inverted demand-for-money function in the levels of M, y, and p. If the income velocity of money is subject to permanent shifts over time, with the precise extent and the timing of such shifts unknown to the econometrician, then estimation of an ordinary least squares equation in terms of levels runs into difficulties. Thus, many researchers prefer to work with rates of growth of M, y, and p, since permanent shifts in velocity have a transitory influence only on the rate of change of velocity. (see Plosser and Schwert (1978) for additional arguments in favor of working with differenced data).

However, a specification in terms of growth rates makes the estimation of any lagged effects much more difficult. To see this, assume, for example, that the time series model that connects two variables x and y is:

$$y_t = x_{t-1} + u_t$$

with u_t a serially uncorrelated residual term

let x_t be a pure random walk:

$$x_t = x_{t-1} + v_t$$

with v_t a serially uncorrelated white noise term.

If a researcher is unaware of the one period lag and mistakingly regresses y_t on x_t instead of x_{t-1}, he will still obtain a consistent and unbiased estimate of the true regression coefficient, since

$$y_t = x_t + u_t - v_t$$

However, if the regression is run in first-difference form, then the coefficient of Δx will only be 1 if the lag is specified correctly. A regression of Δy_t on the contemporaneous change in x, Δx_t, will produce a regression coefficient that tends to 0:

$$\Delta y_t = 0.\Delta x_t + v_{t-1} + u_t - u_{t-1}$$

As in the case of the demand for money, we are often confronted with both problems at the same time: permanent shifts of unknown magnitude and timing in the "constant term" of our model if it is specified in terms of levels, and lack of guidance from economic theory with respect to time lags that makes it hazardous to work with differenced data. With the Recursive Prediction Error method that will be discussed below we are able to work with levels, because the "constant term" in our model can incorporate permanent shifts that may occur.

Equation (1) explains the evolution over time of the "constant term" in the inverted demand for money function. θ_6 represents a trend term about which agents learn as time goes on. The term $\theta_5 \Delta y^e_{t+1}$ indicates whether the expected path of velocity is income dependent. If θ_5 equals 0, then the elasticity of the expected demand for real balances with respect to expected real income is unity. The final term in equation (1), $\theta_4 \epsilon_t$, has to be seen in connection with the error term in equation (3). It shows which proportion of the unexplained surprises in the price level is relevant for the prediction of next period's price level.

Together, equations (1) and (2) are the "state equations" in our model. They indicate the movement of the unobservable variables c^e_{t+1} and p^e_{t+1}. An anchor is provided by equation (3), which is called an "observation equation" in the terminology of Kalman filtering. This equation shows how the observed value of the current price level p_t is related to the expectation p^e_t that was formed at the end of the previous period. Part of the discrepancy can be explained by the current surprises in the two exogenous variables of the model: $(M_t - M^e_t)$ and $(y_t - y^e_t)$. ϵ_t represents that portion of $(p_t - p^e_t)$ which cannot be explained by these two variables.

Our model does not explore the interconnections between current surprises in money, output, and all the relevant opportunity costs in the demand for money. Therefore it is impossible to deduce on theoretical grounds the signs of the coefficients of $M - M^e$ and $y - y^e$ in equation (3). If the predominant effect of an unanticipated monetary surprise is to shift the aggregate demand curve outward in an aggregate $p - y$ diagram, then such a surprise raises both output and prices beyond their expected values, and the sign of θ_1 should be positive. If, however, the expectational errors $(y_t - y^e_t)$ are primarily caused by unforeseen changes in productivity, these should be represented by unforeseen shifts of the aggregative supply-curve along the aggregative demand-curve and the coefficient of $y_t - y^e_t$ should be negative. θ_1 and θ_2 do not have an interpretation as elasticities and do not shed light upon the nature of the transmission process. The observation equation has been extended with terms in $(M_t - M^e_t)$ and $(y - y_e)$ only to sharpen the estimates of the state variable p^e_t and the parameters θ_5, θ_6, θ_7, and θ_4 in the state equations. The discrepancies $(p_{t+1} - p^e_{t+1})$ are the final measure of the usefulness of this model in predicting future price levels; the residuals of the observation equation ϵ_t serve to steer the evolution of the state variables and the parameters. Details of the computational procedure are given in section IV below.

Equations (4) and (5) are auxiliary equations that connect the expected *levels* and the expected *growth rates* of M and y. Levels of variables are used in the two state equations and in the observation equation that are to be estimated with the Recursive Prediction Error method. Since the univariate models for money and output compute expected growth rates of these variables (equations (6) and (7)), the auxiliary equations (4) and (5) are needed to connect the two.

Equations (6) and (7) govern the evolution of expectations with respect to money growth and output growth. This particular type of simple error-learning process is appropriate if the actual growth rates of M and y follow an ARIMA (0, 1, 1) process. As will be explained below, the Multi-State Kalman Filter not only allows us to learn about the current value of the state variables $\dot{M}^e_{t+1}$ and $\dot{y}^e_{t+1}$ but also allows for learning with respect to the parameter values ψ_M and ψ_y. Moreover, the method allows for the values of ψ_M and ψ_y to differ according to the magnitude of the current error $\dot{M} - \dot{M}^e_t$ and $\dot{y}_t - \dot{y}^e_t$. In other words, the MSKF algorithm can cope with situations in which small errors in predicting money growth or output growth have to be incorporated almost one-to-one into the revisions of expected future growth rates (ψ close to zero), whereas any exceptionally large prediction errors that might occur should not be incorporated one-to-one into the expected growth rates, being more temporary (ψ close to one).

As the layout of the model in Exhibit I shows, the problem of forming rational forecasts of the two input variables $\dot{M}$ and $\dot{y}$ is considered prior to the problem of computing optimal forecasts of the expected future price level. An alternative procedure would be to compute forecasts of the input variables jointly with forecasts of the endogenous variables of the model. I have not followed the simultaneous approach, since to do so would violate to some extent the assumption that $\dot{M}$ and $\dot{y}$ are exogenous input variables. The problem to be considered in this paper is that of forecasting price levels for given projections of money and output. This setting does not allow for feedback from prices to money and output, and therefore the expectations of future money growth and output growth have not been calculated in a simultaneous multivariate context.

III. DIFFERENT WAYS IN WHICH THE WORLD MAY CHANGE

Throughout this section I shall assume that economic agents are aware of the current values of the parameters θ_1, θ_2, θ_4, θ_5, θ_6, θ_7 and the parameters ψ_M and ψ_y in the models for $\dot{M}^e$ and $\dot{y}^e$. Agents are also aware of the current variance-covariance matrix of these parameters and of the variance in the error term ϵ_t. They do not possess any insight into the future plans of the monetary authorities, and all the new information they get is limited to the current values of p, M, and y. Consequently, they can learn only about any changes that may have occurred through deducing their nature from the patterns over time of the prediction errors $\hat{M}_t - \hat{M}_t^e$ and $\hat{y}_t - \hat{y}_t^e$ and the model errors ϵ_t.

Within the context of our simple model, the following three types of unexpected events can be profitably studied:

1. *A purely temporary change in the mixture of transitory and permanent shocks that determine the evolution of $\hat{M}_t$ and/or $\hat{y}_t$.*

2. *A permanent change in the relative importance of the transitory and permanent shocks that govern the behavior of $\hat{M}_t$ and/or $\hat{y}_t$.*

3. *A permanent change in the relative importance of temporary and permanent shifts in the inverted demand for money function, equation (2).*

These three types of events will be studied in some detail, and the empirical work in sections IV and V will show how the learning processes necessitated by these events can be modelled. The present section will conclude with a brief discussion of some radical types of events, such as the money supply process getting into a "higher gear," or a qualitative change in the inflationary process. Such events are frequently discussed in the literature; for example, in connection with the dynamic stability of the demand for money, but I shall argue that they are out of bounds within the context of the present model, because they would involve a learning process that requires changes not only in the value of the parameters but also in the specification of the demand for money function.

Type 1

A purely temporary change in the mixture of transitory and permanent shocks that determines the evolution of $\hat{M}_t$ and/or $\hat{y}_t$. This type of event is best studied when writing the ARIMA (0, 1, 1) model in the Kalman filter way:

$$\bar{\hat{M}}_t = \bar{\hat{M}}_{t-1} + \eta_{M,t} \tag{8}$$

$$\hat{M}_t = \bar{\hat{M}}_t + \epsilon_{M,t} \tag{9}$$

$\epsilon_{M,t}$ and $\eta_{M,t}$ are mutually independent and serially uncorrelated error terms. A similar two-equation model can be written down for $\bar{\hat{y}}_t$. In these Kalman filter models, equation (8) is the state equation that shows how the unobservable expected growth-rate of money moves as a pure random walk over time. Equation (9) is the observation equation that shows how the actual growth-rate of money anchors the expectational values. The connection between equations (8) and (9) with the first order moving average model becomes clear when we shift equation (9) one period backwards and subtract the result from the original equation (9). We get:

$$\Delta\hat{M}_t = (\bar{\hat{M}}_t - \bar{\hat{M}}_{t-1}) + \epsilon_{M,t} - \epsilon_{M,t-1} = \eta_{M,t} + \epsilon_{M,t} - \epsilon_{M,t-1} \qquad (10)$$

It follows from equation (10) that the autocorrelation function (ACF) of $\Delta\hat{M}_t$ must be zero for all lags greater than one. The corresponding time series model, therefore, must be the first order moving average model:

$$\Delta\hat{M}_t = (1 - \psi B)a_t \qquad (11)$$

Here, ψ is a constant moving average parameter and the a_t are the noises that drive this time series model. Box and Jenkins (1970, 122-125) show how a comparison of the ACFs of (10) and (11) results in an equation expressing the value of ψ as a function of the variances of $\epsilon_{M,t}$ and $\eta_{M,t}$.

As long as the ratio of the variances of $\epsilon_{M,t}$ and $\eta_{M,t}$ does not change, both the Box-Jenkins model (11) and the Kalman filter model (8) and (9) produce serially uncorrelated errors. Assume now that there is a purely temporary deviation from the usual patterns of the shocks. A useful example would be the simultaneous occurrence of an exceptionally large value for the permanent shock η, and a zero value for the transitory shock ϵ, after which both noise terms return to normal. This particular case of a temporary aberration is studied extensively by Brunner, Cukierman and Meltzer (1980) within the context of a model that incorporates more markets than the one-market model of section II but is constructed out of building blocks similar to those I have employed.

period	η	ϵ
0	normal	normal
1	large	0
2	normal	normal
3	normal	normal
"	"	"

With ψ between zero and one, the above sequence of shocks must result in serially correlated forecast errors for $\hat{M}$, because agents are not aware that the large shock in period 1 is of a purely permanent nature. Thus, as is stressed by

Brunner, Cukierman, and Meltzer, a single event gives rise to a series of correlated errors:

> *Evidence of ex post serial correlation in a particular sample is not evidence of inefficient use of information. Rational agents, looking back on the period, find support for the hypothesis that a large permanent shock occurred but was misperceived at the time.* (Brunner et al., p. 486)

Brunner, Cukierman, and Meltzer show how an unavoidable confusion between transitory and permanent shocks helps to explain why real wages appear to be "sticky," and why a single unexpected event that is misinterpreted by economic agents can lead to a lengthy period during which the actual unemployment rate deviates from the natural rate.

A learning process takes place, of course, with respect to the correct value of the underlying, permanent growth-rate of the money stock $\hat{\bar{M}}_t$. However, as long as the event described above occurs in isolation, there is no need for agents to update their estimate of the moving average parameter ψ. Only if the frequency with which this type of event occurs increases will agents have to revise their estimates of the variance of the permanent shocks and will their estimate of ψ change accordingly. But then we have reached the next item on our agenda: an event that permanently alters agents' perception of the law of motion for an exogenous variable.

Type 2

A permanent change in the relative importance of the transitory and permanent shocks that govern the behavior of $\hat{M}_t$ and/or $\hat{y}_t$. This second type of change can occur in at least three ways:

First, it is possible that agents perceive that the variance of their forecast errors in predicting the growth-rates of money and/or output has increased; *second,* they may have doubts about the purely incidental character of a change of the first type described above and, therefore, wish to adapt their prior probabilities regarding the expected future variances of ϵ and η; *third,* agents may find that their current estimate of ψ is no longer optimal and that a different value for the parameter results in forecast errors that better approximate serially uncorrelated white noise.

In all three cases, it remains correct to describe the evolution of $\hat{M}_t$ and $\hat{y}_t$ by means of an ARIMA (0, 1, 1) model. There is a *quantitative* change in that the optimal value of the moving average parameter and/or the estimate of the residual variance changes, but there is no *qualitative* change to another time series model. Similarly, the simple Kalman filter of equations (2) and ([illegible]) can still describe the laws of motion of $\hat{M}_t$ and $\hat{y}_t$; it is only the variances of the transitory and/or permanent noises that have changed. As a consequence, one important property of this particular time series model has been preserved, namely, the fact that there is a constant term structure of expectations for $\hat{M}^e$ and $\hat{y}^e$. It follows that a constant term structure of expectations for $\hat{p}_t^e$ is also preserved (see Bomhoff, 1980). Agents are unable to foresee any changes in the expected rates of growth of money, output, and prices.

This feature of a flat term structure of inflationary expectations leads to important and welcome simplifications in the demand for money function. In theory the current demand for money holdings should depend on the expected rate of price change between now and period $t+1$, between periods $t+1$ and $t+2$, between periods $t+2$ and $t+3$, etc. (see Motley, 1967; Brock, 1972, 1974). With a constant term structure of expectations, this whole string of expectational variables collapses into the single representative expectation $\hat{p}_{t+1}^e$. Thus, we can limit the substitution margin in the demand for money function to just the expected rate of change of prices between the current and next period, and it is only this single term that has been included in our inverted demand for money equation (2).

The specification of the inverted demand for money function remains the same, but an event of this second type will affect the magnitude of the estimated parameters, both in equation (2) for the expected values of the price level and in equation (3) for the differences between expected and actual price levels. As a matter of principle, it should be possible to deduce the changes in these coefficients from the changes in the stochastic processes for $\hat{M}_t$ and of $\hat{y}_t$. One would have to formalize the appropriate intertemporal maximization problem and to view the parameters that govern the law of motion of the exogenous variables as constraints upon this optimization problem. Within this context, the demand for money function would have the status of a lower-level "decision rule" (Sargent, 1981), and it should - in theory - be possible to deduce the parameters of the demand for money as functions of the parameters of preferences, technologies, and constraints in the intertemporal maximization problem.

At the present state of the art, it is feasible to perform exercises in comparative statics with stochastic equilibrium models, but it does not yet seem possible to simulate the learning process that must occur during the transition from one state to another (see Sargent, 1981, for the reasons why learning is hard to incorporate within the stochastic equilibrium models). However convenient it may be for econometric and theoretical reasons to abstract from learning processes, it may still be worthwhile for economic reasons to incorporate a suboptimal learning mechanism if the behavior of [illegible]us variables over time shows that there is much to be learned. Th[illegible] estimates of the models for $\hat{M}_t^e$ and $\hat{y}_t^e$ to be discussed in the next se[illegible] that rational agents would continuously have to adapt their estim[illegible] and ψ_y and the residual variances in the models for $\hat{M}_t^e$ and $\hat{y}_t^e$. [illegible]s of the money supply and of real output were changing, and th[illegible]ameters in the inverted demand for money function (2) and t[illegible]tion equation (3) must have been changing as well. Something ha[illegible] either the cross-equation restrictions between equations (6) and [illegible]quations (1) - (3), or the incorporation of learning processes. I have [illegible] to neglect the cross-equation restrictions, because the data for the two exogenous variables show so clearly that the laws of motion for $\hat{M}$ and $\hat{y}$ were changing during the period under review (1961-1978).

The parameters θ_1 - θ_7 must change over time as agents learn about the laws of motion of the exogenous variables. However, the general specification of equations (2) and (3) remains correct; in particular, the restriction of the term structure of inflationary expectations to the single term $\hat{p}_{t+2}^e$. Events of this second type are thus admissible within the context of the model.

Type [illegible]

A permanent change in the relative importance of the temporary and permanent shifts in the inverted demand for money function, equation (2). It is unavoidable in empirical macroeconomics that not all factors determining the movements of a macroeconomic variable can be modelled satisfactorily. The demand for money, for example, is influenced by changing payment habits and by technological developments in the financial sector, but it is far from easy to find quantitative time series data that represent these developments well. The researcher has to take recourse in less-than-perfect proxy variables, or he has to assume that the constant term and/or any deterministic trend terms in his model will serve as stand-ins for the omitted variables.

Assume, for instance, that one wishes to explain variable y and that quantitative data are available for two important exogenous influences, x_1 and x_2. All other factors that influence y have been subsumed in the constant term c and a linear deterministic trend t, so that the researcher estimates the following regression equation:

$$y_t = c + a_1 x_{1,t} + a_2 x_{2,t} + a_3 . t + \epsilon_t$$

If this equation is estimated by simple or generalized least squares, then the constant term c is treated as just one more parameter, namely, the coefficient of a fictitious variable that takes on the value 1 at all times. The role of c, however, is to stand in for all the nonspecified influences on y, and, therefore, it would be more appropriate if c were regarded as an unobservable exogenous variable. Proceeding further, a natural assumption would be that the time series behavior of c is similar to that of the observed exogenous variables x_1 and x_2. If, for example, x_1 and x_2 are nonstationary time series, then it would seem logical to assume that the unobservable exogenous influences proxied by c are nonstationary also.

The Recursive Prediction Error algorithm is capable of estimating both the values of regression parameters, such as a_1, a_2 and a_3, and of producing an estimate of the current positions of one or more unobservable variables such as c. The time series properties of the "constant term" can be specified to conform either with the time series properties of the observable exogenous variables, or to agree with the researcher's prior notions of the evolution over time of the unobservable variables that c is meant to represent. Here I have modelled the "constant term" as a random walk, augmented with a trend term plus a term that allows for an influence of the expected increase in real income:

$$c^e_{t+1} = c^e_t + \theta_6 + \theta_5 \Delta y^e_{t+1} + \theta_4 \epsilon_t \qquad (1)$$

The RPE algorithm provides recursive estimates of θ_4, θ_5, and θ_6 that reflect ongoing learning about the correct values of these parameters. θ_4 indicates the relative weights of the transitory and permanent shifts in expected velocity that are not explained by y^e_{t+1} and $\hat{p}^e_{t+1}$. If θ_4 is zero, than any current inexplicable errors in predicting the price level p_t are of a purely temporary nature and do not lead to adjustments in the predictions of future price levels. If θ_4 is equal to 1, then the error in the observation equation ϵ_t is incorporated fully into the path of expected future price levels. With $\theta_4 > 1$ the errors ϵ_t lead to more than proportional corrections for p^e_{t+1}.[1]

Just as in the case of type 2, changes in the estimated value θ_4 should be accompanied by changes in the other coefficients of equations (2) and (3). But, since the change is not of a qualitative nature - the unobservable variable c^e_t remains a pure random walk plus trend - the original *specification* of equations (2) and (3) should still be correct. Only if the time series process for c^e_t did change - for example, if c^e_t became constant over time - would it become

[1] Our algorithm constrains θ_4 to lie between 0 and 2 in order to ensure stability (see Moore and Weiss, 1979).

necessary to return to first principles to investigate whether the forms of equations (2) and (3) were still correct or whether the intertemporal maximization problem that underlies the decision rule for money holdings would require a different specification for the demand for money. A learning process with respect to θ_4 is not such a break with the past that it would require a different demand for money function and is, therefore, admissible within our model. Changes in θ_4 are accompanied by changes in θ_1, θ_2, θ_5, θ_6, and θ_7; once again the cross-equation restrictions emphasized by Sargent and others have not been imposed because these are not suitable in a context of learning.

Inadmissible Experiments

The demand for money function is - in the terminology of Lucas and Sargent - a "decision rule," derivable at least in principle from some higher-level intertemporal maximization problem. The specification of this decision rule already tells us much about the constraints under which economic agents maximize their utility. If, for example, consumers live in constant fear of hyperinflation, then their demand for money holdings would depend not only on the expected rate of inflation in the immediate future, $\dot{p}^e_{t+1}$, but also on a whole string of expectations with respect to the price level in later periods. As soon as a researcher decides to limit the term structure of expectations in the demand-for-money function to just the expected rate of inflation between now and the next period, he has implicitly decided already that agents base their decisions on a term structure of inflationary expectations that can be represented by the single expectation, $\dot{p}^e_{t+1}$. Having opted for that particularly simple form of the demand for money, the analyst has to abstain, in my opinion, from certain thought experiments, such as what would happen if agents came to expect a long-term systematic acceleration in the money supply or agents became fearful of a self-propelling hyperinflation (see Bomhoff, 1980, Chapter 5). If the prior probability of such events were different from zero, then one could only conclude that the demand-for-money function had been specified incorrectly and should contain a more complete term structure of inflationary expectations.

The intertemporal stability of the relationship between money and prices has to be investigated within the context of Motley (1969) and Brock (1972) and not with a given demand-for-money function that is confronted with different assumptions about the money supply or the dynamics of inflationary expectations. A demand-for-money function that omits the term structure of inflationary expectations is incompatible with assumptions about the money supply that allow for the possibility of explosive growth over any length of time, and is incompatible also with speculation about self-propelling take-offs into hyperinflation.

IV. LEARNING ABOUT TRANSITORY AND PERMANENT SHOCKS WITH THE MSKF-METHOD

Expectations of the growth rates of the two exogenous variables M and y have been computed using Kalman Filters and the auxiliary equations (4) and (5) of the model in Exhibit I subsequently connect these expected *growth rates* to expected future *levels* of the money stock and real output that are needed for the state and observation equations of the model. The expected growth rates are, as is customary with Kalman filtering, determined recursively; the forecasts for period t are computed without using in any way the realizations of the time series for periods $t+1$ and beyond. In this respect Kalman filter methods are comparable to methods such as adaptive expectations with a fixed coefficient or to moving average methods. All such "on-line" algorithms imitate the actual formation of forecasts by economic agents - who have to base their predictions on the past and cannot make use of future observations - better than "off-line" methods such as autoregressive least-squares or Box-Jenkins models that use data from the complete sample period.

The so-called Multi-State-Kalman filter (MSKF) (Harrison and Stevens, 1971, 1976) goes beyond other "on-line" methods, since it allows for feedback from the data to the forecasting algorithm. A number of separate fixed filters are applied to the data, and the forecasts are computed as a weighted average of the forecasts from the individual filters, with weights that are adjusted over time according to the success of each separate filter over the recent past. The composite forecasts therefore are both recursively determined and adapt to new information about the law of motion of the exogenous variable: The MSKF-method can cope with changes over time in the probability mixture of permanent and transitory shocks. Brunner et al (1980), Meltzer (1981) and Cukierman and Meltzer (1981) have emphasized the relevance of the simultaneous occurrence of permanent and transitory shocks in their theoretical work; empirical applications in economics include papers by Lawson (1980) and Bomhoff and Korteweg (1983).

Table 1 illustrates the working of the MSKF algorithm. The first column contains a segment of one of the time series used in this paper: the actual growth rates of the French money supply ($M2$) between late 1968 and mid-1975. The complete input series starts in 1961 I, and expectations are computed with the MSKF algorithm beginning in 1961 IV. Thus, the segment

Table 1

An Illustration of the MSKF Algorithm
Expected Growth Rates of the French Money Supply

	Actual	Expected	Error	Prior Probabilities for the next shock (percent)			
				small permanent	small temporary	outlier permanent	outlier temporary
1968-IV	2.11	3.13	-1.02	3.4	91.6	0.1	4.9
1969-I	2.02	2.76	-0.74	9.9	85.1	0.4	4.6
II	2.33	2.50	-0.18	6.7	88.3	0.3	4.7
III	1.85	2.44	-0.58	9.5	85.5	0.3	4.7
IV	0.24	2.27	-2.04	9.3	85.7	0.4	4.6
1970-I	1.43	2.10	-0.66	8.9	86.1	0.1	4.9
II	2.82	1.97	0.86	4.9	90.1	0.1	4.9
III	2.77	2.30	0.47	10.4	84.6	0.2	4.8
IV	5.13	2.48	2.65	8.2	86.8	0.2	4.8
1971-I	4.58	2.62	1.96	8.2	86.8	3.5	1.5
II	4.22	4.38	-0.16	12.5	82.5	1.6	3.4
III	3.89	4.62	-0.72	13.9	81.1	3.0	2.0
IV	4.24	4.28	-0.03	12.9	82.1	2.6	2.4
1972-I	3.75	4.26	-0.51	13.7	81.3	2.7	2.3
II	5.02	4.12	0.90	9.5	85.5	2.4	2.6
III	4.98	4.37	0.61	15.1	79.9	2.8	2.2
IV	3.84	4.53	-0.70	10.1	84.9	2.5	2.5
1973-I	0.93	4.31	-3.38	15.5	79.5	2.8	2.2
II	4.84	2.75	2.09	13.1	81.9	0.2	4.8
III	2.96	4.35	-1.39	10.7	84.3	0.7	4.3
IV	3.86	4.01	-0.15	11.0	84.0	0.5	4.5
1974-I	3.75	3.99	-0.24	11.2	83.8	0.6	4.4
II	4.34	3.93	0.40	11.0	84.0	0.6	4.4
III	2.28	4.01	-1.74	9.6	85.4	0.5	4.5
IV	4.68	3.67	1.01	6.7	88.3	0.4	4.6
1975-I	3.65	3.94	-0.29	7.3	87.7	0.4	4.6
II	2.79	3.88	-1.10	7.5	87.5	0.4	4.6

in Table 1 shows the operation of the method at a time when any transient influences from the initialization conditions have disappeared.

The second column shows the expected growth rates as calculated with the filter, and column three indicates the resulting forecast errors. The predictions are weighted averages of four distinct forecasting models. Each model can be written in the form of equations (8) and (9) as on page 13.

$$\bar{\hat{M}}_t = \bar{\hat{M}}_{t-1} + \eta_{M,t}$$

$$\hat{M}_t = \bar{\hat{M}}_t + \epsilon_{M,t}$$

The models differ in the values that have been assumed for the variances of $\epsilon_{M,t}$ and $\eta_{M,t}$. The first two of these four models are appropriate to "normal" situations; the remaining two models are designed to deal with outlier situations. The following values have been assumed for the variances of ϵ and η in the four Kalman filters:

state	ψ	Var η	Var ϵ	Var a_t
small permanent	0.05	0.9025	0.05	1
small temporary	0.95	0.0025	0.95	1
outlier permanent	0.01	15.6816	0.16	16
outlier temporary	0.99	0.0016	15.84	16

The first column indicates the values for the parameter ψ in the corresponding ARIMA (0, 1, 1) model:

$$\Delta \bar{M}_t = (1 - \psi B) a_t \tag{11}$$

I have assumed that the variance of the outliers is sixteen times as large as the variance of the process during "normal" periods. The "normal" value of the variance has been set equal to 1.0 in the chart above to facilitate inspection of the relative variances of ϵ and η; during estimation a robust estimate of the variance of a_t is adaptively computed from the forecast errors. Statistical Appendix A gives further details about the computation of the univariate expectations with the MSKF method.

The final four columns of Table 1 indicate the prior probabilities of the four different simple Kalman filters as they are recomputed each period after the observation of that period's growth-rate of the money supply. The priors have been constrained so that the sum of the prior probabilities of the two "normal" models is constant at 95%, which leaves a 5% probability for the two outlier models. The numbers on the first line of the Table indicate, for example, that after the final quarter of 1968 both normal-sized shocks to the growth-rate of the money stock as well as any exceptionally large disturbances were considered to be largely temporary. A large forecast error is made, for example, in 1969 IV, with the actual growth-rate 2 percentage points on a quarterly basis below the expected growth rate. When the priors have to be adjusted after this shock, it is too early to tell whether this outlier will prove to be permanent or transitory, and therefore we see little change in the prior probabilities of the two outlier models.

When the actual growth-rate for 1970 I has been observed, it becomes clear that the outlier in 1969 IV was temporary: the prior probabilities of the two outlier models change with an even larger weight being attached now to the temporary outlier model. The four prior probabilities subsequently change little until a new outlier occurs during 1970 IV. The forecast for 1971 I confirms that the outlier is assumed to be more temporary then permanent, since less than 10% of the forecast error in 1970 IV is incorporated into the expected growth-rate, which increases only from 2.48 to 2.62. However, when agents observe the actual growth rate in 1971 I, they realize that the change in money-growth that occurred during 1970 IV was of a permanent nature, and thus the weight of the permanent outlier model increases sharply. The prediction for 1971 II reflects this, and so does the prediction for 1973 II, when it is assumed that the sharp drop in 1973 I is partly permanent: the expected growth decreases from 4.31 to 2.75 only.

This data segment shows nicely the changes in the prior probabilities of the two outlier models over time according to whether the most recent outliers have been permanent or transitory. Less visible to the naked eye are the causes for shifts in the priors of the two Kalman filters for normal-sized errors, but it is obvious that a learning process is going on with respect to these priors, too.

Table 2 shows the success of the MSKF filter in predicting the quarterly growth-rates of money (*M2*) and real output in the six countries studied in this paper. All European data have been taken from the recent study by Den Butter and Fase (1981) of the demand for money in eight European countries. Den Butter and Fase also investigate the demand for money in Denmark, Ireland, and the United Kingdom, but the time series data for these countries cover considerably shorter periods of time. For that reason they have been discarded for the current analysis which is limited to the five countries listed in Table 2. The United States data were provided by the Federal Reserve Bank of St Louis.

Growth-rates have been computed as first differences of natural logarithms, but all error statistics have been multiplied by 100 in order to achieve

Table 2

Residuals of the Univariate Models

	(1) ex ante forecast errors MSKF-algorithm	(2) ex-ante errors $\hat{x}^e_t = \hat{x}_{t-1}$	(3) ex-ante errors of $\hat{x}^e_t$ = mean ($\hat{x}_t$) (pre-sample period)	(4) ex-post errors ordinary least squares (no correction for degrees of freedom)	(5) ex-post errors $\hat{x}^e_t$ = mean ($\hat{x}_t$) (sample period)
Money					
Belgium	1.54	1.96	2.21	1.52	1.77
France	1.13	1.32	1.16	1.02	1.15
Germany	1.82	2.13	1.91	1.55	1.82
Italy	1.11	1.41	1.65	1.06	1.39
Netherlands	2.09	2,23	2.23	1.82	1.99
U.S.A.	0.50	0.52	0.82	0.41	0.53
Output					
Belgium	1.45	2.12	1.42	1.34	1.40
France	1.46	1.67	1.40	1.28	1.34
Germany	1.35	1.53	1.21	1.20	1.19
Italy	1.83	2.25	1.66	1.52	1.64
Netherlands	1.89	2.83	1.78	1.77	1.77
U.S.A.	1.00	1.15	1.11	0.90	0.98

comparability with errors that are expressed as a percentage. Data on the levels of money and output are available from 1961 I, so that growth-rates can be computed beginning 1961 II. The MSKF filter looks back at the latest two observations when the prior probabilities are updated, so that the first expected growth-rate is that computed for 1961 IV (see Harrison and Stevens, 1971, 1976, for details on the computation of the posterior probabilities; and Lawson, 1980, for the Bayesian manner in which the priors are updated). The period terminates with the final quarter of 1978 for the European countries; the statistics for the United States refer to this same period, with an exception for Table 8 below.

During the first part of the period, the expectations are influenced by the way in which the prior probabilities are set initially and by the initial estimate of the "normal variance" of the process. In our implementation the filters are initialized automatically and in an identical way for all time series (see Bomhoff and Kortweg, forthcoming 1983, appendix 2, for details). The initial estimate of the normal variance of the process contains an ex-post element, since it is based on the first ten observations.

In order to minimize the transient effects of the manner in which the filter is initialized, I have disregarded the first five years of the resulting series for the expected growth-rates when computing the standard errors of the forecasts in the first column of Table 2. The numbers in all the remaining columns of the Table refer also to the final 49 observations of the sample. Columns 2 and 3 show the ex-ante errors of two simple naive models: column 2 shows the errors made when the last observed value for the growth-rate is taken as the expectation for the next period, and column three shows the errors made if one uses the mean of the first 22 observations as an estimate of the growth-rates for the remaining 49 periods. Column four shows the ex-post residual errors of an ordinary least squares regression, estimated over the whole sample period, in which the growth-rates of money and output are regressed on a constant term plus their own growth rates, lagged 1-6 periods. The final column of Table two shows the root mean square error of a naive ex-post model that puts each expected growth rate equal to the mean growth rate over the period under investigation. The Table shows that our version of the MSKF filter produces reasonable forecasts for the growth-rates of the money supply in the six countries. The MSKF algorithm leads both ex-ante naive models and performs roughly as well as the ex-post naive model.

Less satisfactory are the results of the six series of the growth-rates of real output. One way to improve the results might be to construct multivariate expectations of the growth-rate of real output that incorporate explicitly one or more of the determinants of the business cycle.

Clemens J.M. Kool and I are currently working on these problems. We also plan to investigate whether one additional "level" (Jacobs and Jones, 1980) needs to be incorporated in the Kalman filter models. We presently limit the types of shocks to transitory and permanent shifts in the growth-rates, but it may be necessary to allow for transitory shocks to the level of the series. More experiments are needed to determine the ideal form of the MSKF algorithm for macroeconomic forecasting; the effort should be worthwhile, since the MSKF method produces forecasts that (1) are purely ex-ante, (2) incorporate a learning mechanism, and (3) can deal with situations in which most small shocks are transitory and most large shocks permanent, or vice versa.

V. LEARNING ABOUT VELOCITY WITH THE RPE-ALGORITHM

In this section I discuss the way in which expected future values of the price level are computed for given expectations of future levels of the money supply and real output. The estimates are made recursively, and predictions are thus purely ex-ante. In the course of each period t agents discover the current values of the money supply, the level of real output, and the price level. A fraction of the ex-post prediction error, which they now know was made when predicting the price level p_t, can be assigned to the prediction errors in the two exogenous variables: $(M_t - M_t^e)$ and $(y_t - y_t^e)$. The remaining unexplained part of the prediction error is called ϵ_t (see equation (3) in section II above). Each non-zero value for ϵ_t leads to adaptations of the current values of the parameters of the model. The adjusted values of the parameters $\theta_1, \theta_2, \theta_4, \theta_5$, θ_6, and θ_7 are used to compute a forecast for $t+1$ that is based also on the expectations at time t of the levels of money and output in that next period (M_{t+1}^e, y_{t+1}^e). The prediction is made with the two state equations from the model, equations (1) and (2):

$$c_{t+1}^e = c_t^e + \theta_4 \epsilon_t + \theta_5 \Delta y_{t+1}^e + \theta_6 \tag{1}$$

$$p_{t+1}^e = c_{t+1}^e + M_{t+1}^e - y_{t+1}^e + \theta_7 (\hat{p}_{t+2}^e) \tag{2}$$

The ex-ante prediction error $p_{t+1} - p_{t+1}^e$ is a measure of the success of the model in predicting price levels. However, it is the ex-post residual ϵ_t from the observation equation that is used to steer the evolution of the model parameters over time:

$$p_t = p_t^e + \theta_1(M_t - M_t^e) - \theta_2(y_t - y_t^e) + \epsilon_t \qquad (3)$$

The Recursive Prediction Error method has been developed by Ljung (1977, 1978), and by Moore and Weiss (1979). It is closely related to the Extended Kalman Filter method (see Anderson and Moore (1979), Goodwin and Payne (1977)).[2] Statistical Appendix B contains the principal features of the algorithm that I have used for this paper. The only other application of the RPE method to a macroeconomic problem that I am currently aware of is a paper by Burmeister and Wall (1980). These authors are not concerned with ex-ante predictions or adaptive learning processes, however, and focus attention upon parameter estimates based on repeated runs through the data, both forward and backward.

The RPE method requires initial estimates of all parameters and of their variance-covariance matrix. Clemens J.M. Kool and I have used an ordinary least squares regression:

$$p_t = c_t + 1.M_t^e - 1.y_t^e + \theta_1(M_t - M_t^e) - \theta_2(y_t - y_t^e) + \theta_6.t$$

over the period 1961 IV - 1966 I in order to find initial estimates of θ_1, θ_2, θ_6, and c. The initial variances of θ_1, θ_2, and θ_6 are taken also from this least squares regression. θ_4 is initialized at a value of 1.0 with a standard error of 0.5; θ_5 has an initial value of zero with an initial standard error of 0.25. Finally, θ_7 has been initialized both at 0.0 and at 0.19, the latter value being the average interest elasticity of the demand-for-money functions estimated by Den Butter and Fase for the five countries under review. In both cases, the standard error of θ_7 has been initialized at 0.095. As the starting value of θ_7 had a minimal effect on the results, I report only on the estimates that used a starting value of 0 for θ_7.

The covariance terms between the parameters were initially set at zero. In the course of the RPE estimation, all parameters and the variance-covariance matrix are updated after each period, according to the formulas given in Statistical Appendix B, taken from Moore and Weiss (1979). The required partial derivatives of the state variables and the residuals with respect to the parameters are calculated analytically. Estimation with the RPE algorithm starts at 1963 IV and continues until the end of the estimation period in 1978

[2]The early work by Peter Young and associates is very useful on the relationship between recursive least squares and Kalman filtering; see Young (1974), Young, Shellswell, and Neethling (1971), and Young and Whitehead (1975).

IV. All error statistics have been computed over the period 1966 IV through 1978 IV, during which the algorithm generates purely ex-ante forecasts.

The Results

Table 3 shows how the parameters θ_5, θ_1, θ_2, and θ_7 change between the starting points of the RPE algorithm and the final year of the estimation. Printed below each coefficient is the square root of the corresponding element on the main diagonal of matrix I^{-1}, the inverse of the Information Matrix (see Anderson and Moore, 1979). The numbers in the Table can thus serve to indicate the degree of variability of the estimated parameters. Some noteworthy features of this Table are:

The final estimates for the coefficients of $(M_t - M_t^e)$ and $(y_t - y_t^e)$ are closer together than the initial estimates. Apparently there are similarities between the six countries that are not captured by the initial least squares estimates but become visible as learning proceeds.

The coefficient of $\hat{p}^e_{t+2}$ turns negative for all six countries if $\hat{p}^e_{t+2}$ is measured as:

$$\hat{p}^e_{t+2} = E_t p_{t+2} - E_t p_{t+1} = 4x \left\{ \theta_6 + \hat{M}^e_{t+2} - (1-\theta_5)\hat{y}^e_{t+2} \right\} \qquad \text{I}$$

which is the original specifications of the model. The coefficient θ_7 is sometimes positive and generally insignificant if $\hat{p}^e_{t+2}$ is measured over a 4-quarter span and proxied by:

$$\begin{aligned} \hat{p}^e_{t+2} &= E_t p_{t+2} - p^e_{t-2} \\ &= c^e_t + \theta_5(y^e_{t+2} - y^e_t) + 2\theta_6 + M^e_{t+2} - y^e_{t+2} - p^e_{t-2} \end{aligned} \qquad \text{II}$$

Possibly these results tell us something about the difference between the short-run liquidity effect and the longer-run inflationary effect of a surge in money growth, but any interpretation is hazardous, both because of the presence of $(y_t - y^e_t)$ and $(M_t - M^e_t)$ in the observation equations and because the model does not include a theory about changes in the real rate of interest.

Tables 4 and 5 show the learning processes that take place with respect to the trend parameter θ_6 and the "constant term" c^e. Values are presented for 10-quarter intervals. Note that the trend θ_6 has been assumed to be representable by a parameter about which agents learn more as time goes by, whereas

Table 3

Adaptively Estimated Parameters

	coeff. of Δy_{t+1} θ_5		coeff. of $(M\text{-}M^e)_t$ θ_1		coeff. of $-(y\text{-}y^e)_t$ θ_2		coeff. of p_{t+2} θ_7	
country	**start**	**1978-III**	**start**	**1978-III**	**start**	**1978-III**	**start**	**1978-III**
Belgium	0.0 (0.2500)	-0.0567 (0.1387)	0.4007 (0.3072)	0.4936 (0.1005)	0.9994 (0.2229)	0.5080 (0.1130)	0.0 (0.0950)	-0.1169 (0.0717)
France	0.0 (0.2500)	0.1862 (0.1280)	0.8542 (0.6596)	0.4953 (0.1107)	0.8160 (0.3352)	0.5171 (0.1112)	0.0 (0.0950)	-0.1367 (0.0677)
Germany	0.0 (0.2500)	0.0733 (0.1051)	0.4822 (0.2360)	0.6071 (0.0743)	0.5416 (0.1003)	0.4799 (0.0716)	0.0 (0.0950)	-0.1751 (0.0525)
Italy	0.0 (0.2500)	-0.0036 (0.1319)	0.0232 (0.4327)	0.3466 (0.1316)	1.5563 (0.4328)	0.5089 (0.1194)	0.0 (0.0950)	-0.1561 (0.0601)
Netherlands	0.0 (0.2500)	-0.0491 (0.1328)	0.8337 (0.3746)	0.5023 (0.1149)	0.9493 (0.1513)	0.6843 (0.0913)	0.0 (0.0950)	-0.1016 (0.0750)
U.S.A.	0.0 (0.2500)	0.1129 (0.1063)	0.6328 (0.4222)	0.5756 (0.1903)	1.0505 (0.2420)	0.5514 (0.1164)	0.0 (0.0950)	-0.1489 (0.0667)

Table 4

Evolution of the Trend Term, θ_6

country	1966-I	1968-III	1971-I	1973-III	1976-I	1978-III
Belgium	0.32	0.30	0.31	0.30	0.29	0.28 (0.08)
France	-0.58	-0.58	-0.56	-0.58	-0.59	-0.59 (0.10)
Germany	0.37	0.36	0.36	0.35	0.35	0.35 (0.05)
Italy	-0.31	-0.32	-0.33	-0.35	-0.35	-0.36 (0.09)
Netherlands	0.80	0.78	0.78	0.78	0.75	0.74 (0.11)
U.S.A.	0.76	0.76	0.76	0.76	0.76	0.76 (0.04)

Note: all entries have been multiplied by 100

Table 5

The Evolution of the "Constant Term" c^e

country	1966-I	1968-III	1971-I	1973-III	1976-I	1978-III
Belgium	5.399	5.359	5.411	5.224	5.112	5.015
France	5.408	5.360	5.398	5.290	5.209	5.200
Germany	6.128	6.013	6.064	5.955	6.001	5.987
Italy	4.887	4.795	4.768	4.637	4.584	4.546
Netherlands	5.710	5.711	5.736	5.812	5.716	5.679
U.S.A.	6.065	6.134	6.160	6.241	6.336	6.435

the "constant" term is an unobservable state variable that does not converge to a final value but continues to behave like a random-walk-plus trend.

Table 6 shows the evolution of parameter θ_4. The activity of this parameter relates to the third type of change discussed in section III above. The final value of the parameter differs considerably from its initial value for Belgium and Germany, which indicates that it is worthwhile to incorporate a learning mechanism about the relative importance of transitory and permanent shocks to the demand for money.

Table 7 shows the prediction errors of the RPE algorithm and compares the errors to the standard errors of two naive models for predicting the price level:

$$p^e_t = p_{t-1} ;$$

$$p^e_t = p_{t-1} + (p_{t-1} - p_{t-2})$$

Part I of the Table gives summary statistics for the two naive models. Part II indicates the forecast errors of the model that was used also in Tables 3 through 6. On average, the errors are 1½ times as large as those of the best naive model: $p^e_t - p_{t-1} = p_{t-1} - p_{t-2}$. This shows that the relationship between the money stock and the price level, although primarily a phenomenon that is relevant in the longer-run, is nevertheless useful for very-short-run forecasts and not that much poorer than forecasts that exploit the inertia in the actually observed rate of change of the price level.

Parts II - V of Table 7 provide information about the marginal contributions of $\hat{y} - \hat{y}^e$ and $\hat{M} - \hat{M}^e$ to the forecasts of p_{t+1}. A comparison between parts III and IV shows that knowledge about $\hat{M} - \hat{M}^e$ is more valuable for the inflation forecasts than information about $\hat{y} - \hat{y}^e$. The final part VI of Table 7 gives the errors made if the expected rate of inflation is proxied by $p^e_{t+2} - p_{t-2}$ instead of $4x\ [p^e_{t+2} - p^e_{t+1}]$ as in parts II-V.

The Table shows the average size of the one-period-ahead errors produced by applying the state equation. If multi-period projections are available for money and output, then the model can be used also to generate multi-period predictions for the price level. The errors in such multi-period forecasts depend also on the magnitude of the residuals in the observation equation, ϵ_t, since $\theta_4 \cdot \epsilon_t$ is incorporated permanently into the expected future path of the price level. The root mean square errors of the observation equations (not shown in Table 7) are similar in magnitude to the errors of the state equations. As-

Table 6

The Evolution of the Degree of Permanence of the Shocks, As Measured By θ_4

country	1966-I	1968-III	1971-I	1973-III	1976-I	1978-III
Belgium	0.985	0.927	1.126	1.233	1.235	1.259 (0.120)
France	1.101	1.082	1.248	1.352	1.305	1.232 (0.151)
W. Germany	0.908	1.312	1.351	1.320	1.389	1.413 (0.127)
Italy	1.234	1.207	1.090	1.167	1.283	1.355 (0.140)
Netherlands	1.139	1.065	1.133	1.097	1.255	1.367 (0.154)
U.S.A.	1.008	1.051	1.128	1.099	1.086	1.114 (0.182)

Table 7

Prediction Errors of the RPE Algorithm (in perc.)

	Belgium	France	W. Germany	Italy	Neth.	U.S.A.
RMSE (M-M^e)	1.541	1.129	1.821	1.114	2.087	0.503
RMSE (y-y^e)	1.451	1.462	1.347	1.826	1.893	0.998
I RMSE (p-p_{-1})	2.053	2.000	1.399	3.006	2.083	1.526
Robust estimate of the standard error (p-p_{-1})	1.958	2.655	1.791	3.518	2.671	2.001
RMSE $\Delta(p$-$p_{-1})$	1.784	0.863	0.914	1.019	1.584	0.437
Robust estimate of the standard error $\Delta(p$-$p_{-1})$	1.382	0.796	1.007	1.065	1.700	0.436
II RMSE state equation (p-p^e)	1.973	1.085	1.379	1.580	1.667	0.651
Robust estimate of the standard error (p-p^e)	2.161	1.120	1.401	1.616	1.947	0.684
III RMSE state equation (p-p^e)	2.144	1.314	1.818	1.690	2.412	0.806
Robust estimate of the standard error (p-p^e)	2.450	1.626	1.979	1.695	2.649	0.733
IV RMSE state equation (p-p^e)	2.551	1.516	2.135	1.763	2.417	0.857
Robust estimate of the standard error (p-p^e)	3.104	1.678	2.148	1.911	2.534	1.007
V RMSE state equation (p-p^e)	2.780	1.656	2.391	1.748	2.888	0.904
Robust estimate of the standard error (p-p^e)	3.383	1.656	2.486	1.531	3.139	0.988
VI RMSE state equation (p-p^e)	1.967	1.105	1.553	1.594	1.647	0.687
Robust estimate of the standard error (p-p^e)	2.158	1.182	1.694	1.714	1.929	0.746

I : two naive models
II : complete model
III : without $\hat{y}$ - $\hat{y}^e$
IV : without $\hat{M}$ - $\hat{M}^e$
V : without $\hat{y}$ - $\hat{y}^e$ and $\hat{M}$ - $\hat{M}^e$
VI : model with p^e_{+2} - p^e_{-2} as proxy for expected inflation.

Note: All entries have been multiplied by 100.

suming equality between the RMSEs of the state and observation equations, plus a value of one for the parameter θ_4, which is about average for the six countries, we can compute a rough estimate of the standard error in a n-period-ahead-prediction with the help of the formula:

$$E_t(p_{t+n} - E_t(p_{t+n}))^2 = n E_t(p_{t+1} - E_t(p_{t+1}))^2$$

If, for example, the residual errors of the one-period-ahead predictions are about 1.5 percent - as is roughly the situation for the five European countries studied - and if we assume that projections for M and y are available for the next three years, then the estimated standard error of the corresponding price level three years from today would be:

$$1.5 \sqrt{12} = 5 \text{ percent}$$

The residual errors in the United States are considerably smaller; in that country, the estimated standard error of a conditional forecast of the price level three years into the future would be $0.7 \sqrt{12} = 2½$ percent. Parameter uncertainty would add something to this estimate, but not much, because the parameters tend to be well determined and change comparatively little over periods that do not extend beyond a few years.

Finally, Table 8 shows outcomes for the United States over the second half of the sample period. All summary statistics in Tables 2-7 refer to a period that terminates in 1978 IV, but in Table 8 I have continued the computations up to 1981. The results show that the RPE algorithm is capable of tracking the actual path of the United States price level quite well. There is no evidence of persistent runs of positive or negative forecast errors.

VI. CONCLUSIONS

During the early seventies, exciting work was done in the field of adaptive parameter estimation by Cooley, Prescott, and others (see Swamy and Tinsley, 1980, for a recent review of this literature). Interest in adaptive estimation waned somewhat in recent years for two reasons: first, there were severe technical problems in implementing adaptive estimation, particularly in an "on-line" context, and second, because the important advances in formulating and estimating rational expectations equilibrium models did not fit in well with the assumption that agents are unsure of and have much to learn about relevant parameters as time goes on.

Table 8

U.S.A. - Quarterly Average Data For the Expected Levels of the Money Stock, Real GNP and the GNP Deflator

	$\ln p_t$ (GNP deflator)	$\ln p_t^e$ 'state equation'	$\ln p_t^e$ 'observation equation'	$\ln M_t^e$ (M1B)	$\ln y_t^e$ (GNP-1972 dollars)	$\ln c_t^e$
1970- I	4.499	4.496	4.504	5.332	6.992	6.159
II	4.512	4.516	4.515	5.345	6.984	6.160
III	4.520	4.529	4.522	5.357	6.988	6.165
IV	4.533	4.535	4.549	5.370	7.001	6.170
1971- I	4.548	4.556	4.535	5.389	6.987	6.160
II	4.561	4.570	4.571	5.409	7.014	6.181
III	4.570	4.588	4.580	5.438	7.021	6.177
IV	4.579	4.585	4.577	5.448	7.031	6.173
1972- I	4.592	4.588	4.585	5.450	7.041	6.182
II	4.599	4.607	4.600	5.479	7.064	6.197
III	4.608	4.608	4.611	5.496	7.088	6.203
IV	4.620	4.621	4.621	5.517	7.098	6.206
1973- I	4.634	4.633	4.625	5.543	7.118	6.212
II	4.651	4.640	4.647	5.563	7.149	6.229
III	4.668	4.666	4.666	5.570	7.141	6.241
IV	4.688	4.680	4.680	5.578	7.144	6.250
1974- I	4.706	4.701	4.716	5.589	7.152	6.267
II	4.730	4.730	4.726	5.609	7.135	6.264
III	4.756	4.747	4.751	5.613	7.137	6.277
IV	4.784	4.777	4.785	5.622	7.128	6.289
1975- I	4.809	4.812	4.819	5.635	7.110	6.296
II	4.822	4.840	4.828	5.640	7.081	6.292
III	4.840	4.838	4.828	5.659	7.108	6.295
IV	4.858	4.849	4.847	5.681	7.143	6.317
1976- I	4.867	4.866	4.861	5.684	7.149	6.336
II	4.876	4.873	4.881	5.699	7.176	6.353
III	4.888	4.889	4.889	5.718	7.177	6.355
IV	4.903	4.900	4.903	5.727	7.181	6.361
1977- I	4.917	4.919	4.916	5.748	7.190	6.370
II	4.934	4.930	4.931	5.775	7.220	6.382
III	4.947	4.947	4.945	5.792	7.231	6.394
IV	4.962	4.959	4.967	5.808	7.249	6.406
1978- I	4.976	4.981	4.982	5.831	7.248	6.408
II	5.001	4.993	4.988	5.849	7.255	6.410
III	5.020	5.018	5.020	5.874	7.282	6.435
IV	5.043	5.037	5.035	5.893	7.289	6.443
1979- I	5.064	5.060	5.058	5.910	7.303	6.462
II	5.082	5.078	5.091	5.921	7.311	6.477
III	5.101	5.108	5.106	5.950	7.300	6.473
IV	5.121	5.122	5.121	5.973	7.314	6.477

Table 8 continued

	$\ln p_t$ (GNP deflator)	$\ln p_t^e$ 'state equation'	$\ln p_t^e$ 'observation equation'	$\ln M_t^e$ (M1B)	$\ln y_t^e$ (GNP-1972 dollars)	$\ln c_t^e$
1980 - I	5.143	5.139	5.133	5.980	7.313	6.485
II	5.166	5.162	5.166	5.997	7.321	6.499
III	5.188	5.183	5.196	5.973	7.285	6.502
IV	5.214	5.218	5.216	6.032	7.300	6.504
1981 - I	5.237	5.241	5.227	6.061	7.311	6.510
II	5.253	5.249	5.260	6.061	7.336	6.534

Fortunately, recent work by Ljung, Moore Weiss, and Anderson has resulted in a new form of Kalman filtering that produces "on-line" estimates of both parameters and state variables without the convergence problems that marred earlier versions of extended Kalman filtering. Furthermore, the work of Brunner, Cukierman, and Meltzer has rightly stressed the enormous importance of the transitory/permanent confusion and the need for agents to become familiar with changes in the relative importance of transitory and permanent shocks. For these two reasons I opted to neglect the cross-equation restrictions and investigated the potential of adaptive estimation for a simple problem in monetary economics (see Sargent, 1981; Hansen and Sargent, 1980; Turner and Whiteman, 1981; for the importance of cross-equation restrictions and their apparently unavoidable neglect in a learning context).

The results in section IV of the paper show that univariate expectations of future levels of money and output must take into account that the relative weights of permanent and transitory shocks change continually. A static past during which the laws of motion of such variables did not change, and which would provide, therefore, a solid basis for analysis of future changes in these laws of motion simply does not exist. We may be able to roll back our theoretical models to the levels where parameters of technologies and preferences remain invariant, but the world does not oblige by offering an historical base period during which the constraints also remained constant. The world changes all the time; if we do not learn, we are lost.

Recursive and adaptive estimation of a simple three-equation model in section V shows that the RPE method is capable of producing well-behaved estimates of both the model parameters and the unobservable state variables. It follows that it is no longer necessary to regard each and every unpredicted shift in the demand for money as prima facie evidence of the impossibility of computing price level paths that correspond to medium-term targets for money growth. The relationship between monetary actions and the price level is predictable and can incorporate rational learning about recent shifts in the demand for money. With recursive and adaptive estimation of the link between money and prices, it becomes easier to implement "rules rather than discretion" (see Kydland and Prescott, 1977, who suggest that no changes in monetary policy be executed until after a two-year waiting period). With that perspective in mind, the results of the paper indicate that Kalman filter methods can be profitably put to work on the two major problems which Karl Brunner mentions as predominantly confronting monetary policymaking at this time: choosing, and adhering to a monetary strategy; and the reliable interpretation of monetary events (Brunner, 1981).

DATA APPENDIX

The data for this study have been taken from Den Butter and Fase (1981), apart from the U.S. data that were kindly provided by the Federal Reserve Bank of St. Louis. The series from the money stock consists of quarterly averages of monthly $M2$ data, apart from Italy where Den Butter and Fase used end-of-quarter figures, and the United States where M_1 is used. y refers to real g.n.p. apart from Italy and France, where Den Butter and Fase work with data on g.d.p. Den Butter and Fase employ seasonal dummy variables in their estimation of the demand-for-money function; I have used the "fixed-multiplicative" method to deseasonalize the money data so that the least possible damage is done to the underlying time series structure of the money supply process. All the real g.n.p. or g.d.p. data are seasonally adjusted in the original sources. Details about three minor corrections to the data follow:

I have made one change in the real output series for the Netherlands. The value which Den Butter and Fase give for the first quarter of 1970 differs considerably from that given by the Central Planning Bureau (CPB), an agency of the Dutch Ministry of Economic Affairs. Both series consist of estimates only, since no official quarterly national accounts data exist for the Netherlands. As the estimate by the Dutch Central Bank for 1970 used by Den Butter and Fase appears implausible, I have substituted the CPB estimate for real g.n.p. in that quarter.
Den Butter and Fase and I have corrected a discontinuity in the money ($M2$) data for Belgium in 1969.
Finally, I have presumed that agents were aware at the time of the exceptional nature of the negative shock to real output in France during the second quarter of 1968. In order to avoid letting this exceptional event unduly influence the expectations regarding real output in France, I have replaced the value for 1968 II by a straight interpolation of the values for 1968 I and 1968 III. The Multi-State-Kalman filter method has been applied to the French output series after correction for this episode.