1.2. Nonhomogeneous Heat Equation $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \Phi(x,t)$

1.2-1. Domain: $-\infty < x < \infty$. Cauchy problem for the nonhomogeneous heat equation.

An initial condition is prescribed:

$$w = f(x) \quad \text{at} \quad t = 0.$$

Solution:

$$w(x,t) = \int_{-\infty}^{\infty} f(\xi)G(x,\xi,t)\,d\xi + \int_0^t \int_{-\infty}^{\infty} \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau,$$

where

$$G(x,\xi,t) = \frac{1}{2\sqrt{\pi a t}} \exp\left[-\frac{(x-\xi)^2}{4at}\right].$$

1.2-2. Solutions of boundary value problems in terms of the Green's function.

We consider boundary value problems for the heat equation* on an interval $0 \le x \le l$ with the general initial condition

$$w = f(x) \quad \text{at} \quad t = 0$$

and various homogeneous boundary conditions. The solution can be represented in terms of the Green's function as

$$w(x,t) = \int_0^l f(\xi)G(x,\xi,t)\,d\xi + \int_0^t \int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau.$$

1.2-3. Domain: $0 \le x < \infty$. First boundary value problem for the heat equation.

A boundary condition is prescribed:

$$w = 0 \quad \text{at} \quad x = 0.$$

Green's function:

$$G(x,\xi,t) = \frac{1}{2\sqrt{\pi a t}} \left\{ \exp\left[-\frac{(x-\xi)^2}{4at}\right] - \exp\left[-\frac{(x+\xi)^2}{4at}\right] \right\}.$$

1.2-4. Domain: $0 \le x < \infty$. Second boundary value problem for the heat equation.

A boundary condition is prescribed:

$$\frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = 0.$$

Green's function:

$$G(x,\xi,t) = \frac{1}{2\sqrt{\pi a t}} \left\{ \exp\left[-\frac{(x-\xi)^2}{4at}\right] + \exp\left[-\frac{(x+\xi)^2}{4at}\right] \right\}.$$

* Hereinafter we shell used the term “heat equation” to mean “nonhomogeneous heat equation”.

1.2-5. Domain: $0 \le x < \infty$. Third boundary value problem for the heat equation.

A boundary condition is prescribed:

$$\frac{\partial w}{\partial x} - kw = 0 \quad \text{at} \quad x = 0.$$

Green's function:

$$G(x,\xi,t) = \frac{1}{2\sqrt{\pi a t}}\left\{\exp\left[-\frac{(x-\xi)^2}{4at}\right] + \exp\left[-\frac{(x+\xi)^2}{4at}\right] - 2k\int_0^\infty \exp\left[-\frac{(x+\xi+\eta)^2}{4at} - k\eta\right]d\eta\right\}.$$

1.2-6. Domain: $0 \le x \le l$. First boundary value problem for the heat equation.

Boundary conditions are prescribed:

$$w = 0 \quad \text{at} \quad x = 0, \qquad w = 0 \quad \text{at} \quad x = l.$$

Two forms of representation of the Green's function:

$$G(x,\xi,t) = \frac{2}{l}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi\xi}{l}\right)\exp\left(-\frac{an^2\pi^2 t}{l^2}\right)$$
$$= \frac{1}{2\sqrt{\pi a t}}\sum_{n=-\infty}^{\infty}\left\{\exp\left[-\frac{(x-\xi+2nl)^2}{4at}\right] - \exp\left[-\frac{(x+\xi+2nl)^2}{4at}\right]\right\}.$$

The first series converges rapidly at large t and the second series at small t.

1.2-7. Domain: $0 \le x \le l$. Second boundary value problem for the heat equation.

Boundary conditions are prescribed:

$$\frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = 0, \qquad \frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = l.$$

Two forms of representation of the Green's function:

$$G(x,\xi,t) = \frac{1}{l} + \frac{2}{l}\sum_{n=1}^{\infty}\cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi\xi}{l}\right)\exp\left(-\frac{an^2\pi^2 t}{l^2}\right)$$
$$= \frac{1}{2\sqrt{\pi a t}}\sum_{n=-\infty}^{\infty}\left\{\exp\left[-\frac{(x-\xi+2nl)^2}{4at}\right] + \exp\left[-\frac{(x+\xi+2nl)^2}{4at}\right]\right\}.$$

The first series converges rapidly at large t and the second series at small t.

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