Exact Solutions > Linear Partial Differential Equations >
Second-Order Parabolic Partial Differential Equations > Heat Equation with Central Symmetry

### 1.6. Heat Equation with Central Symmetry $\frac{\partial w}{\partial t}=a\left(\frac{\partial^{2} w}{\partial r^{2}}+\frac{2}{r} \frac{\partial w}{\partial r}\right)$

This is the heat (diffusion) equation with central symmetry, $r=\sqrt{x^{2}+y^{2}+z^{2}}$ is the radial coordinate.

## 1.6-1. Particular solutions of the heat equation with central symmetry:

$$
\begin{aligned}
w(r) & =A+B r^{-1}, \\
w(r, t) & =A+B\left(r^{2}+6 a t\right), \\
w(r, t) & =A+B\left(r^{4}+20 a t r^{2}+60 a^{2} t^{2}\right), \\
w(r, t) & =A+B\left[r^{2 n}+\sum_{k=1}^{n} \frac{(2 n+1)(2 n) \ldots(2 n-2 k+2)}{k!}(a t)^{k} r^{2 n-2 k}\right], \\
w(r, t) & =A+2 a B t r^{-1}+B r, \\
w(r, t) & =A r^{-1} \exp \left(a \mu^{2} t \pm \mu r\right)+B, \\
w(r, t) & =A+\frac{B}{t^{3 / 2}} \exp \left(-\frac{r^{2}}{4 a t}\right), \\
w(r, t) & =A+\frac{B}{r \sqrt{t}} \exp \left(-\frac{r^{2}}{4 a t}\right), \\
w(r, t) & =A r^{-1} \exp \left(-a \mu^{2} t\right) \cos (\mu r+B)+C, \\
w(r, t) & =A r^{-1} \exp (-\mu r) \cos \left(\mu r-2 a \mu^{2} t+B\right)+C, \\
w(r, t) & =\frac{A}{r} \operatorname{erf}\left(\frac{r}{2 \sqrt{a t}}\right)+B,
\end{aligned}
$$

where $A, B, C$, and $\mu$ are arbitrary constants, $n$ is an arbitrary positive integer.

## 1.6-2. Reduction to a constant coefficient equation. Some formulas.

$1^{\circ}$. The substitution $u(r, t)=r w(r, t)$ brings the original equation with variable coefficients to the constant coefficient equation $u_{t}=a u_{r r}$, which is discussed in Subsection 1.1.
$2^{\circ}$. Suppose $w=w(r, t)$ is a solution of the heat equation with central symmetry. Then the functions

$$
\begin{aligned}
& w_{1}=A w\left( \pm \lambda r, \lambda^{2} t+C\right)+B, \\
& w_{2}=\frac{A}{|\delta+\beta t|^{3 / 2}} \exp \left[-\frac{\beta r^{2}}{4 a(\delta+\beta t)}\right] w\left( \pm \frac{r}{\delta+\beta t}, \frac{\gamma+\lambda t}{\delta+\beta t}\right), \quad \lambda \delta-\beta \gamma=1,
\end{aligned}
$$

where $A, B, C, \beta, \delta$, and $\lambda$ are arbitrary constants, are also solutions of this equation. The second formula may usually be encountered with $\beta=1, \gamma=-1$, and $\delta=\lambda=0$.

## 1.6-3. Boundary value problems for the heat equation with central symmetry.

For solutions of various boundary value problems, see Subsection 1.7 with $\Phi(r, t) \equiv 0$.

## References

Carslaw, H. S. and Jaeger, J. C., Conduction of Heat in Solids, Clarendon Press, Oxford, 1984.
Polyanin, A. D., Handbook of Linear Partial Differential Equations for Engineers and Scientists, Chapman \& Hall/CRC, 2002.

