3.2. Poisson Equation $\Delta w + \Phi(\mathbf{x}) = 0$

The two-dimensional Poisson equation has the following form:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \Phi(x, y) = 0 \quad \text{in the Cartesian coordinate system,}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2} + \Phi(r, \varphi) = 0 \quad \text{in the polar coordinate system.}$$

3.2-1. Domain: $-\infty < x < \infty$, $-\infty < y < \infty$.

Solution:

$$w(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(\xi, \eta)\ln\frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}}\,d\xi\,d\eta.$$

3.2-2. Domain: $-\infty < x < \infty$, $0 \le y < \infty$. First boundary value problem.

A half-plane is considered. A boundary condition is prescribed:

$$w = f(x) \quad \text{at} \quad y = 0.$$

Solution:

$$w(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y f(\xi)\,d\xi}{(x-\xi)^2 + y^2} + \frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty}\Phi(\xi, \eta)\ln\frac{\sqrt{(x-\xi)^2 + (y+\eta)^2}}{\sqrt{(x-\xi)^2 + (y-\eta)^2}}\,d\xi\,d\eta.$$

3.2-3. Domain: $0 \le x < \infty$, $0 \le y < \infty$. First boundary value problem for the Poisson equation.

A quadrant of the plane is considered. Boundary conditions are prescribed:

$$w = f_1(y) \quad \text{at} \quad x = 0, \qquad w = f_2(x) \quad \text{at} \quad y = 0.$$

Solution:

$$w(x, y) = \frac{4}{\pi}xy\int_0^{\infty}\frac{f_1(\eta)\eta\,d\eta}{[x^2+(y-\eta)^2][x^2+(y+\eta)^2]} + \frac{4}{\pi}xy\int_0^{\infty}\frac{f_2(\xi)\xi\,d\xi}{[(x-\xi)^2+y^2][(x+\xi)^2+y^2]}$$
$$+ \frac{1}{2\pi}\int_0^{\infty}\int_0^{\infty}\Phi(\xi, \eta)\ln\frac{\sqrt{(x-\xi)^2+(y+\eta)^2}\sqrt{(x+\xi)^2+(y-\eta)^2}}{\sqrt{(x-\xi)^2+(y-\eta)^2}\sqrt{(x+\xi)^2+(y+\eta)^2}}\,d\xi\,d\eta.$$

3.2-4. Domain: $0 \le x \le a$, $0 \le y \le b$. First boundary value problem for the Poisson equation.

A rectangle is considered. Boundary conditions are prescribed:

$$w = f_1(y) \quad \text{at} \quad x = 0, \qquad w = f_2(y) \quad \text{at} \quad x = a,$$
$$w = f_3(x) \quad \text{at} \quad y = 0, \qquad w = f_4(x) \quad \text{at} \quad y = b.$$

Solution:

$$w(x, y) = \int_0^a\int_0^b \Phi(\xi, \eta)G(x, y, \xi, \eta)\,d\eta\,d\xi$$
$$+ \int_0^b f_1(\eta)\left[\frac{\partial}{\partial \xi}G(x, y, \xi, \eta)\right]_{\xi=0} d\eta - \int_0^b f_2(\eta)\left[\frac{\partial}{\partial \xi}G(x, y, \xi, \eta)\right]_{\xi=a} d\eta$$
$$+ \int_0^a f_3(\xi)\left[\frac{\partial}{\partial \eta}G(x, y, \xi, \eta)\right]_{\eta=0} d\xi - \int_0^a f_4(\xi)\left[\frac{\partial}{\partial \eta}G(x, y, \xi, \eta)\right]_{\eta=b} d\xi.$$

Two forms of representation of the Green's function:

$$G(x,y,\xi,\eta) = \frac{2}{a}\sum_{n=1}^{\infty}\frac{\sin(p_n x)\sin(p_n\xi)}{p_n\sinh(p_n b)}H_n(y,\eta) = \frac{2}{b}\sum_{m=1}^{\infty}\frac{\sin(q_m y)\sin(q_m\eta)}{q_m\sinh(q_m a)}Q_m(x,\xi),$$

where

$$p_n = \frac{\pi n}{a},\quad H_n(y,\eta) = \begin{cases}\sinh(p_n\eta)\sinh[p_n(b-y)] & \text{for } b \ge y > \eta \ge 0,\\ \sinh(p_n y)\sinh[p_n(b-\eta)] & \text{for } b \ge \eta > y \ge 0,\end{cases}$$

$$q_m = \frac{\pi m}{b},\quad Q_m(x,\xi) = \begin{cases}\sinh(q_m\xi)\sinh[q_m(a-x)] & \text{for } a \ge x > \xi \ge 0,\\ \sinh(q_m x)\sinh[q_m(a-\xi)] & \text{for } a \ge \xi > x \ge 0.\end{cases}$$

3.2-5. Domain: $0 \le r \le R$. First boundary value problem for the Poisson equation.

A circle is considered. A boundary condition is prescribed:

$$w = f(\varphi) \quad \text{at} \quad r = R.$$

Solution in the polar coordinates:

$$w(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} f(\eta)\frac{R^2 - r^2}{r^2 - 2Rr\cos(\varphi-\eta) + R^2}\,d\eta + \int_0^{2\pi}\int_0^R \Phi(\xi,\eta)G(r,\varphi,\xi,\eta)\xi\,d\xi\,d\eta,$$

where

$$G(r,\varphi,\xi,\eta) = \frac{1}{4\pi}\ln\frac{r^2\xi^2 - 2R^2 r\xi\cos(\varphi-\eta) + R^4}{R^2[r^2 - 2r\xi\cos(\varphi-\eta) + \xi^2]}.$$

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