3.3. Helmholtz Equation $\Delta w + \lambda w = -\Phi(\mathbf{x})$

Many problems related to steady-state oscillations (mechanical, acoustical, thermal, electromagnetic) lead to the two-dimensional Helmholtz equation. For $\lambda < 0$, this equation describes mass transfer processes with volume chemical reactions of the first order.

The two-dimensional Helmholtz equation has the following form:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \lambda w = -\Phi(x, y) \quad \text{in the Cartesian coordinate system,}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2} + \lambda w = -\Phi(r, \varphi) \quad \text{in the polar coordinate system.}$$

3.3-1. Particular solutions of the homogeneous Helmholtz equation with $\Phi \equiv 0$.

1°. Particular solutions of the homogeneous Helmholtz equation in the Cartesian coordinate system:

$$\begin{aligned}
&w = (A\cos\mu_1 x + B\sin\mu_1 x)(C\cos\mu_2 y + D\sin\mu_2 y), && \lambda = \mu_1^2 + \mu_2^2,\\
&w = (A\cos\mu_1 x + B\sin\mu_1 x)(C\cosh\mu_2 y + D\sinh\mu_2 y), && \lambda = \mu_1^2 - \mu_2^2,\\
&w = (A\cosh\mu_1 x + B\sinh\mu_1 x)(C\cos\mu_2 y + D\sin\mu_2 y), && \lambda = -\mu_1^2 + \mu_2^2,\\
&w = (A\cosh\mu_1 x + B\sinh\mu_1 x)(C\cosh\mu_2 y + D\sinh\mu_2 y), && \lambda = -\mu_1^2 - \mu_2^2,
\end{aligned}$$

where A, B, C, and D are arbitrary constants.

2°. Particular solutions of the homogeneous Helmholtz equation in the polar coordinate system:

$$\begin{aligned}
&w = [AJ_m(\mu r) + BY_m(\mu r)](C\cos m\varphi + D\sin m\varphi), && \lambda = \mu^2,\\
&w = [AI_m(\mu r) + BK_m(\mu r)](C\cos m\varphi + D\sin m\varphi), && \lambda = -\mu^2,
\end{aligned}$$

where $m = 1, 2, \ldots;$ A, B, C, D are arbitrary constants; the $J_m(\mu)$ and $Y_m(\mu)$ are the Bessel functions; and the $I_m(\mu)$ and $K_m(\mu)$ are the modified Bessel functions.

3.3-2. Domain: $-\infty < x < \infty$, $-\infty < y < \infty$.

1°. Solution for $\lambda = -s^2 < 0$:

$$w(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(\xi, \eta)K_0(s\varrho)\,d\xi\,d\eta, \qquad \varrho = \sqrt{(x-\xi)^2 + (y-\eta)^2}.$$

2°. Solution for $\lambda = k^2 > 0$:

$$w(x, y) = -\frac{i}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(\xi, \eta)H_0^{(2)}(k\varrho)\,d\xi\,d\eta, \qquad \varrho = \sqrt{(x-\xi)^2 + (y-\eta)^2},$$

where $H_0^{(2)}(z)$ is the Hankel function of the second kind of order 0. The radiation conditions (Sommerfeld conditions) at infinity were used to obtain this solution.

3.3-3. Domain: $-\infty < x < \infty$, $0 \le y < \infty$. First boundary value problem.

A half-plane is considered. A boundary condition is prescribed:

$$w = f(x) \quad \text{at} \quad y = 0.$$

Solution:

$$w(x,y)=\int_{-\infty}^{\infty} f(\xi)\left[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta)\right]_{\eta=0} d\xi+\int_0^{\infty}\int_{-\infty}^{\infty}\Phi(\xi,\eta)G(x,y,\xi,\eta)\,d\xi\,d\eta.$$

1°. The Green's function for $\lambda=-s^2<0$:

$$G(x,y,\xi,\eta)=\frac{1}{2\pi}\bigl[K_0(s\varrho_1)-K_0(s\varrho_2)\bigr],$$

$$\varrho_1=\sqrt{(x-\xi)^2+(y-\eta)^2},\qquad \varrho_2=\sqrt{(x-\xi)^2+(y+\eta)^2}.$$

2°. The Green's function for $\lambda=k^2>0$:

$$G(x,y,\xi,\eta)=-\frac{i}{4}\bigl[H_0^{(2)}(k\varrho_1)-H_0^{(2)}(k\varrho_2)\bigr].$$

3.3-4. Domain: $0\le x\le a$, $0\le y\le b$. First boundary value problem.

A rectangle is considered. Boundary conditions are prescribed:

$$w=f_1(y)\quad\text{at}\quad x=0,\qquad w=f_2(y)\quad\text{at}\quad x=a,$$
$$w=f_3(x)\quad\text{at}\quad y=0,\qquad w=f_4(x)\quad\text{at}\quad y=b.$$

1°. Eigenvalues of the homogeneous problem with $\Phi\equiv 0$ (it is convenient to label them with a double subscript):

$$\lambda_{nm}=\pi^2\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}\right);\qquad n=1,\,2,\,\dots;\quad m=1,\,2,\,\dots$$

Eigenfunctions and the norm squared:

$$w_{nm}=\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right),\qquad \|w_{nm}\|^2=\frac{ab}{4}.$$

2°. Solution for $\lambda\neq\lambda_{nm}$:

$$\begin{aligned}w(x,y)&=\int_0^a\int_0^b\Phi(\xi,\eta)G(x,y,\xi,\eta)\,d\eta\,d\xi\\&+\int_0^b f_1(\eta)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta)\right]_{\xi=0}d\eta-\int_0^b f_2(\eta)\left[\frac{\partial}{\partial\xi}G(x,y,\xi,\eta)\right]_{\xi=a}d\eta\\&+\int_0^a f_3(\xi)\left[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta)\right]_{\eta=0}d\xi-\int_0^a f_4(\xi)\left[\frac{\partial}{\partial\eta}G(x,y,\xi,\eta)\right]_{\eta=b}d\xi.\end{aligned}$$

Two forms of representation of the Green's function:

$$G(x,y,\xi,\eta)=\frac{2}{a}\sum_{n=1}^{\infty}\frac{\sin(p_n x)\sin(p_n\xi)}{\beta_n\sinh(\beta_n b)}H_n(y,\eta)=\frac{2}{b}\sum_{m=1}^{\infty}\frac{\sin(q_m y)\sin(q_m\eta)}{\mu_m\sinh(\mu_m a)}Q_m(x,\xi),$$

where

$$p_n=\frac{\pi n}{a},\quad \beta_n=\sqrt{p_n^2-\lambda},\quad H_n(y,\eta)=\begin{cases}\sinh(\beta_n\eta)\sinh[\beta_n(b-y)] & \text{for } b\ge y>\eta\ge 0,\\ \sinh(\beta_n y)\sinh[\beta_n(b-\eta)] & \text{for } b\ge\eta>y\ge 0,\end{cases}$$

$$q_m=\frac{\pi m}{b},\quad \mu_m=\sqrt{q_m^2-\lambda},\quad Q_m(x,\xi)=\begin{cases}\sinh(\mu_m\xi)\sinh[\mu_m(a-x)] & \text{for } a\ge x>\xi\ge 0,\\ \sinh(\mu_m x)\sinh[\mu_m(a-\xi)] & \text{for } a\ge\xi>x\ge 0.\end{cases}$$

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