5.1. Equation of Transverse Vibration of the Form $\dfrac{\partial^2 w}{\partial t^2} + a^2 \dfrac{\partial^4 w}{\partial x^4} = 0$

Equation of transverse vibration of elastic rods.

5.1-1. Particular solutions:

$$w(x,t) = (Ax^3 + Bx^2 + Cx + D)t + A_1x^3 + B_1x^2 + C_1x + D_1,$$
$$w(x,t) = \bigl[A\sin(\lambda x) + B\cos(\lambda x) + C\sinh(\lambda x) + D\cos(\lambda x)\bigr]\sin(\lambda^2 at),$$
$$w(x,t) = \bigl[A\sin(\lambda x) + B\cos(\lambda x) + C\sinh(\lambda x) + D\cos(\lambda x)\bigr]\cos(\lambda^2 at),$$

where A, B, C, D, A_1, B_1, C_1, D_1, and λ are arbitrary constants.

5.1-2. Domain: $-\infty < x < \infty$. Cauchy problem.

Initial conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0, \qquad \partial_t w = a g''(x) \quad \text{at} \quad t = 0.$$

Boussinesq solution:

$$w(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\bigl(x - 2\xi\sqrt{at}\bigr)\bigl(\cos\xi^2 + \sin\xi^2\bigr)\,d\xi + \frac{1}{a\sqrt{2\pi}} \int_{-\infty}^{\infty} g\bigl(x - 2\xi\sqrt{at}\bigr)\bigl(\cos\xi^2 - \sin\xi^2\bigr)\,d\xi.$$

5.1-3. Domain: $0 \le x < \infty$. Free vibration of a semiinfinite rod.

The following conditions are prescribed:

$$w = 0 \quad \text{at} \quad t = 0, \qquad \partial_t w = 0 \quad \text{at} \quad t = 0 \qquad \text{(initial conditions)},$$
$$w = f(t) \quad \text{at} \quad x = 0, \qquad \partial_{xx} w = 0 \quad \text{at} \quad x = 0 \qquad \text{(boundary conditions)}.$$

Boussinesq solution:

$$w(x,t) = \frac{1}{\sqrt{\pi}} \int_{x/\sqrt{2at}}^{\infty} f\left(t - \frac{x^2}{2a\xi^2}\right)\left(\sin\frac{\xi^2}{2} + \cos\frac{\xi^2}{2}\right) d\xi.$$

5.1-4. Domain: $0 \le x \le l$. Boundary value problems.

For solutions of various boundary value problems, see Subsection 5.2 for $\Phi \equiv 0$.

References

Sneddon, I., *Fourier Transformations*, McGraw-Hill, New York, 1951.

Polyanin, A. D., *Handbook of Linear Partial Differential Equations for Engineers and Scientists*, Chapman & Hall/CRC, 2002.

