5. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + b\sinh(\lambda w).$

Sinh-Gordon equation. It arises in some areas of physics.

1°. Traveling-wave solutions:

$$w(x,t) = \pm\frac{2}{\lambda}\ln\left[\tan\frac{b\lambda(kx+\mu t+\theta_0)}{2\sqrt{b\lambda(\mu^2-ak^2)}}\right],$$

$$w(x,t) = \pm\frac{4}{\lambda}\operatorname{arctanh}\left[\exp\frac{b\lambda(kx+\mu t+\theta_0)}{\sqrt{b\lambda(\mu^2-ak^2)}}\right],$$

where k, μ, and θ_0 are arbitrary constants. It is assumed that $b\lambda(\mu^2-ak^2)>0$ in both formulas.

2°. Functional separable solution:

$$w(x,t) = \frac{4}{\lambda}\operatorname{arctanh}\bigl[f(t)g(x)\bigr], \qquad \operatorname{arctanh} z = \frac{1}{2}\ln\frac{1+z}{1-z},$$

where the functions $f=f(t)$ and $g=g(x)$ are determined by the first-order autonomous ordinary differential equations

$$(f'_t)^2 = Af^4 + Bf^2 + C, \qquad a(g'_x)^2 = Cg^4 + (B-b\lambda)g^2 + A,$$

where A, B, and C are arbitrary constants.

3°. For other exact solutions of the sinh-Gordon equation, see the nonlinear Klein–Gordon equation with $f(w) = b\sinh(\lambda w)$.

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