6. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + b\sin(\lambda w).$

Sine-Gordon equation. It arises in differential geometry and various areas of physics.

1°. Traveling-wave solutions:

$$w(x,t) = \frac{4}{\lambda}\arctan\left\{\exp\left[\pm\frac{b\lambda(kx+\mu t+\theta_0)}{\sqrt{b\lambda(\mu^2-ak^2)}}\right]\right\} \quad \text{if} \quad b\lambda(\mu^2-ak^2)>0,$$
$$w(x,t) = -\frac{\pi}{\lambda}+\frac{4}{\lambda}\arctan\left\{\exp\left[\pm\frac{b\lambda(kx+\mu t+\theta_0)}{\sqrt{b\lambda(ak^2-\mu^2)}}\right]\right\} \quad \text{if} \quad b\lambda(\mu^2-ak^2)<0,$$

where k, μ, and θ_0 are arbitrary constants. The first expression corresponds to a single-soliton solution.

2°. Functional separable solutions:

$$w(x,t) = \frac{4}{\lambda}\arctan\left[\frac{\mu\sinh(kx+A)}{k\sqrt{a}\cosh(\mu t+B)}\right], \quad \mu^2 = ak^2+b\lambda>0;$$
$$w(x,t) = \frac{4}{\lambda}\arctan\left[\frac{\mu\sin(kx+A)}{k\sqrt{a}\cosh(\mu t+B)}\right], \quad \mu^2 = b\lambda-ak^2>0;$$
$$w(x,t) = \frac{4}{\lambda}\arctan\left[\frac{\gamma}{\mu}\,\frac{e^{\mu(t+A)}+ak^2e^{-\mu(t+A)}}{e^{k\gamma(x+B)}+e^{-k\gamma(x+B)}}\right], \quad \mu^2 = ak^2\gamma^2+b\lambda>0,$$

where A, B, k, and γ are arbitrary constants.

3°. An N-soliton solution is given by ($a=1$, $b=-1$, and $\lambda=1$)

$$w(x,t) = \arccos\left[1-2\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial t^2}\right)(\ln F)\right],$$
$$F=\det\left[M_{ij}\right], \quad M_{ij}=\frac{2}{a_i+a_j}\cosh\left(\frac{z_i+z_j}{2}\right), \quad z_i=\pm\frac{x-\mu_i t+C_i}{\sqrt{1-\mu_i^2}}, \quad a_i=\pm\sqrt{\frac{1-\mu_i}{1+\mu_i}},$$

where μ_i and C_i are arbitrary constants.

4°. For other exact solutions of the sine-Gordon equation, see the nonlinear Klein–Gordon equation with $f(w)=b\sin(\lambda w)$.

5°. The sine-Gordon equation is integrated by the inverse scattering method.

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