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Edmond Halley's Life Table and Its Uses*

Edmond Halley (1656-1742) was a remarkable man of science who made important contributions in astronomy, mathematics, physics, financial economics, and actuarial science. Halley was fortunate to have been born into a wealthy family and to have had a father who provided for a first-rate education for his son. Halley enrolled in Oxford University at age 17, stayed for three years and, without a degree in hand, set sail for St. Helena in the south Atlantic to observe and catalogue stars unobservable from Europe. The voyage took two years and, upon his return to London, he was elected to the Royal Society at age 22 for his St. Helena work. Halley became the editor of *Philosophical Transactions* (the journal of the Royal Society), an Oxford professor from 1704-20, and Astronomer Royal at Greenwich from 1720 to his death. Isaac Newton and Halley were friends, and he urged Newton to write what became the *Principia Mathematica* and assisted financially and editorially in its publication. Halley plotted the orbits of several comets. In particular, he conjectured that objects that appeared in 1531, 1607, and 1682 were one and the same comet that would reappear approximately every 75 years. He correctly predicted that the comet would return in 1758, and it was posthumously named in his honor after its reappearance at the predicted time. Halley made two forays into financial economics, demography, and actuarial science. The second work (1705, 1717) was on compound interest. He derived formulae for approximating the annual percentage rate of interest implicit in financial transactions and annuities. His first contribution (1693) was seminal and is the topic of this note. In this work, Halley developed the first life table based on sound demographic data; and he discussed several applications of his life table, including calculations of life contingencies.

Halley obtained demographic data for Breslau, a city in Silesia which is now the Polish city Wroclaw. Breslau kept detailed records of births, deaths, and the ages of people when they died. In comparison, when John Graunt (1620-1674) published his famous demographic work (1662), ages of deceased people were not recorded in London and would not be re-

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corded until the 18th century.[1] Caspar Neumann, an important German minister in Breslau, sent some demographic records to Gottfried Leibniz who in turn sent them to the Royal Society in London. Halley analyzed Newmann's data which covered the years 1687-1691 and published the analysis in the *Philosophical Transactions*. Although Halley had broad interests, demography and actuarial science were quite far afield from his main areas of study. Hald (2003) has speculated that Halley himself analyzed these data because, as the editor of the *Philosophical Transactions*, he was concerned about the *Transactions* publishing an adequate number of quality papers.[2] Apparently, by doing the work himself, he ensured that one more high quality paper would be published.

The Breslau data had the property that annual births were approximately equal to deaths,[3] there was little migration in or out of the city, and age specific death rates were approximately constant; that is, Breslau had an approximately stationary population. After some adjustments and smoothing of the data, Halley produced a combined table of male and female survivors; here reproduced as Table 1. He determined the population was approximately 34,000 people. To explain this table, let l_x denote the size of a population at exact age $x = 0,1,2,\ldots,\omega$, where ω is the youngest age at which everyone in the population has died, then $L_x = .5(l_x + l_{x+1})$ captures the average number alive between ages *x* and *x*+1; or, alternatively, the number of years lived by members of the population between ages *x* and *x*+1. Halley's life table gives L_{x-1}; so, for example, the very first entry (for age *x* = 1) is $L_{x-1} = L_{1-1} = L_0 = .5(l_0 + l_1) = 1000$, the average number of people alive between ages zero and one.[4] Figure 1 is the graph of Halley's table; and, for purposes of comparison, we also show the life table for the US in 2004 (CDCP, 2007). Halley made seven observations and used his life table to exemplify those observations.

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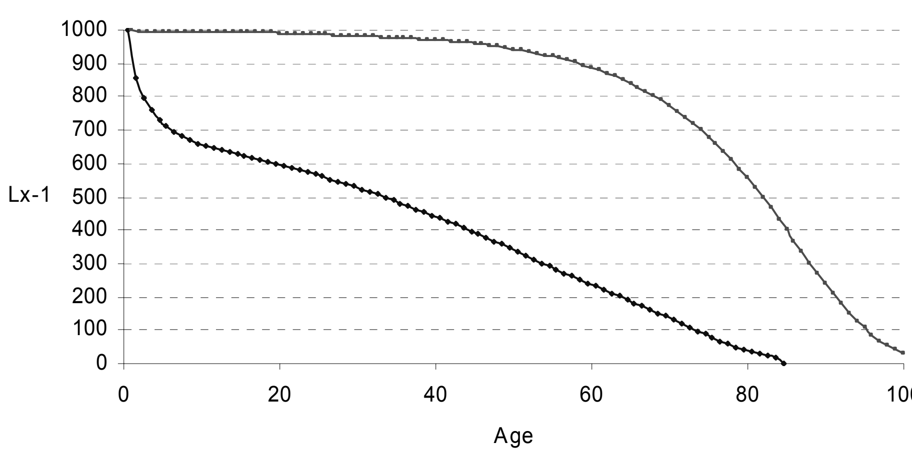
[1]John Graunt developed a life table in 1662 based on London's bills of mortality, but he engaged in a great deal of guess work because age at death was unrecorded and because London's population was growing in an un-quantified manner due to migration.
[2]Without arguing in support or against Hald in this regard, we note that the same issue of *Philosophical Transactions* contained papers by the great chemist/physicist Robert Boyle and the noted mathematician John Wallis.
[3]There was a small increase in population. As Halley put it "an increase of the people may be argued of 64 per annum." Here, Halley mentions that excess births "may perhaps be balanced by the levies of the emperor's service in his wars."
[4]Table 1 has a radix of 1000. The Breslau data had $l_0 = 1238$ and $l_1 = 890$, implying $L_0 = 1064$. Halley seems to have rounded to 1000 for convenience.

Table 1. Halley's Life Table

Age x	L_{x-1}	Age x	L_{x-1}	Age x	L_{x-1}	Age x	L_{x-1}
1	1000	23	579	45	397	67	172
2	855	24	573	46	387	68	162
3	798	25	567	47	377	69	152
4	760	26	560	48	367	70	142
5	732	27	553	49	357	71	131
6	710	28	546	50	346	72	120
7	692	29	539	51	335	73	109
8	680	30	531	52	324	74	98
9	670	31	523	53	313	75	88
10	661	32	515	54	302	76	78
11	653	33	507	55	292	77	68
12	646	34	499	56	282	78	58
13	640	35	490	57	272	79	49
14	634	36	481	58	262	80	41
15	628	37	472	59	252	81	34
16	622	38	463	60	242	82	28
17	616	39	454	61	232	83	23
18	610	40	445	62	222	84	20
19	604	41	436	63	212		
20	598	42	427	64	202	85-100	107
21	592	43	417	65	192		
22	586	44	407	66	182	Total	34000

Figure 1. Halley's 1693 Breslau and 2004 US Population Survivor Functions



First, Halley looked at his table from a military point of view (perhaps because Graunt did exactly the same thing in 1662) and calculated "the proportion of men able to bear arms." He computed the number of people between the ages of 18 and 56, divided by two to estimate the number of men, and expressed the latter number as a fraction of the entire population of 34,000 people. Halley's approximate answer was "9/34" or about .26 of the population (see Table 1). If one were to make a similar calculation using the current US life table illustrated in Figure 1, the corresponding fraction is .24. Little has changed since Halley's time in this regard even though Figure 1 illustrates two very different life tables.

Second, Halley computed survival odds between ages using $L_{x+t}/(L_x - L_{x+t})$. He gave an example of "377 to 68 or 5.5 to 1" for a man age 40 living to age 47 (see Table 1).

Third, Halley computed "the age, to which it is an even wager that a person of the age proposed shall arrive before he die." That is, Halley calculated the median additional years of life. He gave an example for a 30 year old. There are 531 survivors at that age and half that many between ages 57 and 58 (see Table 1). Therefore, Halley's median was between 27 and 28 years. Halley made no life expectancy calculations.

Fourth, in one rather long sentence, Halley mentioned that the price of term insurance "ought to be regulated," and its price related to the odds of survival. He pointed out that the odds of one year survival were "100 to 1 that a man of 20 dies not in a year, and but 38 to 1 for a man of 50 years of age." Halley's point is clear, but there is a typographical error in the paper because the odds of survival for a 50 year old are approximately 30 to 1 (see Table 1).

Fifth, Halley did not give an explicit mathematical formula for a life annuity, but he provided text and example calculations that clearly showed that he used the following formula:[5]

(1) $$a_x = \sum_{t=1}^{\omega-x-1} (1+i)^{-t} (L_{x+t} / L_x).$$

Halley calculated life annuities with a 6% discount rate and provided the expected present values shown in Table 2. The "Years Purchase" Columns are the expected present values of life annuities of one pound. Halley noted that the British government sold annuities for seven years purchase regard-

[5]After some re-writing, Halley's life annuity formula is similar to Jan De Witt's (1671) formula as shown in the Appendix.

less of ages of nominees. Table 2 shows this was about half the value of an annuity on 5, 10, or 15-year-old nominees and poor governmental policy for all nominees under age 60, but the British government did not change its single-price policy after Halley's work.

Table 2. Halley's Life Annuity Table

Age	Years Purchase	Age	Years Purchase	Age	Years Purchase
1	10.28	25	12.27	50	9.21
5	13.40	30	11.72	55	8.51
10	13.44	35	11.12	60	7.60
15	13.33	40	10.57	65	6.54
20	12.78	45	9.19	70	5.32

Sixth, Halley turned his attention to a joint life annuity on two lives. He used a rectangle with length L_x and height L_y to represent lives age *x* and *y*. In contemporary notation, let $L_x \equiv L_{x+t} + {}_tD_x$ and $L_y \equiv L_{y+t} + {}_tD_y$, where ${}_tD_x$ and ${}_tD_y$ denote deaths from L_x and L_y within *t* years. The product of L_x and L_y is

(2a) $$L_xL_y = L_{x+t}L_{y+t} + L_{x+t}\,{}_tD_y + L_{y+t}\,{}_tD_x + {}_tD_x\,{}_tD_y\,.$$

The left side of (2a) represents the area of Halley's rectangle which he calls the total number of "chances." Halley gave the example from Table 1 for *x* = 18 and *y* = 35 and said "[t]here are in all 610 x 490 or 298,900 chances." Halley continued the example for *t* = 8 and said that the number of chances was "50 x 73 or 3650 that they are both dead," which is the last term of the right hand side of (2a). This gives us

(2b) $$L_xL_y - {}_tD_x\,{}_tD_y = L_{x+t}L_{y+t} + L_{x+t}\,{}_tD_y + L_{y+t}\,{}_tD_x$$

(2c) $$(1 - {}_tD_x\,{}_tD_y / L_xL_y) = (1/L_xL_y)(L_{x+t}L_{y+t} + L_{x+t}\,{}_tD_y + L_{y+t}\,{}_tD_x)$$

where (2c) is the probability of at least one life surviving. The life annuity that pays when at least one of two nominees survives becomes

(3) $$a_{xy} = \sum_{t=1}^{\omega-x-1} (1+i)^{-t} (1 - {}_tD_x \, {}_tD_y / L_x L_y) .$$

Halley did not provide any numerical examples of annuity calculations in this part of his paper.

Seventh, Halley considered the problem of annuities on three lives. He drew a complicated looking three dimensional figure which is the extension of the rectangle he previously considered. The dimensions of this new figure, in modern notation, are $L_x \equiv L_{x+t} + {}_tD_x$, $L_y \equiv L_{y+t} + {}_tD_y$, and $L_z \equiv L_{z+t} + {}_tD_z$. The product $L_x L_y L_z = (L_{x+t} + {}_tD_x)(L_{y+t} + {}_tD_y)(L_{z+t} + {}_tD_z)$ has eight terms that correspond to various living and death states for three lives. At this point, Halley computed the value of a life annuity that pays whenever at least one of the three nominees is alive with a formula like

(4) $$a_{xyz} = \sum_{t=1}^{\omega-x-1} (1+i)^{-t} (1 - {}_tD_x \, {}_tD_y \, {}_tD_z / L_x L_y L_z) .$$

He gave an example where $x = 10$, $y = 30$, and $z = 40$ and concluded such an annuity was worth 16. 58 years purchase. Finally, Halley talked about a reversionary annuity on the youngest life age x after the older lives ages y and z. That is, the annuity pays the youngest nominee after the older nominees die. The value of this annuity is

(5) $$a_{yz|x} = \sum_{t=1}^{\omega-x-1} (1+i)^{-t} (L_{x+t} \, {}_tD_y \, {}_tD_z / L_x L_y L_z) .$$

At this point Halley seemed to tire of the laborious calculations involved in formula (5) and he concluded his paper.

To summarize, here is what we can say about Halley's paper: (1) we still use life tables similar to the one he developed and (2) we still make calculations of life contingencies as he did. The main difference between Halley and modern work lies in Halley's use of the average survivors between ages (*i.e.*, $L_x, L_{x+1}, ..., L_{\omega-1}$) rather than survivors at exact ages ($l_x, l_{x+1}, ..., l_{\omega-1}$), although some (*e.g.*, Poitras, 2000) interpret Halley as using survivors at exact ages. In either case, Halley wrote a remarkable paper 300 years ago; formulae (1), (3), (4), and (5) are especially insightful.

Halley reflected on his paper in a postscript. Four additional paragraphs appear like a coda which he entitled "Some Further Considerations on the Breslau Bills of Mortality." Halley mused about "how unjustly we repine at the shortness of our lives" and "think ourselves wronged if we attain not old age." After observing that only about half of Breslau's 1,238 newly born children survive 17 years, Halley added that we should not fret about "untimely death" but rather "submit to that dissolution which is the necessary condition of our perishable materials." He concluded this train of thought by observing the "blessing" we have received if we have lived more than the median years of life at birth. Halley's second, and last, comment dealt with human fertility. He calculated approximately 15,000 persons between ages 16 and 45 (see Table 1) and estimated that at least 7,000 were "women capable to bear children." He reckoned that 1,238 births relative to 7,000 fertile women were "but little more than a sixth part." If all women in this age group were married, Halley thought "four of six should bring a child every year." Celibacy was to be discouraged and large families encouraged because "the strength and glory of a king" was in direct proportion to the "magnitude of his subjects." Halley concluded with a carrot and stick policy prescription: the stick part was that celibacy should be discouraged through "extraordinary taxing and military service," and the carrot was that large families should be encouraged through society finding employment for poor people and through laws such as the "*jus trium liberorum* among the Romans."[6]

— James E. Ciecka

[6]Augustus Caesar granted certain privileges to fathers of three or more children. These privileges were known by the term *jus trium liberorum*. Thomas Malthus mentioned these laws in *An Essay on the Principle of Population* (1798) as being ineffective among poorer classes and of some minor influence on higher classes of Roman citizens.

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Appendix

We can get close to Halley's life annuity formula (1) from Jan de Witt's formula as shown in (A1)-(A5). De Witt (1671; Hendricks, 1852 and 1853) used the distribution of deaths $d_x = l_x - l_{x+1}$ in his formula for the expected present value of a life annuity; here written as the left hand side of formula (A1).

(A1) $$E(a_{\overline{T}|}) = \sum_{t=1}^{\omega-x-1} a_{\overline{t}|}(d_{x+t}/l_x) = \sum_{t=1}^{\omega-x-1}\sum_{j=1}^{t}(1+i)^{-j}(d_{x+t}/l_x)$$

(A2) $$\begin{aligned} &= [(1+i)^{-1}](d_{x+1}/l_x) \\ &+ [(1+i)^{-1}+(1+i)^{-2}](d_{x+2}/l_x) \\ &+ [(1+i)^{-1}+(1+i)^{-2}+(1+i)^{-3}](d_{x+3}/l_x) \\ &\vdots \\ &+ [(1+i)^{-1}+(1+i)^{-2}+\cdots+(1+i)^{-(\omega-x-1)}](d_{\omega-1}/l_x) \end{aligned}$$

(A3) $$\begin{aligned} &= [(1+i)^{-1}](d_{x+1}+d_{x+2}+d_{x+3}+\cdots+d_{\omega-1})/l_x \\ &+ [(1+i)^{-2}](d_{x+2}+d_{x+3}+\cdots+d_{\omega-1})/l_x \\ &+ [(1+i)^{-3}](d_{x+3}+\cdots+d_{\omega-1})/l_x \\ &\vdots \\ &+[(1+i)^{-(\omega-x-1)}](d_{\omega-1})/l_x \end{aligned}$$

(A4) $$\begin{aligned} &= [(1+i)^{-1}](l_{x+1})/l_x \\ &+ [(1+i)^{-2}](l_{x+2})/l_x \\ &+ [(1+i)^{-3}](l_{x+3})/l_x \\ &\vdots \\ &+[(1+i)^{-(\omega-x-1)}](l_{\omega-1})/l_x \end{aligned}$$

(A5) $$= \sum_{t=1}^{w-x-1}(1+i)^{-t}(l_{x+t}/l_x)$$

In going from the left to the right hand side of (A1), we simply use the definition

$$a_{\overline{t|}} \equiv \sum_{j=1}^{t} (1+i)^{-j} .$$

Summations are expanded in (A2) and then re-grouped in (A3). (A4) uses the property that the coefficient of $(1+i)^{-1}$ is

$$(d_{x+1} + d_{x+2} + ... + d_{\omega-1}) / l_x = l_{x+1} / l_x .^{7}$$

Similarly, the coefficient of $(i+i)^{-2}$ sums to $(d_{x+2} + ... + d_{\omega-1}) / l_x = l_{x+2} / l_x$, the coefficient of $(i+i)^{-3}$ is $(d_{x+3} + ... + d_{\omega-1}) / l_x = l_{x+3} / l_x$, and so on, until we get to the last term $(i+i)^{-(w-x-1)}$ with coefficient $(d_{\omega-1}) / l_x = l_{\omega-1} / l_x$. Formula (A5) becomes Halley's life annuity (formula (1)) when we substitute the average number of survivors between ages for the number alive at exact ages. That is, Halley used $L_x, L_{x+1}, ..., L_{\omega-1}$ and de Witt used $l_x, l_{x+1}, ..., l_{\omega-1}$. Halley published in 1693, some 22 years after de Witt; but there is no information that Halley was aware of de Witt's work. De Witt's formulation emphasizes the expected present value nature of a life annuity and uses the distribution of deaths $d_{x+1} / l_x ,\dots,\ d_{\omega-1} / l_x$. Halley uses the survivor distribution $L_{x+1} / L_x ,\dots,\ L_{\omega-1} / L_x$. De Witt's formulation allows one to compute higher order moments but Halley's does not. However, Halley's formulation, with the substitution of $l_x, l_{x+1}, ..., l_{\omega-1}$ for $L_x, L_{x+1}, ..., L_{\omega-1}$, has become the much more widely used method.

[7]The number of people alive at age x+1 is l_{x+1}. Since all must eventually die, we have $(d_{x+1} + d_{x+2} + ... + d_{\omega-1}) = l_{x+1}$. A similar idea holds for all ages; all people alive at a certain age will eventually die and the sum of those deaths equals the number alive at that age.