Chemical Engineering Science, Vol. 36, No. 12, pp. 1941–1954, 1981
Printed in Great Britain.

0009–2509/81/121941–14$02.00/0
Pergamon Press Ltd.

OBSERVABILITY AND REDUNDANCY CLASSIFICATION IN PROCESS NETWORKS
THEOREMS AND ALGORITHMS

G. M. STANLEY† and R. S. H. MAH*
Northwestern University, Evanston, IL 60201, U.S.A.

(*Received* 29 *May* 1980; *accepted* 22 *April* 1981)

Abstract—The utility of observability and redundancy in characterizing the performance of process data estimators was established in previous studies[10]. In this paper two classification algorithms for determining local and global observability and redundancy for individual variables and measurements are presented. The concepts of biconnected components, perturbation subgraphs and feasible unmeasurable perturbations are introduced, and their properties are developed and used to effect classification, simplification and dimensional reduction. Step-by-step application of these algorithms is illustrated by examples.

INTRODUCTION

In a previous paper[10] we developed the concepts of observability and redundancy for constrained steady state systems and demonstrated that these concepts are useful in characterizing the performance of process data estimators with regard to bias and uniqueness of an estimate, convergence of estimation procedures and the feasibility and implications of problem decomposition. We also derived first and second order sufficient conditions for local observability and, in the case of linear constraints and measurements, global observability. However, from the practical point of view the application of these conditions suffers two serious drawbacks. In the first place, they are based on matrix rank tests which are computationally cumbersome. For systems for which local observability classification is of no concern, one approach might be to perform a "structural" rank test[8]. Secondly, and even more seriously, since these tests are applied to the system as a whole, they cannot be used to classify the observability of individual variables. In this paper we shall show that by exploiting the structural characteristics of a process network we can develop theorems and algorithms which would classify individual variables in such a network.

There has been a growing interest recently in developing graph-theoretical controllability criteria, which are closely related to observability criteria, for linear dynamic systems. Göknar[3, 4] developed necessary and sufficient conditions, and an algorithm for testing controllability using signal-flow graphs. Lin[5] introduced the notion of "structural controllability", which is independent of the values of non-zero system parameters. This concept was related to properties of a directed graph derived from the structure of the system equations. In this paper we shall develop graph-theoretical observability and redundancy criteria for steady state process networks.

PROCESS NETWORK AND PERTURBATION SUBGRAPHS

We shall begin with a discussion of some graph-theoretic concepts and terminology needed for the classification algorithms, but it is not our intention to provide a comprehensive introduction to graph theory. The reader is referred to Mah and Shacham[7] for a summary of some very useful properties of graphs and digraphs and to Deo[2] for a fuller treatment of graph theory.

As we have shown in a previous paper[6], the process graph is a very useful representation of the topological structure of a process. We used that representation to obtain certain decomposition results for process data reconciliation, coaptation and fault detection. In this investigation we shall further exploit the structural properties of such a representation. We shall start with the properties of the underlying graph in which the directions of arcs have been erased.

A node v is a cut-node or articulation point of a connected graph G if its removal disconnects G. For instance, in Fig. 1(a) nodes 2 and 5 are cut-nodes. A graph is separable or 1-connected if it contains a cut-node. Now suppose we split a cut-node into two nodes to produce two disjoint subgraphs and let us refer to this operation as splitting. If we repeat this operation until all subgraphs are non-separable, then the resulting subgraphs are called blocks or biconnected components. The four biconnected components derived from the graph in Fig. 1(a) are shown in Fig. 1(b).

Clearly, two arcs belong to the same biconnected component if and only if they belong to a common cycle. Any two biconnected components are either disjoint or have exactly one cut-node in common. Each cut-node lies in at least two biconnected components and all other nodes can belong to only one biconnected component. It can also be shown that the rank and nullity of a graph are preserved in the splitting operation. Further discussion

*Author to whom correspondence should be addressed.
†Present address: Exxon Chemical Company, Linden, New Jersey, U.S.A.

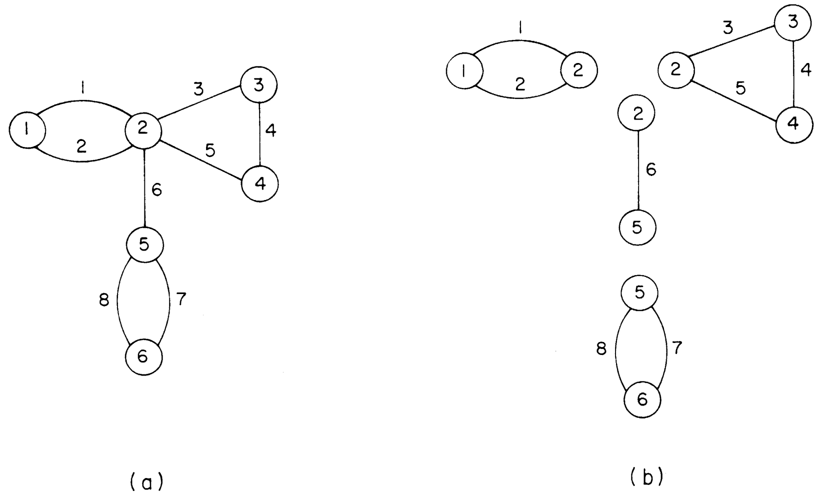


Fig. 1. A graph and its biconnected components.

of the properties of biconnected components is given elsewhere[11, 1].

Another operation which will be carried out in the classification algorithm is the aggregation of two adjacent nodes with the elimination of all arcs between them. We shall refer to this operation simply as aggregation and the aggregated node as a pseudo-node.

By contrast to a process graph we shall use the term "process network" to refer to both the structure of the graph and the attributes of the arcs. Schematically, solid lines and broken lines are used to represent mass flow arcs and pure energy flow arcs, and temperature and flow measurements are indicated by single and double slashes on the respective arcs, as shown in Fig. 2. On the basis of these attributes we can define a hierarchy of subgraphs. G_s is the subgraph of G with all measured pure energy flow arcs deleted. G_m is the subgraph of G with all pure energy flow arcs deleted. G_{mu} is obtained from G_m by deleting all arcs with flow measurements and G_{muu} is obtained from G_{mu} by deleting all arcs with

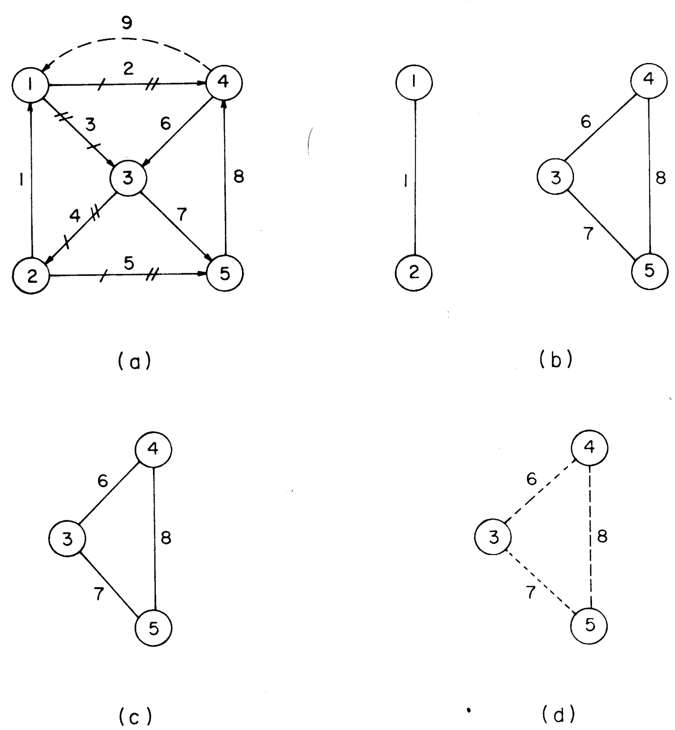


Fig. 2. A process network and some of its subgraphs: (a) a process network; (b) completely unmeasured mass flow subgraph G_{muu}; (c) a perturbation subgraph; (d) another perturbation subgraph.

temperature measurements. Thus, $G_{muu} \subset G_{mu} \subset G_m \subset G_s \subset G$. In this treatment we shall assume a one-to-one correspondence between temperature and enthalpy; measurement of one immediately infers measurement of the other. For a given mass flow arc i the enthalpy is related to the mass flow x_i and energy flow y_i by

$$H_i = \begin{cases} y_i/x_i, & x_i \neq 0 \\ 0, & x_i = 0. \end{cases} \tag{1}$$

For the steady state process network under consideration if the constraints consist of mass balances,

$$A_m x = 0 \tag{2}$$

only, where x is the vector of mass flows and A_m is the incidence matrix of G_m, the process network is termed a mass flow network. If the constraints contain the energy balances

$$\mathbf{A}\mathbf{y} = \mathbf{0} \tag{3}$$

as well, where $\mathbf{y}$ is the vector of energy flows and $\mathbf{A}$ the incidence matrix of G, it is termed a mass-energy flow network.

Now the solution $(\mathbf{x}, \mathbf{y})$ to eqns (2) and (3) is related to the fundamental cycle (mesh) flows $\mathbf{x}^2$ and $\mathbf{y}^2$ by

$$\mathbf{x} = \mathbf{C}_m{}^T\mathbf{x}^2 \tag{4}$$

$$\mathbf{y} = \mathbf{C}^T\mathbf{y}^2 \tag{5}$$

where $\mathbf{C}_m$ and $\mathbf{C}$ are the fundamental cycle matrices of G_m and G, respectively.

It is extremely important to note that the space of feasible solutions $(\mathbf{x}, \mathbf{y})$ for a mass-energy flow network (abbreviated as G) is linear. The nonlinearity of the energy balances appears only when we attempt to calculate the enthalpy $H_i = y_i/x_i$ or the temperature. Thus, if $(\mathbf{x}_1, \mathbf{y}_1)$ and $(\mathbf{x}_2, \mathbf{y}_2)$ are both solutions to eqns (2) and (3), so are their linear combinations, although some of these solutions may be physically infeasible, e.g. solutions giving rise to negative absolute temperatures. In practice this problem will not arise since only solutions close to known feasible solutions will be considered.

It also follows from the linearity of the solution space that linear combinations of feasible solutions for two subgraphs G_1 and G_2 are feasible solutions for $G_1 \cup G_2$ and that the space of feasible solutions for G is spanned by the vectors representing the feasible solutions for all possible subgraphs. In particular, the feasible solution space for G is minimally spanned by the feasible solutions for the subgraphs which are the fundamental cycles of G. The last observation gives rise to the following important definition and result:

Definition. Let $(\mathbf{x}^0, \mathbf{y}^0)$ be a feasible solution in G and let $(\delta\mathbf{x}, \delta\mathbf{y})$ be a feasible solution for G_1, where $G_1 \subset G$, $\delta\mathbf{x} \to \mathbf{0}$ and $\delta\mathbf{y} \to \mathbf{0}$. G_1 is a perturbation subgraph of G at $(\mathbf{x}^0, \mathbf{y}^0)$, if the measurements on G cannot distinguish $(\mathbf{x}^0, \mathbf{y}^0)$ from $(\mathbf{x}^0 + \delta\mathbf{x}, \mathbf{y}^0 + \delta\mathbf{y})$.

Lemma 1. In a process network with the underlying graph G, an arc variable (x_i, y_i or T_i) is locally unobservable at $(\mathbf{x}^0, \mathbf{y}^0)$ if and only if the arc lies in at least one perturbation subgraph of G.

The lemma follows immediately from the definition and the fact that feasible unmeasurable perturbations[10] can occur in a perturbation subgraph. It provides a basis for establishing sufficient conditions for observability and unobservability. However, the required theorems are sometimes more conveniently stated in terms of feasible unmeasurable perturbations which will be studied next. Figure 2 illustrates the subgraph G_{muu} and two perturbation subgraphs derived from a simple process network.

OBSERVABILITY CLASSIFICATION IN MASS FLOW NETWORKS

For mass flow networks the properties needed for observability classification are fairly obvious. However, it is worthwhile stating the principal results in order to bring out the contrast with the more complex situation of mass-energy flow networks. We shall omit the proofs which are given elsewhere[9].

Theorem 1. Let G be the underlying graph of a mass flow network, let v be a cut-node of G and let G_1 and G_2 be two subgraphs such that $G_1 \cup G_2 = G = G_m$ and $G_1 \cap G_2 = v$. Then

(a) No net mass flow can cross node v from G_1 to G_2 (or G_2 to G_1).

(b) No net mass flow can enter or leave a biconnected component of G_m.

(c) Nonzero mass flow in arc i is feasible if and only if arc i lies in a cycle in G_m.

Theorem 2. In a mass flow network mass flow in an arc i is unobservable if and only if arc i lies in a cycle of unmeasured arcs.

An immediate consequence of Theorem 2 is the following corollary which follows from Theorem 8 of our previous paper[10] and was previously proven by a different technique[6].

Corollary. For constrained least-squares estimation in a mass flow network, the flow estimate in an unmeasured arc i is nonunique if and only if arc i lies in an unmeasured cycle.

Note that since the constraints and measurement equations are both linear for mass flow networks, there is no distinction between local and global observability.

OBSERVABILITY CLASSIFICATION IN MASS-ENERGY FLOW NETWORKS

In mass flow networks the unmeasured cycle criterion (Theorem 2) provided both necessary and sufficient conditions for unobservability and observability. In mass-energy flow networks there are two types of measurements (flow and temperature) and three types of variables (temperature, mass and energy flows). With this increase in complexity, it will no longer be possible to write down such simple graph-theoretical conditions. Instead, sufficient conditions for observability and very different sufficient conditions for unobservability will be presented. Furthermore, since the energy balances are nonlinear due to the products of flow rates and enthalpies, it will be necessary to distinguish global observability criteria from local observability criteria.

It is important to note that the sufficient conditions for

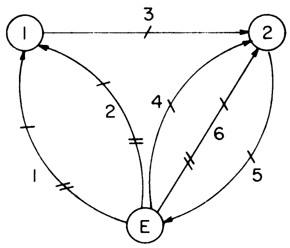


Fig. 3. A blending system.

mass flow observability implied by Theorem 2 for a mass flow network still apply in a mass-energy flow network. That is, if every cycle in G_m containing arc i has at least one flow measurement, then x_i is observable using only mass balances and mass flow measurements. Hence x_i is observable in the mass-energy flow network which includes mass balances and flow measurements as well as energy balances and temperature measurements. However, if arc i is in a cycle of G_m in which all mass flows are unmeasured, it may still be possible to calculate these flows using temperature measurements and energy balances. In fact, the difficulty in classifying observability in a mass-energy flow network arises precisely because of the possibility that a cycle of mass flow arcs might not have any mass flow measurements at all.

The blending system shown in Fig. 3 demonstrates the points just made. The flow x_3 lies in no cycles in G_{mu}, and hence it is observable just using the mass balance equations and flow measurements. However, x_4 and x_5 form a cycle with no mass flow measurements. Only through the use of temperature measurements and the energy balance can x_4 and x_5 be calculated. It will be seen that the possibility of performing this calculation depends on the values of H_4 and H_5.

We now present the basic theorems needed for observability classification. As in the case of the mass flow network, it will be necessary to determine which flows are feasible.

Theorem 3. Let G be the underlying graph of a mass-energy flow network, let v be a cut-node of G, and let G_1 and G_2 be the subgraphs such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = v$. Then

(a) No net mass or energy flow can cross node v from G_1 to G_2 (or G_2 to G_1)

(b) No net mass or energy flow can enter or leave a biconnected component of G.

(c) Nonzero mass flow in an arc i is feasible if and only if arc i lies in a cycle of G_m

(d) Nonzero energy flow in an arc i is feasible if and only if arc i lies in a cycle of G.

The proof of Theorem 3 is very similar to that of Theorem 1 and is omitted here for the same reason.

Theorem 4. Let G be the underlying graph of a mass-energy flow network and let mass flow arcs i and j form a cut-set of G. T_j is globally observable if T_i is measured.

Proof. Let G_1 and G_2 be the two subgraphs of G after deleting arcs i and j and let

$$a_i = \begin{cases} 1, & \text{if arc } i \text{ is directed from } G_1 \text{ to } G_2 \\ -1, & \text{if arc } i \text{ is directed from } G_2 \text{ to } G_1 \end{cases} \tag{6}$$

$$a_j = \begin{cases} 1, & \text{if arc } j \text{ is directed from } G_1 \text{ to } G_2 \\ -1, & \text{if arc } j \text{ is directed from } G_2 \text{ to } G_1. \end{cases} \tag{7}$$

Then the mass and energy balances about G_2 are

$$a_i x_i + a_j x_j = 0 \tag{8}$$

$$a_i x_i H_i + a_j x_j H_j = 0. \tag{9}$$

Multiplying eqn (8) by H_i and subtracting it from eqn (9), we obtain

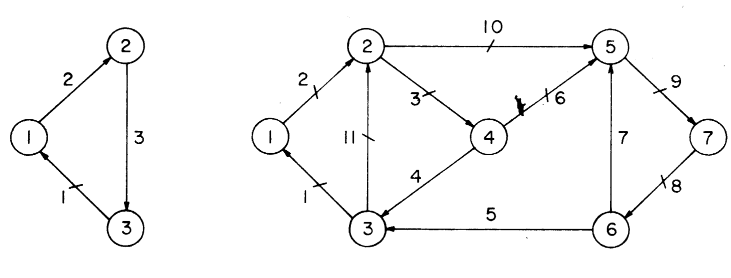
$$a_j x_j (H_j - H_i) = 0. \tag{10}$$

If $x_j \neq 0$, $H_j = H_i$, and H_i cannot be perturbed without affecting the measurement T_i. If $x_j = 0$, then by definition $H_j = 0$ and its value is again fixed. ■

Corollary. Let G be the underlying graph of a mass-energy flow network and let mass flow arcs, j, i_1, i_2, ... and i_k form a cut-set of G. T_j is locally observable, if T_{i_1}, T_{i_2}, ... T_{i_k} are measured and $H_{i_1} = H_{i_2} = \ldots = H_{i_k}$.

We shall omit the proof which follows an analogous development to that of Theorem 4 but illustrate the application with reference to Fig. 4. Theorem 4 shows that T_1, T_2 and T_3 in Fig. 4(a) are globally observable, and by the corollary, if $H_6 = H_{10}$, T_5 is observable and $H_5 = H_6 = H_{10}$ in Fig. 4(b).

The two foregoing results deal with sufficient conditions for global and local observability. The next three



(a) (b)

Fig. 4. Process networks.

theorems stipulate the sufficient conditions for global and local unobservability.

Theorem 5. Let G be the underlying graph of a mass-energy flow network.

(a) if arc i is in a cycle of C of G_{muu}, then x_i, y_i and T_i are globally unobservable;

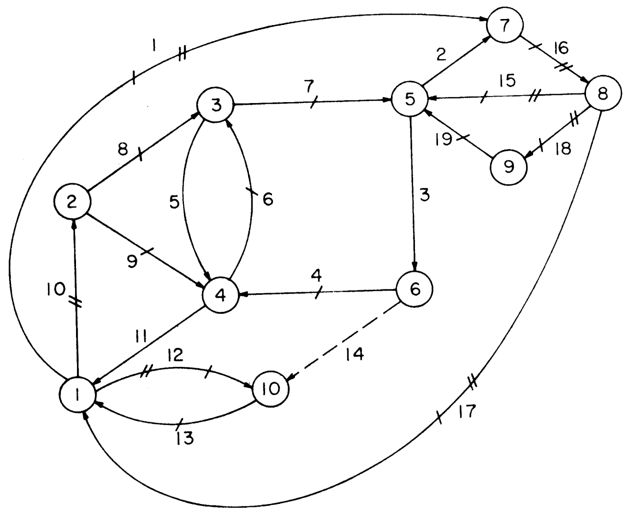
(b) if arc i is in a cycle C of G_{mu} with exactly one temperature measurement, then x_i and y_i are globally unobservable.

Proof. (a) For the cycle C in G_{muu} choose any arc k, $k \neq i$, in C and choose a spanning tree for G_{muu} for which arc k is not a branch. Consider first an arbitrary perturbation δx_k with T_k held constant. Then $\delta y_k = H_k \delta x_k$ and from eqns (4) and (5) we have also perturbed all x_j and y_j in C by δx_k and δy_k, respectively. Since there are no measurements in C, these perturbations are clearly feasible and unmeasurable, and since the argument does not depend on the values of x_j, y_j or H_j in C, x_i and y_i are globally unobservable.

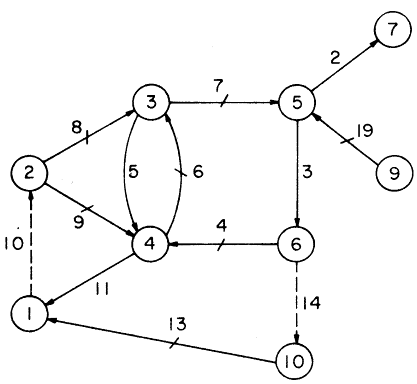
Next consider an arbitrary perturbation δy_k with x_k held constant. From eqn (5) all y_j in C are perturbed by δy_k and $H_j = (y_j \pm \delta y_k)/x_j$ with the sign determined by the direction of arc j. In either case H_j is perturbed to a new feasible value and there are no measurements in C to detect the change. Hence, by the same argument T_i is globally unobservable.

(b) Now use the same construction as before but let arc k be the arc with the temperature measurement. The first type of perturbations (δx_j, δy_j) is still feasible and unmeasured, but the second type (δy_j, δT_k) would be detected. Hence x_i and y_i are globally unobservable, but no conclusion is drawn about the observability of T_i. ■

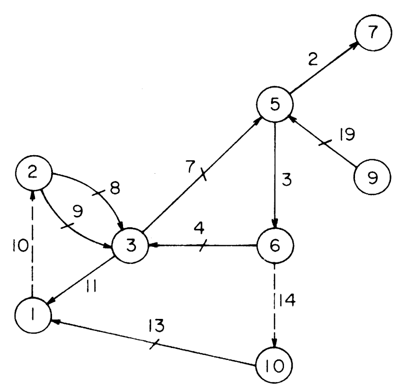
An illustration of the above theorem may be found in Fig. 5(a). Arcs 5 and 6 form a cycle in G_{mu} with exactly one temperature measurement. Hence x_5, x_6, y_5 and y_6 are globally unobservable. Note that even though mass and energy flows are globally unobservable in a cycle of G_{mu} with exactly one temperature measurement, it is still



(a)



(b)



(c)

Fig. 5.

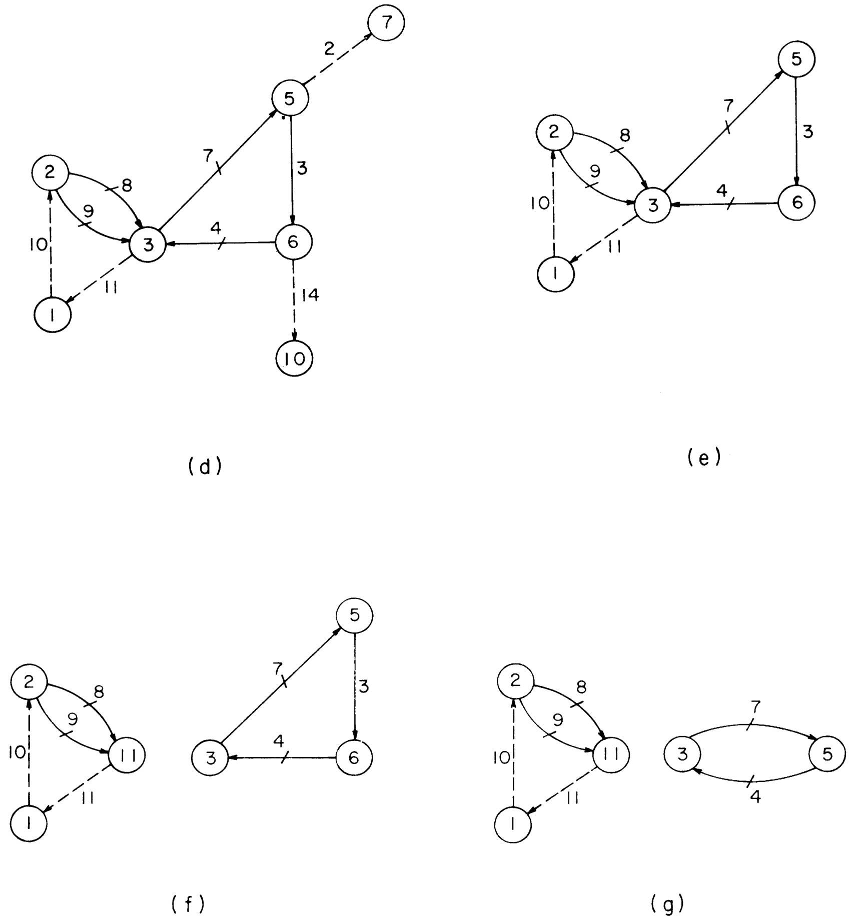


Fig. 5. Illustrations of steps in the observability classification algorithm.

possible for the temperatures to be locally or globally observable. In Fig. 4(b) $x_4 - x_7$, $y_4 - y_7$ are globally unobservable but T_5 is locally observable if $H_6 = H_{10}$ (Corollary to Theorem 4), and in Fig. 4(a) x_1, x_2, x_3, y_1, y_2, y_3 are globally unobservable but T_1, T_2, T_3 are globally observable.

Theorem 6. Let G be the underlying graph of mass-energy flow network and let arc i be in a cycle C in G with no temperature or energy flow measurements. Then y_i is globally unobservable. Furthermore, if arc i is a mass flow arc, then T_i is also globally unobservable.

Proof. Choose any arc k, $k \neq i$, in C and a spanning tree for G for which arc k is not a branch. Perturb y_k arbitrarily by δy_k while holding x_k constant. For each arc j in C, y_j is perturbed by δy_k according to eqn (5). Furthermore, since $H_j = y_j/x_j$ for mass flow arcs, H_j is also perturbed. Since no measurements detect these feasible perturbations, the theorem follows. ∎

One difference between this theorem and the preceding one is that the cycle in Theorem 6 might include a pure energy flow arc. For instance, in Fig. 6 arcs 1 and 2 form a completely unmeasured cycle. Theorem 6 applies but Theorem 5 does not. Another difference is that the cycle in Theorem 6 may contain mass flow measurements which are explicitly excluded in Theorem 5.

Suppose all mass flows are globally observable just using the mass flow measurements and mass balances. Then any temperature or energy flow can be classified as globally unobservable by Theorem 6 or globally observable by the following corollary:

Corollary. Let G be the underlying graph of a mass-energy flow network, let G_{mu} be acyclic, and let arc i be any arc in G. If every cycle containing arc i has a temperature or energy flow measurement, then y_i is globally observable, and so is T_i if arc i is a mass flow arc. The proof of the corollary follows immediately from Theorems 2 and 3(d).

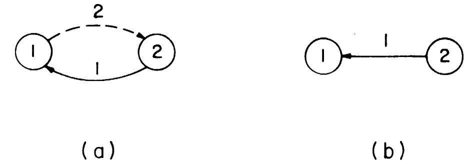


Fig. 6. Erroneous classification.

In many cases observability will depend on the temperature or enthalpy values of some arcs. The following local unobservability theorem is an extension of Theorem 5:

Theorem 7. Let G be the underlying graph of a mass-energy flow network and let arc i be an arc in a cycle C in G_{mu}. If there is at most one value of enthalpy calculated from all temperature measurements on C, then x_i and y_i are locally unobservable. Furthermore, if T_i is unmeasured, and if $H_i \neq H_k$, T_k being a measured temperature, then T_i is also locally unobservable.

Proof. Let arc k, $k \neq i$, be one of the arcs in C with a temperature measurement. Choose a spanning tree for G_{mu} for which arc k is not a branch. Let $\mathbf{x}^0$ and $\mathbf{y}^0$ denote the mass and energy flows before the perturbations. Consider an arbitrary perturbation δx_k with T_k held constant. Then $x_k = x_k^0 + \delta x_k$, $H_k = H_k^0$ and $y_k = y_k^0 + H_k^0 \delta x_k$, and for any other arc j in C, $x_j = x_j^0 + a_j \delta x_k$ and $y_j = y_j^0 + a_j \delta y_k = x_j^0 H_j^0 + a_j H_j^0 \delta x_k$, where $a_j = 1$ (or -1) if arc j is oriented in the same (or opposite) direction as arc k. But for all the arcs in C with measured temperature $H_j^0 = H_k^0$. Hence $y_j = x_j H_k^0$ and $H_j = H_k^0 = H_j^0$. Thus, these temperature measurements would not detect the mass and energy flow perturbations, and x_i and y_i are locally unobservable.

Now on the unmeasured arc i in C $H_i = y_i/x_i = (H_i^0 x_i^0 + a_i H_k^0 \delta x_k)/(x_i^0 + a_i \delta x_i)$. If $H_i^0 \neq H_k^0$, then the value of H_i depends on δx_k which is arbitrarily chosen. Hence, T_i is also locally unobservable. ■

Some illustrations of Theorem 7 can now be given. If $H_4 = H_5$, arcs 4 and 5 in Fig. 3 form a cycle as stipulated in Theorem 7 and x_3, y_4, x_5, y_5 are locally unobservable. In Fig. 5(a), if $H_4 = H_7$, x_3, x_4, x_5, x_7 and y_3, y_4, y_5, y_7 are locally unobservable, and if $H_3 \neq H_4$, then T_3 is also locally unobservable. We note that Theorem 7 applies to cycles with more than one measurement and provides more information than Theorem 5 even for a mass flow cycle with a single temperature measurement. For instance, in the cycle formed by arcs 4, 5, 6 and 7 in Fig. 4(b), if $H_4 \neq H_6$, T_4 is locally unobservable by Theorem 7.

As we pointed out earlier, observability classification based on matrix rank tests can only yield information about the system as a whole, whereas from the viewpoint of process analysis it is often crucial to identify the individual variables with the various levels of observability or unobservability. The matrix rank tests are also computationally cumbersome. For the mass-energy flow network in Fig. 5 the rank of a 38×37 matrix would have to be tested. In this case the test is inconclusive. But in any case the matrix rank test can only establish local system observability or unobservability, since the evaluation must be carried out at a particular set of conditions. By comparison with the matrix rank tests which are basically numerical tests, the potency of graph-theoretical classification criteria becomes very apparent.

Because of these shortcomings matrix rank tests should be used only as a last resort after we have exhausted all the other alternatives of observability classification. In the algorithm to be presented the graph-theoretic theorems will be used to classify as many variables as possible. These variables can then be removed from further consideration. The elimination usually results in the decomposition of the process network into smaller sub-networks. With this reduction in dimensionality the matrix rank tests can then be applied effectively.

It is important to note that in the following theorem which gives both necessary and sufficient conditions for local observability we will only be concerned with subgraphs of the original process graph. The absence or existence of perturbation subgraphs will be the basis for local observability or unobservability.

It should be clear that an arc with both mass flow and temperature measurements would admit no unmeasurable perturbation in mass or energy flow. After its temperature, mass and energy flows have been classified as globally observable, it can be safely deleted from further considerations. Arcs 1, 12, 15, 16, 17 and 18 in Fig. 5(a) fall into this category. Similarly a pure energy arc i with an energy flow measurement would admit no unmeasurable perturbation in energy flow. It too can be deleted after y_i has been classified as globally observable.

Next consider a mass flow arc i with mass flow measurement but no temperature measurement. Clearly it would admit no unmeasured mass flow perturbation, but it could admit unmeasured energy flow perturbation through temperature changes. Since arc i cannot be in the mass flow perturbation subgraph, it may be converted to a pure energy flow arc with the understanding that T_i and y_i of the original arc are observable if and only if y_i in the corresponding pure energy flow arc is observable. Such an arc should be so converted after its mass flow has been classified as globally observable. Arc 10 in Fig. 5(a) is such an example.

It should be evident that after these deletions and conversions, the only arcs remaining will be mass flow arcs with measured temperatures only, unmeasured mass flow arcs, and unmeasured pure energy flow arcs. Let the incidence matrices correspond to these arcs be $\mathbf{A}_{m1}$, $\mathbf{A}_{m2}$ and $\mathbf{A}_e$, respectively, and define a new matrix $\mathbf{B}_1$ by

$$\mathbf{B}_1 = \mathbf{A}_{m1}\mathbf{\Lambda} \tag{11}$$

where $\mathbf{\Lambda}$ is a diagonal matrix whose elements are the enthalpies H_j corresponding to the arcs in $\mathbf{A}_{m1}$. The most general case of a matrix to be rank-tested will contain sub-matrices $\mathbf{A}_{m1}$, $\mathbf{A}_{m2}$, $\mathbf{A}_e$ and $\mathbf{B}_1$.

Theorem 8. Let a mass-energy flow network contain only mass flow arcs with measured temperatures, unmeasured mass flow arcs, and unmeasured pure energy flow arcs, let the incidence matrices corresponding to these arcs be $\mathbf{A}_{m1}$, $\mathbf{A}_{m2}$ and $\mathbf{A}_e$, respectively, and let the energy flow coefficient matrix corresponding to $\mathbf{A}_{m1}$ be $\mathbf{B}_1$. The network is locally observable if and only if the rank of the partitioned matrix

$$\begin{pmatrix} \mathbf{A}_{m1} & \mathbf{A}_{m2} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_1 & \mathbf{0} & \mathbf{A}_{m2} & \mathbf{A}_e \end{pmatrix} \tag{12}$$

is equal to n, the total number of mass and energy flow arcs.

Proof. Let $\delta\mathbf{x}^1$ be the mass flow perturbations in (the subgraph corresponding to) $\mathbf{A}_{m1}$, which give rise to corresponding energy flow perturbations in $\mathbf{B}_1$, let $\delta\mathbf{x}^2$ and $\delta\mathbf{y}^2$ be the mass and energy flow perturbations in $\mathbf{A}_{m2}$, and let $\delta\mathbf{y}^3$ be the energy flow perturbations in $\mathbf{A}_e$. Since these are the only unmeasurable perturbations and since the nodal mass and energy balances are given by

$$\begin{pmatrix} \mathbf{A}_{m_1} & \mathbf{A}_{m_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_1 & \mathbf{0} & \mathbf{A}_{m_2} & \mathbf{A}_e \end{pmatrix} \begin{pmatrix} \delta\mathbf{x}^1 \\ \delta\mathbf{x}^2 \\ \delta\mathbf{y}^2 \\ \delta\mathbf{y}^3 \end{pmatrix} = \mathbf{0} \tag{13}$$

any nonzero solution to eqn (13) would constitute a feasible unmeasurable perturbation. But the theorem stipulates the necessary and sufficient condition for precluding such nonzero solutions, and by Lemma 1 the network is locally observable if and only if it contains no perturbation subgraphs. ■

Note that the rank of the partitioned matrix depends on the enthalpy values used in eqn (11). However, if the number of rows of matrix (12) is less than n, the system is unobservable for any enthalpy values.

OBSERVABILITY CLASSIFICATION ALGORITHM

Starting with a process network with its arc attributes of measured variables and enthalpy values, the algorithm classifies each temperature, mass and energy flow into one of the following categories: globally observable (g.o.), locally observable (l.o.), locally unobservable (l.u.), globally unobservable (g.u.), or an "unobservable block" (u.b.). The last category indicates that the arc is in a perturbation subgraph which cannot be further classified. An unobservable block contains at least one locally unobservable variable.

The basic approach taken in the algorithm is to determine which arcs are in which perturbation subgraphs of G. Initially the entire process graph is considered as a candidate for perturbations. If an arc cannot possibly be in any perturbation subgraph, it is deleted. If a perturbation subgraph can be identified, it is aggregated into a pseudo-node after its arc variables are appropriately classified. At any time during the execution of the algorithm the network under consideration is the unclassified sub-network of the original process network. In this connection it is important to note that the identities of G, G_m etc. change during the execution of the algorithm, referring always to the unclassified subgraphs remaining at any instant.

We shall now state the algorithm and then illustrate the various steps with respect to the network in Fig. 5:

1. *Arc reduction rules*

For each arc i in G call REDUCE (i) which is defined below.

Procedure REDUCE (i):

A. For a mass flow arc i
- i. classify T_i as g.o., if T_i is measured.
- ii. classify x_i as g.o., if x_i is measured. Furthermore,
  - a. delete arc i and classify y_i as g.o., if T_i is also measured.
  - b. convert arc i to an unmeasured pure energy flow arc, if T_i is not measured. The observability of T_i will be the same as that of y_i.

B. For a pure energy flow arc i, delete arc i and classify $y_i(T_i)$ as g.o., if y_i is measured.

C. Exceptions: Variables which have been previously classified should not be reclassified.

2. *Unobservable cycle rules*

A. Find components and biconnected components of G_{muu}.

B. Aggregate all nodes in each biconnected component of G_{muu} with more than one arc. All unmeasured variables in the arcs so eliminated are g.u.

C. For each pure energy flow arc i with incident nodes v and w in the same component of G_{muu} y_i is g.u.
- i. For all arcs j in the cycle formed by arc i and arcs in G_{muu} after y_j and T_j are classified g.u., delete arc i.

D. For each arc k in G_m with a measured temperature and with incident nodes v and w in the same component of G_{muu} aggregate nodes v and w and also aggregate all nodes in the cycle formed by arc k and arcs in G_{muu}. For each arc i in G_{muu} so eliminated do not reclassify T_i if already classified, but
- i. classify T_i as l.u., if $H_i \neq H_k$.
- ii. classify T_i as in an u.b., if $H_i = H_k$.

All other unmeasured variables in the eliminated arcs are g.u.

3. *Feasibility rules*

A. Mass flow feasibility
- i. Find biconnected components of G_m.
- ii. For each biconnected component of G_m with exactly one arc i mark x_i as measured and call REDUCE (i).

B. Energy flow feasibility
- i. Find components, biconnected components and cut-nodes of G.
- ii. For each biconnected component of G with exactly one arc i mark y_i, T_i and also x_i as measured, and call REDUCE (i).

C. Perform splitting operation on each node v which is still a cut-node of G.

4. *Parallel arc rule*

If arcs i and j are parallel pure energy flow arcs, delete arc j and classify y_i and y_j as g.u.

5. *Matrix rules*

Aggregate the nodes in each component of G_{muu}.
For each component of G apply matrix rank test.
Classify each previously unclassified variable as
A. l.o. if the component is l.o.
B. g.u. if the matrix has fewer rows than columns.
C. in an u.b. if the component is l.u.

AN EXAMPLE

We shall now apply the observability classification algorithm step by step to the process network in Fig. 5(a).

1. Application of arc reduction rules results in the deletion of arcs 1, 12, 15, 16, 17 and 18 after their temperatures, mass and energy flows have been classified as g.o., and in the conversion of mass flow arc 10 to a pure energy flow arc after x_{10} has been classified as g.o. The revised G is shown in Fig. 5(b).

2. The unmeasured cycle rules are based on the application of Theorems 5, 6 and 7. Step 2B aggregates the nodes and classifies the temperatures, mass and energy flows as g.u. in a completely unmeasured cycle of mass flow arcs according to Theorem 5(a). The reason for aggregation is to reduce the size of the network to be further classified. The variables on all eliminated arcs have been classified, and consequently, require no further attention. Since aggregation of two nodes is algebraically equivalent to combining two rows in the mass balance equations (2) and combining two rows in the energy balance equations (3), the solution in terms of variables external to the pseudo-node are unaffected.

In the process of aggregating nodes v and w any mass flow arc k with measured temperature connecting the same two nodes will also be eliminated. Since there is at least one path between nodes v and w in G_{muu}, arc k must be in a cycle with exactly one temperature measurement. By Theorem 5(b) x_k and y_k are g.u. An unmeasured pure energy flow arc k between nodes v and w will similarly be eliminated, and by Theorem 6, y_k is also g.u. All these situations are accounted for in step 2B, but since they do not arise in Fig. 5(b) the network is unaltered in this step.

After step 2B G_{muu} consists only of trees. In step 2C we consider each completely unmeasured cycle with exactly one pure energy flow arc. Then by Theorem 6 y_k is g.u., and for each of the other arcs in the cycle (say, arc j in G_{muu}) T_j and y_j are also g.u. In this step the adjacent nodes v and w are not aggregated since the mass flows in the cycle have not yet been classified. Instead arc k is deleted. This deletion will not affect potential unmeasurable energy flow perturbations in arcs external to the cycle. Again step 2C does not affect the network in Fig. 5b.

In step 2D we consider each remaining cycle in G_m with exactly one temperature measurement T_k. For each arc i in the cycle x_i and y_i are g.u. by Theorem 5(b). Furthermore, by Theorem 7 T_i is l.u. if $H_i \neq H_k$. T_i is in an u.b. if $H_i = H_k$, since arc i is in a perturbation subgraph. In either case all nodes in the cycle are aggregated into a pseudo-node with no effect on potential unmeasurable perturbations in arcs external to the pseudo-node. Note that if T_i has already been classified as g.u. in step 2C, it should not be reclassified. In the example in Fig. 5(b) nodes 3 and 4 are aggregated. x_5, x_6 and y_5, y_6 are g.u. T_6 has been previously classified as g.o. in step 1. If $H_5 \neq H_6$, T_5 is l.u., and if $H_5 = H_6$, T_5 is in an u.b.

3. In steps 3A and 3B infeasible mass and energy flow perturbations are identified using feasibility rules based on Theorems 1 and 3. Arcs which do not permit unmeasurable perturbations are tagged and reprocessed by arc reduction rules. We begin with each biconnected component with exactly one arc i in the mass flow sub-network. Since mass flow perturbation in such an arc is impossible by Theorem 1(c), x_i is treated as if it were measured. By tagging it as "measured" we can immediately subject it to reprocessing by arc reduction rules without any other special provisions.

Similarly, unmeasurable mass and energy flow perturbations are ruled out in a biconnected component of G containing exactly one arc i by Theorem 3(c) and (d). To avoid the inconvenience of distinguishing whether arc i is a mass or pure energy flow arc, mark T_i, y_i and x_i as "measured" and invoke the arc reduction rules. Notice that x_i is tagged in this case for convenience.

Applying step 3A to the network in Fig. 5(c) we mark x_2, x_{11}, x_{13} and x_{19} as "measured" and after appropriate arc reductions the network appears as shown in Fig. 5(d). Similarly, y_2, T_2 and y_{14} and T_{14} are tagged as "measured" in step 3B which gives rise to the network in Fig. 5(e). In step 3C we perform splitting on node 3, the cut-node in G. The splitting permits the matrix rank test to be applied to disjoint subgraphs of lower dimensions shown in Fig. 5(f).

At this point the need for avoiding reclassification in step 1C may be illustrated with a simple example. Referring to Fig. 6(a), y_1, y_2 and T_1 are classified as g.u. in step 2C. After the deletion of arc 2, arc 1 is not in any cycle. Hence x_1 is g.o. However, if we do not prohibit reclassification, it might be erroneously concluded from Fig. 7(b) that T_1 and y_1 are also g.o.

4. The parallel arc rule applies to unmeasured pure energy flow arcs whose flows may be classified as g.u. by Theorem 6. All but one of the energy flow arcs may be deleted after classification. Notice that in this step the adjacent nodes v and w must not be aggregated. Since the arcs between v and w can only transmit pure energy flow, aggregation would introduce a mass flow path which does exist in the unaggregated network. For example, Fig. 7(b) would suggest feasible unmeasurable mass flow perturbations through arcs 1 and 2, which are not permitted in Fig. 7(a).

5. At this point we have exhausted all the graph-related reduction and classification rules. A comparison of Fig. 5(a) and 5(f) shows clearly the dramatic reduction in network dimensionality. However, we can make a further reduction by aggregating the nodes in each component of G_{muu} before applying the matrix rank test. Since a completely unmeasured path of mass flow arcs connect these nodes, the aggregation clearly does not

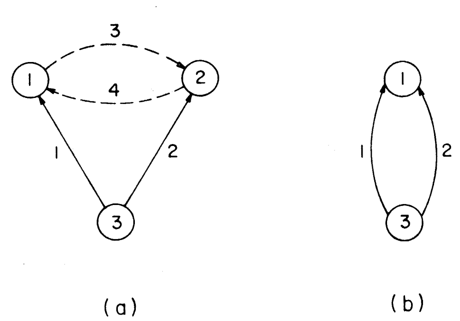


Fig. 7. Erroneous aggregation of nodes.

affect feasible unmeasurable perturbations external to these nodes. If the matrix rank test shows a component of G containing such a pseudo-node to be l.o., then the variables associated with the eliminated arcs in the pseudo-node are also l.o. If the test shows the component to be l.u. we can only classify the eliminated arcs as being in an u.b.

With reference to the example, aggregation of nodes 5 and 6 transforms Fig. 5(f) to Fig. 5(g), and the application of Theorem 8 shows that arcs 8–11 are in a g.u. block and that arcs 3, 4 and 7 are l.o. if $H_7 \neq H_4$ and in a l.u. block if $H_7 = H_4$.

REDUNDANCY CLASSIFICATION IN MASS FLOW NETWORKS

Since redundancy is defined in terms of observability[10], the redundancy of a measured variable can be determined by deleting the measurement and applying the observability classification algorithm. Depending on whether the unmeasured variable is globally (locally) observable or unobservable, the measurement is globally (locally) redundant or non-redundant. This relationship is exploited in proving redundancy theorems, but as a computational procedure, it is not very efficient and should be avoided except as a last resort. In fact much of the redundancy classification algorithm is devoted to strategems of classification without resorting to this "brute-force" procedure. It is important to note that, unlike observability, redundancy is defined only with respect to measured variables, although they both depend on network configuration and measurement placements.

As before, we shall first establish the governing theorems, then present the classification algorithm and finally illustrate the steps in the algorithm with an example.

Since a mass flow network is linear, its properties with respect to observability and redundancy are global. We need simply to characterize a measurement as redundant or non-redundant. The following theorem states three equivalent criteria for redundancy.

Theorem 9. In a mass flow network let arc i connecting nodes v and w be measured. Then the flow measurement x_i is redundant if and only if

(a) nodes v and w are not connected by a path of unmeasured arcs (regardless of directions);

(b) nodes n and w lie in different components of G_{mu};

(c) arc i is in a cut-set of G_m consisting solely of measured arcs.

Proof. (a) To test for non-redundancy delete the measurement at arc i. The deleted measurement was non-redundant if and only if x_1 is now unobservable, which is true if and only if arc i now lies in a cycle of unmeasured arcs (Theorem 2), which is true if and only if nodes v and w are connected by a path of unmeasured arcs other than arc i. Hence the measurement x_i is redundant, if and only if nodes v and w are not connected by a path of unmeasured arcs.

(b) If nodes v and w lie in two different components of G_{mu}, they cannot be connected by a path of unmeasured arcs. Hence the result follows directly from (a).

(c) This is equivalent to case (b). ■

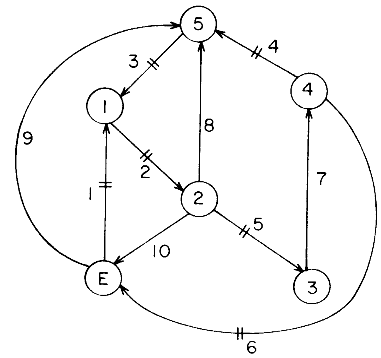


Fig. 8. A mass flow network.

To illustrate the application of the above theorem let us consider Fig. 8. Since no measured arc in Fig. 8 lies in a cycle with only one measurement, all measurements, $x_1 - x_6$, are redundant. Yet the system as a whole is unobservable because of the cycle of unmeasured arcs 8–10. If we switch the measurement from arc 5 to arc 8, the system becomes observable and the measurements x_4, x_6 and x_8, each lying in a cycle with exactly one measurement, become non-redundant.

REDUNDANCY CLASSIFICATION IN MASS-ENERGY FLOW NETWORKS

Theorem 10. In a mass-energy flow network let nodes v and w be connected by arc i. Then

(a) if arc i is a mass flow arc and if v and w are connected by a path in G_{muu}, the mass flow measurement x_i is globally non-redundant;

(b) if n and w are connected by a path with no temperature or energy flow measurements, the temperature measurement T_i or energy flow measurement y_i is globally non-redundant.

Proof. (a) Delete the measurement. By Theorem 5 x_i is now globally unobservable, and hence the measurement was non-redundant. ■

(b) Delete the temperature or energy flow measurement. By Theorem 6 T_i or y_i is now globally unobservable, and hence the measurement was non-redundant.

As an illustration of this theorem, let us consider the network in Fig. 9(a). By Theorem 10 (b) the temperature measurements T_6, T_8 and T_9 are all globally non-redundant, because they each lie in a cycle (arcs 5, 6, arcs, 5, 8, 10, 11, and arcs 9, 10, 11, respectively), with exactly one temperature measurement. Notice that we could not characterize the redundancy of any of the flow measurements on the basis of this theorem, but the next theorem which give sufficient conditions for redundancy of a mass flow measurement in a mass-energy flow network will permit further classification of some flow measurements.

Theorem 11. In a mass-energy flow network let the mass flow of arc i connecting nodes v and w be measured. The measurement x_i is globally redundant if (a) v and w are in different components of G_{mu}, or (b) arc i is in a cut-set of G_m, consisting solely of mass flow arcs with flow measurements.

Proof. This theorem is a simple extension of Theorem 9. The conditions stipulated are sufficient to prove the

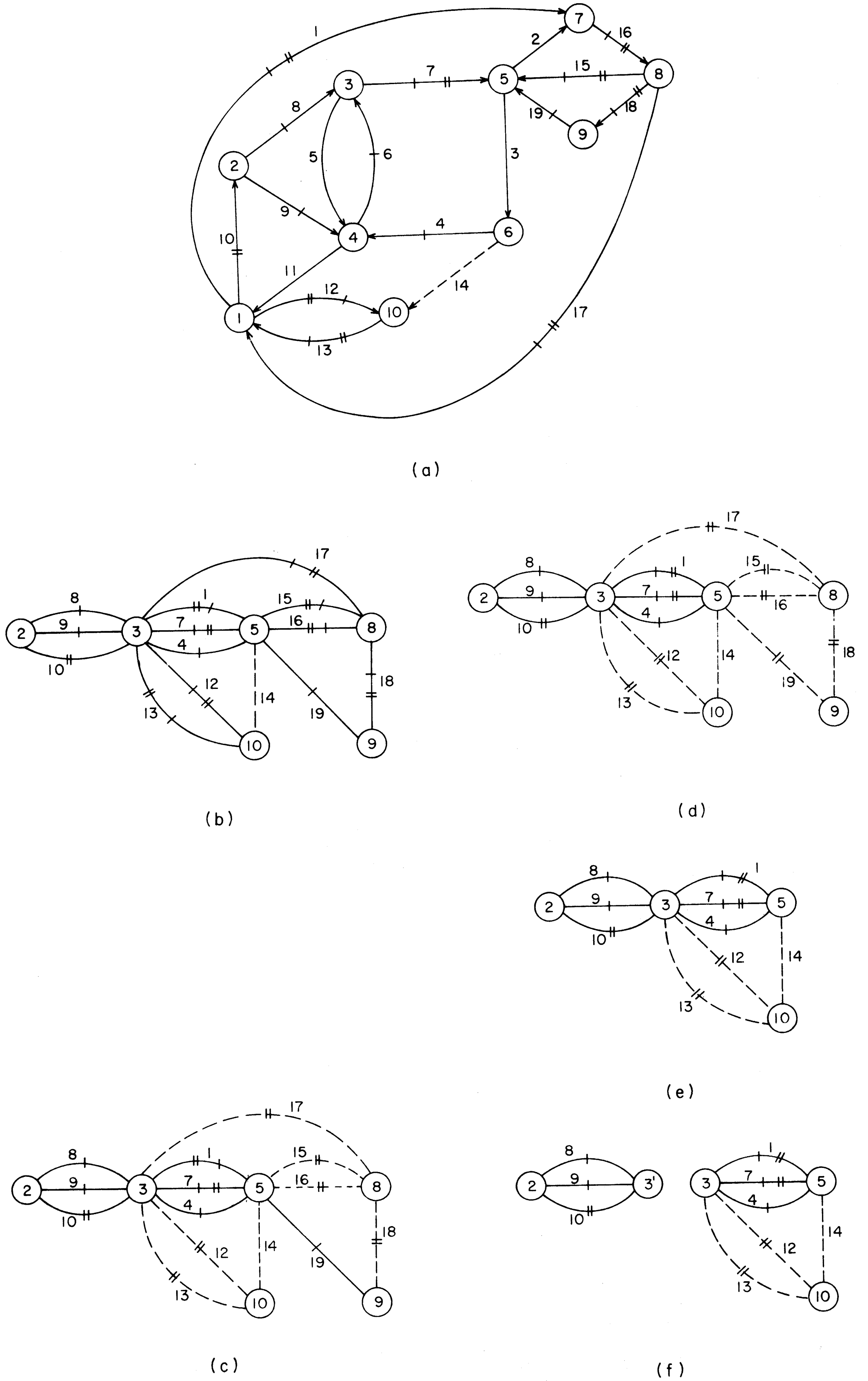


Fig. 9. Illustration of the steps in the redundancy classification algorithm.

redundancy of the flow measurement x_i using mass conservation relations only. Notice that the theorem makes no stipulation on temperature measurements. ■

By this theorem the measurements $x_{15}-x_{18}$ in Fig. 9(a) are all globally redundant. The next theorem will give sufficient conditions for redundancy of an energy flow measurement in a mass-energy flow network.

Theorem 12. In a mass-energy flow network let a pure energy flow arc i connecting nodes v and w be measured. The measurement y_i is globally redundant if (a) v and w are in different components of G_s; or (b) arc i is in a cut-set of G, consisting solely of measured pure energy flow arcs.

Proof. If arc i is the only arc connecting two components of G_s, as in (a), any flow or perturbation is clearly infeasible even without the measurement. Hence y_i is observable and the measurement on y_i is redundant. Case (b) is equivalent to case (a). ■

Applications of Theorem 12 often arise as a result of converting mass-energy flow arcs to pure energy flow arcs during the execution of the redundancy classification algorithm which is presented next.

REDUNDANCY CLASSIFICATION ALGORITHM

The redundancy classification algorithm requires the same input data as the observability classification algorithm. It classifies each measurement as globally redundant (g.r.), globally non-redundant (g.n.)., locally redundant (l.r.), locally non-redundant (l.n.), or as in a locally "non-redundant" block (l.n.b.). The l.n.b. category indicates that the measured variable is in a subgraph which is locally unobservable without the measurement, but that a further classification of the subgraph in terms of redundant and non-redundant measurements is not possible. The l.n.b. classification arises when a variable is classified as in a u.b. by the observability algorithm after the deletion of the measurement.

The general approach taken in the redundancy classification algorithm resembles that of the observability classification algorithm. Aggregation is performed when it does not affect the future classification of arcs external to the resulting pseudo-node and after all measurements internal to the pseudo-node have been classified. An arc is deleted from a process graph or a derived graph after it has been shown that no feasible unmeasurable perturbations can occur in that arc and that all variables associated with that arc can be calculated using only redundant measurements. No measurement is deleted until it has been appropriately classified. After we have exhausted all the measurements which can be directly classified using the redundancy classification theorems, the definition of redundancy is invoked. The measurement is temporarily deleted and the observability classification algorithm is applied to the residual subgraphs.

As with the observability algorithm, at any stage of algorithm execution the network under consideration is the unclassified sub-network of the original process network, with the identities of G, G_s etc. continually undergoing metamorphosis.

We shall now present the algorithm.

1. *Aggregation*

For each completely unmeasured arc (v, w) in G_m aggregate nodes v and w. Classify as g.n. any measured variables in the arcs so eliminated.

2. *Redundancy in mass flow cut-set*

A. Find components of G_{mu}

B. For each arc (v, w) with a mass flow measurement x_i if nodes v and w lie in two components of G_{mu}

i. classify the measurement x_i as g.r.

ii. convert arc i to an unmeasured pure energy flow arc, if T_i is unmeasured, or a measured pure energy flow arc, if T_i is measured. The classification of the T_i measurement will be the same as that of the y_i measurement.

3. *Mass flow feasibility*

A. Find biconnected components of G_m.

B. For each biconnected component of G_m with exactly one arc i, convert arc i to a measured pure energy flow arc. The classification of the T_i measurement will be the same as that of the y_i measurement.

4. *Redundancy in energy flow cut-set*

A. Find components of G_s

B. For each pure energy flow arc (v, w) with measurement y_i if nodes v and w lie in two components of G_s, classify the energy flow measurement y_i as g.r. and delete arc i.

5. *Energy flow feasibility*

A. Find biconnected components, cut-nodes and components of G.

B. For each biconnected component of G with exactly one arc i, classify every measurement on arc i as g.r. and delete arc i.

C. Perform splitting operations on each cut-node v of G to separate its biconnected components.

6. *Parallel arc rules*

A. For each mass flow arc i with a temperature measurement, if it is parallel to an unmeasured pure energy flow arc or an arc in G_m with a mass flow measurement only, classify the measurement T_i as g.n.

B. For each arc i with mass flow measurement only, if it is parallel to an arc in G_{mu}, classify the measurement x_i as g.n.

7. *"Brute force" rules*

For each component of G derived in step 5C temporarily delete each unclassified measurement in turn and apply the observability classification algorithm to determine its status as follows:

If the variable is	classify the measurement
g.o.	g.r.
l.o.	l.r.
g.u.	g.n.
l.u.	l.n.
u.b.	l.n.b.

AN EXAMPLE

We shall now illustrate the step-by-step application of the redundancy classification algorithm to the process network in Fig. 9(a).

1. The aggregation in step 1 is a direct application of Theorem 10. This step results in the elimination of arcs 2, 3, 5, 6 and 11 and the revised network as shown in Fig. 9(b). The measurement T_6 is classified g.n.

2. The classification of mass flow measurement x_i in step 2 is based on the conditions stipulated in Theorem 11. Since it is not yet determined if energy flow perturbations in arc i are detectable by redundant measurements, arc i is not deleted but converted to a pure energy flow arc. Because the mass flow is measured and known, the measurement T_i is equivalent to the measurement y_i and treated equivalently in the classification. It is not deleted because it is not yet classified. The network at the completion of this step is shown in Fig. 9(c), and the measurements x_{12}, x_{13}, x_{15}–x_{18} are classified g.r.

3. After step 2 there may be mass flow arcs left which are no longer in cycles of G_m. These arcs are converted to pure energy flow arcs in step 3, since non-zero mass flows in these arcs are ruled out by Theorem 3(c). Arc 19 in Fig. 9(c) is such an example. Since only redundant measurements were deleted from the network in step 2, the implication is that x_{19} can be calculated using only redundant flow measurements. Note that if arc 19 had a mass flow measurement, it would have belonged to a cut-set in step 2 and eliminated before step 3. Similarly it would have been eliminated in step 1 if it were completely unmeasured. Hence the acyclic arcs of G_m, such as arc 19, must have exactly one temperature measurement each. As in step 2, the temperature measurement is treated equivalently as an energy flow measurement and retained for further processing. The outcome of step 3 is shown in Fig. 9(d).

4. In this step Theorem 12 is applied to the classification of energy flow measurements. Measurements satisfying the sufficient conditions of Theorem 12 are classified as g.r. and the associated arcs can be eliminated from further consideration after the classification. Figure 9(e) shows the outcome of this step. The measurements T_{15}–T_{19} are classified as g.r.

5. In this step we invoke Theorem 3 (d) to eliminate arcs with infeasible perturbations or zero flows. By virtue of the prior processing steps the only acyclic arcs remaining must be energy flow arcs with or without temperature (energy flow) measurements. Such arcs can be eliminated from further considerations after the associated temperature measurements have been classified as g.r. After such arcs, if any, have been eliminated the network is split according to its biconnected components to reduce the dimensionality and computing time in the subsequent processing steps. Since there are no acyclic arcs in Fig. 9(e), we proceed directly to Fig. 9(f).

6. The temperature measurement classification is a direct application of Theorem 10(b), and the mass flow measurement classification is based on Theorem 5(b). In contrast to the earlier steps no node aggregation or arc deletion is performed. In this step the measurements T_8 and T_9 in Fig. 9(f) are classified as g.n.

7. At this point the definition of redundancy is invoked to classify the remaining unclassified measurements. A measurement is redundant if and only if its deletion causes no loss in observability in the system. Each measurement is deleted in turn, and its observability is determined by the application of the observability classification algorithm. On that basis the measurements (a) x_1 is in a l.n. block if and only if $H_1 = H_4$; (b) x_7 is in a l.n. block if and only if $H_7 = H_4$; (c) T_1, T_4, T_7, T_{12} and T_{13} are g.r.

CLOSING REMARKS

In this paper we introduce the concepts of biconnected components, perturbation subgraphs and feasible unmeasurable perturbations and show how their properties may be used to effect observability and redundancy classification, simplification and dimensional reduction. These powerful graph-theoretic techniques are embodied in two classification algorithms for observability and redundancy in process networks. The treatment which assumes only mass and energy conservation constraints may be readily extended to process networks with other additional constraints, e.g. specified split fractions. It can be shown[9] that most of the graph-theoretic results may be generalized, the exceptions being Theorems 2, 6, 7 and 8, and the sufficient conditions of Theorems 1(c), 3(c) and 3(d).

Acknowledgment—This work was supported by the National Science Foundation Grants GK-43430 and ENG 76-18852.

NOTATION

- $\mathbf{A}$ the incidence matrix of graph G
- $\mathbf{A}_m$ the incidence matrix of graph G_m
- $\mathbf{A}_{m_1}$ the incidence matrix of a sub-network of mass flow arcs, with measured temperatures only
- $\mathbf{A}_{m_2}$ the incidence matrix of a sub-network of unmeasured mass flow arcs
- $\mathbf{A}_e$ the incidence matrix of a sub-network of unmeasured pure energy flow arcs
- $\mathbf{B}_1$ an energy flow coefficient matrix defined by eqn (11)
- $\mathbf{C}_m^T$ a fundamental cycle matrix of graph G_m
- $\mathbf{C}^T$ a fundamental cycle matrix of graph G
- G a process graph
- G_1 a subgraph of G
- G_2 a subgraph of G
- G_s the subgraph of G with all measured pure energy flow arcs deleted
- G_m the subgraph of G with all pure energy flow arcs deleted
- G_{mu} the subgraph of G_m with the deletion of all arcs with flow measurements
- G_{muu} the subgraph of G_{mu} with the deletion of all arcs with temperature measurements
- H_i the enthalpy of material stream in arc i
- T_i the temperature of material stream in arc i
- v a node
- x_i the mass flow in arc i
- $\mathbf{x}$ a vector of mass flows
- $\delta\mathbf{x}$ perturbation of vector of mass flows
- $\mathbf{x}^0$ a vector of feasible mass flows in G
- y_i the energy flow in arc i
- $\mathbf{y}$ a vector of energy flows
- $\delta\mathbf{y}$ perturbation of vector of energy flows
- $\mathbf{y}^0$ a vector of feasible energy flows in G
- w a node

Λ a diagonal matrix whose elements are the enthalpies H_j corresponding to the arcs in A_{m_1}

REFERENCES

[1] Aho A. V., Hopcroft J. E. and Ullman J. D., *The Design and Analysis of Computer Algorithms.* Addison Wesley, Reading, Mass. 1974.
[2] Deo N., *Graph Theory with Applications to Engineering and Computer Science.* Prentice-Hall, Englewood Cliffs, New Jersey 1974.
[3] Göknar I. C., *Int. J. Syst. Sci.* 1976 **7** 65.
[4] Göknar I. C., *Int. J. Syst. Sci.* 1976 **7** 121.
[5] Lin C. T., *IEEE Trans. Auto. Control* 1974 **AC19** 201.
[6] Mah R. S., Stanley G. M. and Downing D. M., *Ind. Engng Chem. Proc. Des. Dev.* 1976 **15** 175.
[7] Mah R. S. H. and Shacham M., *Advan. Chem. Engng* 1978 **10** 126.
[8] Shields R. W. and Pearson J. B., *IEEE Trans. Automat. Control* 1976 **AC-21** 203.
[9] Stanley G. M., Ph.D. Thesis, Northwestern University, Evanston, Illinois 1977.
[10] Stanley G. M. and Mah R. S. H., *Chem. Engng Sci.* 1981 **36** 259.
[11] Tarjan R., *SIAM J. Comput.* 1972 **1** 146.

* **To contact author**: see http://gregstanleyandassociates.com/contactinfo/contactinfo.htm