§3. Incompressible surfaces

The majority of 3-manifold theory studies submanifolds of a 3-manifold M, and uses them to gain information about M. This is particularly fruitful because surfaces (i.e. 2-manifolds) are well understood. However, only certain surfaces embedded within M have any relevance. The most important of these are 'incompressible' and are defined as follows.

Definition. Let S be a properly embedded surface in a 3-manifold M. Then a *compression disc* D for S is a disc D embedded in M such that $D \cap S = \partial D$, but with ∂D not bounding a disc in S. If no such compression disc exists, then S is *incompressible*.

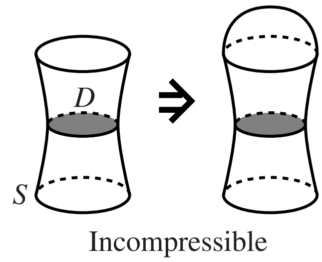

Figure 7.

Of course, a 2-sphere or disc properly embedded in a 3-manifold is always incompressible.

Remark. Suppose that D is a compression disc for S. We may assume that D lies in $\mathrm{int}(M)$. There is then a way of 'simplifying' S as follows. Essentially using Proposition 6.6 (see §2), we may find an embedding of $D \times [-1,1]$ in $\mathrm{int}(M)$ with $(D \times [-1,1]) \cap S = \partial D \times [-1,1]$. Then

$$S \cup (D \times \{-1,1\}) - (\partial D \times (-1,1))$$

is a new surface properly embedded in M. It is obtained *by compressing S along D*.

Denote the Euler characteristic of compact surface S by $\chi(S)$. Define the *complexity* of S to be the sum of $-\chi(S)$, the number of components of S and the number of 2-sphere components of S. Note that this number is non-negative. A compression to S reduces $-\chi(S)$ by two. It either leaves the number of components unchanged or increases it by one. It does not create any 2-sphere components, unless S is a torus or Klein bottle compressing to a 2-sphere. Hence, we have the following.

Lemma 3.1. *Compressing a surface decreases its complexity.*

We will occasionally abuse notation by 'compressing' along a disc D with $D \cap S = \partial D$, but with ∂D bounding a disc in S. Note that in this case, the complexity of the surface is left unchanged.

Definition. A compact orientable 3-manifold is *Haken* if it is prime and contains a connected orientable incompressible properly embedded surface other than S^2.

Note that every compact orientable prime 3-manifold M with non-empty boundary is Haken. For we may pick a disc in ∂M and push its interior into the interior of M so that the disc is properly embedded. This is a connected orientable incompressible properly embedded surface, as required. Of course, it is not a particularly interesting surface, but we will see later that, unless M is a 3-ball, other interesting surfaces also live in M.

Haken was a prominent 3-manifold topologist, and he was the first person to realize the importance of incompressible surfaces. (He also has a number of other mathematical accolades; for example, he proved the famous 4-colour theorem in graph theory.) Haken 3-manifolds are extremely well understood. For example, we will prove the following topological rigidity theorem.

Theorem 3.2. *Let M and M' be closed orientable 3-manifolds, with M Haken and M' prime. If M and M' are homotopy equivalent, then they are homeomorphic.*

Another major result which demonstrates the usefulness of incompressible surfaces is the following.

Theorem 3.3. *Let S be an orientable surface properly embedded in a compact prime orientable 3-manifold M. Then S is incompressible if and only if the map $\pi_1(S) \to \pi_1(M)$ induced by inclusion is an injection.*

In one direction (that π_1-injectivity implies incompressibility) this is quite straightforward, but the converse is difficult and quite surprising. We will prove this theorem later in the course.

We now demonstrate that Haken 3-manifolds are fairly common, by giving plenty of examples of incompressible surfaces in various manifolds.

Definition. A connected surface S properly embedded in a connected 3-manifold M is *non-separating* if $M - S$ is connected.

Lemma 3.4. *Let S be a surface properly embedded in a 3-manifold M. The following are equivalent:*

(i) S is non-separating;

(ii) there is a loop properly embedded in M which intersects S transversely in a single point;

(iii) there is a loop properly embedded in M which intersects S transversely in an odd number of points.

Proof. (i) $\Rightarrow$ (ii). Suppose that S is non-separating. Pick a small embedded arc intersecting S transversely. The endpoints of this arc lie in the same path-component of $M - S$, and so may be joined by an arc in $M - S$. The two arcs join to form a loop, which we may assume is properly embedded. This intersects S transversely in a single point.

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (i). If S separates M into two components, any loop in M intersecting S transversely alternates between these components. Hence, it intersects S an even number of times. $\square$

Example. The 3-torus $S^1 \times S^1 \times S^1$ contains a non-separating torus.

Proposition 3.5. *Let M be a prime orientable 3-manifold containing a non-separating 2-sphere S^2. Then M is homeomorphic to $S^2 \times S^1$.*

Proof. By Proposition 6.6, S^2 has a neighbourhood homeomorphic to $S^2 \times [-1, 1]$. Since S^2 is non-separating, there is a loop ℓ properly embedded in M intersecting S^2 transversely in a single point. For small enough $\epsilon > 0$, $\ell \cap (S^2 \times [-\epsilon, \epsilon])$ is a single arc. Using technology that we will develop in §6, $\ell - (S^2 \times [-\epsilon, \epsilon])$ has a neighbourhood in $M - (S^2 \times (-\epsilon, \epsilon))$ homeomorphic to a ball B such that

$B \cap (S^2 \times \{-\epsilon\})$ and $B \cap (S^2 \times \{\epsilon\})$ are two discs. Then, using an obvious product structure on B, $X = B \cup (S^2 \times [-\epsilon, \epsilon])$ is homeomorphic to $S^2 \times S^1$ with the interior of a closed 3-ball removed. Note that ∂X is a separating 2-sphere in M. Hence, since M is prime, this bounds a 3-ball B' in M. Then $M = X \cup B'$ is homeomorphic to $S^2 \times S^1$. □

A 3-manifold M is known as *irreducible* if any embedded 2-sphere in M bounds a 3-ball. Otherwise, it is *reducible*. By Proposition 3.5, an orientable reducible 3-manifold is either composite or homeomorphic to $S^2 \times S^1$.

Proposition 3.6. *Let M be an orientable prime 3-manifold containing a properly embedded orientable non-separating surface S. Then M is either Haken or a copy of $S^2 \times S^1$.*

Proof. If M contains a non-separating 2-sphere, we are done. If S is incompressible, we are done. Hence, suppose that S compresses to a surface S'. Then S' is orientable. By Lemma 3.3, there is a loop ℓ intersecting S transversely in a single point. By shrinking the product structure on $D \times [-1, 1]$ as in the proof of Proposition 3.5, we may assume that ℓ intersects $D \times [-1, 1]$ in arcs of the form $\{*\} \times [-1, 1]$. Hence, it intersects S' transversely in an odd number of points. So, at least one component of S' is non-separating. By Lemma 3.1, the complexity of this component is less than that of S. Hence, we eventually terminate with an incompressible orientable non-separating surface. □

Example. The above argument gives that any non-separating torus in $S^1 \times S^1 \times S^1$ is incompressible. (We need to know, in addition, that $S^1 \times S^1 \times S^1$ is prime.)

We will prove the following result in §7. In combination with Proposition 3.6, this provides examples of many Haken 3-manifolds.

Theorem 3.7. *Let M be a compact orientable 3-manifold. If $H_1(M)$ is infinite, then M contains an orientable non-separating properly embedded surface.*

The converse of Theorem 3.7 is also true. So this does not in fact create any more examples of Haken manifolds than Proposition 3.6. However, it is often more convenient to calculate the homology of a 3-manifold than to construct an explicit non-separating surface in it.

There is one notable 3-manifold that is not Haken.

Theorem 3.8. *The only connected incompressible surface properly embedded in S^3 is a 2-sphere. Hence, S^3 is not Haken.*

At the same time, we will prove.

Theorem 3.9. (Alexander's theorem) *Any pl properly embedded 2-sphere in S^3 is ambient isotopic to the standard 2-sphere in S^3. In particular, it separates S^3 into two components, the closure of each component being a pl 3-ball. Hence, S^3 is prime.*

Remark. The theorem is not true for topological embeddings of S^2 in S^3. Also, it is remarkable that the corresponding statement for pl or smooth 3-spheres in S^4 remains unproven.

Proof of Theorems 3.8 and 3.9. Let S be a connected incompressible properly embedded surface in S^3. We will show that S is ambient isotopic to the standard 2-sphere in S^3. Let p be some point in $S^3 - S$. Then $S^3 - p$ is pl homeomorphic to $\mathbb{R}^3$. Hence, S is simplicial in some subdivision of a standard triangulation of $\mathbb{R}^3$.

Claim. There is a product structure $\mathbb{R}^2 \times \mathbb{R}$ on $\mathbb{R}^3$, and an ambient isotopy of S, so that after this isotopy, the following is true: for all but finitely many $x \in \mathbb{R}$, $(\mathbb{R}^2 \times \{x\}) \cap S$ is a collection of simple closed curves, and at each of the remaining $x \in \mathbb{R}$, we have one 'singularity' of of one of the following forms:

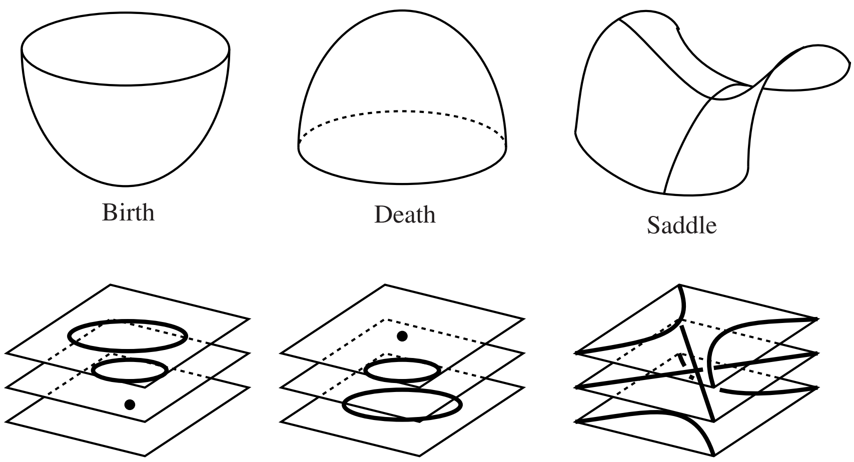

Figure 8.

Proof of claim. Each simplex in the triangulation of $\mathbb{R}^3$ is convex in $\mathbb{R}^3$. The set of unit vectors parallel to 1-simplices of S is finite. We take a product structure $\mathbb{R}^2 \times \mathbb{R}$, so that neither $\mathbb{R}^2 \times \{0\}$ nor $\{0\} \times \mathbb{R}$ contains any of these vectors. We may also assume that, for each x, $\mathbb{R}^2 \times \{x\}$ contains at most one vertex of S. When $\mathbb{R}^2 \times \{x\}$ does not contain a vertex of S, $(\mathbb{R}^2 \times \{x\}) \cap S$ is a collection of simple closed curves. Near the vertices of S, the singularities are a little more complicated than required, and hence we perform an ambient isotopy of S to improve the situation. Let ϵ be the length of the shortest 1-simplex in $\mathbb{R}^3$ that intersects S. Focus on a single vertex v of S. Let B be the polyhedron in $\mathbb{R}^3$ with vertices at precisely the points on the 1-simplices of $\mathbb{R}^3$ at distance $\epsilon/2$ from v. Then we may subdivide $\mathbb{R}^3$ further so that B is simplicial. Then $S \cap \partial B$ is a simple closed curve separating ∂B into two discs. Replace $S \cap B$ with one of these discs, which can be achieved by an ambient isotopy. Performing this operation at each vertex of S results in singularities only of the required form. This proves the claim.

Suppose that the singularities of S occur at the heights $x_1 < \ldots < x_n$. Note that the singularity at x_1 is a birth, and at x_n is a death. We prove the theorem by induction on the number of singularities n. The smallest possible n is two, in which case S is a 2-sphere embedded in the standard way.

Let x_k be the smallest non-birth singularity. If it is a death, then, since S is connected, S is a 2-sphere embedded in the standard way. Hence, we may assume that x_k is a saddle. As x increases to x_k, either

(i) two curves C_1 and C_2 approach to become a single curve C_3, or

(ii) one curve C_4 pinches together form two curves C_5 and C_6.

In (i), we may ambient isotope S, to replace this saddle singularity and the singularities below C_1 and C_2 with a single birth singularity. The theorem then follows by induction.

In (ii), if C_5 and C_6 both lie below death singularities, then S is a 2-sphere ambient isotopic to the standard 2-sphere in S^3.

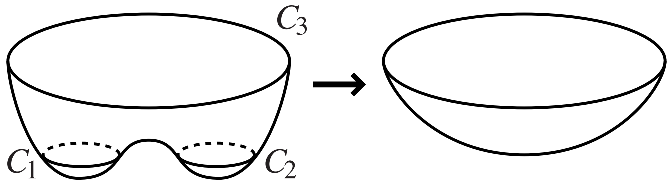

Figure 9.

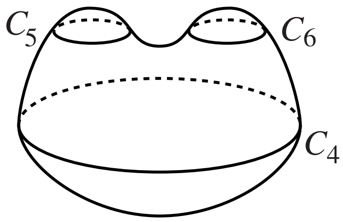

Figure 10.

Suppose therefore that one of these curves (C_5, say) does not lie below a death singularity. The curve C_5 bounds a horizontal disc D. There may be some simple closed curves of $S \cap \text{int}(D)$. But each of these lies above birth singularities. So, we may ambient isotope S, increasing the height of these singularities to above x_k. Hence we may assume that $D \cap S = \partial D = C_5$. By the incompressibility of S, C_5 bounds a disc D' in S. Hence, if we 'compress' S along D, we obtain a surface S' with same genus as S, together with a 2-sphere S^2. Both S^2 and S' have fewer singularities than S. Hence, inductively, S^2 bounds a 3-ball on both sides. One of these 3-balls is disjoint from S'. We may ambient isotope S across this 3-ball onto S'. Thus, S and S' are ambient isotopic. The inductive hypothesis gives that S' (and hence S) is a 2-sphere ambient isotopic to the standard 2-sphere in S^3. □

Using this result, we can prove that any compact 3-manifold M with a single boundary component that is embedded in S^3 is Haken. If S^2 is properly embedded in M, then this 2-sphere separates S^3 into two 3-balls. One of these 3-balls is disjoint from ∂M, and hence lies in M. Therefore M is prime, orientable and compact, and has non-empty boundary. Hence, it is Haken.

Example. Let K be a knot in S^3. We will show in §6 that K has a neighbourhood $\mathcal{N}(K)$ homeomorphic to a solid torus. The 3-manifold $M = S^3 - \text{int}(\mathcal{N}(K))$ is the *exterior* of K. Thus, M is Haken. In fact, it contains an orientable non-separating properly embedded surface, which we now construct.

Pick a planar diagram for the knot K. We view this diagram as lying in $\mathbb{R}^2 \subset \mathbb{R}^3 \subset S^3$. The knot lies in this plane, except near crossings where one arc skirts above the plane, and one below. Pick an orientation of the knot. Remove each crossing of the diagram in the following way:

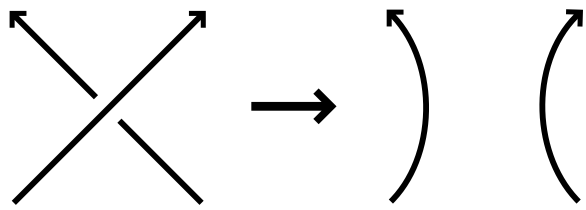

Figure 11.

The result is a collection of simple closed curves in $\mathbb{R}^2$. Attach disjoint discs to these curves, lying above $\mathbb{R}^2$. (Note that the curves may be nested.) Then attach a twisted band at each crossing of K, as in Figure 12. The result is a compact orientable surface S embedded in S^3 with boundary K. Such a surface is known as a *Seifert surface* for K.

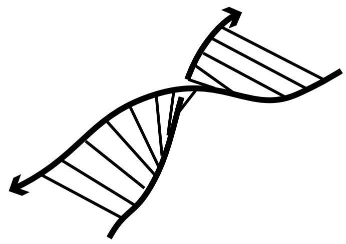

Figure 12.

We may take $\mathcal{N}(K)$ small enough so that $S \cap \mathcal{N}(K)$ is a single annulus. Then $S \cap M$ is an orientable properly embedded surface in M. It is non-separating, since a small loop encircling K intersects the surface transversely in a single point.

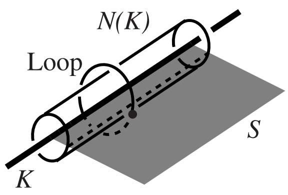

Figure 13.