STRESSES IN A PLATE DUE TO THE PRESENCE OF CRACKS AND SHARP CORNERS.

By C. E. INGLIS, Esq., M.A., Fellow of King's College, Cambridge.

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PART I.

THE methods of investigation employed for this problem are mathematical rather than experimental, and this mathematical treatment is given in some detail in Part II. of this paper. Part I. is a summary of the more important results and conclusions, and in this part mathematics are kept as far as possible in the background.

The main work of the paper lies in the determination of the stresses around a hole in a plate, the hole being elliptic in form. The results are exact, and are consequently applicable to the extreme limits of form which an ellipse can assume. If the axes of the ellipse are equal, a circular hole is obtained; by making one axis very small the stresses due to the existence of a fine straight crack can be investigated.

The destructive influence of a crack is a matter of common knowledge, and is particularly pronounced in the case of brittle non-ductile materials. This influence is turned to useful account in the process of glass cutting. A fine scratch made on the surface produces such a local weakness to tension that a fracture along the line of the scratch can be brought about by applied forces which produce in the rest of the plate quite insignificant stresses.

In ductile materials some easing off of the local stresses at the end of a crack will be effected by plastic yield of the substance, but for the case of alternating loads the protection against breakdown gained thereby must be limited. A small load tending to open the crack will produce overstrain at its ends. On reversing the load the crack closes again, but not before it has set up some reversed stress at the ends of the crack. In this manner a small alternating load may produce in the material an alternating stress far in excess of its elastic range, and under these circumstances, if a crack has once fairly started, no amount of ductility will prevent it spreading through the substance.

In a paper read before the Sheffield Society of Engineers and Metallurgists in January, 1910, Professor B. Hopkinson made some general observations concerning the stresses around a crack. In this paper Professor Hopkinson did not attempt to compute the intensities of these stresses, but the desirability of doing so he strongly emphasised. The present paper is an endeavour to satisfy this requirement.

Fig. 1 (Plate XXVIII.) shows an elliptic hole in a plate, the major and minor axes being $2a$ and $2b$ respectively. The plate is subjected to a tensile stress of intensity R applied in the direction shown. If the hole is nowhere near the edge of the plate and the material is nowhere strained beyond its elastic limit, a tensile stress occurs at A of the value $R\left[1+\frac{2a}{b}\right]$, and a compression stress at B of magnitude R. On exploring the plate in the direction A P, the tensile stress across A P, which has the value $R\left[1+\frac{2a}{b}\right]$ at A, rapidly decreases, and in a short distance attains approximately its normal value R. Advancing along B Q the compression stress across B Q which starts with the value R at B soon changes to a small tensile stress, and this gradually dies out entirely. Stresses in the directions A P, B Q are likewise brought into existence. These radial stresses are zero at A and B. At considerable distances from the edge of the hole the former dies down again to zero, and the latter attains the value R.

The variations in the stresses along and across A B and B Q are shown by the curves of Fig. 2, which are accurately drawn for the case $a = 3b$. An inspection of these curves clearly indicates that the tension at A is by far the greatest stress. In this case it amounts to 7 R. The rapidity with which this stress decreases with the distance from the edge of the hole is also very noticeable. These effects become more marked as the ratio $\frac{a}{b}$ increases.

When $\frac{a}{b} = 100$, the tension at A is 201 times the mean tension.

„ $\frac{a}{b} = 1{,}000$, „ „ 2,001 „ „ „

The ellipse in this latter case would appear as a fine straight crack, and a very small pull applied to the plate across the crack would set up a tension at the ends sufficient to start a tear in the material. The increase in the length due to the tear exaggerates the stress yet further and the crack continues to spread in the manner characteristic of cracks.

So far we have taken the major axis of the ellipse to be perpendicular to the direction of the pull. If the tension is applied in the direction of the major axis, tensile stress of amount $R\left[1+\frac{2b}{a}\right]$ is set up at B, accompanied by a compression stress of magnitude R at A. Hence a crack running in the direction of the pull does not produce great local stress, a conclusion which is almost self-evident.

If the major axis of the ellipse makes an angle θ with the direction of the pull, the tensile stress at the ends of this axis is—

$$R\left[\frac{a}{b}-\cos 2\theta\left(1+\frac{a}{b}\right)\right]$$

For such a case, however, the greatest tension does not occur exactly at these ends, and the value given above may be considerably exceeded. An example of special importance illustrating this point is given in Fig. 3 (Plate XXVIII.). Here the axis of the ellipse is inclined at an angle of 45° to the direction of the pull.

At A the tensile stress is $R\frac{a}{b}$.

At P the tensile stress is—

$$R\left[1+\frac{a}{b}+\frac{b}{a}\right]$$

Between A and P the tensile stress reaches a maximum value, and if $\frac{b}{a}$ is fairly small, a good approximation to this maximum value is—

$$R\frac{a}{2b}\left[1+\frac{\sqrt{2a^2+2b^2}}{a-b}\right]$$

At Q there is compression stress of magnitude R.

For the case $a=3b$—

The tensile stress at A is 3 R;

The tensile stress at P is $4\frac{1}{3}$ R;

The compression stress at Q is R.

The maximum tensile stress which occurs between A and P has the accurate value 4·64 R.

The approximation given above makes it 4·85.

For smaller values of $\frac{b}{a}$ this approximation is yet more accurate.

Extension to the Case of Cracks which are not necessarily Elliptic in Form.

Consider an elliptic hole in the plate modified by cutting away the portion shown shaded in Fig. 4 (Plate XXVIII.).

From the point of view of stress, this is equivalent to applying along the boundaries P Q R and S V W distributions of stress, which in each case is an equilibrium system, and which dies out as the points of contact with the ellipse are approached. The application of such a system does not substantially affect the stress outside the actual region where the distribution is applied, and for the case shown the stresses at A and A′ will not be altered to any appreciable extent, provided that the change in the boundary does not extend to these ends. In other words, the stresses at the ends of a cavity depend almost entirely upon the length of the cavity and the form of its ends.

If the ends of the cavity are approximately elliptic in form, it is legitimate, in calculating the stresses at these points, to replace the cavity by a complete ellipse having the same total length and end formation. If ρ is the radius of curvature at the ends of the major axis of an ellipse, $b = \sqrt{a\rho}$. This substitution for b gives the tensile stress at the ends of the ellipse in the form—

$$R\left[1 + 2\sqrt{\frac{a}{\rho}}\right]$$

This formula will accordingly apply to a cavity of any shape, the length of the cavity being $2a$ and the ends having a radius of curvature ρ; provided that the cavity near its ends merges smoothly into an ellipse.

Thus for the star-shaped hole represented in Fig. 5 (Plate XXVIII.), in which the ends merge into ellipses in the manner indicated, the tension at A will be $R\left[1 + 2\sqrt{\frac{a}{\rho}}\right]$, where ρ is the radius of curvature at the ends of the projecting arms. The compression stress at B will be R.

Case of a Square Hole with Rounded Corners.

To get the square hole with corners rounded to a radius ρ illustrated in Fig. 6, the cutting away process has to be more extended, and this may ease the stress at A somewhat below the value $R\left[1 + 2\sqrt{\frac{a}{\rho}}\right]$, but the error contained in this formula is in all probability quite small.

To this same degree of accuracy the tensile stress at A for the case illustrated in Fig. 7 is

$$R\sqrt{\frac{l}{\rho}}.$$

At P there is a tensile stress

$$R\left[1 + \sqrt{\frac{l}{\rho}} + \sqrt{\frac{\rho}{l}}\right]$$

At Q there is a compression stress of magnitude R. The greatest tensile stress along the rounded corner occurs at a point between A and P, and, if ρ is small compared with l, its approximate value is

$$\frac{R}{2}\sqrt{\frac{l}{\rho}}\left[1 + \frac{\sqrt{2l + 2\rho}}{\sqrt{e} - \sqrt{\rho}}\right].$$

Extension to the Case of a Crack starting from the Edge of a Plate.

Consider once again the plate illustrated in Fig. 1 (Plate XXVIII.). If one half is isolated from the other, the notched plate shown in Fig. 8 is the result. This plate, in addition to a horizontal pull of intensity R, is subjected to normal actions distributed along B Q, B′ Q′, according to one of the curves shown in Fig. 2. If this latter distribution can be neutralised, the stresses in a plate with an elliptic notch in its top edge, subjected to longitudinal tension only, can be deduced.

The tensile stress at A is accordingly $R\left[1+2\sqrt{\frac{a}{\rho}}\right]$, less the tension due to the distribution of normal stress along the top edge. When the notch is narrow and deep, these actions along the top edge produce little or no effect at A. If the pressure at B were maintained all along B Q, the tension produced at B would only amount to R.

Actually this pressure dies out rapidly, and since the total action on B Q amounts to a zero force, the tensile stress set up at A will be much less than R; it may very likely vanish altogether. At any rate, we may conclude that these actions along the top edge are unimportant, so far as the stress at A is concerned, and for the case of a narrow, deep, elliptic notch, the tensile stress at the bottom of the notch lies between the limits—

$$R\left[1+2\sqrt{\frac{a}{\rho}}\right] \text{ and } R\left[2\sqrt{\frac{a}{\rho}}\right],$$

where a is the depth of the notch and ρ is the radius of curvature at its end. The former limit—

$$R\left[1+2\sqrt{\frac{a}{\rho}}\right]$$

is probably the closer approximation, but for the case of a deep sharp-ended notch, the difference between these limits is relatively unimportant.

Extension to the Case of a Notch which is not necessarily Elliptic in Form.

Having deduced the tensile stress at the bottom of a narrow notch of elliptic form, the formula can be seen to have a wider application.

Thus, in Fig. 9 (Plate XXVIII.), the shaded portions can be removed without appreciably modifying the tension at A, which will continue to have the value—

$$R\left[1+2\sqrt{\frac{a}{\rho}}\right],$$

where a is the depth of the notch and ρ the radius of curvature at its end. The argument which leads to this conclusion is identical with that given in connection with Fig. 3, and calls for no repetition.

By similar reasoning the tensile stress at the point A for the cases illustrated in Fig. 10 (Plate XXVIII.) will be—

$$R\left[1 + 2\sqrt{\frac{a}{\rho}}\right]$$

For the rectangular notch represented in Fig. 11—

The tensile stress at A is

$$R\sqrt{\frac{l}{\rho}}.$$

The tensile stress at P is

$$R\left[1 + \sqrt{\frac{l}{\rho}} + \sqrt{\frac{\rho}{l}}\right].$$

The compression stress of Q is R.

The greatest tensile stress along the rounded corner occurs at some point between A and P, and, if ρ is small compared with l, a good approximation to its value is

$$\frac{R}{2}\sqrt{\frac{l}{\rho}}\left[1 + \frac{\sqrt{2l+2\rho}}{\sqrt{l} - \sqrt{\rho}}\right].$$

The last case to be considered is a small crack or notch springing from the side of a hole in the manner shown in Fig. 12 (Plate XXVIII.).

In this case there is a double magnification of stress. The mean stress R is concentrated to the value $R\left[1 + 2\sqrt{\frac{a}{\rho}}\right]$ in the neighbourhood of the crack, and this, again, is magnified to the value

$$R\left[1 + \sqrt{\frac{a}{\rho}}\right]\left[1 + 2\sqrt{\frac{a'}{\rho'}}\right]$$

at the end of the crack, where a and ρ refer to the hole and a' ρ' refer to the crack.

This example offers an explanation of the weakening of a plate which has been punched with holes. Around the edge of the holes fine cracks may be started. These cracks will have every inducement to extend, because the metal round the hole has been rendered hard and brittle by the punch. By the time the crack has extended through this hardened region, it has got such a hold on the plate that no amount of ductility will prevent the crack from advancing. The advisability of removing this hard and probably fractured rim round the edge of a punched hole is very apparent.

PART II.

Determination of the Stresses in a Plate which has an Elliptic Hole.

Throughout the work curvilinear co-ordinates are employed.

Let α = constant and β = constant, define two systems of curves intersecting at right angles.

At any point let u_α, u_β denote the shifts in the direction of the normals to α and β.

Let $e_{\alpha\alpha}$, $e_{\beta\beta}$, $e_{\alpha\beta}$ denote the two stretches and the slide corresponding to these directions.

Then—

$$e_{\alpha\alpha} = h_1 \frac{\partial u_\alpha}{\partial \alpha} + h_1 h_2 u_\beta \frac{\partial}{\partial \beta}\left(\frac{1}{h_1}\right)^*$$

$$e_{\beta\beta} = h_2 \frac{\partial u_\beta}{\partial \beta} + h_1 h_2 u_\alpha \frac{\partial}{\partial \alpha}\left(\frac{1}{h_2}\right)$$

$$e_{\alpha\beta} = \frac{h_1}{h_2}\frac{\partial}{\partial \alpha}(h_2 u_\beta) + \frac{h_2}{h_1}\frac{\partial}{\partial \beta}(h_1 u_\alpha)$$

where—

$$h_1^2 = \left(\frac{\partial \alpha}{\partial x}\right)^2 + \left(\frac{\partial \alpha}{\partial y}\right)^2$$

$$h_2^2 = \left(\frac{\partial \beta}{\partial x}\right)^2 + \left(\frac{\partial \beta}{\partial y}\right)^2$$

The dilation Δ is given by—

$$\Delta = h_1 h_2 \left[\frac{\partial}{\partial \alpha}\left(\frac{u_\alpha}{h_2}\right) + \frac{\partial}{\partial \beta}\left(\frac{u_\beta}{h_1}\right)\right]$$

The rotation ω is given by—

$$2\,\omega = h_1 h_2 \left[\frac{\partial}{\partial \alpha}\left(\frac{u_\beta}{h_2}\right) - \frac{\partial}{\partial \beta}\left(\frac{u_\alpha}{h_1}\right)\right]$$

For the particular problem under consideration the curvilinear co-ordinates α, β, are such that—

$$x = c\cosh\alpha\cos\beta$$
$$y = c\sinh\alpha\sin\beta.$$

α = constant is accordingly the ellipse $\dfrac{x^2}{c^2\cosh^2\alpha} + \dfrac{y^2}{c^2\sinh^2\alpha} = 1.$

β = constant is the hyperbola $\dfrac{x^2}{c^2\cos^2\beta} - \dfrac{y^2}{c^2\sin^2\beta} = 1.$

In this case $h_1^2 = h_2^2 = h^2 = \dfrac{2}{c^2(\cosh 2\alpha - \cos 2\beta)}.$

* See Love's "Elasticity."

General Stress Strain Equations.

These take the form—

$$\left.\begin{aligned}(1-\sigma)\frac{\partial \Delta}{\partial \alpha} - (1-2\sigma)\frac{\partial \omega}{\partial \beta} &= 0\\ (1-\sigma)\frac{\partial \Delta}{\partial \beta} + (1-2\sigma)\frac{\partial \omega}{\partial \alpha} &= 0\end{aligned}\right\}$$

Where σ is the value of Poisson's Ratio.

These two equations state that—

$$(1-\sigma)\Delta + \iota(1-2\sigma)\omega \text{ is a function of } \alpha + \iota\beta$$

Take $$(1-\sigma)\Delta + \iota(1-2\sigma)\omega = \text{constant} \times \frac{e^{-n(\alpha+\iota\beta)}}{\sinh(\alpha+\iota\beta)}$$

So that $$(1-\sigma)\Delta = \text{const.}\frac{-e^{-(n-1)\alpha}\cos(n+1)\beta + e^{-(n+1)\alpha}\cos(n-1)\beta}{\cosh 2\alpha - \cos 2\beta}$$

$$(1-2\sigma)\omega = \text{const.}\frac{e^{-(n-1)\alpha}\sin(n+1)\beta - e^{-(n+1)\alpha}\sin(n-1)\beta}{\cosh 2\alpha - \cos 2\beta}$$

if $$u = \frac{u_\alpha}{h}\alpha \text{ and } v = \frac{u}{h}\beta$$

$$\frac{\delta u}{\delta \alpha} + \frac{\delta v}{\delta \beta} = \frac{\Delta}{h^2} = \frac{a_n}{1-\sigma}\left[e^{-(n-1)\alpha}\cos(n+1)\beta - e^{-(n+1)\alpha}\cos(n-1)\beta\right]$$

$$\frac{\delta u}{\delta \beta} - \frac{\delta v}{\delta \alpha} = -\frac{2\omega}{h^2} = \frac{2a_n}{1-2\sigma}\left[e^{-(n-1)\alpha}\sin(n+1)\beta - e^{-(n+1)\alpha}\sin(n-1)\beta\right]$$

From these two partial differential equations u and v can be determined—

$$u = A_n[(n+p)e^{-(n-1)\alpha}\cos(n+1)\beta + (n-p)e^{-(n+1)\alpha}\cos(n-1)\beta] + \phi$$

$$v = A_n[(n-p)e^{-(n-1)\alpha}\sin(n+1)\beta + (n+p)e^{-(n+1)\alpha}\sin(n-1)\beta] + \psi$$

Where p stands for $3-4\sigma$, and ϕ and ψ are conjugate functions of α and β satisfying Laplace's equation.

Suitable values for ϕ and ψ are, const. $\times\ e^{-n\alpha}\cos n\beta$ and const. $\times\ e^{-n\alpha}\sin n\beta$.

From these general values of u and v values of $e_{\alpha\alpha}\ e_{\beta\beta}\ e_{\alpha\beta}$ can be deduced by means of the relations—

$$e_{\alpha\alpha} = h^2\frac{\delta u}{\delta \alpha} + \frac{u}{2}\frac{\partial h^2}{\partial \beta} - \frac{v}{2}\frac{\delta h^2}{\delta \beta}$$

$$e_{\beta\beta} = h^2\frac{\partial v}{\partial \beta} + \frac{v}{2}\frac{\partial h^2}{\partial \beta} - \frac{u}{2}\frac{\partial h^2}{\partial \alpha}$$

$$e_{\alpha\beta} = \frac{\partial}{\partial \alpha}(h^2 v) + \frac{\partial}{\partial \beta}(h^2 u)$$

The stress components can then be determined by the equations—

$$R_{\alpha\alpha} = \frac{E}{1+\sigma}\left[e_{\alpha\alpha} + \frac{\sigma}{1-2\sigma}\Delta\right]$$

$$R_{\beta\beta} = \frac{E}{1+\sigma}\left[e_{\beta\beta} + \frac{\sigma}{1-2\sigma}\Delta\right]$$

$$S_{\alpha\beta} = \frac{E}{2(1+\sigma)}\left[e_{\alpha\beta}\right]$$

These two sets of calculations are too lengthy to be given in detail; the results are as follows:—

$$\begin{aligned} R_{\alpha\alpha}(\cosh 2\alpha - \cos 2\beta)^2 = {} & \begin{bmatrix} (n+1)e^{-(n-1)\alpha}\cos(n+3)\beta + (n-1)e^{-(n+1)\alpha} & \cos(n-3)\beta \\ -\{4e^{-(n+1)\alpha} + (n+3)e^{-(n-3)\alpha}\} & \cos(n+1)\beta \\ +\{4e^{-(n-1)\alpha} - (n-3)e^{-(n+3)\alpha}\} & \cos(n-1)\beta \end{bmatrix} A_n \\ & + \begin{bmatrix} n e^{-(n+1)\alpha}\cos(n+3)\beta + (n+2)e^{-(n+1)\alpha} & \cos(n-1)\beta \\ -\{(n+2)e^{-(n-1)\alpha} + n e^{-(n+3)\alpha}\} & \cos(n+1)\beta \end{bmatrix} B_n \end{aligned}$$

$$\begin{aligned} R_{\beta\beta}(\cosh 2\alpha - \cos 2\beta)^2 = {} & \begin{bmatrix} -(n-3)e^{-(n-1)\alpha}\cos(n+3)\beta - (n+3)e^{-(n+1)\alpha} & \cos(n-3)\beta \\ +\{(n-1)e^{-(n-3)\alpha} - 4e^{-(n+1)\alpha}\} & \cos(n+1)\beta \\ +\{(n+1)e^{-(n+3)\alpha} + 4e^{-(n-1)\alpha}\} & \cos(n-1)\beta \end{bmatrix} A_n \\ & - \begin{bmatrix} n e^{-(n+1)\alpha}\cos(n+3)\beta + (n+2)e^{-(n+1)\alpha} & \cos(n-1)\beta \\ -\{(n+2)e^{-(n-1)\alpha} + n e^{-(n+3)\alpha}\} & \cos(n+1)\beta \end{bmatrix} B_n \end{aligned}$$

$$\begin{aligned} S_{\alpha\beta}(\cosh 2\alpha - \cos 2\beta)^2 = {} & \begin{bmatrix} (n-1)e^{-(n-1)\alpha}\sin(n+3)\beta + (n+1)e^{-(n+1)\alpha}\sin & (n-3)\beta \\ -(n+1)e^{-(n-3)\alpha}\sin(n+1)\beta - (n-1)e^{-(n+3)\alpha}\sin & (n-1)\beta \end{bmatrix} A_n \\ & + \begin{bmatrix} n e^{-(n+1)\alpha}\sin(n+3)\beta + (n+2)e^{-(n+1)\alpha} & \sin(n-1)\beta \\ -\{(n+2)e^{-(n-1)\alpha} + n e^{-(n+3)\alpha}\} & \sin(n+1)\beta \end{bmatrix} B_n \end{aligned}$$

In these formulæ for $R_{\alpha\alpha}$, $R_{\beta\beta}$, $S_{\alpha\beta}$, n can be any integer positive or negative, and the general expression for these three stresses takes the form of an infinite series, involving a number of arbitrary constants, which have to be determined by the conditions existing at the inner and outer boundaries of the plate.

Case of a Plate subjected to a Tensile Stress R in all Directions, the Plate having an Elliptic Hole defined by $\alpha = \alpha_0$.

The boundary conditions for this case are—

$$R_{\alpha\alpha} = S_{\alpha\beta} = 0, \text{ when } \alpha = \alpha_0,$$

$$\text{and } R_{\alpha\alpha} = R,\ R_{\beta\beta} = R,\ S_{\alpha\beta} = 0, \text{ when } \alpha \text{ is large.}$$

These conditions can be satisfied by making—

$$A_{-1} = -\frac{R}{8},$$

$$A_{+1} = -\frac{R}{8},$$

$$B_{-1} = \frac{R}{2}\cosh 2\alpha_0.$$

Adding together the three terms corresponding to A_{-1}, A_{+1}, and B_{-1}, the exact values of the stress for this case are—

$$R_{\alpha\alpha} = \frac{R\sinh 2\alpha\,[\cosh 2\alpha - \cosh 2\alpha_0]}{[\cosh 2\alpha - \cos 2\beta]^2}$$

$$R_{\beta\beta} = \frac{R\sinh 2\alpha\,[\cosh 2\alpha + \cosh 2\alpha_0 - 2\cos 2\beta]}{[\cosh 2\alpha - \cos 2\beta]^2}$$

$$S_{\alpha\beta} = \frac{R\sin 2\beta\,[\cosh 2\alpha - \cos 2\beta]}{[\cosh 2 - \cos 2\beta]^2}$$

Along the edge of the elliptic hole—

$$R_{\beta\beta} = \frac{2R\sinh 2\alpha_0}{\cosh 2\alpha_0 - \cos 2\beta}$$

Case of a Plate subjected to a Tensile Stress R in the Direction $\beta = \frac{\pi}{2}$, the Plate having an Elliptic Hole defined by $\alpha = \alpha_0$.

The boundary conditions for this case are—

$$R_{\alpha\alpha} = S_{\alpha\beta} = 0, \text{ when } \alpha = \alpha_0,$$

$$\text{and } R_{\alpha\alpha} = \frac{R}{2}(1 - \cos 2\beta), \quad R_{\beta\beta} = \frac{R}{2}(1 + \cos 2\beta), \quad S_{\alpha\beta} = -\frac{R}{2}\sin 2\beta,$$

when α is large.

The conditions can be satisfied by making—

$$A_{-1} = -\frac{R}{16}; \quad B_{-1} = \frac{R}{4}\Big[1 + \cosh 2\alpha_0\Big]$$

$$A_{+1} = -\frac{R}{16} - \frac{R\,e^{2\alpha_0}}{8}; \quad B_{+1} = \frac{R\,e^{4\alpha_0}}{8};$$

$$B_{-3} = -\frac{R}{8}.$$

Adding together the five terms corresponding to A_{-1}, B_{-1}, A_{+1}, B_{-1}, B_{-3}, selected from the general formulæ stated above, the exact expression for $R_{\alpha\alpha}$, $R_{\beta\beta}$, $S_{\alpha\beta}$ can be determined. These expressions are complicated in appearance, and too lengthy to be written out here in full. The most important and interesting information they contain is the

Expression for the Tensile Stress at the Edge of the Elliptic Hole.

This latter takes the comparatively simple form—

$$R_{\beta\beta} = R\frac{\sinh 2\alpha_0 + e^{2\alpha_0}\cos 2\beta - 1}{\cosh 2\alpha - \cos 2\beta}$$

when $\alpha = \alpha_0$.

If the semi-major and minor axes of the ellipse are a and b respectively:—

At the end of the major axis ($\beta = 0$) the tensile stress is—

$$R\left[1 + \frac{2a}{b}\right]$$

At the end of the minor axis $\left(\beta = \frac{\pi}{2}\right)$ there is a compression stress of magnitude R.

Case of a Plate subjected to a Shearing Stress S, the Plate having an Elliptic Hole defined by $\alpha = \alpha_0$.

The boundary conditions for this case are—

$$R_{\alpha\alpha} = S_{\alpha\beta} = 0, \text{ when } \alpha = \alpha_0.$$

If the shearing stress is applied to planes at right angles to the area of the ellipse, the conditions which hold when α is large are—

$$R_{\alpha\alpha} = S\sin 2\beta$$

$$R_{\beta\beta} = -S\sin 2\beta$$

$$S_{\alpha\beta} = S\cos 2\beta$$

To deal with this case, the values of $R_{\alpha\alpha}$, $R_{\beta\beta}$, $S_{\alpha\beta}$, consequent on taking—

$$(1-\sigma)\Delta + \iota(1-2\sigma)\omega = \iota \times \text{constant} \times \frac{e^{-n(\alpha+\iota\beta)}}{\sinh(\alpha+\iota\beta)}$$

have to be employed.

$R_{\alpha\alpha}$, $R_{\beta\beta}$ will be found to have exactly the same form as before, except that sines replace cosines. The new form of $S_{\alpha\beta}$ is obtained by reversing the sign of the original form and replacing sines everywhere with cosines.

The conditions stated above can be satisfied by making—

$$A_1 = \frac{S}{4} e^{2\alpha_0}$$

$$B_{+1} = -\frac{S}{4} e^{4\alpha}$$

$$B_{-3} = -\frac{S}{4}$$

Adding together the three corresponding terms, the value of $R_{\beta\beta}$ at the edge of the hole takes the form—

$$R_{\beta\beta} = -\frac{2 S \sin 2\beta e^{2\alpha_0}}{\cosh 2\alpha_0 - \cos 2\beta}.$$

CASE OF A PLATE HAVING AN ELLIPTIC HOLE DEFINED BY $\alpha = \alpha_0$ SUBJECT TO A TENSILE STRESS R APPLIED IN A DIRECTION MAKING AN ANGLE ϕ WITH THE MAJOR AXIS OF THE ELLIPSE.

This case can be arrived at by a combination of cases previously considered. The expression for the tension along the inside edge of the cavity is—

$$R_{\beta\beta} = \frac{\sinh 2\alpha_0 + \cos 2\phi - e^{2\alpha_0} \cos 2(\phi - \beta)}{\cosh 2\alpha_0 - \cos 2\beta}.$$

If $\phi = \frac{\pi}{4}$

$$R_{\beta\beta} = \frac{\sinh 2\alpha_0 - e^{2\alpha_0} \sin 2\beta}{\cosh 2\alpha_0 - \cos 2\beta}$$

MAXIMUM VALUE OF $R_{\beta\beta}$ WHEN $\phi = \frac{\pi}{4}$.

Let m be the slope of the tangent at the point where $R_{\beta\beta}$ has its maximum value.

m is given by the expression $e^{2\alpha_0} + \sqrt{1 + e^{-4\alpha_0}}$.

The value of β for this point of maximum stress is given by—

$$\tan\beta = -\frac{1}{m} \tanh\alpha_0,$$

and the value of the stress is—

$$-R \cdot \frac{\sin 2\beta}{\cos 2\beta} e^{2\alpha_0}.$$

At the point where the slope of the tangent is + 1

$$R_{\beta\beta} = R\left[1 + \tanh\alpha_0 + \frac{1}{\tanh\alpha_0}\right]$$

At the point where the slope of the tangent is − 1

$$R_{\beta\beta} = -R.$$

The SECRETARY read the following translation of a communication from Professor Mesnager:—

Professor A. MESNAGER (Ingénieur-en-chef des Ponts et Chaussées): Professor Coker's paper is an extremely interesting one. The complete study of the stresses in riveted structures or in the different parts of a plate perforated by rivet holes has never before been worked out so completely or by such interesting methods. Nevertheless the present investigation is limited to the sections perpendicular to the direction of stress, and parallel to the same, both passing through the centre of the rivet hole. These sections are those for which one usually calculates the stress in order to settle the dimensions of the plate. Unfortunately, these are not the sections in which the first permanent deformation usually occurs. In a test piece perforated by rivet holes the first permanent deformation occurs in those sections which are inclined at an angle of 45° to the direction of the stress, although rather closer to the direction perpendicular to the stress. Professor Coker's work forms, therefore, in my opinion, only a beginning of the investigation of this question of the fatigue of riveted plates, and it would be very desirable if this could be extended to other sections, notably to those which are most likely to be subject to permanent deformation. Some remarkable work has, within recent years, been carried out in England on this question of permanent deformation. It has been shown that the importance of the shearing stresses is very great, and we should therefore seek to find the locality in which they reach their maximum values (see in *Engineering*, December 24, 1909, a description of the work of Messrs. Guest, Mason, and Smith, and the discussion thereon). For such work polarised light offers considerable advantages, as the greatest shearing stress is exactly equal to half the difference between the extreme principal stresses. Now the stress perpendicular to the ends of the plate can only be very small; therefore, when the principal stresses which are parallel to the plane of stress are of contrary sign, a simple reading of the calibrator enables us to obtain the required difference. When they are of the same sign, the difference is equal to the greatest of these two stresses.

Professor W. E. DALBY, M.A., B.Sc., F.R.S. (Associate Member of Council): Mr. Chairman and Gentlemen, I thought that possibly the discussion on the two papers would be taken together. If the Chairman will allow a reference to Mr. Inglis' paper, I should like to say that seldom have two more remarkable papers been presented to the Institution in one evening. On the one hand, Mr. Inglis gives a rigid mathematical solution of the problem of finding the stress distribution caused by an elliptical hole in a plate under tension, including the particular case of the circular hole. On the other hand, Professor Coker shows how the stress distribution caused by applying tension through a rivet in a plate (or strictly speaking a pin) can be determined experimentally by his optical method. Comparing the results obtained by the mathematical method of Mr. Inglis with the results found optically by Professor Coker, so far as can be seen from the diagrams and from the details in the papers, the stress distributions as determined agree with one another. The one paper is a complement to the other, and the agreement in comparison enables us to use Professor Coker's method with increasing confidence to solve problems in stress distribution which elude the mathematical machinery at present available for their solution.

I think I am right in saying that one of the main features of Professor Coker's paper—and I hope he will correct me if I am wrong—is the description of an extension of his optical method in order to find the magnitude of one of the two vectors which is implicitly used by Rankine in his investigations of the stress distribution in a plane. Rankine's method consists essentially in reducing the principal stresses to two vectors; one equal in magnitude to half the sum of the principal stresses, and the other equal in magnitude to half the difference. The sum of these two vectors gives the direction of stress in a plane to which the vector which is half the sum of the principal stresses is at right angles. In Professor Coker's former paper to the Institution he showed how to determine optically the vector which is the difference of the principal stresses. To-night he shows how the vector which is the sum of the principal stress can be found. Hence the principal stresses can be found, and also the direction of the normal and shearing stresses at any point in a plane. I think you will all agree that a great advance has been made, an advance which will greatly increase the utility of Professor Coker's method. Professor Coker has shown on a slide the distribution of stress caused by a rivet or pin near enough to the margin of the plate to cause the indications of bending stress to appear. Has he determined how far the margin was from the centre of the hole when bending began to appear, and, if so, does the distance bear any relation to the usual rule used in designing riveted joints, namely, that the edge of the plate must be at least at a distance equal to the diameter of the rivet from the edge of the hole ?

I should like to conclude my remarks by expressing my admiration for the very beautiful results Professor Coker has obtained by his optical method.

Professor B. HOPKINSON, F.R.S. (Associate): Mr. Chairman and Gentlemen, I have not much more to say than that I echo Professor Dalby's admiration of the work done here by Professor Coker. This is the first time I have had the opportunity of seeing the beautiful coloured diagrams that he gets when he strains his pieces and brings to the eye, in a way mathematics cannot do, the manner in which the stress is distributed. I have only one remark in the way of criticism to make about the paper, and it merely concerns the title. I think that what Professor Coker has determined is not the stress in the neighbourhood of a rivet, but rather the stress in the neighbourhood of a pin joint. I believe that when two plates are riveted together the strength of the joint against pull is not the shear strength of the rivet itself, but they are pinched together by tension in the rivet which may not fill the hole at all, and, practically speaking, the joint holds by friction. I cannot cite any evidence on that at the moment, but I think that is the general impression of people who have had to do with riveting joints of the ordinary kind such as are found in structural steel work, and I am pretty sure there is a good deal of experimental evidence in favour of it. Of course, that does not in the least detract from the value of the paper, but it seems to me that the title is perhaps not quite correct. Mr. Inglis just now suggested to me that the real boundary condition, looking at it as a problem in mathematical elasticity, in the neighbourhood of a riveted joint is that the metal just round the rivet is held so that it cannot move radially. It is the same as though the rivet were completely inelastic, and filled the hole, adhering to the metal of the plate all round, so as to prevent all radial displacement. That seems to me to be the correct boundary condition, expressing, as far as mathematics can, what happens in the neighbourhood of a riveted joint.

Turning now to Mr. Inglis' paper, it is, I think, one of great importance. Most failures in engineering structures originate, I suppose, in a crack of some sort, whether they be started, as he suggests, by the presence of some hard material or possibly by a cavity containing some impurity such as slag. They originate in a centre of high stress such as exists at the end of a crack. Mathematicians, of course, have told us before now that the stress in the neighbourhood of a circular hole is as stated in Professor Coker's paper, and we have all learned that if the edge of the crack is an absolutely sharp corner, then you get an infinite stress, but the intermediate condition of a crack that has a very high curvature at the edge but is not infinitely sharp, has, so far as I am aware, been tackled by no one until now, and that, of course, is the important case. Mr. Inglis has shown us exactly how the stress at the end of the crack varies with its curvature and size. He has shown that it is proportional to the square root of the length of the crack, and inversely proportional to its radius of curvature. He has pointed out, however, that his results being based on the mathematical theory of elasticity apply only within the limits of elasticity of the material. I hope that he will carry the investigation a stage further in the near future, and tell us what happens in a ductile material when the elastic limit is locally exceeded. Taking an elliptical cavity in mild steel, the material will begin to yield and stretch at the end of the cavity as soon as the stress there exceeds the yield point, and, as Mr. Inglis points out, that will lead to a re-distribution of stress, the result of which can be represented by taking off the top of the sharp-pointed curve shown in Fig. 2 (Plate XXVIII.) and raising the stress at points from there to the right so as to keep the total area the same. Unless the curvature of the crack is more than a certain amount this will, no doubt, save the material and the crack will not go any further. At the same time, it is obvious that that process of stretching puts an excessive local tax on the ductility of the material in the immediate neighbourhood of the end of the cavity. Whether the crack spreads or not depends on the relation of the amount of the local stretch to the ductility of the material. It would be a valuable sequel to Mr. Inglis' work to find out the dimensions of a cavity which would cause the material to stretch, say, 30 per cent. at the ends when the metal as a whole is stressed to near its elastic limit. Such a cavity would presumably spread as a crack even in mild steel. It is known that good mild steel cannot be torn by any ordinary mechanical means. If a very thin slot be cut with a hacksaw at the edge of a mild steel plate, at right angles to the length of the plate, and the plate be broken in a testing machine, although, of course, the stretch is very much localised, the plate does not break by a tear starting from the slit, but the metal in and near the plane of the slot pulls out just as a longer piece would pull out with a good reduction of area. On the other hand, it is known that ductile material does under some circumstances break by tearing, and Mr. Inglis in his paper cites the commonest instance of that, viz., where a crack starts from a punched rivet hole, or from the edge of a sheared plate. Owing to the greater elastic stress which exists in the neighbourhood of the hole, a crack is started there in the hardened material, and, as he says, that material has sufficient hold on the ductile material beneath it to cause the crack to spread. Once started in that way a crack may spread to any extent even in ductile material, as has sometimes happened in the plates of boilers.

The effect of case-hardening in making mild steel brittle is another instance of the same thing on which I have made a few simple experiments. The Wolseley Tool and Motor Company were good enough to supply me with some bars case-hardened according to their ordinary process for motor car parts. The bars were of good mild steel (about ·12 per cent. of carbon) $\frac{3}{4}$ of an inch in diameter, and they were case-hardened, some to a depth of 1-32nd of an inch, and some to 1-16th of an inch. They were broken in a 5-ton machine by bending. A bar case-hardened like that, although the central part is entirely ductile, snaps like a carrot under bending stress, and does not yield at all. That, I suppose, is a common experience. I then took a bar that had been case-hardened to a depth of 1-16th of an inch and had all the hard metal ground off. The piece so treated was first bent in the testing machine, and then finally bent double in a vice without showing any defect. That shows that the interior still possesses all the ductility of mild steel. This was one half of the same bar which, before the case-hardening was removed, had broken quite short by bending. It broke without a trace of permanent set. Then I tried the experiment of grinding off most of the case-hardening, but not all of it. I left a little strip of hard material on the tension side of the bar when it was bent. When a bar so treated is loaded by bending the process of tearing in ductile material is very well shown. At a certain load the hardened tension part at the bottom cracks with an audible snap, and when the load is a little increased the ductile stuff tears across like a bit of paper. That is, as far as I know, the only way in which mild steel can be made to tear. I have not tried the same experiment with iron instead of steel, but it is generally believed that iron is not affected by case-hardening in the same way. If so, it is perhaps due to its greater ductility, in virtue of which it can stretch sufficiently to relieve the stress even at the base of a fine crack starting from the hard part. A full experimental and mathematical examination of this problem of the ductility necessary to prevent cracks from spreading would, I think, be of great scientific and practical interest. Mr. Inglis has taken the first step, and a very important one, towards its solution, and I hope that he will be able to carry it further.

Mr. C. E. STROMEYER (Member of Council): Mr. Chairman and Gentlemen, like the previous speakers, I very much admire the work that Dr. Coker is doing in regard to the examination of stresses. I attempted to deal with a similar subject (*Proceedings Royal Society*, vol. lv. p. 374) by utilising the interference of monochromatic light for measuring the cross contraction of test pieces. I should imagine that this method might be more safely applied to Professor Coker's problems, using metal, of course, instead of celluloid for determining the value of $p_r + p_t$. The instrument was an interferometer, of which there are now several designs, which allowed changes of dimensions to be measured by counting the number of interference bands which passed a cross wire in the microscope. This instrument had to be securely clamped to the test piece, and other precautions had to be taken to guard against outside disturbances; and I cannot understand how an instrument which differs from mine in having levers should be not only ten times more sensitive than a simple interferometer, but also more reliable in its readings.

Professor Dalby has already mentioned Guest's Law, which seems in general to be reasonably correct, for mild steel seems to break down, not when a single stress or the sum of two principal stresses exceeds a certain limit, but when their difference exceeds a certain limit, and, therefore, from an engineering point of view, the weakness of a structure is measured by the differences of two stresses, which are measured by the polariscope. The measuring of $p_t + p_r$ is of far less importance. Experiments, which to a certain extent confirm Professor Coker's results, have been carried out on curved and on holed plates by E. Preuss (*Zeitsch. d. Vereines Deutscher Ingenieure*, 1912, vol. xliv. pp. 1349 and 1780). The radii of the curvatures of the inner sides of the curved beams or hooks varied from 0 to 35 mm. on a width of 40 mm., and the ratios of holes to widths of the perforated plates were 1 to 9, to 4, to 2·5 and to 1·71 respectively; the ratios of maximum to mean strains being 2·35, 2·34, 2·13, and 2·25. I should like to suggest to Professor Coker, when he continues the experiments, as, of course, I assume he will, if he would perhaps map out or plot on a chart the differences and the sums of the stresses, in order that, instead of having simply one line across the rivet hole, we should have a regular chart of the whole neighbourhood. It will help us very considerably, I think, in applying to other forms the results which Professor Coker is getting for one form.

As regards Mr. Inglis' paper and the formulæ in Professor Coker's paper, I regret to say that, although I have studied them with Professor Love's book on elasticity at my elbow, I find that the notations, which are doubtless very convenient for the discussion of vibrations and other complicated subjects, are so elaborate that I have not been able to do more than apply some simple tests to the results, and do not now feel satisfied that the mathematical deductions are fair representations of practical cases; in other words, the holes, corners, and cracks with which Mr. Inglis' paper deals are mathematical, and not real ones. For instance, fatigue cracks were mentioned, but far too little is as yet known about this subject for anybody to say whether these mathematical solutions apply in those cases or not. I have personally had to deal with about sixty cracked crank shafts, and I do not know with how many cracked tail shafts; and I remember very well the first which I discovered and condemned, for I became aware that marine engineers do run their crank shafts when they are cracked, without very much danger. If the assumed cracks in the paper truly represent practical cracks, then as soon as a crack showed itself in a fillet of a shaft it ought rapidly to break in two, and, at any rate, the crack ought to extend; but I have noticed in many cases that fatigue cracks in crank fillets are not single cracks. They usually start in many places about the same time, and gradually merge into each other. It would almost seem that instead of the first crack being the weakest part of the shaft it is merely a centre of relief, and other parts beyond the crack give way. It is, perhaps, not fair to apply mathematics to ductile material like mild steel; but the tests I made on nickel samples (*Journal of the Iron and Steel Institute*, 1907, i. p. 229; iii. p. 93) may be mentioned here, as they have some bearing on the question. I nickelled my samples for the purpose of ascertaining whether the material grew brittle—whether it aged, as it were. I certainly found that those samples which were bent at once after being nickelled were more ductile than those which were bent after waiting; but the interesting part, from a mathematical point of view, is that, although some of these nicks did crack—they did not all crack, for some of them simply tore with a silky fracture—yet those that did so cracked for only short distances into the metal, and the material at the end of this crack tore, so that, instead of the crack being really a source of weakness, it seemed as if the cracking of the end of a nick was merely a question of induced brittleness. Do not let it be thought I wish to favour sharp corners or cracks. I know they are very dangerous, but I do not believe that mathematics can thoroughly represent the case. They should apply, above all things, to hard materials; but even there I do not find that mathematical cracks are practical ones. For instance, anybody who tries the experiment of continuing a crack in a glass plate by producing strains by light hammering will find it exceedingly difficult to draw the crack in the desired direction, as it should do if the stresses were infinitely great at the end of the crack. Then, again, if these mathematical solutions were quite correct, the fine nick which is cut into a glass by means of a diamond should lead to its breaking in two quite easily, and yet it is perfectly well known that a thick glass cannot be cut with a small diamond; it is, in a sense, the external shape or size of the diamond, and not the edge itself, or the nick which it produces, which does the cutting. Diamonds of considerable size have to be used to cut thick plates. Then, again, if you wish to break a granite stone, which is also a hard material, you do not nick it with a sharp instrument, but you hit the block along the desired line of fracture with a blunt but heavy hammer, and the stone breaks as desired. It is the external shape of the hammer, and not the weakness produced by a fine crack, which causes the fracture.

Professor J. B. HENDERSON, D.Sc. (Associate): Mr. Chairman and Gentlemen, the Institution is indeed to be congratulated on having two such splendid papers contributed at one meeting, the one experimental and the other mathematical. I shall deal, first of all, with Professor Coker's paper. I should like to ask Professor Coker about the probable accuracy in the determination of $p_t + p_r$, because, in determining two quantities by a difference method, the percentage error is notoriously low, depending as it does altogether on the relative values of p_t and p_r, and one might, by making a 1 per cent. error in the measurement, have a 20 per cent. error in the resulting values of p_t and p_r. It occurred to me that the determination of $p_t + p_r$ might be liable to an error due to the specimen being slightly bent, either initially or due to the weight of the extensometer, or due to the pull not being rigorously axial. The extensometer grasps the specimen at two points, and the weight of the extensometer tends to bend it. I do not know what the weight was; it may have been quite small, but one knows the difficulties of getting a specimen initially straight, of keeping it straight, and of applying an axial stress. The mirror is attached to one end, so that any straightening of the specimen gives a deflection which corresponds to a variation of $p_t + p_r$. I had a similar measurement to make some time ago, and I found considerable difficulty due to the bending of the specimen. I got over it in a way which I think might improve the accuracy of Professor Coker's measurement, by putting two mirrors at right angles, one attached to the frame and the other to the moving lever, and then viewing the image of the scale formed by double reflection in the two mirrors. Any bending of the specimen then produces no deflection of the scale at all.

I should like to have Professor Coker's opinion as to the elastic properties of xylonite. Is it quite a reliable elastic material? I was making experiments three years ago on the stresses round deck-hatches, using a glass plate, and Brewster's interference band method, just like Professor Coker's. The plate was optically worked, but unfortunately the students who were making the experiment over-stressed it during the first few experiments and broke it, and as the plate costs several pounds, it was not possible to renew it. Xylonite is a beautiful substance if it can be relied on to give accurate results as to elastic properties. As a matter of history, I think Professor Coker is wrong in calling the symbol m Poisson's ratio; Poisson's ratio was really the reciprocal of m.

Now, turning to Mr. Inglis' paper, I can only hold up my hands in admiration of this beautiful piece of mathematics. Looking at the Appendix, I see mathematical tools whose existence I had almost forgotten, and, like the true artist, he has kept his tools in the background. One can only admire the results; these must be very valuable to practical men, because there is no doubt as to their reliability. The tensions and the stresses round cracks have been a puzzle to most engineers, and the method by which a crack spreads has also been a puzzle. Mr. Inglis shows how it is that the crack spreads as soon as it is subjected to an alternating stress exceeding the elastic limit, by the opening and closing at the edge. I should like to ask Mr. Inglis if he has considered the question of the direction in which a crack will develop. If a crack is across the plate, is there not a tendency, when it develops, to turn parallel to the plate? Does not the plate give way along the plane of shear, according to the present theory? In this case it would go at 45° to the direct stress, and would gradually work round till it was practically parallel to the plate, so that the plate would be saved. The method Mr. Inglis has used for dealing with holes which are other than elliptical is extremely ingenious. By the method of superimposed systems of stresses he has shown that we need not bother much about the shape of the hole, except at the corner, which is a great simplification for practical men; and also that the cracks near the edge of a plate can be treated with equal simplicity.

Mr. W. A. Scoble, B.Sc., Wh.Sc.: Perhaps, Sir, you will excuse me if I deal with some of these points in the order in which they have been raised. With reference to the paper by Professor Coker and myself, the first point was raised by Professor Mesnager in his letter, in which he referred to the fact that the greatest stress occurs at an angle of 45° to the direction of the pull. You remember that, in the photographs which Professor Coker showed you, undoubtedly there was a widening out of the colour bands at those angles. I must confess that in making the experiments on one occasion I overstrained the specimen, and the point at which yield took place was immediately below the rivet, at a point which we investigated. Professor Dalby asked about the stress at the outside of the plate, and the relation between that stress and the amount of overlap. You will notice that we dealt with two cases, one in which the amount of the overlap was somewhat considerable compared with the diameter of the hole. The distance below the hole was approximately three times the rivet diameter. In the other case, the space below the hole was equal to the diameter. In the first case the stress acting across the axis of the specimen below the hole was a compression just near the hole, but it became a very small tension to the end of the plate. When the overlap was made smaller, the distribution of stress was more nearly similar to that in the case of a beam.

A point was mentioned which I think we expected, that our case deals rather with the pin than with the rivet. It is sometimes a little difficult to get a title that exactly fits a paper, and this point was raised in giving the title to the paper. I think that you should remember that this is the first case in which the two methods have been combined, and it should be looked upon rather as showing the possibility of the combination of the two methods to distinguish between the principal stresses. Our case was that of a pin rather than a rivet, because there was no force pressing the two plates together, no force in the direction of the axis of the rivet. There is another consideration which is, I think, equally important, and which we were not able to take into account up to the present. There is often an intense radial pressure due to the rivet being pressed hot into the hole, and there will be stresses around the hole in addition to those which we have investigated. It appears to me that there will be stresses somewhat as if the hole was subjected to internal fluid pressure. We have not dealt with the complete case, but when we are discussing the matter, I think the view to take is this, that when we make experiments it is impossible to exactly reproduce the practical conditions. In fact, frequently it is advisable that they should not be so reproduced. Therefore we have dealt with the first case of a rivet in a hole where the stress in the direction of the axis of the rivet has not been considered, nor the radial stress, which can be compared to an internal fluid pressure; those can be added later.

Mr. Stromeyer referred to the instrument for measuring the sum of the principal stresses, and he said that he could hardly believe that such a simple instrument would measure the stresses. I attribute the accuracy of the results to the simplicity of the instrument. I have read his paper, but I forget the details of the construction of his apparatus. It struck me that the lag which he observed might have been due to the instrument, and not to the lag in the material. He also said that since we pretty much agreed that, according to Guest's Law, the shear stress determines the failure of a plate, therefore all we need to measure is the difference in the principal stresses, which, of course, is double the shear stress. I think we need much more in all cases where we have an unequal distribution of stress; and I think that also has a bearing on the experiment by Professor Hopkinson, in which he took a tension specimen, put a saw-cut into it, and found under test that the specimen fractured very much like an ordinary tension specimen. I must confess that is just what I should expect it to do; and I think that unless all these results which are given are considered in relation to yield—though it is difficult to experiment after yield—we shall get quite wrong impressions of what happens.

Referring to Mr. Inglis' diagram—because his case is very much like ours—and considering a tension specimen with a hole, which may be either a circular or an elliptical one, we have at the hole a very intense stress, which diminishes towards the edge of the plate. Supposing the tension be gradually increased, the material will yield a little at the hole. When it has yielded to the order of, say, 1-1,000th of an inch, all this other material across the section is under the full stress, so that after a little yield we have practically a uniform distribution of stress across that section. I think Professor Hopkinson's experiment confirms that, because he tells us that with a slot across his specimen the appearance of a fracture across the remainder of the section is very much as in the ordinary tension test.

Professor Henderson mentioned the bending action due to the instrument for measuring the sum of the principal stresses. It was made so light that the bending action was entirely negligible. I suppose the weight of the whole instrument was not more than three ounces, or something of that order. The bar, for instance, which you see carrying the supports for the lever, is made of $\frac{1}{16}$ in. steel, and is about $\frac{1}{4}$ in. wide. That will give an idea of the dimensions of the whole apparatus. The clamp which supports the measuring portion is made of $\frac{1}{16}$ in. steel strip about $\frac{1}{2}$ in. wide. With reference to the properties of xylonite, we have made tests on the xylonite at least to the stresses we have had to deal with in this paper. Personally, I was surprised to find it behaved as well as it did. From the nature of the material I did not expect it to have a very extensive elastic range. I was afraid it would have a considerable amount of creep, but I was surprised to find it behaved very much like a specimen of steel

Professor E. G. COKER, M.A., D.Sc. (Associate): Mr. Chairman and Gentlemen, the questions raised in connection with the paper by Mr. Scoble and myself are somewhat numerous, and I will endeavour to reply briefly to those which have not been dealt with by Mr. Scoble. With regard to Professor Mesnager's view of the distribution, the pictures which we exhibit show that there must be a very considerable amount of truth in what he says, and we hope, very soon, to investigate the problem in much greater detail. Professor Dalby has very happily pointed out the connection between an optical measurement and Rankine's ellipse of stress construction. As he points out, the methods described here give the sum and difference of the principal stresses, and these are the vectors Rankine uses. The angular positions of the planes of principal stress can also be determined experimentally, but not in quite so easy and direct a manner as in the Rankine construction.

Professor Hopkinson has also mentioned a matter which we felt sure would be raised in this discussion, namely, the function of the rivet in our experiments. We quite agree that it acts as a pin in these cases, and not as a true rivet. In order to get the tension of the rivet it would be necessary to obstruct the view and hide some of the neighbouring parts we wish to see. We have not, therefore, attempted the rivet problem in its entirety. Mr. Stromeyer mentions a very interesting fact that some years ago he measured the thickness of plates to get the values of the sum of the principal stresses. That is a new fact to us, and a most interesting one. I think he made a good deal of the difficulty with regard to the question of the measurements. There is no real difficulty, because in the ordinary extensometers, like the Ewing instrument for compression, for example, the unit of the scale is one division for a quarter of a millionth of an inch over a length of from one to two inches. The required increase in delicacy is not very great in order to measure accurately the change in a thickness of $\frac{1}{8}$th or $\frac{3}{16}$ths of an inch to as nearly as is required. Another point was raised by Professor Henderson in regard to the accuracy of the measurements. It is perfectly true that it is a severe test to bring two entirely different types of measurement together, and then add and subtract them to obtain the quantities required. Any experimental error you make is bound to assume a large proportion of the difference value you obtain, and in the paper we have not concealed the fact that we did not get absolutely concordant results. I think we were really surprised to find them agree so nearly as they did. They were brought together at the last, and from the way in which they are given it was impossible for us to tell whether they would agree or not.

In this connection I might refer to Mr. Inglis' paper. We are delighted to see that we agree with him, without knowing anything beforehand of what he was going to say. For a particular problem Mr. Inglis has an accurate solution, of which there can be no doubt whatever. It complies with all the conditions of an accurate solution, and, therefore, it must be correct. You will observe that the readings agree fairly well, so far as we have got them for cases in which the two papers can be compared. Professor Henderson also referred to the question of the properties of xylonite. I am glad that my colleague, Mr. Scoble, has mentioned this point independently, because one is always biassed in favour of one's own particular way of doing things, but Mr. Scoble was not. He did not at first believe that the material would give measurements as accurate as we required them, but he is now quite converted from actual experience. If sufficient care is taken, and everything is measured in the very best way, it is possible to get an accuracy of within $\pm$ 2 per cent. You cannot always be sure of getting as close as this, unless you are making very careful measurements, and everything is in first-class order. In previous work we have, without knowing exactly what results ought to be obtained until the very end, got within these limits of accuracy. I do not think the work in the present paper can be taken as close as this all through. There is some creep with time, but not very much. The material is not so perfectly elastic as glass, but then it has this great advantage over glass, that you can put a very considerable stress on it without fracture taking place, and this gives the opportunity of a larger range of measurement than with glass. I dealt with that matter somewhat fully in a previous paper at the Spring Meeting of 1911,* and it is shown there that glass is a superior body for this purpose as regards its elastic properties, but it is inconvenient in almost every other way. With regard to Poisson's ratio, we have used it here in the sense which most engineers use it, fully recognising that we are not in perfect agreement with mathematicians on the point. It is a small matter, but we thought it better to give it in the way we have done.

Mr. C. E. Inglis, M.A.: Mr. Chairman and Gentlemen, I am fortunate in not having to answer many difficult questions, so my reply can be brief. The only adverse criticism comes from Mr. Stromeyer, and it takes the form of a general, though somewhat vague, distrust of mathematics. Mathematical reasoning is the soundest possible form of argument, and the conclusions arrived at are perfectly reliable, provided that the fundamental conditions laid down are properly respected. This Mr. Stromeyer fails to do. The paper deals with cracks in plates subjected to a two dimensional distribution of stress. Mr. Stromeyer apparently thinks it ought to explain the behaviour of cracks in crank shafts. From the mathematical standpoint these cases have almost nothing in common. To discredit the validity of this work because it fails to explain cases which are definitely excluded by the fundamental conditions laid down, is, to put it mildly, somewhat unreasonable. Concerning the direction in which a crack will spread, theory, I think, tells us little or nothing. Professor Henderson suggests that the plate gives way along the plane of greatest shear. I think this is even more than we can at present claim to know. Certainly, it seems fairly well established that the first indication of loss of elasticity occurs in this plane, but whether the ultimate rupture of the material is decided by tension or shear remains to be found out. To settle this point, a knowledge of the stress in a body strained beyond its elastic limit is required, and this takes us into a region at present almost entirely unexplored.

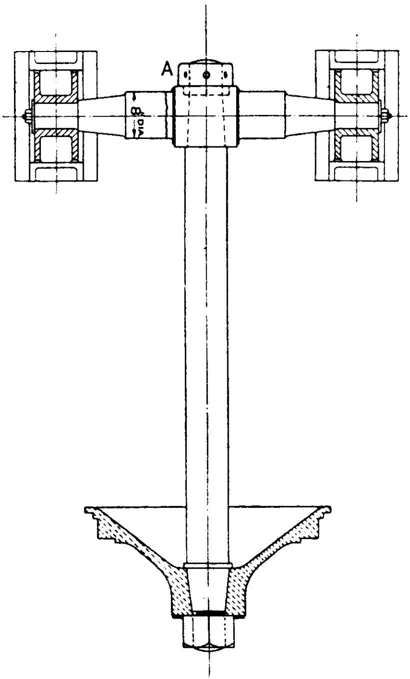
The Chairman (Professor J. H. Biles, LL.D., D.Sc., Vice-President): Gentlemen, I am sure you will allow me to propose a very hearty vote of thanks to Professor Coker and Mr. Scoble for their paper, and to Mr. Inglis for his admirable explanation of what his paper is intended to convey; it is most interesting from the theoretical and purely mathematical side to see this elegant analysis, and from the experimental side to see the results which Professor Coker and his colleague have brought before us. Those who have been away to-night, attracted by other engagements, cannot, I am sure, have had a better entertainment than you have had this evening. At any rate, those who have remained faithful have had their reward. I ask you to give a very hearty vote of thanks to the readers of both these papers.

Written Contributions to the Discussion.

Mr. H. M. Rounthwaite (Member): Referring to Professor Hopkinson's remarks in the above discussion, it occurs to me that the case described below may be of interest; it is one that has been a puzzle to me ever since it occurred. I have mentioned it to many engineers and metallurgists, and have never obtained any helpful suggestion in reference to the cause. The engines of H.M.S. *Gladiator* were being erected in Messrs. Maudslay's shops (1895-6), and the piston, piston-rod, crosshead, and guide blocks (as shown in the sketch) were suspended from a crane, the piston being in the cylinder, and the crosshead in its working position between the two pairs of guides attached to the four columns

* Trans. I.N.A., Vol. LIII. (i.) p. 265.

which carried the cylinder. The piston was at some point above the half-stroke when the sling chain snapped, and the parts shown in the sketch fell until the piston came to violent contact with the bottom of the cylinder. The cylinder was not in any way injured, but one arm of the crosshead broke short off at "A" in the sketch. The fracture resembles a fine tool steel, that is to say, it showed a very fine crystalline structure all over and quite uniform, excepting round the circumference, where the brighter line of the case-hardening showed for perhaps 1-20th inch in. Apparently the only force acting to break the crosshead was the momentum of the portion beyond the break. The forging was specified and had been ordered as Swedish iron, and the two connecting-rod journals had been case-hardened. After the accident a piece of the metal was sent to Professor Arnold, who at once demonstrated by means of micrographs that the material was not Swedish iron at all, but ordinary mild steel. In my apprentice days I was often told by old workmen that one should never case-harden steel, as the process would make it quite rotten, but I could never obtain scientific confirmation of this view, and in the last twenty years I have noticed that many manufacturers make a practice of case-hardening steel pins, &c.



Professor B. Hopkinson, F.R.S. (Associate): I am very much interested in Mr. Rounthwaite's account of the accident to the crosshead of H.M.S. *Gladiator*. I do not think that the mass of the part to the right of the fracture would be sufficient to break the case-hardening, if it were perfectly sound, especially as there would be a certain amount of spring on the piston. Of course, a minute crack in the case on the top side would be quite sufficient to account for the breakage. The character of the fracture described is very interesting, and is precisely similar to that of the smaller case-hardened bars which I described at the Meeting. It is evidently characteristic of case-hardening of this kind that it causes the material within to break short when the latter is mild steel. The statement that it is a common opinion among workmen that mild steel breaks in this way, whereas iron does not, is in accordance with my own experience, but I have never actually tried the experiment with iron.

Mr. Lloyd Woollard, R.C.N.C. (Member): I think the Institution is to be congratulated on this paper by Mr. Inglis, as it is one of the comparatively few attempts that have been made to adapt the mathematical theory of elasticity to the practical problems met with in Naval Architecture. As one of the speakers remarked, there is a tendency in text-books dealing with this subject to limit its application to such problems as are more readily susceptible to mathematical treatment; *e.g.*, the vibration of elastic spheres and kindred investigations having little application to actual structures are very fully treated. In this case, however, the author has selected, on its practical merits, a problem which by no means readily lends itself to theoretical investigation, and has applied the results in a way that is directly useful and interesting to us. It is to be hoped that he may be able to continue his researches to the many problems affecting the strength of ships and other structures that are still unsolved; *e.g.*, the stresses in plate web girders, the stiffening and riveting of bulkheads, &c. I have only one point to raise, which has already been dealt with in part by some of the speakers in the discussion. In a ductile material, like mild steel, how far can a slight overstrain, due to the limit of elasticity being exceeded over a limited region of material, be regarded as really dangerous? There are two instances, well known in naval architecture, where such overstrain is constantly produced and accepted. The first is found in the flat plating near the bottom of some ships, under the action of the water pressure and longitudinal bending moment. In many battleships the thickness of such plating is $\frac{5}{8}$ in.; with a 4-ft. frame spacing and a momentary head of 35 ft. of water, it can be shown (see paper read in 1902,* by Mons. I. G. Boobnoff) that a stress is caused between 7 and 18 tons per square inch, probably nearer the upper limit. On combining this with the stress due to longitudinal bending, it is evident that the elastic limit of mild steel must be exceeded; yet, as far as is known, no sign of weakness has been observed in such cases. A second instance is furnished by an iron chain ring, whose internal diameter is considerable, say, 8 times the diameter of the iron. The ring can be proof tested, and the stress caused thereby estimated fairly readily by a simple mathematical process. It is found that the maximum intensity of stress on the assumptions becomes 30 tons per square inch.

I should like to ask Mr. Inglis whether he has considered this point; and if he can indicate the extent to which the factor of safety ought to be increased in practice to allow for possible small defects in the material.

Mr. C. E. Inglis, M.A.: Concerning the two instances of high resistance to stress mentioned by Mr. Woollard, a possible explanation may be found in the remarkable fortification sometimes produced in a material which has been partially overstrained. For instance, if a thick cylinder is subjected to a large internal pressure, the inner layers will be overstrained. When the pressure is released the strains do not entirely disappear. The elasticity of the outer material produces compression in the overstrained under layers, and this initial state of stress fortifies the material to withstand the next application of pressure. The initial stresses produced by the overstrain are, in fact, very similar to those set up by wire winding on guns. Something of this sort may very likely occur in the two cases mentioned by Mr. Woollard, and the material may be able to withstand large added stresses owing to the existence of beneficial initial stresses brought about in the first instance by a partial overstrain of the material.

* Trans. I.N.A., Vol. XLIV. page 15.