Two Notes on Notation

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Mathematical notation evolves like all languages do. As new experiments are made, we sometimes witness the survival of the fittest, sometimes the survival of the most familiar. A healthy conservatism keeps things from changing too rapidly; a healthy radicalism keeps things in tune with new theoretical emphases. Our mathematical language continues to improve, just as "the d-ism of Leibniz overtook the dotage of Newton" in past centuries [**4**, Chapter 4].

In 1970 I began teaching a class at Stanford University entitled Concrete Mathematics. The students and I studied how to manipulate formulas in continuous and discrete mathematics, and the problems we investigated were often inspired by new developments in computer science. As the years went by we began to see that a few changes in notational traditions would greatly facilitate our work. The notes from that class have recently been published in a book [**15**], and as I wrote the final drafts of that book I learned to my surprise that two of the notations we had been using were considerably more useful than I had previously realized. The ideas "clicked" so well, in fact, that I've decided to write this article, blatantly attempting to promote these notations among the mathematicians who have no use for [**15**]. I hope that within five years everybody will be able to use these notations in published papers without needing to explain what they mean.

The notations I'm talking about are (1) Iverson's convention for characteristic functions; and (2) the "right" notation for Stirling numbers, at last.

1. IVERSON'S CONVENTION. The first notational development I want to discuss was introduced by Kenneth E. Iverson in the early 60s, on page 11 of the pioneering book [**21**] that led to his well known *APL*.

> "If α and β are arbitrary entities and $\mathscr{R}$ is any relation defined on them, the *relational statement* $(a\mathscr{R}b)$ is a logical variable which is true (equal to 1) if and only if α stands in the relation $\mathscr{R}$ to β. For example, if x is any real number, then the function
>
> $$(x > 0) - (x < 0)$$
>
> (commonly called the *sign function* or sgn x) assumes the values 1, 0, or -1 according as x is strictly positive, 0, or strictly negative."

When I read that, long ago, I found it mildly interesting but not especially significant. I began using his convention informally but infrequently, in class discussions and in private notes. I allowed it to slip, undefined, into an obscure corner of one of my books (see page 117 of [**16**]). But when I prepared the final manuscript of [**15**], I began to notice that Iverson's idea led to substantial improvements in exposition and in technique.

Before I can explain why the notation now works so well for me, I need to say a few words about the manipulation of sums and summands. I realized long ago that "boundary conditions" on indices of summation are often a handicap and a waste of time. Instead of writing

$$(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k, \tag{1.1}$$

it is much better to write

$$(1+z)^n = \sum_{k} \binom{n}{k} z^k; \tag{1.2}$$

the sum now extends over all integers k, but only finitely many terms are nonzero. The second formula (1.2) is instantly converted to other forms:

$$(1+z)^n = \sum_{k} \binom{n}{k} z^k = \sum_{k} \binom{n}{k+1} z^{k+1} = \sum_{k} \binom{n}{\lfloor n/2 \rfloor - k} z^{\lfloor n/2 \rfloor - k}; \tag{1.3}$$

by contrast, we must work harder when dealing with (1.1), because we have to think about the limits:

$$(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k = \sum_{k=-1}^{n-1} \binom{n}{k+1} z^{k+1} = \sum_{k=-\lceil n/2 \rceil}^{\lfloor n/2 \rfloor} \binom{n}{\lfloor n/2 \rfloor - k} z^{\lfloor n/2 \rfloor - k}. \tag{1.4}$$

Furthermore, (1.2) and (1.3) make sense also when n is not a positive integer.

Even when limits are necessary, it is best to keep them as simple as possible. For example, it's almost always a mistake to write

$$\sum_{k=2}^{n-1} k(k-1)(n-k) \quad \text{instead of} \quad \sum_{k=0}^{n} k(k-1)(n-k); \tag{1.5}$$

the additional zero terms are more helpful than harmful (and the former sum is problematical when $n = 0$, 1, or 2).

Finally it dawned on me that Iverson's convention allows us to write *any* sum as an infinite sum without limits: If $P(k)$ is any property of the integer k, we have

$$\sum_{P(k)} f(k) = \sum_{k} f(k)[P(k)]. \tag{1.6}$$

For example, the sums in (1.5) become

$$\sum_{k} k(k-1)(n-k)[0 \le k \le n] = \sum_{k} k(k-1)(n-k)[k \ge 0][k \le n]. \tag{1.7}$$

(At the time I made this observation, I had forgotten that Iverson originally defined his convention only for single relational operators enclosed in parentheses; I began to put *arbitrary* logical statements in square brackets, and to assume that this would produce the value 0 or 1.) In this particular case nothing much has been gained when passing from (1.5) to (1.7), although we might be able to make use of identities like

$$k[k \ge 0] = k[k \ge 1]. \tag{1.8}$$

But in general, the ability to manipulate "on the line" instead of "below the line" turns out to be a great advantage.

For example, in my first book [**25**] I had found it necessary to include the rule

$$\sum_{k \in A} f(k) + \sum_{k \in B} f(k) = \sum_{k \in A \cup B} f(k) + \sum_{k \in A \cap B} f(k) \tag{1.9}$$

as a separate axiom for Σ manipulation. But this axiom is unnecessary in [**15**], because it can be derived easily from other basic laws: The left-hand side is

$$\sum_{k\in A} f(k) + \sum_{k\in B} f(k) = \sum_k f(k)[k\in A] + \sum_k f(k)[k\in B]$$
$$= \sum_k f(k)([k\in A] + [k\in B])$$

and the right-hand side is the same, because we have

$$[k\in A] + [k\in B] = [k\in A\cup B] + [k\in A\cap B]. \tag{1.10}$$

The interchange of summation order in multiple sums also comes out simpler now. I used to have trouble understanding and/or explaining why

$$\sum_{j=1}^{n}\sum_{k=1}^{j} f(j,k) = \sum_{k=1}^{n}\sum_{j=k}^{n} f(j,k); \tag{1.11}$$

but now it's easy for me to see that the left-hand sum is

$$\sum_{j,k} f(j,k)[1\le j\le n][1\le k\le j] = \sum_{j,k} f(j,k)[1\le k\le j\le n]$$
$$= \sum_{j,k} f(j,k)[1\le k\le n][k\le j\le n],$$

and this is the right-hand sum.

Here's another example: We have

$$[k \text{ even}] = \sum_m [k=2m] \quad\text{and}\quad [k \text{ odd}] = \sum_m [k=2m+1]; \tag{1.12}$$

therefore,

$$\sum_k f(k) = \sum_k f(k)([k \text{ even}] + [k \text{ odd}])$$
$$= \sum_{k,m} f(k)[k=2m] + \sum_{k,m} f(k)[k=2m+1]$$
$$= \sum_m f(2m) + \sum_m f(2m+1). \tag{1.13}$$

The result in (1.13) is hardly surprising; but I like to have mechanical operations like this available so that I can do manipulations reliably, without thinking. Then I'm less apt to make mistakes.

Let lg stand for logarithms to base 2. Then we have

$$\sum_{k\ge1}\binom{n}{\lfloor \lg k\rfloor} = \sum_{k\ge1}\sum_m \binom{n}{m}[m=\lfloor \lg k\rfloor]$$
$$= \sum_{k,m}\binom{n}{m}[m\le \lg k < m+1][k\ge1]$$
$$= \sum_{m,k}\binom{n}{m}[2^m\le k<2^{m+1}][k\ge1]$$
$$= \sum_m \binom{n}{m}(2^{m+1}-2^m)[m\ge0]$$
$$= \sum_m \binom{n}{m}2^m = 3^n. \tag{1.14}$$

If we are doing infinite products we can use Iversonian brackets as exponents:

$$\prod_{P(k)} f(k) = \prod_k f(k)^{[P(k)]}. \qquad (1.15)$$

For example, the largest squarefree divisor of n is

$$\prod_p p^{[p \text{ prime}][p \text{ divides } n]}.$$

Everybody is familiar with one special case of an Iverson-like convention, the "Kronecker delta" symbol

$$\delta_{ik} = \begin{cases} 1, & i = k; \\ 0, & i \neq k. \end{cases} \qquad (1.16)$$

Leopold Kronecker introduced this notation in his work on bilinear forms [**30**, page 276] and in his lectures on determinants (see [**31**, page 316]); it soon became widespread. Many of his followers wrote δ_j^k, which is a bit more ambiguous because it conflicts with ordinary exponentiation. I now prefer to write $[j = k]$ instead of δ_{jk}, because Iverson's convention is much more general. Although '$[j = k]$' involves five written characters instead of the three in 'δ_{jk}', we lose nothing in common cases when '$[j = k + 1]$' takes the place of '$\delta_{j(k+1)}$'.

Another familiar example of a 0-1 function, this time from continuous mathematics, is Oliver Heaviside's unit step function $[x \geq 0]$. (See [**44**] and [**37**] for expositions of Heaviside's methods.) It is clear that Iverson's convention will be as useful with integration as it is with summation, perhaps even more so. I have not yet explored this in detail, because [**15**] deals mostly with sums.

It's interesting to look back into the history of mathematics and see how there was a craving for such notations before they existed. For example, an Italian count named Guglielmo Libri published several papers in the 1830s concerning properties of the function 0^{0^x}. He noted [**32**] that 0^x is either 0 (if $x > 0$) or 1 (if $x = 0$) or ∞ (if $x < 0$), hence

$$0^{0^x} = [x > 0]. \qquad (1.17)$$

But of course he didn't have Iverson's convention to work with; he was pleased to discover a way to denote the discontinuous function $[x > 0]$ without leaving the realm of operations acceptable in his day. He believed that "la fonction $0^{0^{x-n}}$ est d'un grand usage dans l'analyse mathématique." And he noted in [**33**] that his formulas "ne renferment aucune notation nouvelle... Les formules qu'on obtient de cette manière sont très simples, et rentrent dans l'algèbre ordinaire."

Libri wrote, for example,

$$(1 - 0^{0^{-x}})(1 - 0^{0^{x-a}})$$

for the function $[0 \leq x \leq a]$, and he gave the integral formula

$$\frac{2}{\pi}\int_0^\infty \frac{dq \cos qx}{1+q^2} = e^x \cdot 0^{0^{-x}} + e^{-x}(1 - 0^{0^{-x}}) = \frac{e^x}{0^{-x}+1} + \frac{e^{-x}}{0^x+1}.$$

(Of course, we would now write the value of that integral as $e^{-|x|}$, but a simple notation for absolute value wasn't introduced until many years later. I believe that the first appearance of '$|z|$' for absolute value in Crelle's journal—the journal containing Libri's papers [**32**] and [**33**]—occurred on page 227 of [**56**] in 1881. Karl Weierstrass was the inventor of this notation, which was applied at first only to complex numbers; Weierstrass seems to have published it first in 1876 [**55**].)

Libri applied his 0^{0^x} function to number theory by exhibiting a complicated way to describe the fact that x is a divisor of m. In essence, he gave the following recursive formulation: Let $P_0(x) = 1$ and for $k > 0$ let

$$P_k(x) = 0^{0^{x-k}}P_0(x) - 0^{0^{x-k+1}}P_1(x) - \cdots - 0^{0^{x-1}}P_{k-1}(x).$$

Then the quantity

$$\frac{1 - m \cdot 0^{0^{x-m}}P_0(x) - (m-1)0^{0^{x-m+1}}P_1(x) - \cdots - 2 \cdot 0^{0^{x-2}}P_{m-2}(x) - 0^{0^{x-1}}P_{m-1}(x)}{x}$$

turns out to equal 1 if x divides m, otherwise it is 0. (One way to prove this, Iverson-wise, is to replace $0^{0^{x-k}}$ in Libri's formulas by $[x > k]$, and to show first by induction that $P_k(x) = [x \text{ divides } k] - [x \text{ divides } k-1]$ for all $k > 0$. Then if $a_k(x) = k[x > k]$, we have

$$\begin{aligned}\sum_{k=0}^{m-1} a_{m-k}(x)P_k(x) &= \sum_{k=0}^{m-1} a_{m-k}(x)([x \text{ divides } k] - [x \text{ divides } k-1])\\ &= \sum_{k=0}^{m-1} [x \text{ divides } k](a_{m-k}(x) - a_{m-k-1}(x)).\end{aligned}$$

If the positive integer x is not a divisor of m, the terms of this new sum are zero except when $m - k = m \bmod x$, when we have $a_{m-k}(x) - a_{m-k-1}(x) = 1$. On the other hand if x is a divisor of m, the only nonvanishing term occurs for $m - k = x$, when we have $a_{m-k}(x) - a_{m-k-1}(x) = 0 - (x - 1)$. Hence the sum is $1 - x\ [x \text{ divides } m]$. Libri obtained his complicated formula by a less direct method, applying Newton's identities to compute the sum of the mth powers of the roots of the equation $t^{x-1} + t^{x-2} + \cdots + 1 = 0$.)

Evidently Libri's main purpose was to show that unlikely functions can be expressed in algebraic terms, somewhat as we might wish to show that some complicated functions can be computed by a Turing Machine. "Give me the function 0^{0^x}, and I'll give you an expression for $[x \text{ divides } m]$." But our goal with Iverson's notation is, by contrast, to find a simple and natural way to express quantities that help us solve problems. If we need a function that is 1 if and only if x divides m, we can now write $[x \text{ divides } m]$.

Some of Libri's papers are still well remembered, but **[32]** and **[33]** are not. I found no mention of them in *Science Citation Index*, after searching through all years of that index available in our library (1955 to date). However, the paper **[33]** did produce several ripples in mathematical waters when it originally appeared, because it stirred up a controversy about whether 0^0 is defined. Most mathematicians agreed that $0^0 = 1$, but Cauchy [**5**, page 70] had listed 0^0 together with other expressions like $0/0$ and $\infty - \infty$ in a table of undefined forms. Libri's justification for the equation $0^0 = 1$ was far from convincing, and a commentator who signed his name simply "S" rose to the attack **[45]**. August Möbius **[36]** defended Libri, by presenting his former professor's reason for believing that $0^0 = 1$ (basically a proof that $\lim_{x \to 0+} x^x = 1$). Möbius also went further and presented a supposed proof that $\lim_{x \to 0+} f(x)^{g(x)} = 1$ whenever $\lim_{x \to 0+} f(x) = \lim_{x \to 0+} g(x) = 0$. Of course "S" then asked **[3]** whether Möbius knew about functions such as $f(x) = e^{-1/x}$ and $g(x) = x$. (And paper **[36]** was quietly omitted from the historical record when the collected works of Möbius were ultimately published.) The debate stopped there, apparently with the conclusion that 0^0 should be undefined.

But no, no, ten thousand times no! Anybody who wants the binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \tag{1.18}$$

to hold for at least one nonnegative integer n *must* believe that $0^0 = 1$, for we can plug in $x = 0$ and $y = 1$ to get 1 on the left and 0^0 on the right.

The number of mappings from the empty set to the empty set is 0^0. It *has* to be 1.

On the other hand, Cauchy had good reason to consider 0^0 as an undefined *limiting form*, in the sense that the limiting value of $f(x)^{g(x)}$ is not known *a priori* when $f(x)$ and $g(x)$ approach 0 independently. In this much stronger sense, the value of 0^0 is less defined than, say, the value of $0 + 0$. Both Cauchy and Libri were right, but Libri and his defenders did not understand why truth was on their side.

Well, it's instructive to study mathematical history and to observe how tastes change as progress is made. But let's come closer to the present, to see how Iverson's convention might be useful nowadays. Today's mathematical literature is, in fact, filled with instances where analogs of Iversonian brackets are being used—but the concept must be expressed in a roundabout way, because his convention is not yet established. Here are two examples that I happened to notice just before writing this paper:

(1) Hardy and Wright, in the course of proving the Staudt-Clausen theorem about the denominators of Bernoulli numbers [**20**, §7.9], consider the sum

$$\sum_{p-1 \text{ divides } k} \frac{1}{p}$$

where p runs through primes. They define $\varepsilon_k(p)$ to be 1 if $p - 1$ divides k, otherwise $\varepsilon_k(p) = 0$; then the sum becomes

$$\sum_p \frac{\varepsilon_k(p)}{p}.$$

They proceed to show that $\sum_{m=1}^{p-1} m^k \equiv -\varepsilon_k(p) \pmod p$ whenever p is prime, and the theorem follows with a bit more manipulation.

(2) Mark Kac, introducing the relation of ergodic theory to continued fractions [**24**, §5.4], says: "Let now $P_0 \in \Omega$ and $g(P)$ the characteristic function of the measurable set A; i.e.,

$$g(P) = \begin{cases} 1, & p \in A, \\ 0, & p \overline{\in} A. \end{cases}$$

It is now clear that $t(\tau, P_0, A)$ is given by the formula

$$t(\tau, P_0, A) = \int_0^\tau g(T_t(P_0))\,dt,$$

and... ".

I hope it is now clear why my students and I would find it quite natural to say directly that

$$t(\tau, P_0, A) = \int_0^\tau [T_t(P_0) \in A]\,dt.$$

Also, in the context of Hardy and Wright, we would evaluate $(\sum_{m=1}^{p-1} m^k) \bmod p$ and discover that it is $(p-1)[p-1 \text{ divides } k]$.

If you are a typical hard-working, conscientious mathematician, interested in clear exposition and sound reasoning—and I like to include myself as a member of that set—then your experiences with Iverson's convention may well go through several stages, just as mine did. First, I learned about the idea, and it certainly seemed straightforward enough. Second, I decided to use it informally while solving problems. At this stage it seemed too easy to write just '$[k \geq 0]$'; my natural tendency was to write something like '$\delta(k \geq 0)$', giving an implicit bow to Kronecker, or '$\tau(k \geq 0)$' where τ stands for truth. Adriano Garsia, similarly, decided to write '$\chi(k \geq 0)$', knowing that χ often denotes a characteristic function; he has used χ notation effectively in dozens of papers, beginning with **[10]**, and quite a few other mathematicians have begun to follow his lead. (Garsia was one of my professors in graduate school, and I recently showed him the first draft of this note. He replied, "My definition from the very start was

$$\chi(\mathscr{A}) = \begin{cases} 1 & \text{if } \mathscr{A} \text{ is true} \\ 0 & \text{if } \mathscr{A} \text{ is false} \end{cases}$$

where $\mathscr{A}$ is any statement whatever. But just like you, I got it by generalizing from Iverson's APL. ... I don't have to tell you the magic that the use of the χ notation can do.")

If you go through the stages I did, however, you'll soon tire of writing δ, τ, or χ, when you recognize that the notation is quite unambiguous without an additional symbol. Then you will have arrived at the philosophical position adopted by Iverson when he wrote **[21]**. And I had also reached that stage when I completed the first edition of **[15]**; I adopted Iverson's original suggestion to enclose logical statements in ordinary parentheses, not square brackets.

Unfortunately, not all was well with that first edition. Students found cases where I had parenthesized a complicated logical statement for clarity, for example when I wrote something of the form 'α and (β or γ)'; they pointed out that the simple act of putting parentheses around 'β or γ' automatically caused it to be evaluated as either 0 or 1, according to a strict interpretation of Iverson's rule as I had extended it.

Worse yet, as I began to read the first edition of **[15]** with fresh eyes, I found that the formulas involved too many parentheses. It was hard for me to perceive the structure of complex expressions that involved Iversonian statements; the statements had been clear to me when I wrote them down, but they looked confusing when I came back to them several months later. A computer could readily parse each expression, but good notation must be engineered for human beings.

Therefore in the second and subsequent printings of **[15]**, my co-authors and I now use square brackets instead of parentheses, whenever we wish to transform logical statements into the values 0 or 1. This resolves both problems, and we now believe that the notation has proved itself well enough to be thrust upon the world. Square brackets are used also for other purposes, but not in a conflicting way, and not so often that the multiple uses become confusing.

One small glitch remains: We want to be able to write things like

$$\sum_p [p \text{ prime}][p \leq x]/p \tag{1.19}$$

to denote the sum of all reciprocals of primes $\leq x$. But this summand unfortunately reduces to $0/0$ when $p = 0$. In general, when an Iverson-bracketed statement is false, we want it to evaluate into a "very strong 0," namely a zero so strong that it annihilates anything it is multiplied by—even if that other factor is undefined.

Similarly, in formulas like (1.2) it is convenient to regard $\binom{n}{k}$ as strongly zero when k is negative, so that, for example, $\binom{n}{-10}z^{-10} = 0$ when $z = 0$.

The strong-zero convention is enough to handle 99% of the difficult situations, but we may also be using $1 - [P(k)]$ to stand for the quantity [not $P(k)$]; then we want $[P(k)]$ to give a "strong 1." And paradoxes can still arise, whenever irresistible forces meet immovable objects. (What happens if a strong zero appears in the denominator? And so on.)

In spite of these potential problems in extreme cases, Iverson's convention works beautifully in the vast majority of applications. It is, in fact, far less dangerous than most of the other notations of mathematics, whose dark corners we have learned to avoid long ago. The safe use of Iverson's simple and convenient idea is quite easy to learn.

2. STIRLING NUMBERS. The second plea I wish to make for perspicuous notation concerns the famous coefficients introduced by James Stirling at the beginning of his *Methodus Differentialis* in 1730 [**52**]. The lack of a widely accepted way to refer to these numbers has become almost scandalous. For example, Goldberg, Newman, and Haynsworth begin their chapter on Combinatorial Analysis in the NBS Handbook [**1**] by remarking that notations for Stirling numbers "have never been standardized... We feel that a capital S is natural for Stirling numbers of the first kind; it is infrequently used for other notation in this context. But once it is used we have difficulty finding a suitable symbol for Stirling numbers of the second kind. These numbers are sufficiently important to warrant a special and easily recognizable symbol, and yet that symbol must be easy to write. We have settled on a script capital $\mathscr{S}$ without any certainty that we have settled this question permanently."

The present predicament came about because Stirling numbers are indeed important enough to have arisen in a wide variety of applications, yet they are not quite important enough to have deserved a prominent place in the most influential textbooks of mathematics. Therefore they have been rediscovered many times, and each author has chosen a notation that was optimized for one particular application.

The great utility of Stirling numbers has become clearer and clearer with time, and mathematicians have now reached a stage where we can intelligently choose a notation that will serve us well in the whole range of applications.

I came into the picture rather late, having never heard of Stirling numbers until after receiving my Ph.D. in mathematics. But I soon encountered them as I was beginning to analyze the performance of algorithms and to write the manuscript for my books *The Art of Computer Programming*. I quickly realized the truth of Imanuel Marx's comment that "these numbers have similarities with the binomial coefficients $\binom{n}{k}$; indeed, formulas similar to those known for the binomial coefficients are easily established" [**35**]. In order to emphasize those similarities and to facilitate pattern recognition when manipulating formulas, Marx recommended using bracket symbols $\left[{n \atop k}\right]$ for Stirling numbers of the first kind and brace symbols $\left\{{n \atop k}\right\}$ for Stirling numbers of the second kind. A similar proposal was being made at about the same time in Italy by Antonio Salmeri [**46**].

I was strongly motivated by Charles Jordan's book, *Calculus of Finite Differences* [**23**], which introduced me to the important analogies between sums of factorial powers and integrals of ordinary powers. But I kept getting mixed up when I tried to use Stirling numbers as he defined them, because half of his "first kind" numbers were negative and the other half were positive. I had similar problems with Marx's suggestions in [**35**]; he made all Stirling numbers of the first kind positive, but then he attached a minus sign to half the numbers of the *second* kind. I decided that I'd never be able to keep my head above water unless I worked with Stirling numbers that were entirely signless.

And I soon learned that the signless Stirling numbers have important combinatorial significance. So I decided to try a definition that combined the best qualities of the other notations I'd seen; I defined the quantities $\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$ as follows:

$$\left[{n \atop k}\right] = \text{the number of permutations of } n \text{ objects having } k \text{ cycles};$$

$$\left\{{n \atop k}\right\} = \text{the number of partitions of } n \text{ objects into } k \text{ nonempty subsets}.$$

For example, $\left[{4 \atop 2}\right] = 11$, because there are eleven different ways to arrange four elements into two cycles:

[1, 2, 3] [4]	[1, 2, 4] [3]	[1, 3, 4] [2]	[2, 3, 4] [1]
[1, 3, 2] [4]	[1, 4, 2] [3]	[1, 4, 3] [2]	[2, 4, 3] [1]
[1, 2] [3, 4]	[1, 3] [2, 4]	[1, 4] [2, 3].	

And $\left\{{4 \atop 2}\right\} = 7$, because the partitions of {1, 2, 3, 4} into two subsets are

{1, 2, 3}{4}	{1, 2, 4}{3}	{1, 3, 4}{2}	{2, 3, 4}{1}
{1, 2}{3, 4}	{1, 3}{2, 4}	{1, 4}{2, 3}.	

Notice that this notation is mnemonic: The meaning of $\left\{{n \atop k}\right\}$ is easily remembered, because braces { } are commonly used to denote sets and subsets. We could also adopt the convention of writing cycles in brackets, as in my examples above, where [1, 2, 3] = [2, 3, 1] = [3, 1, 2] is a typical three-cycle; that would make the notation $\left[{n \atop k}\right]$ equally mnemonic. But I don't insist on this.

I have never decided how to pronounce '$\left[{n \atop k}\right]$' and '$\left\{{n \atop k}\right\}$' when I'm reading formulas aloud in class. Many people have begun to verbalize '$\binom{n}{k}$' as "n choose k"; hence I've been saying "n cycle k" for $\left[{n \atop k}\right]$ and "n subset k" for $\left\{{n \atop k}\right\}$. But I have also caught myself calling them "n bracket k" and "n brace k."

One of the advantages of these notational conventions is that binomial coefficients and Stirling numbers can be defined by very simple recurrence relations having a nice pattern:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}; \qquad (2.1)$$

$$\left[{n+1 \atop k}\right] = n\left[{n \atop k}\right] + \left[{n \atop k-1}\right]; \qquad (2.2)$$

$$\left\{{n+1 \atop k}\right\} = k\left\{{n \atop k}\right\} + \left\{{n \atop k-1}\right\}. \qquad (2.3)$$

Moreover—and this is extremely important—these identities hold for all integers n and k, whether positive, negative, or zero. Therefore we can apply them in the midst of any formula (for example, to "absorb" an n or a k that appears in the context $n\left[{n \atop k}\right]$ or $k\left\{{n \atop k}\right\}$), without worrying about exceptional circumstances of any kind.

I introduced these notations in the first edition of my first book [**25**], and by now my students and I have accumulated some 25 years of experience with them; the conventions have served us well. However, such brackets and braces have still not become widely enough adopted that they could be considered "standard." For example, Stanley's magnificent book on *Enumerative Combinatorics* [**51**] uses $c(n,k)$ for $\left[{n \atop k}\right]$ and $S(n,k)$ for $\left\{{n \atop k}\right\}$. His notation conveys combinatorial significance, but it fails to suggest the analogies to binomial coefficients that prove helpful in manipulations. Such analogies were evidently not important enough in his mind to warrant an extravagant two-line notation—although he does use $\left(\left({n \atop k}\right)\right)$ to denote $\binom{n+k-1}{k} = (-1)^k\binom{-n}{k}$, the number of combinations with repetitions permitted. (In a sense, Stanley's $\left(\left({n \atop k}\right)\right)$ is a signless version of the numbers $\binom{-n}{k}$.)

When I wrote *Concrete Mathematics* in 1988, I explored Stirling numbers more carefully than I had ever done before, and I learned two things that really clinch the argument for $\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$ as the best possible Stirling number notations. Ron Graham sent me a preview copy of a memorandum by B. F. Logan [**34**], which presented a number of interesting connections between Stirling numbers and other mathematical quantities. One of the first things that caught my attention was Logan's Table 1, a two-dimensional array that contained the numbers $\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$ simultaneously—implying that there really is only one "kind" of Stirling number. Indeed, when I translated Logan's results into my own favorite notation, I was astonished to find that his arrangement of numbers was equivalent to a beautiful and easily remembered law of duality,

$$\left\{{n \atop k}\right\} = \left[{-k \atop -n}\right]. \tag{2.4}$$

Once I had this clue, it was easy to check that the recurrence relations (2.2) and (2.3) are equivalent to each other. And the boundary conditions

$$\left[{0 \atop k}\right] = \left\{{0 \atop k}\right\} = [k=0] \quad \text{and} \quad \left[{n \atop 0}\right] = \left\{{n \atop 0}\right\} = [n=0] \tag{2.5}$$

yield unique solutions to (2.2) and (2.3) for all integers k and n, when we run the recurrences forward and backward; the "negative" region for Stirling numbers of one kind turns out to contain precisely the numbers of the other kind. For example, the following subset of Logan's table gives the values of $\left[{n \atop k}\right]$ when $|n|$ and $|k|$ are at most 4:

	$k=-4$	$k=-3$	$k=-2$	$k=-1$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$n=-4$	1	0	0	0	0	0	0	0	0
$n=-3$	6	1	0	0	0	0	0	0	0
$n=-2$	7	3	1	0	0	0	0	0	0
$n=-1$	1	1	1	1	0	0	0	0	0
$n=0$	0	0	0	0	1	0	0	0	0
$n=1$	0	0	0	0	0	1	0	0	0
$n=2$	0	0	0	0	0	1	1	0	0
$n=3$	0	0	0	0	0	2	3	1	0
$n=4$	0	0	0	0	0	6	11	6	1

The reflection of this matrix about a 45° diagonal gives the value of $\left\{{n \atop k}\right\} = \left[{-k \atop -n}\right]$.

Naturally I wondered how I could have been working with Stirling numbers for so many years without having been aware of such a basic fact. Surely it must have been known before? After several hours of searching in the library, I learned that identity (2.4) had indeed been known, but largely forgotten by succeeding generations of mathematicians, primarily because previous notations for Stirling numbers made it impossible to state the identity in such a memorable form. These investigations also turned up several things about the history of Stirling numbers that I had not previously realized.

During the nineteenth century, Stirling's connection with these numbers had been almost entirely forgotten. The numbers themselves were studied, in the role of "sums of products of combinations of the numbers $\{1, 2, \ldots, n\}$ taken k at a time." Let $C_k(n)$ and $\Gamma_k(n)$ denote those sums, when the combinations are respectively without or with repetitions; thus, for example,

$$\begin{aligned} C_4(4) &= 1 \cdot 2 \cdot 3 + 1 \cdot 2 \cdot 4 + 1 \cdot 3 \cdot 4 \cdot + 2 \cdot 3 \cdot 4 = 50; \\ \Gamma_3(3) &= 1 \cdot 1 \cdot 1 + 1 \cdot 1 \cdot 2 + 1 \cdot 1 \cdot 3 + 1 \cdot 2 \cdot 2 + 1 \cdot 2 \cdot 3 \\ &\qquad + 1 \cdot 3 \cdot 3 + 2 \cdot 2 \cdot 2 + 2 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 3 + 3 \cdot 3 \cdot 3 = 90. \end{aligned}$$

It turns out that

$$C_k(n) = \left[{n+1 \atop n+1-k} \right] \quad \text{and} \quad \Gamma_k(n) = \left\{ {n+k \atop n} \right\}. \tag{2.6}$$

Christian Kramp [**28**] proved near the end of the eighteenth century that

$$C_k(n) = \sum \binom{n+1}{k+l} \frac{(k+l)!}{j_1!\, 2^{j_1}\, j_2!\, 3^{j_2} j_3!\, 4^{j_3} \cdots}, \tag{2.7}$$

$$\Gamma_k(n) = \sum \binom{n+k}{k+l} \frac{(k+l)!}{j_1!\, 2!^{j_1}\, j_2!\, 3!^{j_2} j_3!\, 4!^{j_3} \cdots}, \tag{2.8}$$

where the sums are over all sequences of nonnegative integers $\langle j_1, j_2, j_3, \ldots \rangle$ such that we have $j_1 + 2j_2 + 3j_3 + \cdots = k$ (i.e., over all partitions of k), and where $l = j_1 + j_2 + j_3 + \cdots$. For example,

$$C_2(n) = \binom{n+1}{4}\frac{1}{8} + \binom{n+1}{3}\frac{1}{3}; \quad \Gamma_2(n) = \binom{n+2}{4}\frac{1}{8} + \binom{n+2}{3}\frac{1}{6}.$$

Notice that $C_k(n)$ and $\Gamma_k(n)$ are polynomials in n, of degree $2k$. The duality law (2.4) and the notational transformations of (2.6) are equivalent to the amazing polynomial identity

$$C_k(n-1) = \Gamma_k(-n); \tag{2.9}$$

but hardly anybody was aware of this surprising fact, otherwise we would almost certainly find it mentioned explicitly in the comprehensive surveys compiled in the 1890s [**19**, **38**].

On the other hand, a rereading of Stirling's original treatment [**52**] makes it clear that Stirling himself would not have found the duality law (2.4) at all surprising. From the very beginning, he thought of the numbers as two triangles hooked together in tandem. Indeed, his entire motivation for studying them was the general identity

$$z^n = \sum_k \left\{ {n \atop k} \right\} z^{\underline{k}}, \tag{2.10}$$

which expresses ordinary powers in terms of falling factorial powers. When n is positive, the nonzero terms in this sum occur for positive values of $k \le n$; but when n is negative, the nonzero terms occur for negative $k \le n$. Stirling presented his tables by displaying $\left\{ {n \atop k} \right\}$ with k as the row index and $\left[{n \atop k} \right]$ with k as the column index; thus, he visualized a tandem arrangement exactly as in the matrix of numbers above, with each column containing a sequence of coefficients for (2.10).

I need to digress a bit about factorial powers. If n is a positive integer and z is a complex number, I like to write

$$z^{\underline{n}} = z(z-1)\ldots(z-n+1), \qquad (2.11)$$

which I call "z to the n falling," and

$$z^{\overline{n}} = z(z+1)\ldots(z+n-1), \qquad (2.12)$$

which is "z to the n rising." More generally, if α is any complex number, factorial powers are defined by

$$z^{\underline{\alpha}} = z!/(z-\alpha)! \quad \text{and} \quad z^{\overline{\alpha}} = \Gamma(z+\alpha)/\Gamma(z), \qquad (2.13)$$

unless these formulas reduce to ∞/∞ (when limiting values are used). My use of underlined and overlined exponents is still controversial, but I cannot resist mentioning a curious fact: Many people (e.g., specialists in hypergeometric series) have become accustomed to the notation $(z)_n$ for rising factorial powers, while many other people (e.g., statisticians) use the same notation for *falling* powers. The curious fact is that this notation is called "Pochhammer's symbol," but Pochhammer himself [**43**] used $(z)_n$ to stand for the binomial coefficient $\binom{z}{n}$. I prefer the underline/overline notation because it is unambiguous and mnemonic, especially when I'm doing work that involves factorial powers of both kinds. (Moreover, I know that $z^{\underline{n}}$ and $z^{\overline{n}}$ are easy to typeset, using macros available in the file `gkpmac.tex` in the standard UNIX distribution of TEX.)

In the special case $n = 3$, Stirling's formula (2.10) gives

$$z^3 = \left\{ {3 \atop 3} \right\} z^{\underline{3}} + \left\{ {3 \atop 2} \right\} z^{\underline{2}} + \left\{ {3 \atop 1} \right\} z^{\underline{1}} = z(z-1)(z-2) + 3z(z-1) + z.$$

And in the special case $n = -1$, it reduces to the infinite sum

$$\begin{aligned} \frac{1}{z} &= \sum_k \left\{ {-1 \atop k} \right\} z^{\underline{k}} \\ &= \sum_k \left[{k \atop 1} \right] z^{\underline{-k}} \\ &= \frac{0!}{z+1} + \frac{1!}{(z+1)(z+2)} + \frac{2!}{(z+1)(z+2)(z+3)} + \cdots, \end{aligned} \qquad (2.14)$$

because

$$\left[{n \atop 1} \right] = (n-1)!\,[n > 0]. \qquad (2.15)$$

(Stirling did not discuss convergence; he was, after all, writing in 1730. We have the partial sum

$$\frac{1}{z} = \sum_{k=1}^{n} \frac{(k-1)!}{(z+1)\ldots(z+k)} + \frac{n!}{z(z+1)\ldots(z+n)};$$

this is a special case of the general identity

$$\frac{1}{z} = \sum_{k=1}^{n} \frac{z_1 \ldots z_{k-1}}{(z+z_1)\ldots(z+z_k)} + \frac{z_1 \ldots z_n}{z(z+z_1)\ldots(z+z_n)} \tag{2.16}$$

discovered by François Nicole [**39**] a few years before Stirling's treatise appeared. Therefore the infinite series (2.14) converges if and only if $\mathrm{Re}(z) > 0$. By induction on n, the same condition is necessary and sufficient for (2.10) when n is any negative integer. See [**41**, §30] for further discussion of (2.10).)

We noted above that the numbers $\left[{m \atop k}\right]$ can be regarded as sums of products of combinations. The first identity in (2.6) is equivalent to the formula

$$z^{\overline{n}} = \sum_k \left[{n \atop k}\right] z^k, \tag{2.17}$$

when n is a nonnegative integer, if we expand the product $z^{\overline{n}}$ and sum the coefficients of each power of z. Similarly, we have

$$z^{\underline{n}} = \sum_k \left[{n \atop k}\right] (-1)^{n-k} z^k. \tag{2.18}$$

These equations are valid also when n is a negative integer; in that case both infinite series converge for $|z| > |n|$. Notice that (2.10) and (2.18) tell us how to convert back and forth between ordinary powers and factorial powers.

Let's turn now to the nineteenth century. Kramp [**29**] decided to explore a slightly generalized type of factorial power, for which he used the notations

$$a^{n|r} = a(a+r)\ldots(a+(n-1)r) \tag{2.19}$$

$$a^{-n|r} = 1/(a-r)(a-2r)\ldots(a-nr) \tag{2.20}$$

when n is a positive integer. Then he considered the expansion

$$a^{n|r} = a^n + n\mathfrak{f}1.a^{n-1}r + n\mathfrak{f}2.a^{n-2}r^2 + \cdots, \tag{2.21}$$

where the coefficients $n\mathfrak{f}m$ are independent of a and r [**29**, §§539–540]; thus $n\mathfrak{f}m$ was his notation for $\left[{n \atop n-m}\right]$. He obtained [**29**, §557] a series of formulas equivalent to

$$m\left[{n \atop n-m}\right] = \sum_{k=0}^{m-1} \binom{n-k}{m+1-k} \left[{n \atop n-k}\right], \tag{2.22}$$

thereby giving a new proof that $\left[{n \atop n-m}\right]$ is a polynomial in n of degree $2m$. This proof, independent of his earlier formulas (2.7) and (2.8), works for both positive and negative values of n.

Kramp implicitly understood the duality principle (2.4), in the sense that he regarded the coefficients $\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$ as the positive and negative portions of a doubly infinite array of numbers. In fact, he assumed that equation (2.21) would hold for arbitrary real values of n. He differentiated $a^{x|r}$ with respect to x and gave formal derivations of several interesting series. However, his expansion (2.21) is equivalent to

$$z^{\overline{n}} = \sum_k \left[{n \atop n-k}\right] z^{n-k} \tag{2.23}$$

(a slight variation of (2.17)), and this series is not always convergent for noninteger

n. We can show, for example, that

$$\left\|\left[\begin{matrix}1/2\\1/2-k\end{matrix}\right]\right\| > k!/7^k \quad \text{for infinitely many } k; \tag{2.24}$$

hence (2.23) diverges for all z when $n = 1/2$. Kramp lived before the days when convergence of infinite series was understood. (See [**29**, §574], where he says that the divergent series $\Sigma_{k>0}B_k y^k/k$ is "très convergente pour peu que y soit une petite fraction"!)

Several other nineteenth-century authors developed the theory of factorial powers, notably Andreas von Ettingshausen [**6**], Ludwig Schläfli [**41**, **48**], and Oskar Schlömilch [**49**], who used the respective notations

$$\overset{n}{\mathrm{F}}_m, \ \overset{n}{\mathrm{A}}_m, \quad \text{and} \quad \overset{n}{\mathrm{C}}_m$$

for the coefficients $\left[\begin{smallmatrix}n\\n-m\end{smallmatrix}\right]$. All of these authors considered both positive and negative integers n. Thus, for example, Ettingshausen's notation for a Stirling number such as $\left\{\begin{smallmatrix}n+m\\n\end{smallmatrix}\right\} = \left[\begin{smallmatrix}-n\\-n-m\end{smallmatrix}\right]$ was

$$\overset{-n}{\mathrm{F}}_m$$

(see [**6**, §151]).

Incidentally, these works of Kramp and Ettingshausen proved to be important in the history of mathematical notations. Kramp's book introduced the notation $n!$ for factorials [**29**, pages V and 219], and Ettingshausen's book introduced the notation $\binom{n}{k}$ for binomial coefficients [**6**, page 30]. Ettingshausen wrote his book shortly after Fourier [**8**] had invented Σ-notation for sums; Ettingshausen tried a German variation, writing $\mathfrak{S}^k_{a,b}$ for what has evolved into $\Sigma^b_{k=a}$. He also wrote $(a,r)^n$ for Kramp's $a^{n|r}$; thus, for example, Ettingshausen [**6**, §153 and §156] gave the equations

$$(a,d)^n = \overset{w}{\underset{0}{\mathfrak{S}}} \overset{n}{\mathrm{F}}_w a^{n-w} d^w \quad \text{and} \quad a^n = \overset{r}{\underset{0}{\mathfrak{S}}} (-1)^r \overset{-n+r}{\mathrm{F}_r} (a,d)^{n-r} d^r$$

as equivalents of Kramp's (2.21) and Stirling's (2.10). He presented Kramp's (2.22) in the form

$$v\overset{n}{\mathrm{F}}_v = \overset{w}{\underset{0,v-1}{\mathfrak{S}}} \binom{n-w}{v+1-w} \overset{n}{\mathrm{F}}_w,$$

and remarked [**6**, §154] that this holds for both negative and positive n. Ettingshausen had related the F coefficients to sums of products of combinations with and without repetition; thus he implicitly confirmed (2.9).

The first person to attach Stirling's name to the numbers we now call Stirling numbers was Niels Nielsen in 1904 [**40**]; he said that this new nomenclature had been suggested to him by T. N. Thiele. (The numbers may have been studied before Stirling's time; for example, I once found the values of $\left[\begin{smallmatrix}n\\k\end{smallmatrix}\right]$ for $1 \le n \le 7$ in some unpublished manuscripts of Thomas Harriot, dating from about 1600, in the British Museum [**26**, page 241]. But Stirling almost surely deserves the credit for being first to deduce nontrivial facts about $\left[\begin{smallmatrix}n\\k\end{smallmatrix}\right]$ and $\left\{\begin{smallmatrix}n\\k\end{smallmatrix}\right\}$.)

Nielsen wrote C^k_n for $\left[\begin{smallmatrix}n\\n-k\end{smallmatrix}\right]$, which he called a "Stirling number of rank n"; and he wrote $\mathfrak{C}^k_n$ for $\left\{\begin{smallmatrix}n+k-1\\n-1\end{smallmatrix}\right\}$, which he called a "Stirling number of rank $-n$." (He should really have defined its rank to be $1-n$). In equation (41) of his paper, Nielsen obtained a rigorous proof of the duality law (2.4); but he had to state it in a peculiar way, because he had defined C_n^k and $\mathfrak{C}_n^k$ only for nonnegative n and k. Thus, he could not write $C_n^k = \mathfrak{C}_{1-n}^k$; he had to say instead that $f_k(n) = g_k(1-n)$, where $f_k(n)$ and $g_k(n)$ were the polynomials defined by C_n^k and $\mathfrak{C}_n^k$. Tweedie [**54**] expressed (2.4) with similar circumlocutions.

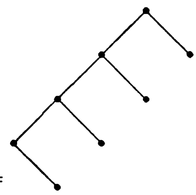
When Jordan took up Stirling numbers [**22**], he wrote S_n^k for $(-1)^{n-k}\left[{n \atop k}\right]$ and $\mathfrak{S}_n^k$ for $\left\{{n \atop k}\right\}$. He does not seem to have known the duality law (2.4), probably because he had learned about Stirling numbers from Nielsen's book [**41**], which omitted some of the details in Nielsen's paper [**40**]. And as far as I know, the duality law largely disappeared from mathematicians' collective consciousness during most of the twentieth century; it seems to have been mentioned explicitly only in a few scattered places: (1) Hansraj Gupta, "working in a small township away from what was then the only University in the Panjab" [**18**, page 5], rediscovered Stirling numbers and Stirling duality by himself, in the early 1930s. This became part of his Ph.D. dissertation [**17**], and he included it in a book on number theory prepared many years later [**18**, Chapter 5]. (2) H. W. Gould [**12**] was probably the first twentieth-century mathematician to observe that we can use the polynomials $\left[{n \atop n-k}\right]$ and $\left\{{n \atop n-k}\right\}$ to extend the domain of Stirling numbers to negative values of n. Gould's way of writing (2.4) was $S_1(-n-1,k) = S_2(n,k)$; and shortly thereafter [**13**], he mentioned the equivalent formula

$$S_{-k}^{-n} = (-1)^{n-k}\mathfrak{S}_n^k,$$

in Jordan's notation. (3) R. V. Parker [**42**], like Gupta, displayed both of Stirling's triangles in tandem, presenting them in a single table as Logan later did. (4) In 1976, Ira Gessel and Richard Stanley investigated some of the deeper structure underlying the Stirling polynomials $f_k(n) = \left\{{n+k \atop n}\right\}$ and $g_k(n) = \left[{n \atop n-k}\right]$. They noted in particular [**11**, equation (3)] that $f_k(-n) = g_k(n)$. This fact is equivalent to the duality law (2.4).

Stanley had discovered a beautiful theorem in his Ph.D. thesis a few years earlier [**50**, Proposition 13.2(i)], now called the reciprocity theorem for order polynomials: If P is any finite partially ordered set, let $\Omega(P,n)$ be the number of order-preserving mappings from P into the totally ordered set $\{1,2,\ldots,n\}$; and let $\overline{\Omega}(P,n)$ be the number of such mappings that are strictly order-preserving. Thus, if $x \prec y$ in P, the mappings f enumerated by $\Omega(P,n)$ must satisfy $f(x) \le f(y)$, and the mappings g enumerated by $\overline{\Omega}(P,n)$ must satisfy $g(x) < g(y)$. Stanley's theorem states that, in general, we have $f(-n) = (-1)^p g(n)$, where p is the number of elements of P. For example, if P consists of p isolated points with no order constraints whatever, we have $\Omega(P,n) = \overline{\Omega}(P,n) = n^p$. And if the points of P are themselves totally ordered, then $\Omega(P,n)$ is $\binom{n+p-1}{p}$, the number of combinations of n things p at a time with repetitions permitted, and $\overline{\Omega}(P,n)$ is $\binom{n}{p}$, the combinations without repetition. In both cases we have $\Omega(P,-n) = (-1)^p\overline{\Omega}(P,n)$.

I showed Stanley the first draft of this note and asked him whether the Stirling duality law (2.4) could be derived as a special case of his general reciprocity law. Sure enough, he replied that Gessel had noticed a simple way to do exactly that, shortly after the paper [**11**] was written. Let P_k be the partial order on $2k$ points typified by

$P_4 =$  ;

then

$$\Omega(P_k, n) = \sum_{1 \le x_1, \dots, x_k, y_1, \dots, y_k \le n} [x_1 \le \cdots \le x_k][x_1 \ge y_1] \dots [x_k \ge y_k]$$
$$= \sum_{1 \le x_1, \dots, x_k \le n} [x_1 \le \cdots \le x_k] x_1 \dots x_k,$$

and

$$\overline{\Omega}(P_k, n) = \sum_{1 \le x_1, \dots, x_k, y_1, \dots, y_k \le n} [x_1 < \cdots < x_k][x_1 > y_1] \dots [x_k > y_k]$$
$$= \sum_{2 \le x_1, \dots, x_k \le n} [x_1 < \cdots < x_k](x_1 - 1) \dots (x_k - 1)$$
$$= \sum_{1 \le x_1, \dots, x_k \le n-1} [x_1 < \cdots < x_k] x_1 \dots x_k.$$

Thus the sums are, respectively, $\Gamma_k(n)$ and $C_k(n-1)$; by (2.6) we have $\Omega(P_k, n) = \left\{ {n+k \atop n} \right\}$ and $\overline{\Omega}(P_k, n) = \left[{n \atop n-k} \right]$, hence (2.4) is indeed an instance of Stanley's theorem.

Now we are ready to discuss the second reason why I became convinced that $\left[{n \atop k} \right]$ is the right symbolism for these coefficients after I had translated Logan's memo [**34**] into that notation: We know that $\left[{n \atop n-k} \right]$ is a polynomial in n, when k is an integer; hence, as Kramp knew, we can sensibly define the quantity $\left[{\alpha \atop \alpha - k} \right]$ for arbitrary complex α and integer k, using that same polynomial. Then—and here comes the punch line—Logan noticed that the fundamental equations (2.17) and (2.18) generalize to *asymptotic formulas*, valid for arbitrary exponents α: If $z \to \infty$ and if m is any nonnegative integer, we have

$$z^{\overline{\alpha}} = \sum_{k=0}^{m} \left[{\alpha \atop \alpha - k} \right] z^{\alpha - k} + O(z^{\alpha - m - 1}); \qquad (2.25)$$

$$z^{\underline{\alpha}} = \sum_{k=0}^{m} \left[{\alpha \atop \alpha - k} \right] (-1)^k z^{\alpha - k} + O(z^{\alpha - m - 1}). \qquad (2.26)$$

(See [**15**, exercise 9.44]; equation (2.25) is a correct way to formulate Kramp's divergent series (2.23). These equations are special cases of a still more general result proved by Tricomi and Erdélyi [**53**, **9**].) The easily remembered expansions in (2.25) and (2.26) were quite a revelation to me. I had often spent time laboriously calculating approximations to ratios such as $z^{\overline{1/2}} = \Gamma(z + 1/2)/\Gamma(z)$, the hard way: I took logarithms, then used Stirling's approximation, and then took exponentials. But equations (2.25) and (2.26) produce the answer directly.

Moreover Stirling's original identity (2.10) can be generalized in a similar way: If α is any complex number, we have

$$z^{\alpha} = \sum_k \left\{ \alpha \atop \alpha - k \right\} z^{\underline{\alpha - k}}, \qquad \operatorname{Re}(z) > 0. \tag{2.27}$$

When I wrote the first draft of this note, I knew only that the series (2.27) was convergent, and that it was asymptotically correct as $z \to \infty$; so I conjectured that equality might hold. Soon afterward, B. F. Logan found the following proof (although he naturally stated it in his own notation): Suppose first that $\operatorname{Re}(\alpha) < 1$. Then we have the well known identity

$$z^{\alpha - 1} = \frac{1}{\Gamma(1-\alpha)} \int_0^{\infty} e^{-zt} t^{-\alpha}\, dt, \qquad \operatorname{Re}(z) > 0, \tag{2.28}$$

and we can substitute $e^{-t} = 1 - u$ to get

$$z^{\alpha - 1} = \frac{1}{\Gamma(1-\alpha)} \int_0^1 (1-u)^{z-1} u^{-\alpha} \left(\frac{1}{u} \ln \frac{1}{1-u} \right)^{-\alpha} du.$$

Now it turns out that the powers of $(1/u) \ln 1/(1-u)$ generate the Stirling numbers $\left\{ \alpha \atop \alpha - k \right\} = \left[k - \alpha \atop -\alpha \right]$, in the sense that

$$\left(\frac{1}{u} \ln \frac{1}{1-u} \right)^{-\alpha} = \sum_k \left\{ \alpha \atop \alpha - k \right\} \frac{u^k}{(k-\alpha)\ldots(1-\alpha)}, \tag{2.29}$$

a series that converges for $|u| < 1$ (see [**15**, equations (6.45), (6.53), (7.50)]). Therefore

$$\begin{aligned} z^{\alpha} &= \sum_k \left\{ \alpha \atop \alpha - k \right\} \frac{z}{\Gamma(k+1-\alpha)} \int_0^1 (1-u)^{z-1} u^{k-\alpha}\, du \\ &= \sum_k \left\{ \alpha \atop \alpha - k \right\} \frac{\Gamma(z+1)}{\Gamma(z+1+k-\alpha)} = \sum_k \left\{ \alpha \atop \alpha - k \right\} \frac{z!}{(z+k-\alpha)!}, \end{aligned}$$

and (2.27) is verified when $\operatorname{Re}(\alpha) < 1$. To complete the proof, we need only show that (2.27) holds for $\alpha + 1$ if it holds for α; but this is easy, because

$$\begin{aligned} z^{\alpha+1} &= \sum_k \left\{ \alpha \atop \alpha - k \right\} z \cdot z^{\underline{\alpha - k}} \\ &= \sum_k \left\{ \alpha \atop \alpha - k \right\} \left(z^{\underline{\alpha + 1 - k}} + (\alpha - k) z^{\underline{\alpha - k}} \right) \\ &= \sum_k \left\{ \alpha \atop \alpha - k \right\} z^{\underline{\alpha + 1 - k}} + \sum_k \left\{ \alpha \atop \alpha + 1 - k \right\} (\alpha + 1 - k) z^{\underline{\alpha + 1 - k}} \\ &= \sum_k \left\{ \alpha + 1 \atop \alpha + 1 - k \right\} z^{\underline{\alpha + 1 - k}} \end{aligned}$$

by the basic recurrence equation (2.3).

Notice that in all of the general identities (2.25)–(2.27), as in the original formulas (2.10), (2.17), and (2.18) that inspired them, the lower index within the braces or brackets is the same as the exponent of z. This makes the relations easy to remember, by analogy with the binomial theorem

$$(1+z)^{\alpha} = \sum_k \binom{\alpha}{k} z^k, \quad \text{when } |z| < 1. \tag{2.30}$$

Some readers will have been thinking, "This all looks fairly plausible, but unfortunately Knuth is overlooking a key point that ruins the whole proposal: We *can't* use the notation $\left[{n \atop k}\right]$ for Stirling numbers, because it has already been used for more than a century as the standard notation for Gauss's generalized binomial coefficients."

Well, there is a down side to every good idea, but this objection is not really severe. For one thing, the standard notation for Gaussian binomial coefficients involves a hidden parameter q, and it's not unusual for modern researchers to make transformations that change q. Therefore Gauss's notation is incomplete, and Andrews (for example) has used the notation $\left[{n \atop k}\right]_{q^2}$ for the Gaussian coefficient with q^2 as the hidden parameter [**2**, page 49]. Such examples suggest that it is appropriate to denote Gaussian binomials as $\binom{n}{k}_q$, especially since they reduce to ordinary binomials when $q = 1$. This notation also generalizes nicely to such things as Fibonomial coefficients $\binom{n}{k}_{\mathscr{F}}$; see [**27**]. We can then reserve the notation $\left[{n \atop k}\right]_q$ for a q-generalization of $\left[{n \atop k}\right]$. (This reverse strategy was unfortunately adopted in [**14**].) Secondly, I do not believe that any existing mathematical works, including books like [**2**] which use Gaussian coefficients extensively, would become seriously cluttered if the Gaussian $\left[{n \atop k}\right]$ were changed everywhere to $\binom{n}{k}_q$. Even so, such changes are not necessary; there is obviously no harm in beginning a mathematical paper or a book chapter or an entire book with a statement to the effect that "$\left[{n \atop k}\right]$ will denote a Gaussian binomial coefficient with parameter q in what follows." All notation can be redefined for special purposes. Therefore Stirling number enthusiasts are not encroaching on Gaussian territory when they write $\left[{n \atop k}\right]$, if they also mumble something about Stirling in order to set the context.

One further point is worth noting in conclusion: As soon as the notations $\left[{n \atop k}\right]$ and/or $\left\{{n \atop k}\right\}$ are adopted, there will no longer be a need to speak about Stirling numbers "of the first and second kind," except as a concession to history. Nielsen wrote a superb book [**41**], but he did the world a disservice by originating the *Erster Art* and *Zweier Art* terminology, because that terminology has no mnemonic value and is historically inaccurate. Stirling introduced the numbers $\left\{{n \atop k}\right\}$ first and brought in $\left[{n \atop k}\right]$ second. Indeed, practical applications have always tended to involve the numbers $\left\{{n \atop k}\right\}$ much more often than their $\left[{n \atop k}\right]$ counterparts. It seems far better to speak of $\left\{{n \atop k}\right\}$ as a Stirling subset number, and to call $\left[{n \atop k}\right]$ a Stirling cycle number. Then the names are tied to intuitive, student-friendly concepts, not to arbitrary and offputting concepts of the kth kind.

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