e is transcendental

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e is irrational and transcendental numbers exist

The irrationality of e is straightforward to prove, and has been known since at least Euler (who first called e, "e").

Theorem. *e is irrational.*

Proof. Let $H_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$, $H_n = H_n(1)$. Then

$$\begin{aligned} e - H_n &= \sum_{k=n+1}^{\infty} \frac{1}{k!} \\ &= \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \dots\right) \\ &< \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \dots\right) \\ &= \frac{1}{(n+1)!}\frac{1}{1-\frac{1}{n+1}} = \frac{1}{(n+1)!}\frac{n+1}{n} = \frac{1}{n}\frac{1}{n!}. \end{aligned}$$

If $e = \frac{p}{q}$, then $q!(e - H_q) \in \mathbb{Z}$, but

$$0 < q!(e - H_q) < \frac{1}{q},$$

a contradiction. □

[Fun fact: $H_n(x) \in \mathbb{Q}[x]$ is irreducible for all n].

The moral of the proof is that H_n (a rational number) approximates e too well. Consider the following

Lemma (Liouville, 1844)**.** *If ξ is a real algebraic number of degree $n > 1$, then there is a constant $A > 0$ (depending on ξ) such that*

$$\left|\xi - \frac{h}{k}\right| > \frac{A}{k^n}.$$

Proof. Suppose $p(x) \in \mathbb{Z}[x]$ is irreducible of degree n with $p(\xi) = 0$. Then

$$p(\xi) - p(h/k) = (\xi - h/k)p'(\alpha)$$

for some α between ξ and h/k by the mean value theorem. The left hand side is a non-zero rational number ($p(\xi) = 0$ and p is irreducible so h/k is not a root) with denominator less than k^n so that we get

$$\frac{1}{k^n} \leq \left|\xi - \frac{h}{k}\right| \sup\{p'(x) : x \in (\xi - 1, \xi + 1)\}.$$

□

The above result can be improved to

Theorem (Thue-Siegel-Roth)**.** *For all $\epsilon > 0$, there are only finitely many rational solutions to*

$$\left|\xi - \frac{h}{k}\right| < \frac{1}{k^{2+\epsilon}}$$

if ξ is algebraic and irrational.

The $2 + \epsilon$ exponent is the best possible since we have the following

Proposition. *If $\xi \in \mathbb{R}$ is irrational then there are infinitely many rationals p/q such that*

$$|\xi - p/q| < 1/q^2.$$

Proof. This is an application of the pigeonhole principle. Two of the $n + 1$ numbers $1, \{k\xi\}$ (the fractional part of $k\xi$) for $1 \leq k \leq n$ must lie in one of the n subintervals $(i/n, (i + 1)/n], 0 \leq i \leq n - 1$ of $(0, 1]$. Hence there is a p and $1 \leq q \leq n$ such that

$$|q\xi - p| < 1/n, \text{ i.e } |\xi - p/q| < 1/nq \leq 1/q^2.$$

Infinitely many of these p/q must be distinct, else $|\xi - p/q|$ takes on a minimum value, say larger than $1/n$ for some n, and the above construction gives a contradiction. □

Transcendental numbers exist (by cardinality arguments - thanks Cantor!), but let's exhibit one explicitly (as Liouville did).

Proposition. $\xi = \sum_{n=0}^{\infty} 10^{-n!}$ *is transcendental.*

Proof. Let $k_j = 10^{j!}, h_j = 10^{j!} \sum_{n=0}^{j} 10^{-n!}$. Then $(h_j, k_j) = 1$ (as $h_j \equiv 1 \mod 10$) and

$$\begin{aligned}\left|\xi - \frac{h_j}{k_j}\right| &= \sum_{n=j+1}^{\infty} 10^{-n!} < \sum_{n=(j+1)!}^{\infty} 10^{-n} \\ &= 10^{-(j+1)!}\frac{1}{1 - 1/10} = \frac{10}{9 \cdot 10^{j!}}(10^{j!})^{-j} \\ &< A_j k_j^{-j},\end{aligned}$$

where $A(j) \to 0$ as $j \to \infty$ so that ξ is transcendental by the lemma above. □

e is transcendental

We now begin the proof that e is transcendental (Hermite, 1873). We have to be able to simultaneously approximate e^x at different values to obtain a contradiction similar to that given above for the irrationality of e.

For a polynomial $f(x)$, let $F(x) = \sum_{i=0}^{\infty} f^{(i)}(x)$. Integrating by parts a bajillion times, we get

$$e^x \int_0^x f(t)e^{-t}dt = F(0)e^x - F(x) \text{ (the "Hermite identity").}$$

Consider the specific polynomial

$$f(x) = \frac{x^{p-1}(x-1)^p \cdot \dots \cdot (x-n)^p}{(p-1)!}.$$

(n will be the degree of the fictitious minimal polynomial for e over $\mathbb{Q}$ and p will be a large prime).

We have the following estimate for $0 \le k \le n$:

$$\begin{aligned} |e^k F(0) - F(k)| &= \left| e^k \int_0^k f(t)e^{-t}dt \right| \\ &\le ne^n \sup_{t\in[0,n]} \{f(t)\} \\ &= \frac{n^{p-1}(n^p)^n}{(p-1)!} \end{aligned}$$

which goes to zero as $p \to \infty$ for a fixed n. [Mildly interesting: this proof requires the existence of infinitely many primes.] We now show that such an estimate is impossible if e is algebraic by showing that $\sum_{k=1}^{n} F(0)e^k - F(k)$ is an integer between 0 and 1.

1. $F(0) \in \mathbb{Z}\backslash p\mathbb{Z}$ for $p > n$:

   We have $f(x) = a(x)b(x)$ where

   $$a(x) = \frac{x^{p-1}}{(p-1)!}, b(x) = (x-1) \cdot \dots \cdot (x-n)^p,$$

   so that

   $$f^{(N)}(x) = \sum_{i=0}^{N} a^{(i)}(x)b^{(N-i)}(x)\binom{N}{i}.$$

   Note that $a^{(i)}(0) = 0$ unless $i = p-1$ in which case $a^{(p-1)}(0) = 1$. Hence

   $$\begin{aligned} F(0) = \sum_{N=0}^{\infty} f^{(N)}(0) &= \sum_{N=0}^{\infty}\sum_{i=0}^{N} a^{(i)}(0)b^{(N-i)}(0)\binom{N}{i} \\ &= \sum_{N=p-1}^{\infty} b^{(N-(p-1))}(0)\binom{N}{p-1} \\ &= b(0) + p(\dots) = (-1)^{p^2} n!^p + p(\dots) \in \mathbb{Z}\backslash p\mathbb{Z} \text{ (remember that } p > n). \end{aligned}$$

2. $F(k) \in p\mathbb{Z}$ for $1 \le k \le n$:

   We have $f(x) = c(x)d(x)$ where

$$c(x) = \frac{(x-k)^p}{(p-1)!}, d(x) = \frac{x^{p-1}(x-1)^p \cdot \dots \cdot (x-n)^p}{(x-k)^p}.$$

   Note that $c^{(i)}(k) = 0$ unless $i = p$ in which case $c^{(p)}(k) = p$. Hence

$$\begin{aligned} F(k) &= \sum_{N=0}^{\infty} \sum_{i=0}^{N} c^{(i)}(k) d^{(N-i)}(k) \binom{N}{i} \\ &= p \sum_{N=p}^{\infty} d^{(N-p)}(k) \binom{N}{p} \in p\mathbb{Z}. \end{aligned}$$

Now, if e were algebraic, say $\sum_{k=0}^{n} c_k e^k = 0, c_k \in \mathbb{Z}, c_0 \neq 0$, and $p > |c_0|$, then

$$\begin{aligned} 1 &\le \left| \sum_{k=0}^{n} c_k F(k) \right| \text{ (because } c_0, F(0) \in \mathbb{Z} \backslash p\mathbb{Z}, F(k) \in p\mathbb{Z}) \\ &= \left| \sum_{k=0}^{n} c_k F(k) - \left(\sum_{k=0}^{n} c_k e^k \right) F(0) \right| \\ &= \left| \sum_{k=0}^{n} c_k \left(F(k) - e^k F(0) \right) \right| \\ &\le M \sum_{k=0}^{n} |F(k) - e^k F(0)| \text{ (where } M = \max_k \{|c_k|\}) \end{aligned}$$

which is less than 1 for p large as shown above. Hence e is transcendental.

What about π?

Here is a proof that π is irrational in the spirit of Hermite.

For a polynomial $f(x)$, let $F(x) = \sum_{n=0}^{\infty} (-1)^n f^{(2n)}(x)$ (this mimics $\sin x$ in the way we mimicked e^x before). We have

$$\frac{d}{dx} \left(F'(x) \sin x - F(x) \cos x \right) = f(x) \sin x, \int_0^{\pi} f(x) \sin x dx = F(0) + F(\pi).$$

If $\pi = a/b$ were rational, consider the polynomial

$$f(x) = \frac{b^n}{n!} x^n (\pi - x)^n = \frac{1}{n!} x^n (a - bx)^n \in \mathbb{Q}[x].$$

We have bounds

$$0 < \int_0^{\pi} f(x) \sin x dx \le \frac{b^n \pi^{2n}}{n!} \int_0^{\pi} \sin x dx = \frac{2(\pi^2 b)^n}{n!} \to 0 \text{ as } n \to \infty.$$

Note that $f^{(k)}(0) = 0$ for $0 \le k < n$ as f has a zero of order n at zero. We also have $f^{(k)}(0) \in \mathbb{Z}$ for $k \ge n$ by the following (easy) lemma.

Lemma. *For $p(x) \in \mathbb{Z}[x]$, $k!$ divides all the coefficients of $p^{(k)}(x)$.*

Proof. $\frac{d^k}{dx^k}x^n = k!\binom{n}{k}x^{n-k}$ for $k \le n$ and higher derivatives are zero. □

Hence $f^{(k)}(0) \in \mathbb{Z}$ for all k. Finally note that

$$f(x) = f(\pi - x), f^{(k)}(x) = (-1)^k f^{(k)}(\pi - x)$$

so that $f^{(k)}(\pi) = (-1)^k f^{(k)}(0) \in \mathbb{Z}$ for all k. Therefore $F(0) + F(\pi) \in \mathbb{Z}$, a contradiction, and π is irrational.

We can prove that π is transcendental using the methods we used for e, although the details are slightly more tedious. We start with the identity $e^{\pi i} + 1 = 0$. If πi were algebraic (degree n), we would have

$$0 = \prod_{i=1}^{n}(1 + e^{\gamma_i}) = \sum_{\epsilon_i \in \{0,1\}} e^{\sum_i \epsilon_i \gamma_i} = a + \sum_{i=1}^{m} e^{\alpha_i}$$

where the γ_i are the galois conjugates of πi, $a = 2^n - m$ are the number of zero exponents in the first sum (note that $a \ge 1$), and the α_i are the non-zero exponents in the first sum.

Thinking about symmetric functions for a while (details omitted), we see that

$$\phi(x) = \prod_{\epsilon_i \in \{0,1\}} \left(x - \sum_{i=1}^{n} \epsilon_i \gamma_i\right) \in \mathbb{Q}[x].$$

Divide by x^a and clear denominators to get a polynomial

$$\psi(x) = \sum_{i=0}^{m} b_i x^i \in \mathbb{Z}[x], b_m > 0, b_0 \ne 0$$

whose roots are exactly the α_i. Furthermore, assume $b_m \alpha_i$ is an algebraic integer for all i.

Once again we apply the "Hermite identity," this time to the polynomial

$$f(x) = \frac{b_m^{(m-1)p}}{(p-1)!}x^{p-1}\psi^p(x) = \frac{b_m^{mp}}{(p-1)!}x^{p-1}\prod_{i=1}^{m}(x - \alpha_i)^p.$$

Plug in $x = \alpha_i$ and sum over i to get

$$-aF(0) - \sum_{i=1}^{m} F(\alpha_i) = \sum_{i=1}^{m} e^{\alpha_i} \int_0^{\alpha_i} f(t)e^{-t}dt.$$

Our goal, as before, is to show that the LHS is a non-zero integer but that the RHS can be made arbitrarily small. We have

$$F(0) = (-1)^{mp} b_m^{mp} \left(\prod_i \alpha_i\right)^p \in \mathbb{Z} \backslash p\mathbb{Z}$$

for large p. We also have

$$\sum_{i=1}^{m} F(\alpha_i) = p b_m^{mp} \sum_i \alpha_i^{p-1} \prod_{j \neq i} (\alpha_i - \alpha_j)^p \in p\mathbb{Z}$$

for large p because it is symmetric in α_i and the denominator is cleared by b_m^{mp}.

We now estimate the integral on the RHS:

$$\left| e^{\alpha_i} \int_0^{\alpha_i} f(t) e^{-t} dt \right| \leq \left(|\alpha_i| |b_m^{m-1}| |\psi|(|\alpha_i|) \right)^p / (p-1)! \to 0$$

as $p \to \infty$.

Generalizations

Theorem (Lindemann-Weierstrass, 1885)**.** *If $\alpha_1, \ldots, \alpha_k$ are distinct algebraic numbers, then $e^{\alpha_1}, \ldots, e^{\alpha_k}$ are linearly independent over $\overline{\mathbb{Q}}$.*

We also have the solution of Hilbert's seventh problem

Theorem (Gelfond-Schneider, 1934)**.** *For algebraic $a \notin \{0, 1\}$ and irrational algebraic b, a^b is transcendental.*

So numbers such as $2^{\sqrt{2}}, i^i$ are transcendental.

Another generalization due to Lang (an axiomatization of Schneider's methods) is

Theorem. *Suppose K is a number field, $\{f_i\}_{i=1}^n$ meromorphic functions of order $\leq \rho$ such that $K(\{f_i\}_i)$ has transcendence degree ≥ 2 over K and $K[\{f_i\}_i]$ is closed under differentiation. If $\{w_j\}_{j=1}^m$ are distinct complex numbers such that $f_i(w_j) \in K$ for all i, j then $m \leq 20\rho[K : \mathbb{Q}]$.*

Theorem (Hermite-Lindemann)**.** *e^α is transcendental for all $\alpha \in \overline{\mathbb{Q}} \backslash \{0\}$.*

Proof. The proof that π is transcendental directly generalizes to this. Or, take the meromorphic functions in the theorem above to be z, e^z and K to be $\mathbb{Q}(\alpha, e^\alpha)$. Theses function take values in K for z any integer multiple of α. □

Theorem (Schneider)**.** *If $\wp$ is a Weierstrass a function with g_2, g_3 algebraic, then $\wp(\alpha)$ is transcendental for all $\overline{\mathbb{Q}} \backslash \{0\}$.*

Sketch. First the relevant definitions. If $\Lambda \subseteq \mathbb{C}$ is a rank two lattice, define

$$\wp(z; \Lambda) = \frac{1}{z^2} + \sum_{w \in \Lambda \backslash \{0\}} \frac{1}{(z-w)^2} - \frac{1}{w^2}.$$

Then $\wp$ satisfies the algebraic differential equation

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3$$

where

$$g_2 = 60 \sum_{w \in \Lambda \backslash \{0\}} \frac{1}{w^4}, g_3 = 140 \sum_{w \in \Lambda \backslash \{0\}} \frac{1}{w^6}.$$

Addition formula, etc.????? □

A far-reaching generalization of the theorem of Gelfond-Schneider is

Theorem (Baker, 1966)**.** *If* $\{\alpha_i\}_{i=1}^n$ *and* $\{\beta_i\}_{i=0}^n$ *are algebraic (and* $\alpha_i \neq 0$*) of degree at most* d *and with heights at most* A, B *(for* $\{\alpha_i\}_{i=1}^n$ *and* $\{\beta_i\}_{i=0}^n$ *respectively) then*

$$\Lambda := \beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

is either zero or $|\Lambda| > B^{-C}$ *for an effectively computable constant* C *depending only on* $n, d, A,$ *and* $\{\log \alpha_i\}_i$*. [The height of an algebraic number* γ *is* $\max_i\{|c_i|\}$ *where* $\sum_i c_i x^i$ *is the minimal polynomial of* γ *over* $\mathbb{Z}$*.]*

For related results and applications, such as the class number one problem for imaginary quadratic fields:

$$\mathbb{Q}(\sqrt{-d}) \text{ with } d > 0 \text{ has class number one iff } d \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\},$$

Baker was awarded a Fields medal in 1970.

References

[1] Baker, Alan, *Transcendental Number Theory*

[2] Niven, Ivan, *Irrational Numbers*

[3] `http://math.stanford.edu/~ksound/TransNotes.pdf`