Magical Applications of Mathematics

Book Review of Magical Mathematics: The Mathematical Ideas That Animate Great Magic Tricks, by Persi Diaconis and Ron Graham. Reviewed by Arthur Benjamin.

Magical Mathematics exemplifies how mathematics should be taught. Imagine that you begin your class in discrete mathematics (or number theory or mathematical modeling) with the following trick. You bring out a deck of cards, and ask someone in the audience to cut the cards as often as they like, then take the card on top of the deck. Pass the deck to someone else have them take the next card, and have a third person take the following card. Ask each person to concentrate on their card. You explain that you will determine one person's suit with just a few hints. The first person tells you their value (Ace or two or … or King) and the second person tells you their suit (Spades or Hearts or Diamonds or Clubs), and the third person tells you nothing. With that information, and a little extra concentration, you reveal everyone's card. Most likely everyone in the class is now wondering "How did you do that?" This is precisely the reaction that we strive for in our classrooms.

Let's think about this trick. Since the cards were cut, but not shuffled, the cards are still in the same cyclic order. Since there are 13 x 4 = 52 possible responses from the first two audience members, that should be enough information to reveal the three cards. If the cards were stacked in an obvious order (say beginning with Ace of Spades, Two of Spades, …) and you heard that the first card had value Seven and the second card was Spades then you know the identities of all three cards (Seven of Spades, Eight of Spades and Nine of Spades). But of course, this ordering of the deck would be obvious to your audience once they hear you name the cards (or if they glance the deck while the cards are being cut). Is there a less obvious way to arrange the cards? Here's a very effective stack (known to magician's as Si Stebbins). The card values are arranged starting with the King (with numerical value thirteen) and adding 3 (mod 13) to each value producing the order (K, 3, 6, 9, Q, 2, 5, 8, J, A, 4, 7, 10). The suits are arranged in CHaSeD order (Clubs, Hearts, Spades, Diamonds) so the deck begins (KC, 3H, 6S, 9D, QC, …). You could casually show the cards before the trick begins and they will appear to be in random order. Now if the first audience member says "5" and the second member says "hearts" you can combine those pieces of information to determine that the three cards must be 5 of Clubs, 8 of Hearts, and Jack of Spades, in that order.

One could easily imagine going off on tangents to explore why adding 3 (mod 13) results in cycles of length 13, and that adding any other constant would work just as well because 13 is prime. The trick described here is just one of many from the book's chapter on Universal Cycles. More subtle variations are provided where you ask five people to take a card like before, and ask everyone to concentrate. Explain that you are getting mixed signals. Who has the largest valued card? (They compare cards and someone raises their hand.) Then ask who has the second largest, then the third largest card? Once these people are identified, you tell everyone what their card is. Again, this requires that the cards are ordered in such a way that the answers to your innocent sounding questions uniquely determines the card chosen by the first (and subsequent) person. Other tricks are presented that utilize de Bruijn sequences: binary sequences of length 2^k, where every string of length k appears exactly once when the string is written in a circle. The authors generalize these ideas (for example to higher dimensions), and offer applications, most of which lead to more magical effects.

Although *Magical Mathematics* is written for a general audience, and both authors have considerable experience as performers, they are first and foremost mathematicians of the highest caliber, so their exposition should be especially enjoyed by readers of this journal. Many of the tricks described in this book require some thinking on the part of the performer, requiring more "sleight of mind" than sleight of hand, but some of the tricks are self-working. Many of the tricks are surprising, even after you know the secret.

My favorite chapter "dealt" with the Gilbreath Principle and its extensions. The development of this concept and its generalizations and applications have exactly the same appeal as watching a beautiful number pattern turn into a complex and satisfying theorem. Here's Gilbreath's first principle in action. Suppose you have a deck of cards where the cards are secretly arranged so that the colors alternate (black, red, black, red, …). Ask your volunteer to deal about half of these cards into a pile, then riffle shuffle the two halves together. Surprisingly, even after one riffle shuffle, the cards still retain an interesting structure: the top two cards are still black and red (in some order); the next two cards are still black and red; the next two cards are still black and red; and so on. Not obvious, right?

How about this? Give your spectator 25 (secretly pre-arranged) cards. First have the spectator demonstrate that they can deal cards, by dealing about half the cards into second pile, then riffle shuffle both halves together. The spectator then deals five 5-card poker hands, and everything seems normal. Then the spectator deals five new 5-card poker hands, only to see that one of the players was dealt a straight, and another player was dealt a flush.

A Gilbreath shuffle on n cards consists of choosing a number j between 0 and n, dealing j cards into one pile, then riffle shuffling the two piles together. Among the $n!$ possible permutations of n cards, how many of them are achievable after one Gilbreath shuffle? Remarkably, the answer is 2^{n-1}. The material is presented in such a way that the reader (especially one who is already interested in mathematics) will be just as interested in the proof of that statement as understanding how a magic trick based on that principle works. The general Gilbreath principle says that if the numbers 1 through n are given a Gilbreath shuffle, then for every integer j, the top j numbers (and the next j numbers and the next j numbers, …) are all distinct mod j. These theorems are then applied to create truly magical moments that will appeal to many an audience, especially those sitting in a discrete mathematics classroom. If that weren't exciting enough, the authors then go on to show how this Gilbreath shuffle is intimately connected to real periodic points of the Mandelbrot set. The fact that there is any connection at all is amazing as any magic trick.

The Gilbreath shuffle is "imperfect" since there is no way to predict the exact order of cards after one shuffle. But other shuffles, such as perfect shuffles (also known as Faro shuffles), are not random at all, since they correspond to an exact permutation. In a perfect shuffle, the cards are cut exactly in half, then the two halves are interleaved perfectly. There are two types of perfect shuffles: an Outshuffle keeps the top card on top; an Inshuffle moves the top card to the second position. The authors describe ways to control cards using these shuffles. For example, to move the top card to the *n*th position, you simply express *n-1* in binary and follow instructions. For instance, to bring the top card to position 42, we see that 41 = 32 + 8 + 1 = $(101001)_2$ and then perform In-Out-In-Out-Out-In. (Even if you can't perform a perfect shuffle, the result is beautiful!) There are still many open questions about these shuffles. It was only recently discovered how to easily invert the process to bring a card from a desired position to the top using perfect shuffles.

Another shuffle with attractive mathematical properties is the "Down and Under" shuffle (or Australian shuffle) which magicians sometimes use to reveal a chosen card. From a packet of cards, the magician deals the top card to the table, then one card to the bottom of the packet, then one card to the table, then one card to the bottom of the packet, and continues this until only one card remains in the hand. The elegant mathematical theorem is that if the packet contains *n* cards, then and if $n = 2^k + x$, where $1 \le x \le n/2$, then the last card to be dealt is the card originally in position $2x$.

The book provides an interesting history of mathematical magic, which has been in existence for a very long time. Some tricks are described in one of the very first books published on magic in 1584, as well as one of the first books on arithmetic (Fibonacci's *Liber Abaci*) published in 1202. The authors even go back thousands of years to discuss some of the mathematics that arises in the ancient Chinese text, the *I Ching,* and suggest some performance ideas based on them.

The authors devote a chapter to the mathematics of juggling. Both magic and juggling have a long history, dating back thousands of years. As the authors point out, "There is also a strong connection between mathematics and juggling. Mathematics is often described as the science of patterns. Juggling can be thought of as the art of controlling patterns in time and space."

Most of the mathematics and tricks in the book have a combinatorial or number theoretic flavor, but other branches of mathematics are covered. There is a clever topological swindle involving a simple loop of chain, as well as some discrete probability. (The authors are still in search of a magic trick based on calculus.) Here is a random trick that surprised me. Imagine you have 5 cards, two of which say Win and three of which say Lose. The spectator shuffles the cards and they will be turned up one at a time. You explain that you will place bets as follows. Each time you bet, your next bet will be half as much if you win, and will be 1.5 times as much if you lose. (For instance, if you place a $10 bet and win, your next bet is $5 if you win and it's $15 if you lose.) Starting with a bet of $16, what will be your profit or loss after your 5 bets? Surprisingly, regardless of how the cards are shuffled, you end up with a $5 profit. I can see this being turned into an interesting classroom exercise. I'll save that exercise (and the elegant generalization) to the reader.

Towards the end of the book, we meet some of the most important contributors to mathematical magic. Diaconis describes his visit to the home of Stewart James, a reclusive genius who has made vast contributions to card magic and mathematical magic. When Diaconis asked James if he could borrow a deck of cards, he was shocked to learn that James had not had a deck of cards in his house for years, even though he wrote a monthly column on card magic. James explained, "This may sound strange but I don't have a real deck of cards in the house – haven't had one for four or five years…After all, when Agatha Christie writes a murder mystery, she doesn't have to go out and kill somebody." The authors show special reverence to Martin Gardner, who wrote the foreword to this book, just before he passed away in 2010. Their style of writing is very reminiscent of Gardner's, with most of the material presented at a high school mathematics level, and with lots of examples to make the exposition go down smoothly.

The book, published by Princeton University Press, is beautifully illustrated, printed on high quality paper and loaded with color photos. Both authors have many great stories and anecdotes to tell, but I have one small quibble. In relating a story the authors most often used the pronoun "we" or "us" to describe a situation that clearly only happened to one of them (e.g., "When the time came for us to apply to graduate school, Martin Gardner was one of our letter writers."). Although clearly the authors have had many joint experiences together, it would have been less distracting for there to be more use of expressions like "one of us (P.D.)" when it applied.

Mathematical Magic is a truly magical book, containing ample amounts of mathematics and magic that will amaze and amuse. Diaconis and Graham are both first-rate mathematicians and performers and offer insights and ideas that could not have been expressed by anyone else. This book is destined to be a classic on the subject.