

**TACL 2019**



**ABSTRACTS**

## Preface

This volume contains the papers presented at TACL 2019: Topology, Algebra, and Categories in Logic held June 17–21, 2019 in Nice.

The volume includes the abstracts of 91 accepted contributed talks and of 10 invited talks (a few contributions were rejected after a light refereeing process and a couple of papers were withdrawn by the authors themselves due to the impossibility of their attendance at the conference). We thank all Program Committee members for their precious work in reading the submitted abstracts and giving useful suggestions to the authors.

TACL 2019 is the ninth conference in the series Topology, Algebra, and Categories in Logic (TACL, formerly TANCL). Earlier instalments of the series have been held in Tbilisi (2003), Barcelona (2015), Oxford (2007), Amsterdam (2009), Marseille (2011), Nashville (2013), Ischia (2015), and Prague (2017).

The program of the conference TACL 2019 focuses on three interconnecting mathematical themes central to the semantic study of logic and its applications: topological, algebraic, and categorical methods.

Our main sponsors are the European Research Council, the CNRS, and the Université Côte d’Azur. In particular, the conference has received funding from the European Research Council under the European Union’s Horizon 2020 research and innovation program through the DualL project (grant agreement No. 670624); from the CNRS, which is supporting the TACL 2019 school as one of its Écoles Thématiques; from the Université Côte d’Azur through its International Conferences program, its International Schools program, through support from the interdisciplinary academies Réseaux, Information, et Société Numérique and Systèmes Complexes, respectively, and finally through the Laboratoire J. A. Dieudonné and the Laboratoire I3S, both of the Université Côte d’Azur.

We thank the EasyChair development team for providing their conference management platform.

May 6, 2019  
Nice

Silvio Ghilardi  
Ramon Jansana  
Mai Gehrke

## Table of Contents

### Invited talks

Simulations of quantum resources and the degrees of contextuality.....	1
<i>Samson Abramsky, Rui Soares Barbosa, Martti Karvonen and Shane Mansfield</i>	
Generalized continuous closure spaces: a topological approach to domain theory.....	3
<i>Marcel Ern�</i>	
Possibility Semantics.....	5
<i>Wesley Holliday</i>	
Non-finitely axiomatisable canonical varieties of BAOs with infinite canonical axiomatisations .....	6
<i>Agi Kurucz</i>	
The poset of all logics .....	7
<i>Tommaso Moraschini</i>	
Some Applications of Stone Duality to Automata Theory.....	8
<i>Daniela Petrisan</i>	
Snapshots of duality theory, from 2019 and fifty years earlier .....	9
<i>H. A. Priestley</i>	
Modal Logics of Dependence .....	11
<i>Johan van Benthem</i>	
Logic, Automata, and Model Companions .....	13
<i>Samuel J. van Gool</i>	
Anabelian geometry in model theory setting.....	15
<i>Boris Zilber</i>	

### Contributed talks

Norm complete abelian $\ell$ -groups: topological duality .....	17
<i>Marco Abbadini, Vincenzo Marra and Luca Spada</i>	
Norm complete Abelian $\ell$ -groups: equational axiomatization.....	19
<i>Marco Abbadini, Vincenzo Marra and Luca Spada</i>	
Polyhedral Completeness of Intermediate and Modal Logics .....	21
<i>Sam Adam-Day, Nick Bezhanishvili, David Gabelaia and Vincenzo Marra</i>	
Product of neighborhood frames with additional common modality.....	23
<i>Rajab Aghamov and Andrey Kudinov</i>	
Predicative Implications: A Topological Approach.....	25
<i>Amirhossein Akbar Tabatabai</i>	
Ranges of functors and elementary classes via topos theory .....	27
<i>Peter Arndt</i>	

When is the frame of nuclei spatial: A new approach . . . . .	29
<i>Francisco Avila, Guram Bezhanishvili, Patrick Morandi and Angel Zaldivar</i>	
Analysis of the $\Sigma_1^1$ -Fragment of First Order Gödel Logic extended with Propositional Quantifiers . . . . .	31
<i>Matthias Baaz and Norbert Preining</i>	
Enriched Lawvere Theories for Operational Semantics . . . . .	33
<i>John Baez and Christian Williams</i>	
Enriched distributivity over finite commutative residuated lattices . . . . .	35
<i>Adriana Balan, Peter Jipsen and Alexander Kurz</i>	
Proofs and surfaces . . . . .	37
<i>Dorđe Baralić, Pierre-Louis Curien, Marina Milićević, Jovana Obradović, Zoran Petrić, Mladen Zekić and Rade Živaljević</i>	
A new logic arising from a scattered Stone space . . . . .	39
<i>Guram Bezhanishvili, Nick Bezhanishvili, Joel Lucero-Bryan and Jan van Mill</i>	
A generalization of Gelfand-Naimark-Stone duality to completely regular spaces . . . . .	41
<i>Guram Bezhanishvili, Patrick Morandi and Bruce Olberding</i>	
The van Benthem characterisation theorem for descriptive models . . . . .	43
<i>Nick Bezhanishvili and Tim Henke</i>	
Nearness Posets . . . . .	45
<i>Tristan Bice</i>	
Towards completeness of logics of information and common belief . . . . .	47
<i>Marta Bilkova</i>	
Counting finite involutive bisemilattices . . . . .	49
<i>Stefano Bonzio, Michele Pra Baldi and Diego Valota</i>	
Difference hierarchies over lattices . . . . .	51
<i>Célia Borlido</i>	
Time-reversal homotopical properties of concurrent systems . . . . .	53
<i>Cameron Calk, Eric Goubault and Philippe Malbos</i>	
A temporal interpretation of intuitionistic quantifiers . . . . .	55
<i>Luca Carai and Guram Bezhanishvili</i>	
Semantic analysis and proof theory for monotone modal logic . . . . .	57
<i>Jinsheng Chen, Giuseppe Greco, Alessandra Palmigiano and Apostolos Tzimoulis</i>	
Completeness properties in abstract algebraic logic . . . . .	59
<i>Petr Cintula and Carles Noguera</i>	
Orders on Groups: an Approach through Spectral Spaces . . . . .	61
<i>Almudena Colacito</i>	
The logic of categories and informational entropy . . . . .	63
<i>Willem Conradie, Andrew Craig, Alessandra Palmigiano and Nachoem Wijnberg</i>	

Goldblatt-Thomason for LE-logics .....	65
<i>Willem Conradie, Alessandra Palmigiano and Apostolos Tzimoulis</i>	
Intuitionistic and classical non-normal modal logics: An embedding .....	67
<i>Tiziano Dalmonde, Charles Grellois and Nicola Olivetti</i>	
Lifting functors from Pos to Pries .....	69
<i>Jim de Groot</i>	
A Sahlqvist theorem for subordination algebras .....	71
<i>Laurent De Rudder and Georges Hansoul</i>	
Extensions of the Stone Duality to the category of zero-dimensional Hausdorff spaces .....	73
<i>Georgi Dimov and Elza Ivanova-Dimova</i>	
Semi-reflective Extensions of Dualities and a New Approach to the Fedorchuk Duality ....	74
<i>Georgi Dimov, Elza Ivanova-Dimova and Walter Tholen</i>	
Extensions of dualities and a new approach to de Vries' Duality Theorem .....	76
<i>Georgi Dimov, Elza Ivanova-Dimova and Walter Tholen</i>	
Resource Reasoning in Duality-theoretic Form: Stone-type Dualities for Bunched and Separation Logics .....	77
<i>Simon Docherty and David Pym</i>	
Amalgamating poset extensions .....	79
<i>Rob Egrot</i>	
Artin glueings as semidirect products .....	81
<i>Peter Faul and Graham Manuell</i>	
Generic Models for Topological Evidence Logics .....	82
<i>Saul Fernandez Gonzalez, Alexandru Baltag and Nick Bezhanishvili</i>	
On non-distributive lattices with involution .....	84
<i>Wesley Fussner</i>	
Characterization of flat polygonal logics .....	86
<i>David Gabelaia, Levan Uridia, Mamuka Jibladze and Evgeny Kuznetsov</i>	
Undecidability methods for residuated lattices .....	88
<i>Nikolaos Galatos and Gavin St. John</i>	
A Glivenko theorem for lattice-ordered groups .....	90
<i>José Gil-Férez, Frederik Lauridsen and George Metcalfe</i>	
Nonclassical first order logics: semantics and proof theory .....	92
<i>Giuseppe Greco, Peter Jipsen, Alexander Kurz, M. Andrew Moshier, Alessandra Palmigiano and Apostolos Tzimoulis</i>	
Algebraic proof theory for LE-logics .....	94
<i>Giuseppe Greco, Peter Jipsen, Fei Liang, Alessandra Palmigiano and Apostolos Tzimoulis</i>	
Proof theory and semantics for structural control .....	96
<i>Giuseppe Greco, Apostolos Tzimoulis and Michael Moortgat</i>	

On the variety of Goedel-MV-algebras .....	98
<i>Revaz Grigolia, Antonio Di Nola and Gaetano Vitale</i>	
Some theorems concerning Grzegorzcyk contact lattices .....	100
<i>Rafal Gruszczynski and Andrzej Pietruszczak</i>	
Computing the validity degree in Łukasiewicz logic .....	102
<i>Zuzana Hanikova</i>	
Remarks on the Topos Approach to Quantum Mechanics .....	104
<i>John Harding and Chris Heunen</i>	
Hyper-MacNeille completions of Heyting algebras .....	106
<i>John Harding and Frederik Lauridsen</i>	
Comprehension bicategories and fibrations of toposes .....	108
<i>Sina Hazratpour and Steve Vickers</i>	
Twist products arising from residuated bimodules .....	110
<i>Pengfei He, Adam Přenosil and Constantine Tsinakis</i>	
Relational semantics for extended contact algebras .....	112
<i>Tatyana Ivanova</i>	
Colimits of effect algebras via a reflection .....	114
<i>Gejza Jenča</i>	
Partially ordered varieties of involutive residuated posets .....	116
<i>Peter Jipsen</i>	
On the structure of finite (commutative) idempotent involutive residuated lattices .....	118
<i>Peter Jipsen, Olim Tuyt and Diego Valota</i>	
From MV-semirings to Involutive semirings .....	120
<i>Peter Jipsen and Sara Vannucci</i>	
Presenting de Groot duality of stably compact spaces by entailment relations .....	122
<i>Tatsuji Kawai</i>	
Projective unification in $\text{NExt}(\mathbf{K4})$ .....	124
<i>Sławomir Kost</i>	
From intuitionism to Brouwer's modal logic .....	126
<i>Zofia Kostrzycka</i>	
Stone dualities between étale categories and restriction semigroups .....	128
<i>Ganna Kudryavtseva</i>	
The spectrum of a localic semiring .....	129
<i>Graham Manuell</i>	
A topos for piecewise-linear geometry, and its logic .....	130
<i>Vincenzo Marra and Matias Menni</i>	
Frege's Basic Law V via Partial Orders .....	132
<i>Giovanni Marco Martino</i>	

B-frame Representations for Complete Lattices . . . . .	134
<i>Guillaume Massas</i>	
Free Kleene algebras with domain . . . . .	136
<i>Brett McLean</i>	
Singly generated quasivarieties and residuated structures . . . . .	138
<i>Tommaso Moraschini, James Raftery and Jamie Wannenburg</i>	
Epimorphisms in Varieties of Heyting Algebras . . . . .	140
<i>Tommaso Moraschini and Jamie Wannenburg</i>	
Exact and Fitter Sublocales . . . . .	142
<i>Michael Andrew Moshier, Rick Ball and Ales Pultr</i>	
The undecidability of profiniteness . . . . .	144
<i>Anvar Nurakunov and Michal Stronkowski</i>	
Overview and perspectives of the general construction of spectra . . . . .	146
<i>Axel Osmond</i>	
Completion of pseudo-orthomodular posets . . . . .	148
<i>Jan Paseka</i>	
Axiom $T_D$ and the relation between sublocales and subspaces of a space . . . . .	150
<i>Jorge Picado and Aleš Pultr</i>	
Topological representations of congruence lattices . . . . .	152
<i>Miroslav Ploščica</i>	
Axiomatising categories of spaces: the case of compact Hausdorff spaces . . . . .	154
<i>Luca Reggio</i>	
Duality for two-sorted lattices . . . . .	156
<i>Umberto Rivieccio and Achim Jung</i>	
Quantale semantics for Lambek calculus with subexponentials . . . . .	158
<i>Daniel Rogozin</i>	
Derivations on bounded pocrimms and MV-algebras with product . . . . .	160
<i>Dana Salounova, Jan Kuhr and Jiri Rachunek</i>	
Decidability of the equational theory of the natural join and inner union . . . . .	162
<i>Luigi Santocanale</i>	
Mix $\star$ -autonomous quantales and the continuous weak order . . . . .	163
<i>Luigi Santocanale and Maria João Gouveia</i>	
Partial frames . . . . .	164
<i>Anneliese Schauerte and John Frith</i>	
Two Approaches to Substructural Modal Logic . . . . .	166
<i>Igor Sedlar</i>	
On global algebraic completeness of the Gödel-Löb provability logic . . . . .	168
<i>Daniyar Shamkanov</i>	

Glivenko’s theorem, finite height, and local finiteness . . . . .	170
<i>Ilya Shapirovsky</i>	
Simplicial semantics and one-variable fragments of modal predicate logics . . . . .	172
<i>Valentin Shekhtman</i>	
Order-enriched solid functors . . . . .	174
<i>Lurdes Sousa and Walter Tholen</i>	
Divisibility and diagonals . . . . .	176
<i>Isar Stubbe</i>	
The coproduct of frames as encoding d-frame structure . . . . .	178
<i>Anna Laura Suarez and Achim Jung</i>	
Gluing residuated lattices . . . . .	180
<i>Sara Ugolini and Nikolaos Galatos</i>	
Point-free theories of space and time: a sort history, models and representation theory . . . .	182
<i>Dimiter Vakarelov</i>	
Axiomatizing the crisp Gödel modal logic . . . . .	184
<i>Amanda Vidal Wandelmer and Ricardo Oscar Rodriguez</i>	
Correspondence, Canonicity, and Model Theory for Monotonic Modal Logics . . . . .	186
<i>Kentaro Yamamoto</i>	
The Bohr compactification of an abelian group as a quotient of its Stone-Cech compactification . . . . .	188
<i>Pavol Zlatos</i>	
Pierce stalks in preprimal varieties . . . . .	190
<i>William Javier Zuluaga Botero and Diego Vaggione</i>	



# Simulations of quantum resources and the degrees of contextuality

Samson Abramsky, Rui Soares Barbosa, Martti Karvonen, and Shane Mansfield

A key objective in the field of quantum information and computation is to understand the advantage which can be gained in information processing tasks by the use of quantum resources. While a range of examples have been studied, to date a systematic understanding of quantum advantage is lacking.

Our focus here is on quantum resources which take the form of *non-local*, or more generally *contextual*, correlations. Contextuality is one of the key signatures of non-classicality in quantum mechanics [7, 4], and has been shown to be a necessary ingredient for quantum advantage in a range of information processing tasks [8, 6, 5, 2]. In previous work [1], we introduced a notion of simulation between quantum resources, and more generally between resources described in terms of contextual correlations, in the “sheaf-theoretic” framework for contextuality introduced in [3]. The notion of simulation is expressed as a morphism of empirical models, in a form which allows the behaviour of one set of correlations to be simulated in terms of another using classical processing and shared randomization. Mathematically, this is expressed as coKleisli maps for a comonad of “measurement protocols” on the category of empirical models. This setting is expressive, and allows for a number of variations, e.g. grading the simulation by the number of copies of the simulating resource or by the depth of measurement adaptivity in the protocol, and also allows for a natural relaxation to a notion of approximate simulation.

As with classical notions of reducibility in computability and complexity theory, the existence of simulation maps allows us to compare different contextual behaviours in a fine-grained, mathematically robust way. We can define a *degree of contextuality* as an equivalence class of empirical models under two-way simulability. These degrees are then partially ordered by the existence of simulations between representatives. Existing results from the study of non-locality can be interpreted as showing the richness of this order, and there are many natural further questions which arise.

The property of (non)contextuality itself can be equivalently formulated as the existence of a simulation by an empirical model over the empty scenario [1]. This suggests that much of contextuality theory can be generalized to a “relativized” form, i.e. essentially working in slice categories.

As an example, consider the classic theorem of Vorob'ev [9]. It characterizes those scenarios over which all empirical models are noncontextual, in terms of an acyclicity condition on the underlying simplicial complex. This can be formulated as characterizing those scenarios such that every model over them can be simulated by a model over the empty scenario. More generally, we can ask for conditions on scenarios  $(X, \Sigma, O)$  and  $(Y, \Delta, P)$  such that every empirical model over  $(Y, \Delta, P)$  can be simulated by some empirical model over  $(X, \Sigma, O)$ .

## References

- [1] Samson Abramsky, Rui Soares Barbosa, Martti Karvonen, and Shane Mansfield. A comonadic view of simulation and quantum resources. In *Proceedings of the 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LiCS 2019)*, 2019. To appear, available at <https://www.cs.ox.ac.uk/people/rui.soaresbarbosa/lics2019.pdf>.
- [2] Samson Abramsky, Rui Soares Barbosa, and Shane Mansfield. Contextual fraction as a measure of contextuality. *Physical Review Letters*, 119(5):050504, 2017.
- [3] Samson Abramsky and Adam Brandenburger. The sheaf-theoretic structure of non-locality and contextuality. *New Journal of Physics*, 13(11):113036, 2011.
- [4] John S Bell. On the problem of hidden variables in quantum mechanics. *Reviews of Modern Physics*, 38(3):447–452, 1966. doi:10.1103/RevModPhys.38.447.
- [5] Juan Bermejo-Vega, Nicolas Delfosse, Dan E Browne, Cihan Okay, and Robert Raussendorf. Contextuality as a resource for models of quantum computation with qubits. *Physical Review Letters*, 119(12):120505, 2017.
- [6] Mark Howard, Joel Wallman, Victor Veitch, and Joseph Emerson. Contextuality supplies the ‘magic’ for quantum computation. *Nature*, 510(7505):351, 2014.
- [7] Simon Kochen and Ernst P Specker. The problem of hidden variables in quantum mechanics. *Journal of Mathematics and Mechanics*, 17(1):59–87, 1967.
- [8] Robert Raussendorf. Contextuality in measurement-based quantum computation. *Physical Review A*, 88(2):022322, 2013.
- [9] Nikolai Nikolaevich Vorob'ev. Consistent families of measures and their extensions. *Theory of Probability & Its Applications*, 7(2):147–163, 1962.

**GENERALIZED CONTINUOUS CLOSURE SPACES:  
A TOPOLOGICAL APPROACH TO DOMAIN THEORY**

MARCEL ERNÉ

Domain theory in the narrow sense is concerned with continuous lattices and domains, and was developed in the comprehensive monograph by Gierz et al. [7]. In a wider sense, it may be regarded as the theory of

<b>Topological</b>	<b>Treatment of</b>	<b>Transitive</b>	<b>Terms and relations</b> [1, 3, 5]
<b>Algebraic</b>	<b>Aspects of</b>	<b>Approximating</b>	<b>Auxiliary relations</b> [1, 7]
<b>Categories of</b>	<b>Core generated</b>	<b>Closure and</b>	<b>Convergence spaces</b> [1, 4]
<b>Lattices in the</b>	<b>Logic of</b>	<b>Languages and</b>	<b>Lambda calculus</b> [8].

An auxiliary relation on a poset  $(X, \leq)$  is a relation  $\prec$  on  $X$  such that  $w \leq x \prec y \leq z$  implies  $w \prec z$ , and the latter implies  $w \leq z$ . If each of the sets  $\prec y = \{x : x \prec y\}$  is an ideal (a directed lower set) or an  $\omega$ -ideal (an ideal in the sense of Frink [6]), we speak of an ideal relation or  $\omega$ -ideal relation, respectively. Further,  $\prec$  is separating iff for  $x \not\leq y$  there is a  $z$  such that  $\prec z \subseteq \leq x$  but not  $\prec z \subseteq \leq y$ , and approximating iff each  $y \in X$  is the supremum of  $\prec y$ ; we say  $\prec$  has the weak interpolation property if for  $x \prec z$  there is a finite  $F \subseteq \prec z$  so that  $y \leq u$  for all  $y \in F$  implies  $x \prec u$ ; for ideal relations, this is the usual interpolation property.

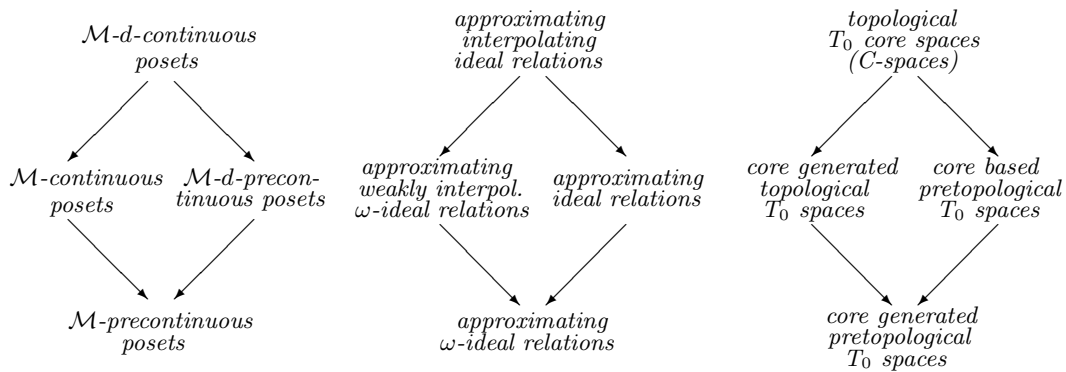
Two useful relations are the following: for a preclosure operator  $p$  (an extensive and inclusion-preserving map on  $\mathcal{P}X$ ), the specialization order  $\leq_p$  on  $X$  is defined by  $x \leq_p y \Leftrightarrow p\{x\} \subseteq p\{y\}$ , and the interior relation  $\ll_p$  by  $x \ll_p y \Leftrightarrow y \notin p(X \setminus \uparrow x)$ ; and  $(X, p)$  is called a precore space if  $p\downarrow = \downarrow p = p$  and  $y \in p(\ll_p y)$  for all  $y \in X$  (the upset and downset operators,  $\uparrow$  and  $\downarrow$ , refer to  $\leq_p$ ). A core space [1] is then a precore space with idempotent (closure operator)  $p$ . A fundamental observation is that the category of  $T_0$  precore spaces is concretely isomorphic to that of posets with separating auxiliary relations, by sending  $(X, p)$  to  $(X, \leq_p, \ll_p)$ . The classical continuous domains and their way-below relations are obtained by taking for  $p$  the Scott preclosure operator, assigning to each subset  $Y$  of a dcpo (directed complete poset) the lower set generated by all suprema of directed subsets of  $\downarrow Y$ .

But also two classes of convergence spaces play a crucial role in the present context: a convergence space is core based resp. core generated iff each filter converging to a point contains another one that has a base resp. subbase of cores, where a core is the intersection of all neighborhoods of some point in the associated topological space. Hence, the topological core spaces are the C-spaces [1] or worldwide web spaces [3, 5], which are characterized by complete distributivity of their topology.

As demonstrated in [2], the theory of continuous lattices and domains admits flexible extensions to so-called  $\mathcal{M}$ -precontinuous posets  $X$ ; here,  $\mathcal{M}$  is a subset of  $\mathcal{P}X$ , and each  $y \in X$  is the supremum of the set  $\ll_{\mathcal{M}} y = \bigcap \{\downarrow Z : Z \in \mathcal{M}, y \in \Delta Z\}$ , where  $\Delta Z$  is the cut generated by  $Z$ . If the  $\mathcal{M}$ -below relation  $\ll_{\mathcal{M}}$  has the weak interpolation property, we speak of  $\mathcal{M}$ -continuous posets, and if moreover the sets  $\ll_{\mathcal{M}} y$  are directed, of  $\mathcal{M}$ - $d$ -precontinuous resp.  $\mathcal{M}$ - $d$ -continuous posets.

The Scott convergence [7], alias (lower) lim-inf convergence, is generalized in the present setting to the notion of  $\mathcal{M}$ -convergence: a net  $\mathcal{M}$ -converges to a point  $x$  iff there is a set  $Z \in \mathcal{M}$  of eventual lower bounds of the net such that  $x$  belongs to  $\Delta Z$  (which means  $x \leq \bigvee Z$  if  $Z$  has a supremum). Passing from nets to filters, a filter  $\mathcal{M}$ -converges to  $x$  iff there is a  $Z \in \mathcal{M}$  such that  $x \in \Delta Z$  and  $\uparrow z$  lies in the filter for all  $z \in Z$ . The convergence-theoretical relevance of  $\mathcal{M}$ -precontinuity resp.  $\mathcal{M}$ -continuity for arbitrary sets  $\mathcal{M}$  of  $\omega$ -ideals is then manifested by the equivalence to the condition that  $\mathcal{M}$ -convergence is pretopological resp. topological.

The aforementioned connections between relational and topological structures in domain theory are made precise by concrete equivalences between the following four triples of categories (where  $\mathcal{M}$  runs through all collections of  $\omega$ -ideals):



Replacing posets and their cut operators with arbitrary closure spaces, equipped with their specialization order, one finally arrives at the definition of generalized continuous closure spaces, which provide a purely topological environment for domain theory, as demanded by leading workers in the field.

#### REFERENCES

- [1] Erné, M.: The ABC of order and topology, in: H. Herrlich and H.-E. Porst (Eds.), *Category Theory at Work*, Heldermann, Berlin, 1991, 57–83.
- [2] Erné, M.:  $\mathcal{Z}$ -continuous posets and their topological manifestation. *Appl. Cat. Struct.* **7** (1999) 31–70.
- [3] Erné, M.: Infinite distributive laws versus local connectedness and compactness properties. *Top. and Appl.* **156** (2009), 2054–2069.
- [4] Erné, M.: Generalized continuous closure spaces. I: meet preserving closure operators, II: pretopological and topological convergence, III:  $\mathcal{M}$ -quasicontinuity and  $\mathcal{M}$ -lim-inf convergence. Preprints, University of Hannover, 2019.
- [5] Erné, M.: Web spaces and worldwide web spaces: topological aspects of domain theory. *Logical Methods in Computer Science* **15** (2019), 23:1–23:38. <https://arxiv.org/abs/1802.08170> (2018).
- [6] Frink, O.: Ideals in partially ordered sets. *Amer. Math. Monthly* **61** (1954), 223–234.
- [7] Gierz, G., Hofmann, K. H., Keimel, K., Lawson, J. D., Mislove, M., and Scott, D. S.: *Continuous Lattices and Domains*. Oxford University Press, 2003.
- [8] Scott, D.S.: Continuous lattices. F.W. Lawvere (Ed.), *Toposes, Algebraic Geometry and Logic. Lecture Notes in Math.* **274** (1971), 97–136.

FACULTY FOR MATHEMATICS AND PHYSICS, LEIBNIZ UNIVERSITY HANNOVER, D-30167 HANNOVER, GERMANY

*E-mail address:* [erne@math.uni-hannover.de](mailto:erne@math.uni-hannover.de)

# Possibility Semantics

Wesley H. Holliday

University of California, Berkeley  
wesholliday@berkeley.edu

I will survey a recent research program of investigating “possibility semantics”, a generalization of possible world semantics, for modal, superintuitionistic, and inquisitive logics. Relevant references include the following:

G. Bezhanishvili and W. H. Holliday, “A semantic hierarchy for intuitionistic logic,” *Indagationes Mathematicae*, 2019 (<https://escholarship.org/uc/item/2vp2x4rx>).

G. Bezhanishvili and W. H. Holliday, “Locales, nuclei, and Dragalin frames,” *Advances in Modal Logic*, 2016 (<https://escholarship.org/uc/item/2s0134zx>).

N. Bezhanishvili and W. H. Holliday, “Choice-free Stone duality,” *The Journal of Symbolic Logic*, forthcoming (<https://escholarship.org/uc/item/00p6t2v4>).

N. Bezhanishvili, G. Grilletti, and W. H. Holliday, “Algebraic and topological semantics for inquisitive logic via choice-free duality,” Proceedings of WoLLIC 2019 (<https://escholarship.org/uc/item/69f4t1wg>).

W. H. Holliday, “Algebraic semantics for S5 with propositional quantifiers,” *Notre Dame Journal of Formal Logic*, forthcoming (<https://escholarship.org/uc/item/303338xr>).

W. H. Holliday, “Partiality and adjointness in modal logic,” *Advances in Modal Logic*, 2014 (<https://escholarship.org/uc/item/9pm9t4vp>).

W. H. Holliday, “Possibility frames and forcing for modal logic,” UC Berkeley Working Paper, 2015 (<https://escholarship.org/uc/item/0tm6b30q>).

W. H. Holliday and T. Litak, “Complete additivity and modal incompleteness,” *The Review of Symbolic Logic*, forthcoming (<https://escholarship.org/uc/item/01p9x1hv>).

# Non-finitely axiomatisable canonical varieties of BAOs with infinite canonical axiomatisations

Agi Kurucz  
Department of Informatics  
King's College London

The abstract can be found on the TACL web page.

## THE POSET OF ALL LOGICS

T. MORASCHINI

Universal algebra and abstract algebraic logic are two disciplines that study, respectively, general algebraic structures and propositional logics. One of their main achievements is the development of two parallel taxonomies, one of varieties (a.k.a. equational classes) of algebras, and the other one of propositional logics.

More precisely, the *Maltsev hierarchy* of universal algebra is a classification of varieties in terms of syntactic principles (called *Maltsev conditions*) intended to describe the structure of the congruence lattices of algebras. The first, and perhaps most celebrated, example of a Maltsev condition is the requirement that a variety  $K$  is congruence permutable, equivalent to the syntactic requirement of the existence of a *minority term* for  $K$ , i.e. a ternary term  $\varphi(x, y, z)$  such that

$$K \models \varphi(x, x, y) \approx y \approx \varphi(y, x, x).$$

Similarly, in abstract algebraic logic, the *Leibniz hierarchy* is a taxonomy of propositional logics in terms of rule schemata (called *Leibniz conditions*) whose aim is to govern the interplay between lattices of deductive filters (a.k.a. theories) of logics and lattices of congruences of algebras. One of the most fundamental examples of a Leibniz condition is the requirement that a logic  $\vdash$  possesses a set  $\Delta(x, y)$  of binary formulas satisfying the rules

$$\emptyset \triangleright \Delta(x, x) \text{ and } x, \Delta(x, y) \triangleright y,$$

which generalize the behavior of most implication connectives. This requirement is equivalent to the property that the Leibniz operator of the logic  $\vdash$  is monotone.

From this point of view, it is natural to wonder whether the Maltsev and Leibniz hierarchies are two faces of the same coin. In this talk we investigate this and some related questions.

Part of the work I will report on is joint with R. Jansana.

# Some Applications of Stone Duality to Automata Theory

Daniela Petrişan<sup>1</sup>

Université de Paris, IRIF, CNRS, F-75013 Paris, France  
petrisan@irif.fr

In this talk I will give an overview of several instances where Stone duality is underpinning important constructions in automata theory.

A first such example is a proof of correctness of Brzozowski’s minimization algorithm. Several constructions featured in this algorithm can be understood as liftings of well known adjunctions to categories of automata: determinization of automata can be understood via a lifting of the Kleisli adjunction between the category `Rel` of sets and relations and the category `Set` of sets and functions; while reversing nondeterministic automata can be understood via a lifting of the self-duality of `Rel`.

A second application is a methodology for extending notions of algebraic recognition to their topological counterparts. I will discuss a notion of syntactic Boolean space associated to a language and extensions of well-known constructions on regular languages to a non-regular setting.

The talk is based on the papers [1] and [2, 3].

## References

- [1] Thomas Colcombet and Daniela Petrişan. Automata minimization: a functorial approach. In *CALCO*, volume 72 of *LIPICs*, pages 8:1–8:16. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2017.
- [2] Mai Gehrke, Daniela Petrisan, and Luca Reggio. The schützenberger product for syntactic spaces. In *ICALP*, volume 55 of *LIPICs*, pages 112:1–112:14. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2016.
- [3] Mai Gehrke, Daniela Petrisan, and Luca Reggio. Quantifiers on languages and codensity monads. In *LICS*, pages 1–12. IEEE Computer Society, 2017.



## Snapshots of duality theory, from 2019 and fifty years earlier

Hilary Priestley

Sugihara monoids and Sugihara algebras provide complete algebraic semantics for the relevance logics  $R$ -mingle, with or without Ackermann's truth constant. The variety  $\mathcal{SA}$  of Sugihara algebras is generated by  $\mathbf{Z} = (Z, \wedge, \vee, \neg, \rightarrow)$  having the chain of integers as lattice reduct,  $\neg a = -a$  and  $a \rightarrow b = (-a) \vee b$  if  $a \leq b$  and  $(-a) \wedge b$  otherwise. For  $\mathcal{SM}$ , Sugihara monoids, the constant 0 is added to the language. Both varieties are locally finite, with their finite subdirectly irreducibles having size  $k$ , for  $k = 2, 3, \dots$

The 2019 snapshot will showcase joint work with Leonardo Cabrer [1]–[4]. This work demonstrates the power of a range of duality techniques. We exploit dual equivalences with strong properties and capitalise on what the theory delivers at the finite level: 'logarithmic' behaviour, pictorial representations, and transparent access to finitely generated free algebras. Our aim is salesmanship: the underlying natural duality theory will be locked away in a black box.

Stone duality and Priestley duality have very special features. They are *strong dualities*: dual equivalences set up by hom-functors into a generating algebra and an alter ego, with both functors converting embeddings (surjections) to surjections (embeddings); moreover, duals of free algebras are given by concrete products. A duality of this type has been developed for any finitely generated quasivariety of Sugihara algebras [3]. By moving to a framework allowing multisorted dual structures, one can embrace any finitely generated quasivariety or variety, and likewise for Sugihara monoids. Additionally, the multisorted approach facilitates transition to the Priestley duality for the lattice reducts. The relational structure on the dual side is supplied by homomorphisms and partial endomorphisms. The varieties  $\mathcal{SA}$  and  $\mathcal{SM}$  can be treated together. [To the initiated: no restriction to the odd case.]

### Free algebras [4]

Sugihara algebras and monoids have the property that any finitely generated free algebra  $F$  can be calculated within some finitely generated (quasi)variety and so can be accessed easily using a multisorted natural duality for that (quasi)-variety. From the natural dual of  $F$  one can pass by a quotienting process to the Birkhoff dual  $Y$  of  $F$ 's lattice reduct. This provides a picture of  $Y$  which lends itself to combinatorial analysis. Moreover, the pictures display evidence of finite level Esakia duality in play. (Affinities between odd Sugihara monoids and Heyting algebras are known to exist in general.)

### Admissibility algebras: a route to admissible rules for $R$ -mingle

An algebraic method due to Metcalfe and Röthlisberger for testing a rule for admissibility is only computationally feasible for varieties whose free algebras are small—a rare occurrence. Cabrer *et al.* [1] proved that, when a strong duality is available, the problem translates into dual form and this leads to an algorithm to find, for any  $n$ , a minimal 'admissibility algebra' on which to test

rules (*qua* quasi-equations) in  $n$  variables. Cabrer and Priestley [2] applied the algorithm to each finitely generated Sugihara algebra quasivariety, so providing a solution for any  $k$  to a problem which is insoluble algebraically by computer when  $k = 5$  (3 variables).

We contrast our methods and results with those obtainable from dual equivalences between a category of expansions of distributive lattices and some category of structured Priestley spaces, of which there is a bewildering array in the literature. Equivalences of this type may yield valuable concrete representations and, by forgetting the topology, discrete dualities providing Kripke-style semantics for associated propositional logics. But there is no guarantee that algebraic, or logical, problems can be recast so that they become easier to solve in the dual setting.

This year marks the 50th anniversary of the submission to the Bulletin of the London Mathematical Society of *Representation of distributive lattices by means of ordered Stone spaces*, introducing what has become known as Priestley duality. The second, brief, snapshot will acknowledge a number of other contributions from long ago which have not received due recognition. (Here, the publication in 2019 of an edited English translation of Leo Esakia's classic monograph on duality for Heyting algebras is most welcome.) The opportunity will also be taken to highlight what, with hindsight, have proved to be major landmarks in the evolution of duality theory in a TACL context.

## References

- [1] L.M. Cabrer, B. Freisberg, G. Metcalfe and H.A. Priestley, Checking admissibility using natural dualities *Trans. Comput. Logic* **20**, no.1, Art. 2 (2019) (preprint available at [www.arXiv:1801.02046v2](http://www.arXiv:1801.02046v2))
- [2] L.M. Cabrer and H.A. Priestley, Sugihara algebras: admissibility algebras via the test spaces method (submitted; preprint available at [www/arxiv.org/abs/1809.07816v2](http://www.arxiv.org/abs/1809.07816v2))
- [3] L.M. Cabrer and H.A. Priestley, Sugihara algebras and Sugihara monoids: multisorted dualities (submitted; preprint available at <https://arxiv.org/abs/1901.09533v2>)
- [4] L.M. Cabrer and H.A. Priestley, Sugihara algebras and Sugihara monoids: free algebras (manuscript)

## MODAL LOGICS OF DEPENDENCE

Johan van Benthem, Amsterdam, Stanford & Tsinghua – joint work with Alexandru Baltag

**1 Dependence** The notion of *dependence* is important in such diverse areas as probability, natural language, databases, causality, or interactive games. Of course, senses may differ. In a database, dependence means we can predict the value of one variable from that of others, in statistics, dependence is correlation of values, in logic, dependence arises from quantifier scoping (to see that  $\forall x \exists y Rxy$ , we pick  $y$  dependent on  $x$ ), in causal models, dependence is about real determination, and so on. A useful distinction: ontic dependence in the world vs. epistemic dependence: having information about  $x$  implies having information about  $y$ . The former notion is central, for instance, in Situation Theory, the latter in Epistemic Logic.

**2 Dependence logic** Dependence has caught the attention of logicians, and various systems have been proposed in recent decades, from simple to quite complex. Our framework: Models  $\mathbf{M} = (W, \{=_{x}\}_{x \in \text{VAR}}, V)$  with  $W$  a set of assignments of values to variables,  $s =_x t$  iff  $s(x) = t(x)$ , and  $V$  a valuation function from atomic formulas  $Px$  and  $s \in W$  to truth values. (For sequences  $\mathbf{x}$ ,  $=_{\mathbf{x}}$  means equality for all  $x \in \mathbf{x}$ .) If  $W$  is the space of all maps from VAR to a value domain  $D$ , all variables are independent, while ‘gaps’ encode dependencies. These models, proposed in the 1990s (cf. van Benthem 1996), cover both ontic and epistemic dependence (van Benthem 2001).

**3 CRS and decidable predicate logic** Technical background: CRS style relativized cylindric algebra from the 1980s, supporting decidable systems of predicate logic (cf. Andréka, van Benthem & Némethi 1998). But the above models also support richer first-order languages with irreducible polyadic quantifiers  $\exists \mathbf{x} \varphi$  and substitution modalities. The modal perspective even suggests abstract ‘state models’ that still allow for compositional semantics without buying into undue set theory.

**4 From implicit to explicit** CRS predicate logic is ‘implicit’ (van Benthem 2018): dependence is not made explicit in the language, the logic is non-classical. Instead, we now define an ‘explicit’ classical modal language with two key operators:

$$\begin{aligned} \mathbf{M}, s \models D_{\mathbf{x}}\varphi & \text{ iff } \text{for all } t \text{ with } t =_{\mathbf{x}} s, \mathbf{M}, t \models \varphi \\ \mathbf{M}, s \models D_{\mathbf{x}y} & \text{ iff } \text{for all } t \text{ with } t =_{\mathbf{x}} s, t(y) = s(y) \end{aligned}$$

This language can express a variety of notions of local and global dependence. Key semantic feature: invariance of truth values for formulas  $\varphi$  under shifts to assignments agreeing on the ‘fixing variables’ of  $\varphi$  (this is dual to first-order quantifiers).

**5 Epistemic view** Dependence models induce epistemic models,  $D_{\mathbf{x}}\varphi$  expresses ‘distributed knowledge’ in group  $\mathbf{x}$ , and  $D_{\mathbf{x}y}$  informational dependence of agents.

Vice versa, epistemic models can be represented as assignment models (cf. van Benthem 1996), suggesting generalized dependence models in more abstract style. The epistemic connection makes sense of an interrogative intuition of dependence:  $D_{xy}$  if answers to questions about all of the  $x$  imply an answer to a question about  $y$ .

**6 Proof system** The axioms and rules of modal dependence logic  $LFD$  consist of S5 for the modalities  $D_x\varphi$ , analogues of the Armstrong Axioms for dependence  $D_{xy}$ , and a transfer axiom  $(D_{xy} \wedge D_y\varphi) \rightarrow D_x\varphi$ . This is the core calculus of dependence. The axiom system is sound. There is also an illuminating sequent calculus version.

**7 Completeness Theorem:** A modal formula is derivable in  $LFD$  iff it is valid in dependence models. The proof goes from a standard modal Henkin model to quasi-models which are then represented eventually as the above assignment models.

**8 Decidability Theorem:** Validity in  $LFD$  is decidable. The proof uses a quasi-model construction similar to that known for the Guarded Fragment, but with additional twists in the representation technique in order to deal with dependence atoms.

**9 Correspondence** Additional axioms express constraints on dependence models. First-order axioms  $\exists x \forall y \varphi \rightarrow \forall y \exists x \varphi$  express confluence, axiom  $D_{xyz} \rightarrow (D_{zy} \vee D_{zyx})$  (van Lambalgen 1994) imposes extra structure as in linear vector spaces.

**10 Richer expressive power** \* Decidability remains with added function symbols. \* Equality  $s=t$  leaves axiomatizability but endangers decidability. \* *Theorem:* Modalities for *independence* make  $LFD$  undecidable. \* Updating models with new information: Dynamic logic of learning value of variables  $[x]\varphi$  is decidable, via recursion axioms. Public announcements  $[\alpha]\varphi$  change global dependence structure, requiring conditional dependence modalities, and this system may well be undecidable.

**11 Extended similarity types** Structure beyond our models needed with games, causal graphs, topological spaces for approximating values, and for probabilistic dependence. No results to report yet, but the modal perspective offers new angles.

**12 Related work** There are dependence logics of various sorts in the literature. Time permitting, we will make a brief comparison with the extended epistemic logics of Wang 2016 and the dependence logic of Väänänen 2003.

**References** H. Andréka, J. van Benthem & I. Németi, JPL 1998, ‘Modal Logics and Bounded Fragments of Predicate Logic’. A. Baltag, AIML 2016, ‘To Know is to Know the Value of a Variable’. J. van Benthem: CSLI Pub’s 1996, *Exploring Logical Dynamics*, ILLC 2001, ‘Information as Range or as Correlation’, JPL 2018, ‘Implicit vs. Explicit Stances in Logic’. B. ten Cate & Ph. Kolaitis, 2015, ‘Schema Mappings in Database Theory’. M. van Lambalgen, JSL 1994, ‘Independence, Randomness & the Axiom of Choice’. J. Väänänen, CUP 2003, *Dependence Logic*. Y. Wang, 2016, ‘Beyond Knowing That: New Generation of Epistemic Logics’.

# Logic, Automata, and Model Companions

Sam van Gool, Utrecht University

The aim of this talk is to show a connection between temporal logics and monadic second order (MSO) logics on discrete structures, mediated by model theory.

In formal language theory, logic can be used as a descriptive formalism which measures the complexity of a computational problem. For example, the MSO-definable sets of finite words are exactly the regular languages from automata theory. Büchi and Rabin established such translations between MSO logic and automata on many more structures, including omega-indexed words and various types of trees.

In model theory, logic can be used to give a general account of a mathematical construction; particularly relevant to this talk is the construction of the algebraic closure of a field. A fundamental insight due to Robinson is that the notion of algebraically closed field can be generalized to a purely logical notion of “existentially closed model”. The “model companion” of a first order theory, if it exists, gives a first order description of the class of existentially closed models.

This talk’s main thesis will be that MSO logic ‘is’ the model companion of temporal logic. That is, monadic second order logic is to temporal logic as algebraically closed fields are to fields. Indeed, in joint work with Ghilardi, we proved such a model companion result both for words [1] and for trees up to bisimulation [2]. Extending these results to full MSO on trees is the subject of ongoing work. In the remainder of this abstract, I will make the statement of the result for words [1] more precise.

Monadic second order logic on  $\omega$ , also known as S1S (second order logic of one successor), is defined by adding to the first-order logic of the successor function quantification over unary predicates, i.e., subsets of  $\omega$ . S1S can be used to define sets of *streams* over a finite alphabet  $\Sigma$ , i.e., functions  $S: \omega \rightarrow \Sigma$ . Let  $\varphi$  be a formula of S1S, all of whose free second-order variables are in a finite set  $V$ . Then valuations  $v: V \rightarrow \mathcal{P}(\omega)$  are in one-to-one correspondence with  $\mathcal{P}(V)$ -streams  $S_v: \omega \rightarrow \mathcal{P}(V)$ , and the formula  $\varphi$  thus defines a set of  $\mathcal{P}(V)$ -streams:

$$L_\varphi := \{S_v : \omega, v \models \varphi\}.$$

A different way of arriving at languages of  $\Sigma$ -streams, where now  $\Sigma$  is any finite alphabet, is by using non-deterministic finite automata (NFA). A language of  $\Sigma$ -streams is  $\omega$ -regular if it is recognized by some NFA. Here, the definitions of automata are the same as for finite words, except that the acceptance condition now says that there is a run which visits a final state infinitely often. Büchi proved that the stream languages of the form  $L_\varphi$  are exactly the  $\omega$ -regular languages. In fact, he gave an effective procedure which transforms an S1S formula into an automaton, and vice versa. The S1S formula which is associated to an automaton in this procedure has a special form: all the second-order quantifications are *existential*. From

Büchi’s result, one may thus obtain a ‘normal form’ for S1S, by translating into an automaton and back.

This normal form result is what allows us to prove that a certain first-order theory,  $T^*$ , closely related to S1S, is model complete. Here, a first-order theory  $T^*$  is called *model complete* if for every formula  $\varphi$ , there is an existential formula  $\varphi'$  such that  $T^* \vdash \varphi \leftrightarrow \varphi'$ . Thus, a model complete theory ‘almost’ has quantifier elimination, up to the last layer of quantifiers. The *model companion* of a universal first-order theory  $T$  is the model complete theory  $T^*$  which has the same universal consequences as  $T$ . The model companion of  $T$  is unique if it exists, in which case it is the first order theory of the class of existentially closed models for  $T$ .

Finally, what is the temporal logic that S1S is a model companion of? Denote by  $\mathcal{L}$  the one-way, discrete, linear temporal logic with unary operators ‘next’ and ‘future’, and a constant ‘initial moment’. That is, the syntax of  $\mathcal{L}$  is the propositional language is enriched with unary symbols  $\mathbf{X}$  and  $\mathbf{F}$ , and a nullary symbol  $\mathbf{I}$ . The algebraic models for  $\mathcal{L}$  are Boolean algebras with operators in this signature, subject to axioms expressing that (i)  $\mathbf{X}$  is a Boolean endomorphism; (ii)  $\mathbf{F}a$  is the least fixpoint of the function  $x \mapsto a \vee \mathbf{X}x$ ; (iii)  $\mathbf{I}$  is an atom such that  $\mathbf{X}\mathbf{I} = \perp$ , and  $\mathbf{I} \leq \mathbf{F}a$  whenever  $a \neq \perp$ . Write TA for the universal first order theory axiomatizing this class of temporal algebras. The prototypical example of a temporal algebra is the Boolean algebra  $\mathcal{P}(\omega)$ , equipped with temporal operators  $\mathbf{X}a := \{x \in \omega \mid x + 1 \in a\}$ ,  $\mathbf{F}a := \{x \in \omega \mid \exists y \geq x, y \in a\}$  and  $\mathbf{I} := \{0\}$ . Write TA\* for the first order theory of this particular temporal algebra. Clearly, looking at first order formulas in the algebra  $\mathcal{P}(\omega)$  is almost the same thing as looking at monadic second order formulas interpreted in  $\omega$ , thus,  $\text{TA}^* \approx \text{S1S}$ . Our main result in [1] is:

**Theorem.** The theory TA\* is the model companion of the theory TA.

The proof of this result involves two parts: the first is the normal form procedure mentioned above, the second is a completeness result for the logic  $\mathcal{L}$  with respect to the intended model  $\omega$ , for which we give a short proof based on a Stone-Jónsson-Tarski style duality for temporal algebras.

The above theorem establishes that S1S, when viewed as the first-order theory of the temporal algebra  $\mathcal{P}(\omega)$ , ‘is’ the model companion of the linear temporal logic  $\mathcal{L}$  described above. In [2], we extend this result to monadic second order logic S2S of two successors, which is interpreted on binary trees, and we also treat arbitrarily branching trees, but there we restrict MSO to its bisimulation-invariant fragment. The temporal logics involved in [2] are more involved: we had to design an extension of computation tree logic (CTL) with binary ‘fairness’ operators, and prove a completeness result for that. This led us into a complex but interesting study of the completeness of certain fragments of the  $\mu$ -calculus. In current work in progress, we plan to extend this work to a *graded* temporal logic, towards obtaining a model companion result for full MSO on trees.

## References

- [1] S. Ghilardi and S. J. van Gool, *A model-theoretic characterization of monadic second-order logic on infinite words*, Journal of Symbolic Logic, vol. 82, no. 1, 62-76 (2017).
- [2] S. Ghilardi and S. J. van Gool, *Monadic second order logic as the model companion of temporal logic*, Proc. LICS 2016, 417-426 (2016).

# Anabelian geometry in model theory setting

B.Zilber

University of Oxford

Anabelian geometry is a relatively new branch of algebraic / arithmetic geometry introduced by A.Grothendieck. Its standard language is the language of category theory. The first part of the project reported here is the reformulation of main notions and conjectures of anabelian geometry in the formalism of model theory. Translations between model theory and category theory have proved useful in some other contexts, in particular allowing application of deep results of model theoretic classification theory (categoricity and stability in a broad sense).

The first step (jointly with R.Abdolazadi) is the introduction of a canonical language and a construction of (a) structures  $\tilde{X}^{et}$  in this language that correspond to pro-étale covers of a smooth algebraic variety (reduced scheme)  $X$  over a number field  $k$  and (b) structures  $\tilde{X}^{an}$  that correspond to the universal cover of the complex variety  $X(\mathbb{C})$ . Both are multisorted structures with sorts corresponding to étale covers (unramified covers) and the language allows to express the algebraic geometry on the sorts along with étale morphisms between the sorts.

We prove that, given  $X$  over  $k$ ,

$$\tilde{X}^{et} \equiv \tilde{X}^{an},$$

that is the two structures have the same first-order theory, which we call  $T_X$ . are indistinguishable in the first-order setting.

Moreover, one can give quite a good description of the first-order theory of the structures.

Further on we establish that

$$\pi_1^{et}(X) \cong \text{Aut}(\tilde{X}^{et}),$$

that is the étale fundamental group of  $X$  is the automorphism group of the structure  $\tilde{X}^{et}$ .

This allows to reformulate main conjectures of anabelian geometry in model theory setting. In particular, Grothendieck's section conjecture for hyperbolic curves is proved to be equivalent to the statement on elimination of imaginaries of certain type.

The second part of the project concentrates on model theoretic analysis of structure  $\tilde{X}^{an}$ , the universal cover of  $X(\mathbb{C})$  in analytic category. An essential step here is the extending the first order axioms  $T_X$  to an  $L_{\omega_1, \omega}$ -axiomatisation  $\Sigma_X$  (allowing countable conjunctions and disjunctions) which **conjecturally** may characterise  $\tilde{X}^{an}$  as the only model of cardinality continuum.

The formulation and study of  $\Sigma_X$ , for  $X$  smooth algebraic variety, generalises the similar work for  $X$  a semi-abelian variety carried out by a number of authors in 2002-2015, see a survey in the first part of [1]. Model-theoretic techniques used in these studies is an adaptation of Shelah's theory of *abstract elementary classes* (AEC). The main innovation is the introduction of the language (see above) adequate to the cases of the  $X$  with non-abelian topological fundamental group  $\pi_1(X)$ . Another novelty is the understanding of the crucial role played by

special sets (a generalisation of special subvarieties of mixed Shimura varieties, see also [1]) in formulating axioms  $\Sigma_{\mathbb{X}}$ .

The above mentioned studies resulted in two important type of statements for  $\mathbb{X}$  a non-CM elliptic curve or the algebraic torus  $\mathbb{G}_m$  (M.Bays, M.Gavrilovich and the author) :

1. The abstract elementary class defined by  $\Sigma_{\mathbb{X}}$  has unique model (is categorical) in all uncountable cardinalities;
2. The above statement of categoricity for a broader class of semi-abelian varieties  $\mathbb{X}$  is equivalent to the conjunction of the following two arithmetic statements:
  - (a)  $\text{Gal}(\bar{k} : k)$  acts on the torsion subgroup  $\mathbb{T}$  of  $\mathbb{X}$  as a finite index subgroup of the full automorphism group  $\text{Aut}(\mathbb{T})$  (known as Serre's theorem for elliptic curves);
  - (b) an appropriate re-statement of the main theorem of Kummer theory (known as Bashmakov's theorem for elliptic curves).

Our main result is the generalisation of statement 2 to the general  $\mathbb{X}$ . In this case the torsion subgroup  $\mathbb{T}$  is replaced by the profinite completion  $\hat{\pi}_1(\mathbb{X})$  of the topological fundamental group of  $\mathbb{X}(\mathbb{C})$  as the group acting on special sets. Respectively  $\text{Gal}(\bar{k} : k)$  acts as  $\text{Aut}_S(\hat{\pi}_1(\mathbb{X}))$  on the group, where the subscript  $S$  indicates that we consider the group together with its action on special sets.

For  $\mathbb{X}$  equal to  $\mathbf{P}^1 \setminus \{0, 1, \infty\}$  (the projective line minus 3 points) the group  $\text{Aut}_S(\hat{\pi}_1(\mathbb{X}))$  is conjecturally the Grothendieck - Teichmüller group which is the subject of studies since its introduction in Grothendieck's "Esquisse d'un programme".

The "Kummer theory" part 2(b) for the general  $\mathbb{X}$  was essentially formulated by Adam Harris, [2].

## References

- [1] B.Zilber, *Model theory of special subvarieties and Schanuel-type conjectures*, Annals of Pure and Applied Logic, v. 167, 10, 2016, pp. 1000–1028;
- [2] C.Daw and A.Harris, *Categoricity of modular and Shimura curves*, Journal of the Institute of Mathematics of Jussieu, v. 16, 5, 2017, pp.1075 –1101



# Norm complete abelian $\ell$ -groups: topological duality

Marco Abbadini<sup>1</sup>, Vincenzo Marra<sup>1</sup>, and Luca Spada<sup>2</sup>

<sup>1</sup> Dipartimento di Matematica, Università di Milano  
{marco.abbadini, vincenzo.marra}@unimi.it

<sup>2</sup> (Presenting author) Dipartimento di Matematica, Università di Salerno  
lspada@unisa.it

## 1 Introduction

Let  $X$  be a compact Hausdorff space. Four main representation theorems concerning the algebra of continuous functions over  $X$  were proved at the beginning of the forties. In 1941, Kakutani [3] gave an order-theoretic characterisation of the unital lattice-ordered real Banach lattices of the form  $C(X, \mathbb{R})$  (= continuous functions from  $X$  into  $\mathbb{R}$ ); in this result the non-algebraic concept of *norm* plays a crucial rôle. In the same year, Yosida showed in the landmark paper [5] that a vector lattice with an order unit is isomorphic to  $C(X, \mathbb{R})$  if, and only if, it is archimedean and norm-complete. Similarly, Stone proved in [4] that an abelian lattice ordered group (henceforth  $\ell$ -group) with an order unit is isomorphic to  $C(X, \mathbb{R})$  if, and only if, it is divisible, archimedean, and norm-complete. In sharp contrast with Kakutani's result, the norm in the two latter results is not a primitive operator, but it is induced by the order unit. Finally, in 1943, on the way to a representation theorem for complex  $C^*$ -algebras, Gelfand and Neumark [1] proved that a complex unital  $C^*$ -algebra can be represented as the family of all continuous  $\mathbb{C}$ -valued functions on a compact Hausdorff space if, and only if, it is commutative. As in Kakutani's representation result, the norm is a primitive element in the structure of a  $C^*$ -algebra. All the aforementioned results extend to dualities with the category of compact Hausdorff spaces and continuous functions among them.

In this work we are concerned with a generalisation of Stone's result to *non divisible*  $\ell$ -groups. A first important result to the effect of a functional representation of every archimedean norm complete (w.r.t. the norm induced by the order unit)  $\ell$ -group, was proved by Goodearl and Handelman [2, Theorem 5.5]:

**Theorem 1.1.** *Let  $X$  be a compact Hausdorff space and  $C(X, \mathbb{R})$  be the set of continuous functions from  $X$  into  $\mathbb{R}$ . For each  $x \in X$ , let  $A_x$  be either  $\mathbb{R}$  or  $\mathbb{Z}_n$  for some positive integer  $n$ . Set*

$$D = \{f \in C(X, \mathbb{R}) \mid f(x) \in A_x \text{ for all } x \in X\},$$

*and give to  $D$  the  $\ell$ -group structure inherited from  $C(X, \mathbb{R})$ . Then  $D$  is an (archimedean) norm complete  $\ell$ -group. Conversely, any such a group is isomorphic to one of this form.*

The crucial restriction to functions such that  $f(x) \in A_x$  can be understood as a *labelling* on the space  $X$  that must be respected by the continuous functions considered. This serves as motivation for the definitions in the next section.

## 2 Main results

**Definition 2.1** (The category of a-spaces).

1. Any tuple  $(X, \tau, \zeta)$ , where  $(X, \tau)$  is a topological space and  $\zeta$  is a function from  $X$  into  $\mathbb{N}$  will be called *arithmetic space* or *a-space*, for short. (The symbol  $\zeta$  stands for знаменатель i.e. denominator)
2. If  $(X, \tau, \zeta)$  and  $(X', \tau', \zeta')$  are two a-spaces and  $f: X \rightarrow X'$  is a function, we will say that  $f$  is an *a-map*, if it is continuous and for any  $x \in X$ ,  $\zeta'(f(x))$  divides  $\zeta(x)$ .
3. We call  $\mathbf{A}$  the category of a-spaces with a-maps among them.

**Example 2.2.** For  $q \in \mathbb{R}$ , we write  $\mathbf{den}(q)$  to denote the denominator of  $q$  (in its irreducible form) if  $q \in \mathbb{Q}$ , and 0 otherwise. Let  $I$  be a set. We extend  $\mathbf{den}$  to a map  $\hat{\mathbf{den}}: \mathbb{R}^I \rightarrow \mathbb{N}$  as follows: Let  $p = (p_i)_{i \in I} \in \mathbb{R}^I$ ,

$$\hat{\mathbf{den}}(p) = \begin{cases} \mathbf{lcm}\{\mathbf{den}(p_i) \mid i \in I\} & \text{if } p \in \mathbb{Q}^I, \\ 0 & \text{otherwise.} \end{cases}$$

Where  $\mathbf{lcm}$  stands for least common multiple and 0 is the top of the lattice  $\mathbb{N}$  under the divisibility order. Then, writing  $\rho$  for the usual Tychonoff topology on  $\mathbb{R}^I$ , the triples  $(\mathbb{R}^I, \rho, \hat{\mathbf{den}})$  and  $([0, 1]^I, \rho, \hat{\mathbf{den}})$  are a-spaces.

For  $n \in \mathbb{N}$  we shall write  $\mathbf{div}(n)$  for the set of natural numbers that divide  $n$ .

**Definition 2.3.** An arithmetic space  $(X, \tau, \zeta)$  is called *a-normal space* if the following conditions hold:

1.  $(X, \tau)$  is compact Hausdorff.
2. For every  $n \in \mathbb{N}$ ,  $\zeta^{-1}[\mathbf{div}(n)]$  is closed in the topology  $\tau$ .
3. For every disjoint closed subsets  $A$  and  $B$  of  $(X, \tau)$ , there exist two open disjoint neighbourhoods  $U$  and  $V$  of  $A$  and  $B$ , respectively, such that for every  $x \in X \setminus (U \cup V)$ ,  $\zeta(x) = 0$ .

**Theorem 2.4.** *The a-space  $([0, 1]^I, \rho, \hat{\mathbf{den}})$  of Example 2.2 is a-normal.*

If  $(X, \tau, \zeta)$  is an a-space, we shall denote by  $\mathbf{A}_b(X, \mathbb{R})$  the unital  $\ell$ -group of bounded a-maps from  $(X, \tau, \zeta)$  into  $(\mathbb{R}, \rho, \hat{\mathbf{den}})$ . The main result of this work is the following:

**Theorem 2.5.** *An a-space  $(X, \tau, \zeta)$  is isomorphic to  $(\mathbf{Max}(\mathbf{A}_b(X, \mathbb{R})), \sigma, \hat{\mathbf{den}})$  (where  $\sigma$  is the Stone-Zariski topology) if, and only if,  $(X, \tau, \zeta)$  is an a-normal space.*

**Corollary 2.6.** *The category of (archimedean) norm complete  $\ell$ -groups with  $\ell$ -group morphisms preserving the order unit is dually equivalent to the full subcategory of  $\mathbf{A}$  given by a-normal spaces.*

## References

- [1] I. Gelfand and M. Neumark. On the imbedding of normed rings into the ring of operators in Hilbert space. *Rec. Math. [Mat. Sbornik] N.S.*, 12(54):197–213, 1943.
- [2] K. R. Goodearl and D. E. Handelman. Metric completions of partially ordered abelian groups. *Indiana Univ. Math. J.*, 29(6):861–895, 1980.
- [3] S. Kakutani. Concrete representation of abstract  $(M)$ -spaces. (A characterization of the space of continuous functions.). *Ann. of Math. (2)*, 42:994–1024, 1941.
- [4] M. H. Stone. Applications of the theory of Boolean rings to general topology. *Trans. Amer. Math. Soc.*, 41(3):375–481, 1937.
- [5] K. Yosida. On vector lattice with a unit. *Proc. Imp. Acad. Tokyo*, 17:121–124, 1941.

# Norm complete Abelian $\ell$ -groups: equational axiomatization

Marco Abbadini<sup>1</sup>, Vincenzo Marra<sup>2</sup>, and Luca Spada<sup>3</sup>

<sup>1</sup> (Presenting author) Dipartimento di Matematica, Università di Milano  
`marco.abbadini@unimi.it`

<sup>2</sup> Dipartimento di Matematica, Università di Milano  
`vincenzo.marra@unimi.it`

<sup>3</sup> Dipartimento di Matematica, Università di Salerno  
`lspada@unisa.it`

## 1 Introduction

An *Abelian lattice-ordered group* (or  $\ell$ -group, for short) is an Abelian group  $G$ , endowed with a partial lattice order  $\leq$  that is translation invariant, i.e., for all  $x, y, z \in G$ , if  $x \leq y$ , then  $x + z \leq y + z$ . An element  $u$  of an  $\ell$ -group is a (strong order) unit if, for all  $x \in G$ , there exists  $n \in \mathbb{N}$  such that  $|x| \leq nu$ . A *unital  $\ell$ -group* is an  $\ell$ -group  $G$  with a designated unit  $u$ , and a *morphism of unital  $\ell$ -groups* is a map that preserves the lattice structure, the group structure, and the unit.

An undesired issue about unital  $\ell$ -groups is that, in their usual presentation, they fail to be an elementary class. The problem is essentially due to the fact that the definition of unit may not be expressed in first-order logic, as an application of the compactness theorem shows. However D. Mundici showed in [4] that the category of unital  $\ell$ -groups is equivalent to a finitary variety finitely axiomatized: the category of MV-algebras. In particular, an MV-algebra  $\langle A, \oplus, \neg, 0 \rangle$  is a set  $A$ , equipped with a binary operation  $\oplus$ , a unary operation  $\neg$  and a distinguished constant  $0$  such that  $\langle A, \oplus, 0 \rangle$  is a commutative monoid,  $\neg\neg x = x$ ,  $x \oplus \neg 0 = \neg 0$ , and  $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$ . We have a functor  $\Gamma$ —which is proved to be an equivalence in [4]—from the category of unital  $\ell$ -groups to the category of MV-algebras: for  $(G, u)$  a unital  $\ell$ -group,  $\Gamma((G, u)) := \{x \in G \mid 0 \leq x \leq u\}$ , where, for  $x, y \in \Gamma((G, u))$ ,  $x \oplus y := (x + y) \wedge u$ , and  $\neg x := u - x$ .

Every unital  $\ell$ -group  $(G, u)$  carries a natural seminorm  $\|x\| := \inf \left\{ \frac{p}{q} \in \mathbb{Q}^+ \mid qx \leq pu \right\}$ , which induces a pseudometric  $d(x, y) := \|x - y\|$ . What is missing for  $d$  to be a metric is the implication  $d(x, y) = 0 \Rightarrow x = y$ . This happens precisely when  $G$  is Archimedean, i.e. when, for all  $x, y \in G$ , if, for all  $n \in \mathbb{N}$ ,  $n|x| \leq y$ , then  $x = 0$ . We write *norm complete  $\ell$ -group* for “Archimedean unital  $\ell$ -group complete in the metric  $d$ ”. A *morphism of norm complete  $\ell$ -groups* is simply a morphism of unital  $\ell$ -groups.

## 2 Main results

Our main result is the following.

**Theorem 2.1.** *Up to an equivalence, the category of norm complete  $\ell$ -groups is an (infinitary) variety of algebras.*

Theorem 2.1 is analogous to the well-known result that, up to an equivalence of categories, the category of norm complete vector lattices is an (infinitary) variety of algebras (see [1], [2] and [3]). The difference here is—roughly speaking—that divisibility is not required.

We obtain Theorem 2.1 in two steps.

## 2.1 First step

As a first step, we notice —as stated in Proposition 2.2 below— that the category of norm complete  $\ell$ -groups is equivalent to the category of norm complete MV-algebras, whose definition we make more precise now. On any MV-algebra we can define a pseudometric  $d$  that coincides with the restriction of the pseudometric  $d$  on  $(G, u)$ , where  $(G, u)$  is a unital  $\ell$ -group such that  $\Gamma((G, u)) \cong A$ ; the pseudometric  $d$  on  $A$  is a metric if, and only if,  $(G, u)$  is Archimedean if, and only if,  $A$  is an *Archimedean MV-algebra*, i.e., for any  $x \in A$ , if, for all  $n \in \mathbb{N}$ ,  $\underbrace{x \oplus \cdots \oplus x}_{n \text{ times}} \leq \neg x$ ,

then  $x = 0$ . We write *norm complete MV-algebra* for “Archimedean MV-algebra complete in the metric  $d$ ”. A *morphism of norm complete MV-algebras* is simply a morphism of MV-algebras.

**Proposition 2.2.** *The functor  $\Gamma$  from unital  $\ell$ -groups to MV-algebras restricts to an equivalence between norm complete  $\ell$ -groups and norm complete MV-algebras.*

## 2.2 Second step

As a second step, we provide an equational axiomatization defining a variety **CMV** which is isomorphic to the category of norm complete MV-algebras. This variety is not finitary: the set of primitive operations that we consider is made of a set of primitive operations of MV-algebras, together with an operation  $\gamma$  of countably infinite arity. The idea is that, in the intended models,  $\gamma(x_0, x_1, x_2, \dots) = \lim_{n \rightarrow \infty} a_n$  whenever  $(x_n)_n$  converges “quickly enough”; precisely, when, for all  $n \in \mathbb{N}$ ,  $d(x_n, x_{n+1}) \leq \frac{1}{2^{n+1}}$ . We discuss a proof of the following

**Theorem 2.3.** *The category of norm complete MV-algebras is isomorphic to the variety **CMV**.*

## 2.3 Conclusion

When coupled with the content of Luca Spada’s talk “Norm complete Abelian  $\ell$ -groups: topological duality”, the result presented here amounts to the following duality theorem, that extends Stone-Gelfand duality to  $a$ -normal spaces.

**Theorem 2.4.** *The following categories are pairwise equivalent.*

1. *The opposite of the category of  $a$ -normal spaces with  $a$ -maps among them.*
2. *The category of norm complete  $\ell$ -groups.*
3. *The category of norm complete MV-algebras.*
4. *The (infinitary) variety **CMV**.*

## References

- [1] J. Duskin. Variations on Beck’s tripleability criterion. In *Reports of the Midwest Category Seminar, III*, pages 74–129. Springer, Berlin, 1969.
- [2] J. Isbell. Generating the algebraic theory of  $C(X)$ . *Algebra Universalis*, 15(2):153–155, 1982.
- [3] V. Marra and L. Reggio. Stone duality above dimension zero: Axiomatizing the algebraic theory of  $C(X)$ . *Advances in Mathematics*, 307:253–287, 2017.
- [4] D. Mundici. Interpretation of AF  $C^*$ -algebras in Lukasiewicz sentential calculus. *J. Funct. Anal.*, 65(1):15–63, 1986.

# Polyhedral Completeness of Intermediate and Modal Logics

Sam Adam-Day<sup>1</sup>, Nick Bezhanishvili<sup>1</sup>, David Gabelaia<sup>2</sup>, and Vincenzo Marra<sup>3</sup>

<sup>1</sup> Institute for Logic, Language and Computation, Universiteit van Amsterdam

<sup>2</sup> Andrea Razmadze Mathematical Institute, Ivane Javakhishvili Tbilisi State University

<sup>3</sup> Dipartimento di Matematica “Federigo Enriques”, Università degli Studi di Milano

The collection of open subpolyhedra of any compact polyhedron forms a Heyting algebra, which leads to the polyhedral semantics of intuitionistic propositional calculus IPC [5]. A similar approach to modal logics was developed in [7]. Precursors of this work are [1], [3] and [2].

In this abstract we investigate polyhedral completeness of intermediate and modal logics. For the lack of space we will only concentrate on intermediate logics. However, all the results can be generalized to modal logics above **S4.Grz**. We will define when a logic is polyhedrally complete and give a criterion for polyhedral completeness of intermediate logics in terms of the class of its finite Kripke frames. We will then use this criterion to show that many well-known intermediate logics (e.g., all stable logics) are polyhedrally incomplete. We will also use this criterion to give examples of logics that are polyhedrally complete. A full classification of polyhedrally complete intermediate and modal logics remains an open problem.

Let  $P$  be an  $n$ -dimensional compact polyhedron. By an *open subpolyhedron* of  $P$  we mean a subset of  $P$  whose complementary subset in  $P$  is a compact polyhedron. Under inclusion order, the poset  $\text{Sub}(P)$  of all open subpolyhedra of  $P$  is a Heyting algebra [5]. For a propositional formula  $\varphi$ , we say that  $P \models \varphi$  if  $\text{Sub}(P) \models \varphi$  (i.e.,  $\varphi$  is valid in the Heyting algebra  $\text{Sub}(P)$ ). For a class  $\mathcal{P}$  of polyhedra we write  $\mathcal{P} \models \varphi$  if  $P \models \varphi$  for each  $P \in \mathcal{P}$ .

**Definition 1.** An intermediate logic  $L$  is *polyhedrally complete* if there is a class  $\mathcal{P}$  of polyhedra such that for each formula  $\varphi$  we have  $L \vdash \varphi$  iff  $\mathcal{P} \models \varphi$ .

It was shown in [5] that IPC and  $\text{BD}_n$  (the intermediate logic of frames of depth  $n$ ) are polyhedrally complete. It was also proved there that  $\text{Sub}(P)$  is a locally finite Heyting algebra, which implies that if  $L$  is polyhedrally complete then it has the finite model property. Therefore, the logics that do not have the f.m.p. are not polyhedrally complete. We will now formulate the criterion of polyhedral completeness. As a key tool, we use a classical notion first introduced by Alexandrov in the first half of the last century in connection with his studies of posets as spaces.

**Definition 2.** The *nerve*,  $\mathcal{N}(P)$ , of a poset  $P$  is the set of all subsets of  $P$  which are linearly ordered (i.e. the set of all chains in  $P$ ). We order  $\mathcal{N}(P)$  by subset inclusion  $\subseteq$ . When  $P$  is rooted, define the *rooted nerve*,  $\mathcal{N}^*(P)$ , of  $P$  to be the set of all chains in  $P$  containing the root element, again ordered by subset inclusion.

**Theorem 3.** *An intermediate logic  $L$  is polyhedrally complete if and only if there is a class of finite rooted frames  $\mathcal{C}$  closed under  $\mathcal{N}^*$  such that  $\text{Logic}(\mathcal{C}) = L$ .*

The usefulness of the above theorem is that it provides a characterisation of polyhedrally complete logics purely in terms of Kripke frames. The results of [5] now follow from this theorem directly (although the original proof uses the same idea).

**Corollary 4.** *IPC and  $\text{BD}_n$  for  $n \in \omega$  are polyhedrally complete.*

Are there further examples of polyhedrally complete logics? The next corollary of Theorem 3 shows that we must search outside the realm of stable logics: a logic  $L$  is *stable* if the class  $\text{Frames}(L)$  of all rooted frames of  $L$  is closed under onto monotone images [4].

**Corollary 5.** *If  $L$  is polyhedrally complete, stable, and of height above 3, then  $L = \text{IPC}$ . Therefore,  $\text{BW}_n$ ,  $\text{BTW}_n$ ,  $\text{LC}$ , and  $\text{KC}$  are polyhedrally incomplete.*

There are continuum many stable logics and all of them have the finite model property. Thus, there are continuum many logics with the f.m.p. that are not polyhedrally complete. In fact, one may wonder whether there are any polyhedrally complete logics of infinite height beyond IPC.

We provide a positive answer: Scott's logic,  $\text{SL} = \text{IPC} + ((\neg\neg p \rightarrow p) \rightarrow (p \vee \neg p)) \rightarrow (\neg p \vee \neg\neg p)$  is an infinite-height polyhedrally complete logic. For a rooted frame  $F$  let  $\chi(F)$  denote the Jankov-Fine formula of  $F$ . It is well known that  $\text{SL} = \text{IPC} + \chi\left(\frac{\circ}{\circ}\right)$  and that it has the f.m.p.

[6]. On frames, the formula  $\chi\left(\frac{\circ}{\circ}\right)$  expresses a type of connectedness. Our proof proceeds by considering a stronger form of this connectedness, which is preserved by the nerve construction  $\mathcal{N}^*$ . We then show that  $\text{SL}$  is the logic of the class  $\mathcal{C}$  of finite rooted frames with this strong connectedness property, by showing that any finite frame of  $\text{SL}$  is the p-morphic image of one in  $\mathcal{C}$ . This then gives that  $\text{SL}$  is polyhedrally complete, by Theorem 3. The proof provides a general method, and we use it to exhibit an infinite class of polyhedrally complete logics, all of which are axiomatised by Jankov-Fine formulas of trees.

## References

- [1] M. Aiello, J. van Benthem, and G. Bezhanishvili. Reasoning about space: the modal way. *J. Logic Comput.*, 13(6):889–920, 2003.
- [2] J. van Benthem and G. Bezhanishvili. Modal logics of space. In *Handbook of Spatial Logics*, pages 217–298. Springer, Dordrecht, 2007.
- [3] J. van Benthem, G. Bezhanishvili, and M. Gehrke. Euclidean hierarchy in modal logic. *Studia Logica*, 75(3):327–344, 2003.
- [4] G. Bezhanishvili and N. Bezhanishvili. Locally finite reducts of Heyting algebras and canonical formulas. *Notre Dame J. Form. Log.*, 58(1):21–45, 2017.
- [5] N. Bezhanishvili, V. Marra, D. McNeill, and A. Pedrini. Tarski's theorem on intuitionistic logic, for polyhedra. *Annals of Pure and Applied Logic*, 169(5):373–391, 2018.
- [6] A. Chagrov and M. Zakharyashev. *Modal logic*, volume 35 of *Oxford Logic Guides*. The Clarendon Press, Oxford University Press, New York, 1997. Oxford Science Publications.
- [7] D. Gabelaia, K. Gogoladze, M. Jibladze, E. Kuznetsov, and M. Marx. Modal logic of planar polygons. Preprint submitted to Elsevier, 2018.

# Product of neighborhood frames with additional common modality.

Rajab Aghamov<sup>1</sup> and Andrey Kudinov<sup>123</sup>

<sup>1</sup> National Research University Higher School of Economics, Russian Federation  
 agamov@phystech.edu

<sup>2</sup> Institute for Information Transmission Problem, RAS  
 kudinov@iitp.ru

<sup>3</sup> Moscow Institute of Physics and Technology (State University), Russian Federation

Neighborhood semantics as a generalization of Kripke semantics for modal logic were invented independently by Dana Scott [1] and Richard Montague [2]. Neighborhood semantics is more general than Kripke semantics and in the case of normal reflexive and transitive logics coincides with topological semantics.

In general topology a product of two topological spaces  $\mathcal{X}_1$  and  $\mathcal{X}_2$  is a topological space with the following base  $\{U_1 \times U_2 \mid U_i \text{ is open in } \mathcal{X}_i, i = 1, 2\}$ . In [4] the authors considered a different product of topological spaces: a product of two topological spaces  $\mathcal{X}_1$  and  $\mathcal{X}_2$  is a bitopological space with two topologies—horizontal and vertical. In a similar fashion product of neighborhood frames was introduced by Sano in [3].

We assume familiarity with basic notion of modal logic such as (normal) modal logic and the minimal modal logic  $\mathbf{K}$ . For details see [8].

Let  $L_1$  and  $L_2$  be two modal logics then *the fusion of these logics* (notation  $L_1 \otimes L_2$ ) is the minimal modal logic containing  $L_1$  and  $L_2'$ , where  $L_2'$  is the logic  $L_2$  after renaming all modalities.

Let us define the following logics:

$\mathbf{T} = \mathbf{K} + \Box p \rightarrow p$ ,  $\mathbf{D} = \mathbf{K} + \Box p \rightarrow \Diamond p$ ,  $\mathbf{D4} = \mathbf{D} + \Box p \rightarrow \Box \Box p$ ,  $\mathbf{S4} = \mathbf{T} + \Box p \rightarrow \Box \Box p$ .

In [5] it was proved that for any two  $L_1, L_2 \in \{\mathbf{D4}, \mathbf{D}, \mathbf{T}, \mathbf{S4}\}$

$$L_1 \times_n L_2 = L_1 \otimes L_2.$$

Here  $L_1 \times_n L_2$  is the product of two logics, based on neighbourhood semantics. More precisely it is the logic (all valid formulas) of the class of all products of neighbourhood frames  $\mathcal{X}_1 \times \mathcal{X}_2$  such that  $L_i$  is valid in  $\mathcal{X}_i$  ( $i = 1, 2$ ).

In section 6 of [4] the authors considered yet another type of product of two spaces such that the result is a space with three topologies: horizontal, vertical and the product topology from the general topology. They proved that the logic of such products of all topological spaces will be  $\mathbf{S4} \otimes \mathbf{S4} \otimes \mathbf{S4} + \Box p \rightarrow \Box_1 p \wedge \Box_2 p$  (modality  $\Box$  corresponds to the classical product topology).

We consider similar construction for neighborhood frames (n-frames for short). We say that for two n-frames  $\mathcal{X}_1$  and  $\mathcal{X}_2$  their *full product* will be an n-frame with 3 neighbourhood functions: horizontal, vertical and the “product”. Here is a precise definition.

*N-frame* is a pair  $\mathcal{X} = (X, \tau)$ , where  $X \neq \emptyset$  and  $\tau : X \rightarrow 2^{2^X}$  is a neighborhood function. In order for logic of  $\mathcal{X}$  to be normal  $\tau(x)$  have to be a filter or the full set of subsets of  $X$ .

Let  $\mathcal{X}_1 = (X_1, \tau_1)$  and  $\mathcal{X}_2 = (X_2, \tau_2)$  be two n-frames. Then the *full product* of these n-frames  $\mathcal{X}_1 \times^+ \mathcal{X}_2$  is defined as follows

$$\mathcal{X}_1 \times^+ \mathcal{X}_2 = (X_1 \times X_2, \tau_1', \tau_2', \tau), \text{ where}$$

$$\tau_1'(x_1, x_2) = \{U \subseteq X_1 \times X_2 \mid \exists V (V \in \tau_1(x_1) \ \& \ V \times \{x_2\} \subseteq U)\},$$

$$\begin{aligned}\tau_2^!(x_1, x_2) &= \{U \subseteq X_1 \times X_2 \mid \exists V(V \in \tau_2(x_2) \ \& \ \{x_1\} \times V \subseteq U)\}, \\ \tau(x_1, x_2) &= \{U \mid \exists V_1 \in \tau_1(x) \ \& \ \exists V_2 \in \tau_2(y)(V_1 \times V_2 \subseteq U)\}.\end{aligned}$$

For two unimodal logics  $L_1$  and  $L_2$  we define *full  $n$ -product* of them as follows

$$L_1 \times_n^+ L_2 = \text{Log}(\{\mathcal{X}_1 \times^+ \mathcal{X}_2 \mid L_i \text{ is valid in an } n\text{-frame } \mathcal{X}_i\}).$$

In the case of logics **S4** and **D4** neighborhood semantics is reduced to the topological semantics. From [4] and [6] it is known that

$$\begin{aligned}\mathbf{S4} \times_n^+ \mathbf{S4} &= \mathbf{S4} \otimes \mathbf{S4} \otimes \mathbf{S4} + \Box p \rightarrow \Box_1 p \wedge \Box_2 p, \\ \mathbf{D4} \times_n^+ \mathbf{D4} &= \mathbf{D4} \otimes \mathbf{D4} \otimes \mathbf{D4} + \Box p \rightarrow \Box_1 p \wedge \Box_2 p.\end{aligned}$$

In this work we prove

**Theorem.** *For normal modal logics **D** and **T***

$$\begin{aligned}\mathbf{T} \times_n^+ \mathbf{T} &= \mathbf{T} \otimes \mathbf{T} \otimes \mathbf{T} + \Box p \rightarrow \Box_1 p \wedge \Box_2 p, \\ \mathbf{D} \times_n^+ \mathbf{D} &= \mathbf{D} \otimes \mathbf{D} \otimes \mathbf{D} + \Box p \rightarrow \Box_1 p \wedge \Box_2 p.\end{aligned}$$

There we use the idea of construction  $T_{6,2,2}$  (see [4]) and modify it to the  $\omega$  branching tree  $T_{\omega,\omega,\omega}$ , such that  $\text{Log}(T_{\omega,\omega,\omega}) = \mathbf{D} \otimes \mathbf{D} \otimes \mathbf{D} + \Box p \rightarrow \Box_1 p \wedge \Box_2 p$ . Then we use the pseudo-infinite paths construction from [5] and build an  $n$ -frame  $\mathcal{N}_\omega(T_\omega)$  and construct a  $p$ -morphism-like map from  $\mathcal{N}_\omega(T_\omega) \times^+ \mathcal{N}_\omega(T_\omega)$  to  $T_{\omega,\omega,\omega}$ . And for logic **T** the proof is similar but the trees should be reflexive.

We can try and extend the technique to the logics **K**, **K4**. These logics don't have seriality axiom, so there will be some additional axioms similar to the infinite set of axioms  $\Delta$  from [7].

## References

- [1] Scott, D., *Advice on modal logic*, in: Philosophical Problems in Logic: Some Recent Developments, D. Reidel, 1970 pp. 143 – 173.
- [2] Montague, R., *Universal grammar*, Theoria 36 (1970), pp. 373–398.
- [3] Sano, K., *Axiomatizing hybrid products of monotone neighborhood frames*, Electr. Notes Theor. Comput. Sci. 273 (2011), pp. 51–67.
- [4] Benthem, J., G. Bezhanishvili, B. Cate and D. Sarenac, *Multimodal logics of products of topologies*, Studia Logica 84 (2006), pp. 369–392.
- [5] Kudinov A. *Modal logic of some products of neighbourhood frames*, in: Advances in Modal Logic Issue 9. L. : College Publications, 2012. P. 286–294.
- [6] Kudinov A. *D-logic of product of rational numbers*, ITIS2013, 2013.
- [7] Kudinov A. *On neighbourhood product of some Horn axiomatizable logics* // Logic Journal of the IGPL. 2018. Vol. 26. No. 3. P. 316-338
- [8] Blackburn, P., M. de Rijke and Y. Venema, *Modal Logic*, Cambridge University Press, 2002.
- [9] Pacuit, E., *Neighborhood Semantics for Modal Logic*, Springer, 2017.



# Predicative Implications: A Topological Approach

Amirhossein Akbar Tabatabai

Faculty of Humanities, Utrecht University  
amir.akbar@gmail.com

As Gödel argued in [3], the intuitionistic implication has an impredicative character inherited from the cycle of the implication's introduction and elimination rules. To solve this problem, some alternative logics have been proposed, such as Visser-Ruitenburg's basic logic **BPC** (see [6] and [1]) and Ardeshir-Hessam's extended basic logic **EBPC** (see [2]), as the logics of the transitive persistent Kripke frames and the transitive persistent serial Kripke frames, respectively. The main point, there, is avoiding the modus ponens rule on the logical side, and the reflexivity condition on the semantical side to break the mentioned impredicative cycle. However, this lack of elimination rule breaks the adjunction pair and makes the systems ill-behaved and inadequate for the foundational role they deserve. In this talk, we will explain a brand new approach based on the combination of the notions of time and space to introduce some more practical conservative extensions of these logics that bring back the symmetry of an adjunction pair (see [4]). This spatio-temporal approach, then leads to a brand new family of modal propositional logics augmented with the predicative implications, and a natural topological semantics, both for these logical systems and the usual systems of modal logics. It can be also interpreted as a formalization for van Atten's independently developed solution for the impredicativity problem (see [5]).

## 1 Modal Spaces

**Definition 1.** A pair  $(X, J)$  is called a modal space if  $X$  is a locale and  $J : X \rightarrow X$  is a monotone function which preserves all joins. A modal space is called temporal if for all  $a \in X$ ,  $Ja \leq a$ . It is called serial if it is temporal, and for any  $a \in X$ , if  $Ja = 0$ , then  $a = 0$ . We will denote the class of all modal spaces, temporal spaces, and serial spaces by **MS**, **TS**, and **SS**, respectively. A modal space is called boolean when its locale is boolean. We will show the class of all modal boolean spaces, boolean temporal spaces, and boolean serial spaces by **MS<sup>b</sup>**, **TS<sup>b</sup>**, and **SS<sup>b</sup>**, respectively.

**Remark 2.** Intuitively speaking, the opens of the locale are the observable propositions and  $J$  is the temporal modality that sends the proposition  $a \in X$  to the proposition  $Ja \in X$  meaning " $a$  happened before". Therefore, the condition that  $J$  preserves the arbitrary joins simply means that the temporal structure respects the coverings, or "if there *was* an observation to show that "one of  $a_i$ 's is true", then there *is* at least one observation to show that one of  $a_i$ 's *was* true.

**Example 3.** There are many important examples of modal spaces. For instance, for any continuous map  $(f^* \dashv f_*) : X \rightarrow X$  on the locale  $X$ ,  $(X, f^*)$  is a modal space. For a more illuminating example, assume that  $(W, \leq)$  is a poset formalizing the ordered structure of time. Then, consider the topology of all upward closed subsets of  $W$  and  $J$  on these opens as  $J(U) = \{x \mid \exists y \in U \ y \leq x\}$ . Therefore,  $(Op(W, \leq), J)$  is a modal space.

**Remark 4.** Note that for any fixed  $a \in X$ , the operation  $J(-) \wedge a$  preserves arbitrary joins, and hence has a right adjoint, which we denote by  $a \rightarrow (-)$ . Intuitively speaking, the adjunction captures the predicative implication, with the properties that  $a \rightarrow b$  is provable by  $c$  iff the

fact that “*c happened before*” together with *a*, implies *b*. This time lag makes a delay between introducing an implication, and using it in its applications. For instance,  $a \wedge (a \rightarrow b)$  does not necessarily imply *b*, but if  $a \rightarrow b$  has been proved before, that is if we have  $a \wedge J(a \rightarrow b)$ , then we can prove *b*.

## 2 Predicative Logics

Let  $\mathcal{L}_J$  be the usual language of propositional logic with a unary modal operator *J*. To introduce some formal systems in this language, consider the following set of natural deduction rules:

**F and Implication Rules:**

$${}^F \frac{C \vdash A}{JC \vdash JA} \quad \rightarrow^E \frac{\Gamma \vdash A \quad \Pi \vdash J(A \rightarrow B)}{\Gamma, \Pi \vdash B} \quad \rightarrow^I \frac{JC, A \vdash B}{C \vdash A \rightarrow B}$$

**Additional Rules:**

$${}^{wCoJ} \frac{JA \vdash \perp}{A \vdash \perp} \quad {}^J \frac{\Gamma \vdash JA}{\Gamma \vdash A}$$

Now, define the minimal predicative logic, **mJ**, as the logic of the system consisting of the usual axioms, the cut rule, the usual conjunction and disjunction rules, and the mentioned *F* and the implication rules. If we add the rule *J* to this system, we will have the logic **J**, and if we also add *wCoJ* to **J**, the system is **wInt**. These logical systems provide *faithful well-behaved* extensions of **BPC** and **EBPC**, respectively. Considering the natural topological semantics for these logics, we have the following, as the main result of the talk:

**Theorem 1.** (*Soundness-Completeness, Embedding Theorem*) *Let  $X$  be an infinite Hausdorff space, and  $\mathcal{C}$  a class of modal spaces. If  $X \models_{\mathcal{C}} \Gamma \Rightarrow A$  means the validity of  $\Gamma \Rightarrow A$  in all  $(X, J) \in \mathcal{C}$ , then:*

- (i)  $\Gamma \vdash_{\mathbf{mJ}} A$  iff  $\mathbf{MS} \models \Gamma \Rightarrow A$ .
- (ii)  $\Gamma \vdash_{\mathbf{J}} A$  iff  $\mathbf{TS} \models \Gamma \Rightarrow A$ . If  $\Gamma \cup \{A\}$  is *J*-free, then we also have  $\Gamma \vdash_{\mathbf{J}} A$  iff  $\Gamma \vdash_{\mathbf{BPC}} A$  iff  $X \models_{\mathbf{TS}} \Gamma \Rightarrow A$ .
- (iii)  $\Gamma \vdash_{\mathbf{wInt}} A$  iff  $\mathbf{SS} \models \Gamma \Rightarrow A$ . If  $\Gamma \cup \{A\}$  is *J*-free, then we also have  $\Gamma \vdash_{\mathbf{wInt}} A$  iff  $\Gamma \vdash_{\mathbf{EBPC}} A$  iff  $X \models_{\mathbf{SS}} \Gamma \Rightarrow A$ .

## References

- [1] Mohammad Ardeshir. *Aspects of Basic Logic*. PhD thesis, 1995.
- [2] Mohammad Ardeshir and Bardia Hesaam. An introduction to basic arithmetic. *Logic Jnl IGPL*, 16(1):1–13, 2008.
- [3] Kurt Gödel. The present situation in the foundations of mathematics. In Feferman et al, editor, *Unpublished Essays and Lectures, Volume 3 of Collected Works*. Oxford University Press.
- [4] Amirhossein Akbar Tabatabai. Geometric modality and weak exponentials. Available at <https://arxiv.org/abs/1711.01736>.
- [5] Mark van Atten. Predicativity and parametric polymorphism of brouwerian implication. Available at <https://arxiv.org/abs/1710.07704>.
- [6] Albert Visser. A propositional logic with explicit fixed points. *Studia Logica*, 40:155–175, 1981.

# Ranges of functors and elementary classes via topos theory

Peter Arndt

Mathematical Institute, University of Düsseldorf, `peter.arndt@uni-duesseldorf.de`

Let  $\mathbb{S}, \mathbb{T}$  be first order theories and let  $F: \mathcal{A} := \text{Mod}(\mathbb{S}) \rightarrow \text{Mod}(\mathbb{T}) =: \mathcal{B}$  be a functor between their categories of models. Many classical questions are of the form “Is every  $B \in \mathcal{B}$  of the form  $F(A)$  for some  $A \in \mathcal{A}$ ?” or in a weakened version “Is every  $B \in \mathcal{B}$  elementarily equivalent to  $F(A)$  for some  $A \in \mathcal{A}$ ?”.

Notable examples are Dilworth’s congruence lattice problem, asking whether every algebraic distributive lattice is the congruence lattice of a lattice (solved in the negative by Wehrung [We]), the representation problems for special groups [DM], asking whether every special group is elementarily equivalent to the square class group of a field, or Efrat’s question [Ef] which  $\kappa$ -structures arise as the Milnor  $K$ -theory of a field.

The present work addresses this kind of question. More precisely, the main result gives criteria for determining whether the structures in the image of  $F$  can be distinguished from general structures in  $\mathcal{B}$  by means of certain formulas of infinitary first order logic, the  $\kappa$ -geometric formulas.

**Definition.** *Let  $\Sigma$  be a first order signature and  $\kappa$  a regular cardinal.*

- (i) *A  $\kappa$ -geometric formula is a formula built from atomic formulas (including  $\top, \perp$ ), using arbitrary (set-indexed) disjunctions, conjunctions over less than  $\kappa$  formulas and existential quantification over less than  $\kappa$  variables.*
- (ii) *A  $\kappa$ -geometric theory is a theory which can be axiomatized by formulas of the form  $\forall \bar{x} \phi \rightarrow \psi$ , where  $\phi, \psi$  are  $\kappa$ -geometric formulas, and the quantification can be over an arbitrary set of variables.*
- (iii) *For a class of  $\Sigma$ -structures  $\mathcal{C}$  we denote by  $Th_{\kappa\text{-geom}}(\mathcal{C})$  the  $\kappa$ -geometric theory of  $\mathcal{C}$ , i.e. the set of all formulas of the form  $\forall \bar{x} \phi \rightarrow \psi$ , with  $\phi, \psi$   $\kappa$ -geometric formulas, that are valid in every member of  $\mathcal{C}$ .*
- (iv) *Denote by  $Th_{\neg\kappa\text{-geom}}(\mathcal{C})$  the set of negations of  $\kappa$ -geometric formulas (i.e. formulas of the form  $\forall \bar{x} \phi \rightarrow \perp$  with  $\phi$   $\kappa$ -geometric), that are valid in every member of  $\mathcal{C}$ .*

In the case  $\kappa = \aleph_0$  the above definition recovers the notions of geometric formula and geometric theory known from topos theory. The  $\kappa$ -geometric theories have recently been studied by Espíndola [Es1], [Es2] who developed a theory of classifying toposes for  $\kappa$ -geometric theories and proved a version of Deligne’s completeness theorem.

**Theorem.** *Let  $\mathbb{S}, \mathbb{T}$  be  $\kappa$ -geometric theories over possibly different signatures such that  $\mathcal{A} := \text{Mod}(\mathbb{S})$  and  $\mathcal{B} := \text{Mod}(\mathbb{T})$  are  $\kappa$ -accessible categories, and denote by  $\mathcal{A}_\kappa, \mathcal{B}_\kappa$  their full subcategories of  $\kappa$ -presentable objects.*

*Let  $F: \mathcal{A} \rightarrow \mathcal{B}$  be a functor that preserves  $\kappa$ -filtered colimits and sends  $\kappa$ -presentable objects to  $\kappa$ -presentable objects, i.e. restricts to a functor  $F_\kappa: \mathcal{A}_\kappa \rightarrow \mathcal{B}_\kappa$ . Then the following hold:*

- (a) *If  $F_\kappa: \mathcal{A}_\kappa \rightarrow \mathcal{B}_\kappa$  is essentially surjective, then  $Th_{\kappa\text{-geom}}(\mathcal{F}(\mathcal{A})) = Th_{\kappa\text{-geom}}(\mathcal{B})$ . More generally, this equality of  $\kappa$ -geometric theories holds if and only if  $F_\kappa$  induces an equivalence between the idempotent completions of  $\mathcal{A}_\kappa$  and  $\mathcal{B}_\kappa$ .*

- (b) If  $F_\kappa: \mathcal{A}_\kappa \rightarrow \mathcal{B}_\kappa$  is fully faithful, then  $F(\mathcal{A}) = \text{Mod}(\mathbb{S}')$  for some axiomatic extension  $\mathbb{S}' \supseteq \mathbb{S}$  (i.e. the essential image  $F(\mathcal{A})$  can be characterized by additional  $\kappa$ -geometric formulas in the language of  $\mathbb{S}$ ).
- (c) If, in the situation of (b), additionally one has that every  $B \in \mathcal{B}$  admits some morphism to an  $F(A)$ , for some  $A \in \mathcal{A}$ , then  $\text{Th}_{\neg\kappa\text{-geom}}(F(\mathcal{A})) = \text{Th}_{\neg\kappa\text{-geom}}(\mathcal{B})$ , i.e. the objects in the essential image of  $F$  and general objects of  $\mathcal{B}$  satisfy exactly the same negations of geometric formulas.

As for the scope of the theorem, one should note that the category of models of a usual finitary first order theory over a countable signature is always  $\aleph_1$ -accessible. Vice versa every  $\kappa$ -accessible category is the category of models of a  $\kappa$ -geometric theory. Thus the theorem is applicable to the above mentioned classical questions, which typically revolve around models of usual finitary first order theories.

An outline of the proof goes as follows: Espíndola in [Es2] developed a theory of classifying toposes for  $\kappa$ -geometric theories, working with  $\kappa$ -geometric morphisms, i.e. adjunctions such that the left adjoint preserves limits of cardinality  $< \kappa$ . It is true that every  $\kappa$ -accessible category is the category of models of a  $\kappa$ -geometric theory of *presheaf type*, i.e. the classifying topos is a presheaf topos. The hypotheses of the theorem ensure that the functor  $F$  is induced by a  $\kappa$ -geometric morphism between the classifying toposes and that this morphism is essential, i.e. induced by a functor between the index categories of the presheaf categories. The statements of the theorem are obtained by factorizing the said  $\kappa$ -geometric morphism into a surjection, followed by a dense inclusion, followed by a closed inclusion and determining what are the theories classified by the intermediate toposes. While such factorizations can in general be hard to compute, factorizing essential geometric morphisms between presheaf toposes is easy, as observed by the author in joint work with E. Ochs: It turns out that all the intermediate toposes are presheaf toposes and the morphisms are essential. The construction of the factorization is what yields the criteria of the theorem.

In the talk we will explain all the involved notions, like  $\kappa$ -presentable objects and  $\kappa$ -accessible categories, with examples and give sample applications of the theorem. We will finish with an outline of the proof, without supposing too much knowledge of topos theory.

## References

- [DM] M.A. Dickmann, F. Miraglia, Special Groups: Boolean-Theoretic Methods in the Theory of Quadratic Forms, *Memoirs of the AMS*, no. 689 (2000)
- [Ef] I. Efrat, Valuations, Orderings, and Milnor K-Theory (*Mathematical Surveys and Monographs*, Vol. 124), AMS (2006)
- [Es1] C. Espíndola, Infinitary first-order categorical logic, *Annals of Pure and Applied Logic*, Volume 170, Issue 2, pp. 137-162 (2019)
- [Es2] C. Espíndola, Infinitary generalizations of Deligne's completeness theorem, preprint, <https://arxiv.org/abs/1709.01967>
- [We] F. Wehrung, A solution to Dilworth's congruence lattice problem, *Advances in Mathematics*, Vol. 216, Issue 2, p. 610-625 (2007)

# When is the frame of nuclei spatial: A new approach

F. Ávila<sup>1</sup>, G. Bezhanishvili<sup>2</sup>, P. J. Morandi<sup>2</sup>, A. Zaldívar<sup>3</sup>

<sup>1</sup> Autonomous University of Ciudad Juárez

<sup>2</sup> New Mexico State University

<sup>3</sup> University of Guadalajara

Nuclei play an important role in pointfree topology as they are in 1-1 correspondence with onto frame homomorphisms, and hence describe sublocales of locales. For each frame  $L$ , let  $N(L)$  be the set of nuclei on  $L$ . There is a natural order on  $N(L)$  given by

$$j \leq k \text{ iff } ja \leq ka \text{ for each } a \in L.$$

With this order  $N(L)$  is also a frame, which we will refer to as the *frame of nuclei* or the *assembly* of  $L$ . The complicated structure of  $N(L)$  has been investigated by many authors.

To describe some of the key results about  $N(L)$ , we recall that a frame  $L$  is *spatial* if it is isomorphic to the frame  $\mathcal{O}S$  of open sets of a topological space  $S$ . For a subspace  $T$  of  $S$ , a point  $x \in T$  is *weakly isolated* in  $T$  if there is an open subset  $U$  of  $S$  such that  $x \in T \cap U \subseteq \overline{\{x\}}$ . It is clear that an isolated point of  $T$  is weakly isolated in  $T$ .

The space  $S$  is *scattered* if each nonempty subspace of  $S$  contains an isolated point. It is easy to see that  $S$  is scattered iff each nonempty closed subspace of  $S$  has an isolated point. The space  $S$  is *weakly scattered* if each nonempty closed subspace has a weakly isolated point. It is well known that  $S$  is scattered iff  $S$  is weakly scattered and  $T_D$ , where  $S$  is  $T_D$  if each singleton is locally closed. Finally,  $S$  is *dispersed* if each nonempty closed subset of  $S$  has a detached point (see [5, Def. 1.2]).

The following are some of the landmark results about  $N(L)$ .

- Beazer and Macnab [1] proved that if  $L$  is boolean, then  $N(L)$  is isomorphic to  $L$ , and gave a necessary and sufficient condition for  $N(L)$  to be boolean.
- Simmons [5] proved that if  $S$  is a  $T_0$ -space, then  $N(\mathcal{O}S)$  is boolean iff  $S$  is scattered; and that dropping the  $T_0$  assumption results in the following more general statement:  $N(\mathcal{O}S)$  is boolean iff  $S$  is dispersed.
- Simmons [5, Thm. 4.4] also gave a necessary and sufficient condition for  $S$  to be weakly scattered. This result of Simmons is sometimes stated erroneously as follows:  $N(\mathcal{O}S)$  is spatial iff  $S$  is weakly scattered. While this formulation is false, Isbell [3] proved that if  $S$  is a sober space, then indeed  $N(\mathcal{O}S)$  is spatial iff  $S$  is weakly scattered.
- Niefield and Rosenthal [4] gave necessary and sufficient conditions for  $N(L)$  to be spatial, and derived that if  $N(L)$  is spatial, then so is  $L$ .

In [2] a new technique was developed to study nuclei on  $L$  utilizing Priestley duality for distributive lattices and Esakia duality for Heyting algebras. It was shown that nuclei on a Heyting algebra  $L$  correspond to *nuclear subsets* of the Esakia space  $X_L$  of  $L$ , where  $F$  is a nuclear set provided  $F$  is closed and for each clopen set  $U$ , the downset  $\downarrow(F \cap U)$  is clopen. If  $N(X_L)$  denotes all nuclear subsets of  $X_L$ , then we utilize the dual isomorphism between  $N(L)$  and  $N(X_L)$  of [2] to give an alternate proof of the results mentioned above.

We single out a subset  $Y_L$  of  $X_L$  consisting of nuclear points of  $X_L$ , where  $y \in Y_L$  iff  $\{y\} \in N(L)$ , iff  $\downarrow y$  is clopen, and show that  $L$  is spatial iff  $Y_L$  is dense in  $X_L$ . We prove that

join-prime elements of  $N(X_L)$  are exactly the singletons  $\{y\}$  where  $y \in Y_L$ . From this we derive a characterization of when  $N(L)$  is spatial in terms of  $X_L$ . This yields an alternate proof of the results of Niefield and Rosenthal [4].

The next two results characterize when  $N(L)$  is spatial or boolean.

**Theorem 1.** *Let  $L$  be a frame and  $X_L$  its Esakia space. The following are equivalent.*

- (1) *The frame  $N(L)$  is spatial.*
- (2) *If  $F \in N(X_L)$  is nonempty, then so is  $F \cap Y_L$ .*
- (3)  *$N(L)$  is isomorphic to the frame of open subsets of  $Y_L$ .*

For a topological space  $X$ , let  $\text{RC}(X)$  be the complete boolean algebra of regular closed subsets of  $X$ .

**Theorem 2.** *Let  $L$  be a frame and  $X_L$  its Esakia space. The following are equivalent.*

- (1)  *$N(L)$  is boolean;*
- (2)  *$N(X_L) = \text{RC}(X_L)$ ;*
- (3)  *$\max(D)$  is clopen for each clopen downset  $D$  of  $X_L$ .*

By specializing to the case  $L = \mathcal{O}S$ , we prove that  $N(\mathcal{O}S)$  is spatial iff the soberification of  $S$  is weakly scattered. As a corollary we obtain the result of Isbell [3] that if  $S$  is sober, then  $N(\mathcal{O}S)$  is spatial iff  $S$  is weakly scattered. We give an example showing that this result is false if  $S$  is not assumed to be sober.

Turning to the results of Simmons [5], one of his main tools is the use of the front topology. We show that if  $S$  is  $T_0$ , then  $X_L$  is a compactification of  $S$  with respect to the front topology on  $S$ . From this, by utilizing the  $T_0$ -reflection, we derive Simmons' characterization [5, Thm. 4.4] of arbitrary (not necessarily  $T_0$ ) weakly scattered spaces. We also derive Simmons' theorem that if  $S$  is  $T_0$ , then  $N(\mathcal{O}S)$  is boolean iff  $S$  is scattered. We generalize this result to an arbitrary space by showing that  $S$  is dispersed iff its  $T_0$ -reflection is scattered. This yields the general form of Simmons' theorem that  $N(\mathcal{O}S)$  is boolean iff  $S$  is dispersed.

## References

- [1] R. Beazer and D. S. Macnab. Modal extensions of Heyting algebras. *Colloq. Math.*, 41(1):1–12, 1979.
- [2] G. Bezhanishvili and S. Ghilardi. An algebraic approach to subframe logics. Intuitionistic case. *Ann. Pure Appl. Logic*, 147(1-2):84–100, 2007.
- [3] J. Isbell. On dissolute spaces. *Topology Appl.*, 40(1):63–70, 1991.
- [4] S. B. Niefield and K. I. Rosenthal. Spatial sublocales and essential primes. *Topology Appl.*, 26(3):263–269, 1987.
- [5] H. Simmons. Spaces with Boolean assemblies. *Colloq. Math.*, 43(1):23–39 (1981), 1980.

# Analysis of the $\Sigma_1^1$ -Fragment of First Order Gödel Logic extended with Propositional Quantifiers

Matthias Baaz<sup>1</sup> and Norbert Preining<sup>2,1</sup>

<sup>1</sup> Vienna University of Technology, Vienna, Austria  
{baaz,preining}@logic.at

<sup>2</sup> Accelia Inc., Tokyo, Japan  
norbert@preining.info

## 1 Introduction

First order Gödel logics form a well established class of many-valued logics with good proof-theoretic properties and extensive theory [4]. Quantified Propositional Gödel Logics have been studied only a few times, and not much is known about these logics besides a few results for specific logics [1–3, 5]. Propositional quantifiers allow for quantification of propositions, which in the setting of Gödel logics boils down to quantification over all truth values of the underlying truth-value set. Thus, they are somewhere between first order and second order quantifiers.

In this abstract we initiate the research program to study the combination of propositional and first-order quantifiers with respect to Gödel logic over  $V = [0, 1]$ .

### 1.1 Syntax and semantics for Gödel logics

The (propositional) operations on Gödel sets which are used in defining the semantics of Gödel logics have the property that they are projecting, i.e. that the operation uses one of the arguments (or 1) as result (we dot the operators to distinguish them from the syntactical elements of the language introduced later): For  $a, b \in [0, 1]$  let  $a \dot{\wedge} b := \min(a, b)$ ,  $a \dot{\vee} b := \max(a, b)$ ,  $a \dot{\rightarrow} b := 1$  if  $a \leq b$  and  $a \dot{\rightarrow} b := b$  if  $a > b$ . We define  $\dot{\neg} a := (a \rightarrow 0)$ , so  $\dot{\neg} 0 = 1$ , and  $\dot{\neg} a = 0$  for all  $a > 0$ .

For the following let us denote with  $\mathcal{L}$  the following language combining first-order and quantified propositional elements: Fix a countably infinite set of object variables  $\mathcal{X}$  (usually written as  $x, y, x_i$ ), a countably infinite set  $\mathcal{Q}$  of propositional variables (usually written as  $p, q, p_i$ ), a countable set  $\mathcal{F}$  of functionals  $F$ , countably many for each arity  $(n, m)$ , and a countable set  $\mathcal{P}$  of predicates  $P$ , countably many for each arity  $n$ . Functionals have two sets of parameters, the propositional quantified level parameters and the object level parameters.

**Definition 1** (Terms, atomic formulas, and formulas). Object variables are terms, and if  $\vec{p}$  are propositional variables, and  $\vec{t}$  are terms, then  $F(\vec{p}, \vec{t})$  is again a term. Propositional variables are atomic formulas, and if  $\vec{t}$  are terms, then  $P(\vec{t})$  is again an atomic formula. Formulas are build from disjunction, conjunction, implication, negation, first-order and quantified propositional quantification in the usual way.

Note that we do *not* allow a formula other than a quantified propositional variable to appear in functionals, and that predicates have other arguments than terms, in particular no quantified propositional variables (that is,  $P(q)$  where  $P$  is a predicate symbol and  $q$  a quantified propositional variable, is not allowed!). This guarantees that in the proof of the main theorem, elimination of these functionals can be achieved.



The semantics of first-order Gödel logics with propositional quantifiers, with respect to a fixed Gödel set as truth value set and  $\mathcal{L}$  is defined using the extended language  $\mathcal{L}^{M,V}$ , where  $M$  is a universe of objects.  $\mathcal{L}^{M,V}$  is  $\mathcal{L}$  extended with symbols for every element of  $M$  as constants, so called  $M$ -symbols, as well as constants  $r$  for each  $r \in V$ , the underlying truth value set. These symbols are denoted with the same letters.

**Definition 2** (Semantics of Gödel logics with propositional quantifiers). A valuation  $v$  into  $V = [0, 1]$  consists of a nonempty set  $M = M^v$ , the ‘universe’ of  $v$ , for each  $p \in \mathcal{Q}$  (set of propositional variables) a value  $p^v \in V$ , for each  $(n, m)$ -ary functional  $F$  a function  $F^v : V^n \times M^m \rightarrow M$ , and for each  $k$ -ary predicate symbol  $P$ , a function  $P^v : M^k \rightarrow V$ .

Given a valuation  $v$ , we can naturally define a value  $v(A)$  for any closed formula  $A$  of  $\mathcal{L}^M$ . Propositional constants are evaluated to elements in  $V$ . For functions  $F(\vec{r}^v, \vec{m}^v)$ , we defined  $v(F) = F^v(\vec{r}, \vec{m})$ . For atomic formulas  $A = P(\vec{m})$ , we define  $v(A) = P^v(\vec{m})$ . For atomic formulas  $A = r$ , we define  $v(A) = r$ . For composite formulas  $A$  we define  $v(A)$  naturally by:  $v(\perp) = 0$ ,  $v(A \rightarrow B) = v(A) \dot{\rightarrow} v(B)$ ,  $v(A \wedge B) = v(A) \wedge v(B)$ ,  $v(A \vee B) = v(A) \vee v(B)$ ,  $v(\forall x A(x)) = \inf\{v(A(m)) : m \in M\}$ ,  $v(\exists x A(x)) = \sup\{v(A(m)) : m \in M\}$ ,  $v(\forall^p q A(q)) = \inf\{v(A(r)) : r \in V\}$ , and  $v(\exists^p q A(q)) = \sup\{v(A(r)) : r \in V\}$ . For any closed formula  $A \in \mathcal{L}$ , we let  $\|A\| := \inf\{v(A) : v \text{ a valuation}\}$ .

**Definition 3** (Gödel logic with propositional quantifiers). The *first order Gödel logic with propositional quantifiers*  $\mathbf{G}_V^{\text{foqp}}$ , as the set of all closed formulas of  $\mathcal{L}$ , such that  $\|A\| = 1$ .

**Theorem 4** (Herbrand Theorem for  $\mathbf{G}_{[0,1]}^{\text{foqp}}$ ). (i) For valid formula  $\exists \vec{p} \exists \vec{x} A(\vec{p}, \vec{x})$  with  $A$  being quantifier free, there is a valid Herbrand expansion of form  $\bigwedge_i (C_i \rightarrow H)$  where  $H = \bigvee_{j=1}^n A^0(\vec{c}, \vec{t}_j)$  and  $A^0(\vec{c}, \vec{t}_j)$  are instances of  $A$ ,  $\vec{c}$  are fixed propositional constants, the  $C_i$  are chains of atoms (see [2]) of  $H$  such that after deletions of  $\vec{c}$  from the chains all possible chains on the remaining atoms of  $H$  occur.

(ii) If a valid Herbrand expansion of the above form exists, then  $\exists \vec{p} \exists \vec{x} A(\vec{p}, \vec{x})$  is valid.

**Corollary 5.** The  $\Sigma_1^1$ -fragment of first order Gödel logic over  $[0, 1]$  extended with propositional quantifiers and functionals is recursively enumerable.

The construction above can be used to provide a Herbrand Theorem for the full first order Gödel logic with infix quantifiers and a skolemization with Skolem functionals.

The main remaining open problem is whether  $\mathbf{G}^{\text{foqp}}$  is recursively enumerable.

## References

- [1] M. Baaz, A. Ciabattini, and R. Zach. Quantified propositional Gödel logic. In Andrei Voronkov and Michel Parigot, editors, *Logic for Programming and Automated Reasoning. 7th International Conference, LPAR 2000*, pages 240–256, Berlin, 2000. Springer. LNAI 1955.
- [2] M. Baaz, A. Ciabattini, N. Preining, and H. Veith. A guide to quantified propositional Gödel logic. IJCAR workshop QBF 2001, June 2001. Siena, Italy.
- [3] M. Baaz and N. Preining. Quantifier elimination for quantified propositional logics on Kripke frames of type omega. *Journal of Logic and Computation*, 18:649–668, 2008.
- [4] M. Baaz and N. Preining. Gödel-Dummett logics. In Petr Cintula, Petr Hájek, and Carles Noguera, editors, *Handbook of Mathematical Fuzzy Logic, volume 2*, chapter VII, pages 585–626. College Publications, 2011.
- [5] M. Baaz and H. Veith. An axiomatization of quantified propositional Gödel logic using the Takeuti-Titani rule. In Samuel Buss, Petr Hájek, and Pavel Pudlák, editors, *Proceedings of the Logic Colloquium '98, Prague*, LNL 13, pages 74–87. ASL, 2000.



# Enriched Lawvere Theories for Operational Semantics

John C. Baez, Christian Williams

May 4, 2019

Formal systems are not always explicitly connected to how they operate in practice. Lawvere theories [3] formalize algebraic structures which obey equations, but they do not specify how to compute in such a structure, meaning how to simplify a compound term using rewrite rules. In a category, the objects are types and the morphisms are terms; as the latter are elements of hom-sets, terms are simply equal or unequal. To represent term rewriting, we need hom-objects with more structure. This is precisely the idea of enriched categories [2].

In operational semantics, program behavior is often specified by labelled transition systems: the terms are vertices, and the rewrites are edges [5]. This can be represented by a Lawvere theory in which there is a graph of morphisms between each pair of objects, or a “Gph-theory”. While directed graphs are standard, there are many representations of semantics, so we consider enrichment by any cartesian closed category.

We use an existing definition of enriched algebraic theory [4]. One subtlety, which the main theorems address, is that enrichment also generalizes the arities of the theory. We simplify this aspect by restricting to a subcategory of arities which acts like the natural numbers.

For any enriching category  $\mathbf{V}$ , a **V-theory** is a  $\mathbf{V}$ -enriched Lawvere theory with natural number arities. Several categories have a canonical semantic meaning, forming a spectrum of enrichment which allows us to examine the semantics of term calculi at various levels of detail.

**Graphs:** Gph-theories represent *small-step* operational semantics

— a hom-graph edge represents a single instance of a rewrite rule.

**Categories:** Cat-theories represent *big-step* operational semantics

— a morphism represents an element of the reflexive-transitive closure of the small-step graph.

**Posets:** Pos-theories represent *full-step* operational semantics

— a hom-poset boolean represents the existence of a big-step morphism.

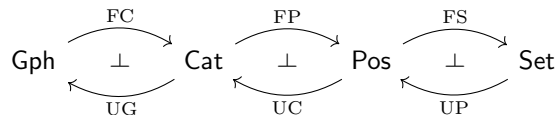
**Sets:** Set-theories represent *denotational* semantics

— a hom-set element represents a connected component of the full-step poset.

The main idea of the paper is that functors between enriching categories enable the translation between different kinds of semantics. We discuss how general monoidal functors induce change-of-base 2-functors between their 2-categories of enriched categories. We prove that functors which preserve finite products induce **change-of-semantics**.

**Theorem 1.** Let  $\mathbf{V}, \mathbf{W}$  be cartesian closed categories with finite coproducts of their terminal objects, and let  $F: \mathbf{V} \rightarrow \mathbf{W}$  be a cartesian functor. Then  $F$  is a “change of semantics” –  $F$  determines a functor from the category of  $\mathbf{V}$ -theories to the category of  $\mathbf{W}$ -theories, and moreover this determines functors between categories of models.

Our main examples arise from a chain of adjunctions which relate these forms of semantics. Right adjoints automatically preserve finite products, but these left adjoints do as well, and they are more important in applications (the last supposes that the language is confluent).



Change of base along  $\text{Gph} \rightarrow \text{Cat}$  maps small-step semantics to big-step semantics.

Change of base along  $\text{Cat} \rightarrow \text{Pos}$  maps big-step semantics to full-step semantics.

Change of base along  $\text{Pos} \rightarrow \text{Set}$  maps full-step semantics to denotational semantics.

Many constructions in operational semantics are encapsulated in this framework, subject only to the simple condition of product preservation. We also have *change-of-theory*, a functor of model categories induced by a functor of  $\mathbf{V}$ -theories: we can modify a theory, such as specifying an evaluation strategy, and it modifies the models functorially. By this and change-of-semantics, we construct the category of all models of all enriched theories using the iterated Grothendieck construction.

We demonstrate these concepts with the SK-combinator calculus, broadening the application of enriched Lawvere theories from computational effects [1] to the semantics of term calculi. The SK calculus can be presented as a Gph-theory.

<i>Th</i> (SK)	
<b>sort</b>	$t$
<b>constructors</b>	$S : 1 \rightarrow t$
	$K : 1 \rightarrow t$
	$(- -) : t^2 \rightarrow t$
<b>rewrites</b>	$\sigma : (((S a) b) c) \Rightarrow ((a b) (a c))$
	$\kappa : ((K a) b) \Rightarrow a$

We show that in the free model on the empty graph, the hom-graph from 1 to  $t$  is precisely the small-step transition system of all SK terms and rewrites. We demonstrate change-of-semantics to big-step, full-step, and denotational. We also suggest that there are many useful change-of-semantics by giving a different example: quotienting by the bisimulation relation.

## References

- [1] M. Hyland and J. Power, Discrete Lawvere theories and computational effects.
- [2] G. M. Kelly, *Basic Concepts of Enriched Category Theory*.
- [3] F. W. Lawvere, Functorial semantics of algebraic theories.
- [4] R. B. B. Lucyshyn-Wright, Enriched algebraic theories and monads for a system of arities.
- [5] G. D. Plotkin, A structural approach to operational semantics.
- [6] J. Power, Enriched Lawvere theories.

# Enriched distributivity over finite commutative residuated lattices

Adriana Balan<sup>1\*</sup>, Peter Jipsen<sup>2†</sup> and Alexander Kurz<sup>2‡</sup>

<sup>1</sup> Department of Mathematical Methods and Models, University Politehnica of Bucharest, Romania

<sup>2</sup> Schmid College of Science and Technology, Chapman University, Orange, CA, USA

It is well-known that there is an adjunction

$$\text{Pre}^{\text{op}} \rightleftarrows \text{CDL}$$

between preorders and completely distributive lattices. This adjunction is induced by homming into the two element chain  $\mathbb{2}$ . Moreover, CDL is the category of algebras for the monad  $[[-, \mathbb{2}], \mathbb{2}]$ . This can be made to work internally in a topos as shown by [9] and, using the same techniques, also to categories enriched over a commutative unital quantale  $\Omega$ , see [1]. The dual adjunction between Pre and CDL then becomes a special case of

$$\Omega\text{-Cat}^{\text{op}} \rightleftarrows \mathcal{DU}\text{-alg}$$

for  $\Omega = \mathbb{2}$ . Here  $\Omega\text{-Cat}$  is the category of categories enriched over  $\Omega$  and  $\mathcal{DU}\text{-alg}$  is the category of algebras for the monad  $[[-, \Omega], \Omega]$ . The notation  $\mathcal{DU}\text{-alg}$  is justified because, as shown by [9, 11], there is a distributive law between the enriched downset monad  $\mathcal{D}$  and the enriched upset monad  $\mathcal{U}$  such that the induced monad  $\mathcal{DU}$  is isomorphic to  $[[-, \Omega], \Omega]$ .

It follows from [9] that the categorical distributive law between  $\mathcal{D}$  and  $\mathcal{U}$  can be written as

$$\int_k [\varphi(k), \int^a G(k)(a) \otimes a] = \int^a \int_k [\varphi(k), \downarrow G(k, a)] \otimes a. \quad (1)$$

where  $k$  ranges over a set  $K$ ,  $a$  over a set  $A$  and  $\varphi : K \rightarrow \Omega$  and  $G : K \times A \rightarrow \Omega$ . The end and coend notation refers to the enriched Kan extensions explained in detail in [7]. The tensor  $\otimes$  is the commutative multiplication in  $\Omega$  and  $[-, -]$  the corresponding internal hom (residuation). The downarrow  $\downarrow$  expresses downset closure.

This can be rephrased in more familiar notation if we introduce operations in the sense of universal algebra that allow us to express the weighted (or indexed) limits and colimits that appear in the Kan-extensions of (1). In more detail, the internal hom  $[w, v]$  in  $\Omega$  is now written as  $w \triangleright v$ , the tensor  $w \otimes v$  in  $\Omega$  is written as  $w \star v$ , the end  $\int_k$  becomes a meet  $\prod_k$  and the coend  $\int^a$  becomes a join  $\bigsqcup_a$ . With this notation (1) now appears as

$$\prod_k \varphi(k) \triangleright (\bigsqcup_a G(k)(a) \star a) = \bigsqcup_a \int_k [\varphi(k), \downarrow G(k, a)] \star a, \quad (2)$$

Unfortunately, the expression  $\int_k [\varphi(k), \downarrow G(k, a)]$  is calculated as a weighted limit in  $\Omega$  and we cannot express it by algebraic operations in general. There are two ways to improve on this.

First, we know from [10] that  $\mathcal{DU}\text{-alg}$  is monadic over **Set** and therefore must have an equational axiomatisation. This can indeed be achieved by extending the signature  $(\prod, \triangleright, \bigsqcup, \star)$  by a constant for each element of  $\Omega$ . Then for each  $\varphi, G$  we obtain an instance of (2) in which  $\int_k [\varphi(k), \downarrow G(k, a)]$  is simply the corresponding constant in  $\Omega$ .

Second, we can explore whether there is a formulation of the distributive law (2) which eliminates  $\int_k [\varphi(k), \downarrow G(k, a)]$  in favour of a more traditional formulation using choice functions, as familiar from the case  $\Omega = \mathbb{2}$ .

---

\*adriana.balan@mathem.pub.ro

†jipsen@chapman.edu

‡akurz@chapman.edu

The more abstract formulation (2) holds, rather surprisingly, for any commutative unital quantale  $\Omega$ , even for those that are not distributive (as a lattice). For a concrete formulation in terms of operations, equations and choice functions, and analogous to the familiar distributive law for lattices, we need additional requirements on the quantale  $\Omega$ . These requirements are satisfied in the lattice case  $\Omega = \mathcal{2}$  and thus do generalise it.

First, while the most famous example of  $\Omega$  in this context is the one given by the real numbers as proposed in [8], we are interested here in finite quantales  $\Omega$ , mainly because this allows us to obtain results of a more conventional finitary algebraic nature. But finite  $\Omega$  are also interesting in their own right as witnessed for example by [4, 5, 2, 6]. Having said this, the theorem below does not depend on assuming that  $\Omega$  is finite as long as we are willing to rely on the axiom of choice and to admit infinitary joins and meets in the signature.

Second, we need to impose that powers preserve finite joins, or equivalently, that we enrich over a finite MTL-algebra [3, 12].

We now have a version of (2) which is universal algebraic and finitary if  $\Omega$  is. It specialises to the familiar distributive law for lattices in the case of  $\Omega = \mathcal{2}$ .

**Theorem.** Let  $\Omega$  be a (finite) commutative unital integral quantale that is completely distributive as a lattice and in which the powers  $v \triangleright -$  for all  $v \in \Omega$  preserve non-empty joins. For an algebra  $A$  in  $\mathcal{DU}\text{-alg}$ , denote by  $F$  the set of functions  $K \rightarrow A$  for some (finite) set  $K$ . Then (2) is equivalent to

$$\prod_{k \in K} \varphi(k) \triangleright \left( \bigsqcup_{a \in A} G(k)(a) \star a \right) = \bigsqcup_{f \in F} \prod_{k \in K} \phi(k) \triangleright (G(k, fk) \star fk). \quad (3)$$

**Conclusion** We have shown that the distributive law arising from enriching over a commutative unital quantale  $\Omega$  can be formulated in terms of operations and equations similar to the familiar distributive law of lattices in case that  $\Omega$  is a finite MTL-algebra.

## References

- [1] O. Băbuş and A. Kurz. On the Logic of Generalised Metric Spaces. In: *CMCS 2016*, Lect. Notes Comput. Sci. 9608, Springer (2016):136–155
- [2] R. Casley, R. F. Crew, J. Meseguer and V. R. Pratt. Temporal Structures, *Math. Structures Comput. Sci.*, 1(2) (1991):179–213
- [3] F. Esteva and L. Godo. Monoidal t-norm based logic: towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems* 124(3)(2001):271–288
- [4] H. Gaifman and V. Pratt. Partial order models of concurrency and the computation of functions. In: *LICS 1987*, IEEE Computer Society Press, Ithaca, NY (1987):72–85
- [5] H. Gaifman. Modeling concurrency by partial orders and nonlinear transition systems. REX Workshop (1988):467–488
- [6] N. Galatos and P. Jipsen. *Residuated Lattices of Size up to 6*. <http://math.chapman.edu/~jipsen/finitestructures/rlattices/RLlist3.pdf>
- [7] M. Kelly. *Basic Concepts of Enriched Category Theory*, London Math. Soc. Lect. Notes Series 64, Cambridge Univ. Press (1982), also republished as *Repr. Theory Appl. Categ.* 10(2005)
- [8] F. Lawvere. Metric spaces, generalized logic and closed categories. *Rend. Semin. Mat. Fis. Milano*, XLIII(1973):135–166. Republished in *Reprints in Theory Appl. Categ.* 1(2002)
- [9] F. Marmolejo, R. Rosebrugh, and R. J. Wood. A basic distributive law. *J. Pure Appl. Algebra*, 168(2) (2002):209–226
- [10] Q. Pu and D. Zhang. Categories Enriched Over a Quantaloid: Algebras. *Theory Appl. Categ.* 30(2015):751–774
- [11] I. Stubbe. The double power monad is the composite power monad. *Fuzzy Sets and Systems* 313(2017):25–42
- [12] M. Ward and R. P. Dilworth. Residuated lattices. *Trans. Amer. Math. Soc.* 45(3)(1939):335–354

# Proofs and surfaces

Dorđe Baralić<sup>1</sup>, Pierre-Louis Curien<sup>2</sup>, Marina Milićević<sup>1</sup>, Jovana Obradović<sup>3</sup>,  
Zoran Petrić<sup>1</sup>, Mladen Zekić<sup>1</sup>, and Rade Živaljević<sup>1</sup>

<sup>1</sup> Mathematical Institute of the Serbian Academy of Sciences and Arts

<sup>2</sup>  $\pi r^2$  team, IRIF (CNRS, University Paris Diderot and INRIA)

<sup>3</sup> Institute of Mathematics of the Czech Academy of Sciences

Incidence theorems in Euclidean or projective geometry state that some incidences follow from other incidences, where an incidence is a pair of a line and a point, together with the information whether the point lies on the line or not. A famous example is Desargue's theorem, which states that if  $ABC$  and  $UVW$  are two triangles such that  $A \neq U$ ,  $B \neq V$  and  $C \neq W$ , if  $BC \cap VW = \{P\}$ ,  $AC \cap UW = \{Q\}$  and  $AB \cap UV = \{R\}$ , then the lines  $AU$ ,  $BV$  and  $CW$  are concurrent if and only if the points  $P$ ,  $Q$  and  $R$  are colinear.

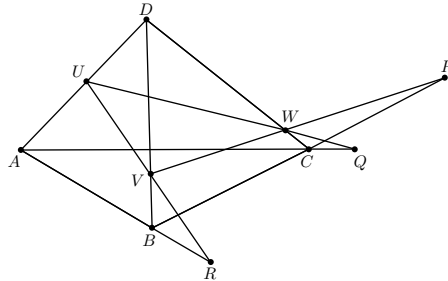
Our intention is to formalise and extend, within proof theory, the idea of Richter-Gebert [1, Section 2.2] on incidence theorems, which we paraphrase as follows:

If  $\mathcal{M}$  is a triangulated manifold that forms a 2-cycle, and therefore is orientable, then the presence of Menelaus configurations on all but one of the triangles automatically implies the existence of a Menelaus configuration on the final triangle.

A sextuple  $(A, B, C, P, Q, R)$  of points in  $\mathbf{R}^2$  makes a *Menelaus configuration* if  $(B, C; P)$ ,  $(C, A; Q)$  and  $(A, B; R)$  are defined and their product is  $-1$ , where, for three mutually distinct points  $X, Y$  and  $Z$  in  $\mathbf{R}^2$ ,  $(X, Y; Z)$  is undefined unless  $X, Y, Z$  are colinear, and is otherwise defined as follows:

$$(X, Y; Z) =_{df} \begin{cases} \frac{XZ}{YZ}, & \text{if } Z \text{ is between } X \text{ and } Y, \\ -\frac{XZ}{YZ}, & \text{otherwise.} \end{cases}$$

The Menelaus theorem states that if  $A, B, C$  are not colinear, then a Menelaus configuration can be equivalently defined purely in terms of incidences, namely:  $P, Q, R$  colinear, as well as  $B, C, P$  colinear,  $C, A, Q$  colinear and  $A, B, R$  colinear. As an example, consider the sphere  $S^2$  triangulated in four triangles (the facets of a tetrahedron) and assume that the vertices  $A, B, C$  and  $D$  of the tetrahedron, as well as the points  $P, Q, R, U, V$  and  $W$ , satisfy all the incidences displayed in the picture below.



By Menelaus theorem, we have Menelaus configurations on the triangles  $BCD$ ,  $CAD$  and  $ABD$ , i.e., we have  $(C, D; W) \cdot (D, B; V) \cdot (B, C; P) = -1$ ,  $(D, C; W) \cdot (A, D; U) \cdot (C, A; Q) = -1$ , and  $(B, D; V) \cdot (D, A; U) \cdot (A, B; R) = -1$ , which, after multiplication and cancellation, delivers  $(B, C; P) \cdot (C, A; Q) \cdot (A, B; R) = -1$ . By Menelaus theorem again,  $P, Q, R$  are colinear.

We introduce a one-sided sequent system, which deals with atomic formulae of the form “this sextuple of points makes a Menelaus configuration”. An intuition (formalised in Proposition 1 below) behind the sequents of our system is that an arbitrary formula in a sequent is entailed by the remaining formulae of the sequent. For an arbitrary countable set  $W$ , let

$$F^6(W) = W^6 - \{(X_1, \dots, X_6) \in W^6 \mid X_i = X_j \text{ for some } i \neq j\}.$$

The atomic formulae of our language are the elements of  $F^6(W)$ . The formulae are built out of atomic formulae by using the connectives  $\bowtie$  (simultaneous conjunction and disjunction) and  $\leftrightarrow$  (the classical equivalence). A sequent is a finite multiset of formulae, and the sequent consisting of a multiset  $\Gamma$  is denoted by  $\vdash \Gamma$ . The axiomatic sequents are formed in the following manner. For every triangulated manifold  $\mathcal{M}$  with 0-cells  $\mathcal{M}_0$ , 1-cells  $\mathcal{M}_1$  and 2-cells  $\mathcal{M}_2$ , such that  $\mathcal{M}_0, \mathcal{M}_1 \subseteq W$ , let  $\nu: \mathcal{M}_2 \rightarrow F^6(W)$  be defined as

$$\nu x = (d_1^1 d_2^2 x, d_0^1 d_2^2 x, d_0^1 d_0^2 x, d_0^2 x, d_1^1 x, d_2^2 x),$$

where  $d_i^j: \mathcal{M}_j \rightarrow \mathcal{M}_{j-1}$ ,  $1 \leq j \leq 2$ ,  $0 \leq i \leq j$ , are the face maps of  $\mathcal{M}$ . Then  $\vdash \{\nu x \mid x \in \mathcal{M}_2\}$  is an axiom of our system. The other axioms are  $\vdash (A, B, C, P, Q, R), (A, B, C, P, Q, R)$  (identity),  $\vdash (A, B, C, P, Q, R), (B, C, A, Q, R, P)$  and  $\vdash (A, B, C, P, Q, R), (A, R, Q, P, C, B)$  (switching of triangles). The rules of inference of the system are the following:

$$\frac{\vdash \Gamma, \varphi \quad \vdash \Delta, \varphi}{\vdash \Gamma, \Delta} \quad \frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta} \quad \frac{\vdash \Gamma, \varphi \quad \vdash \Gamma, \psi}{\vdash \Gamma, \varphi \bowtie \psi} \quad \frac{\vdash \Gamma, \varphi \quad \vdash \Delta, \psi}{\vdash \Gamma, \Delta, \varphi \leftrightarrow \psi}$$

We prove the soundness of our system with respect to Euclidean (resp. projective) *interpretations*, i.e. functions from  $W$  to  $\mathbf{R}^2$  (resp. to  $\mathbf{RP}^2$ ). We say that an interpretation satisfies the atomic formula  $(A, B, C, P, Q, R)$ , when its interpretation as a sextuple of points in  $\mathbf{R}^2$  makes a Menelaus configuration. Let  $\Gamma \models_E \varphi$  (resp.  $\Gamma \models_P \varphi$ ) mean that every Euclidean (resp. projective) interpretation that satisfies every formula in  $\Gamma$  also satisfies  $\varphi$ , where every occurrence of  $\bowtie$  in  $\Gamma$  (resp.  $\phi$ ) is interpreted as disjunction (resp. as conjunction), while  $\leftrightarrow$  is always interpreted as classical equivalence.

**Proposition 1** (Soundness). *If  $\vdash \Gamma, \varphi$  is derivable, then  $\Gamma \models_E \varphi$  (resp.  $\Gamma \models_P \varphi$ ).*

By normalizing in a particular way the derivations of our system, we prove its decidability:

**Proposition 2** (Decidability). *The Menelaus system is decidable.*

We illustrate on examples a general pattern for extracting an incidence result (its formulation and a proof) from derivable sequents of our system: starting from the interpretations that satisfy all but one formulae in a derivable sequent, by the soundness result, such an interpretation satisfies the last formula too. Menelaus theorem is used at both ends to translate from incidences to Menelaus configurations and back.

Finally, we show that the derivable sequents of our system admit a natural cyclic operad structure, thereby answering positively the question of whether cyclic operads appear in general proof-theory, alongside ordinary operads.

## References

- [1] J. RICHTER-GEBERT, *Meditations on Ceva’s theorem, The Coxeter Legacy: Reflections and Projections* (C. Davis and E.W. Ellers, editors), American Mathematical Society and Fields Institute, Providence, 2006, pp. 227-254

# A new logic arising from a scattered Stone space

Guram Bezhanishvili<sup>1</sup>, Nick Bezhanishvili<sup>2</sup>, Joel Lucero-Bryan<sup>3</sup>, and Jan van Mill<sup>4</sup>

<sup>1</sup> Department of Mathematical Sciences, New Mexico State University, Las Cruces, New Mexico, USA  
guram@nmsu.edu

<sup>2</sup> Institute for Logic, Language and Computation, University of Amsterdam, Amsterdam, The Netherlands

N.Bezhanishvili@uva.nl

<sup>3</sup> Department of Mathematics, Khalifa University of Science and Technology, Abu Dhabi, UAE  
joel.bryan@ku.ac.ae

<sup>4</sup> Korteweg-de Vries Institute for Mathematics, University of Amsterdam, Amsterdam, The Netherlands

j.vanMill@uva.nl

Topological semantics for modal logic interprets modal box as topological interior, and thus modal diamond as topological closure. For a topological space  $X$ , the logic  $L(X)$  of  $X$  is the set of modal formulas valid in  $X$ . It is well known that  $L(X)$  is a normal extension of  $S4$ . The famous McKinsey-Tarski theorem states that  $S4$  is the logic of any dense-in-itself metrizable space [4, 7].

In [2], modal logics arising from Stone spaces were studied, and it was shown that each of  $S4$ ,  $S4.1$ ,  $S4.2$ ,  $S4.1.2$ ,  $S4.Gr_z$ ,  $S4.Gr_n$  ( $n \geq 1$ ), and their intersections arises as  $L(X)$  for some Stone space  $X$ . Whether there is a Stone space yielding a logic not among these logics was left open [2, Question 6.2]. We exhibit a scattered Stone space yielding a new logic, thus giving an affirmative answer to the above question. The space is constructed by utilizing the work of Mrowka [5, 6].

We recall that a family  $\mathcal{R}$  of infinite subsets of the natural numbers  $\mathbb{N}$  is *almost disjoint* provided  $R \cap Q$  is finite for any distinct  $R, Q \in \mathcal{R}$ . A *Mrowka* space is  $X := \mathbb{N} \cup \mathcal{R}$  where  $\mathcal{R}$  is almost disjoint and whose topology is generated by the basis consisting of  $O(n) := \{n\}$  for  $n \in \mathbb{N}$  and  $O(R, F) := \{R\} \cup (R \setminus F)$  for  $R \in \mathcal{R}$  where  $F \subset \mathbb{N}$  is finite. If  $\mathcal{R}$  is infinite, then  $X$  is not compact. By [6], there is an almost disjoint family  $\mathcal{R}$  and a Mrowka space  $X$  such that the Čech-Stone compactification  $\beta X$  of  $X$  is the one-point compactification  $\alpha X$  of  $X$ , see Figure 1.

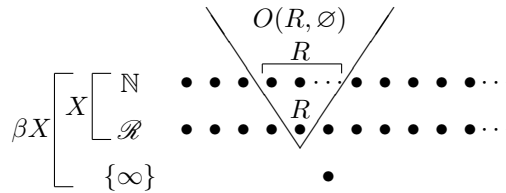


Figure 1: Depiction of  $\beta X = \alpha X$  for a Mrowka space  $X$ , and of  $O(R, \emptyset)$  for  $R \in \mathcal{R}$ .

**Lemma 1.** *Let  $X$  be a Mrowka space such that  $\beta X = \alpha X$ , and let  $\mathfrak{T}_k$  ( $k \in \mathbb{N}$ ) and  $\mathfrak{T}$  be as depicted in Figure 2.*

1.  $\beta X$  is a Stone space of Cantor-Bendixson rank 3.
2. For any  $k \in \mathbb{N}$ , the tree  $\mathfrak{T}_k$  is an interior image of  $\beta X$ .

3. The tree  $\mathfrak{T}$  is not an interior image of any open subspace of  $\beta X$ .

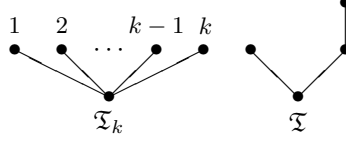


Figure 2: The trees  $\mathfrak{T}_k$  and  $\mathfrak{T}$ .

**Theorem 2.** *Let  $X$  be a Mrowka space such that  $\beta X = \alpha X$  and let  $\chi$  be the Jankov-Fine formula of  $\mathfrak{T}$ . Then the logic of  $\beta X$  is such that  $\mathbf{S4.Gr}_3 + \neg\chi \subseteq \mathbf{L}(\beta X) \subset \mathbf{S4.Gr}_2$ .*

*Proof.* (Sketch) Since  $\beta X$  is of Cantor-Bendixson rank 3, we have that  $\mathbf{S4.Gr}_3 \subseteq \mathbf{L}(\beta X)$ . It follows from Lemma 1.3 that  $\mathbf{L}(\beta X) \vdash \neg\chi$ . Therefore,  $\mathbf{S4.Gr}_3 + \neg\chi \subseteq \mathbf{L}(\beta X)$ . Since  $\mathbf{S4.Gr}_2$  is the logic of  $\{\mathfrak{T}_k \mid k \in \mathbb{N}\}$ , it follows from Lemma 1.2 that  $\mathbf{L}(\beta X) \subseteq \mathbf{S4.Gr}_2$ . The containment is strict because  $\beta X$  is of Cantor-Bendixson rank  $3 > 2$ .  $\square$

**Remark 3.** It is well known (see, e.g., [3, Sec. 9.4]) that in the intuitionistic setting, the negation of the Fine-Jankov formula of the tree  $\mathfrak{T}$  axiomatizes the *Scott logic* obtained by adding to the intuitionistic propositional calculus the *Scott axiom*

$$((\neg\neg p \rightarrow p) \rightarrow (p \vee \neg p)) \rightarrow (\neg p \vee \neg\neg p).$$

Thus, the logic  $\mathbf{S4.Gr}_3 + \neg\chi$  can alternatively be axiomatized by adding to  $\mathbf{S4.Gr}_3$  the Gödel translation of the Scott axiom.

**Remark 4.** A detailed account of these results can be found in [1].

## References

- [1] G. Bezhanishvili, N. Bezhanishvili, J. Lucero-Bryan, and J. van Mill. On modal logics arising from scattered locally compact Hausdorff spaces. *Annals of Pure and Applied Logic*, 170:558–577, 2019.
- [2] G. Bezhanishvili and J. Harding. Modal logics of Stone spaces. *Order*, 29(2):271–292, 2012.
- [3] A. Chagrov and M. Zakharyashev. *Modal logic*. Oxford University Press, 1997.
- [4] J. C. C. McKinsey and A. Tarski. The algebra of topology. *Annals of Mathematics*, 45:141–191, 1944.
- [5] S. Mrówka. On completely regular spaces. *Fund. Math.*, 41:105–106, 1954.
- [6] S. Mrówka. Some set-theoretic constructions in topology. *Fund. Math.*, 94:83–92, 1977.
- [7] H. Rasiowa and R. Sikorski. *The mathematics of metamathematics*. Monografie Matematyczne, Tom 41. Państwowe Wydawnictwo Naukowe, Warsaw, 1963.



# A generalization of Gelfand-Naimark-Stone duality to completely regular spaces

G. Bezhanishvili<sup>1</sup>, P. J. Morandi<sup>1</sup>, B. Olberding<sup>1</sup>

New Mexico State University

Let **bal** be the category of bounded archimedean  $\ell$ -algebras, and let **ubal** be the full subcategory of **bal** consisting of uniformly complete  $\ell$ -algebras. Gelfand-Naimark-Stone duality establishes a dual equivalence between **ubal** and the category **KHaus** of compact Hausdorff spaces. We extend this duality to the category **CReg** of completely regular spaces. This we do by first introducing basic extensions of bounded archimedean  $\ell$ -algebras and generalizing Gelfand-Naimark-Stone duality to a dual equivalence between the category **ubasic** of uniformly complete basic extensions and the category **Comp** of compactifications of completely regular spaces. We then introduce maximal basic extensions and prove that the subcategory **mbasic** of **ubasic** consisting of maximal basic extensions is dually equivalent to the subcategory **SComp** of **Comp** consisting of Stone-Ćech compactifications. This yields the desired dual equivalence for completely regular spaces since **CReg** is equivalent to **SComp**.

To describe the functors establishing Gelfand-Naimark-Stone duality, for a completely regular space  $X$ , let  $C^*(X)$  be the ring of bounded continuous real-valued functions. There is a natural partial order  $\leq$  on  $C^*(X)$  lifted from  $\mathbb{R}$ . Then  $C^*(X) \in \mathbf{ubal}$ . For a continuous map  $\varphi : X \rightarrow Y$  between completely regular spaces let  $\varphi^* : C^*(Y) \rightarrow C^*(X)$  be given by  $\varphi^*(f) = f \circ \varphi$ . Then  $\varphi^*$  is a unital  $\ell$ -algebra homomorphism, and we have a contravariant functor  $C^* : \mathbf{CReg} \rightarrow \mathbf{bal}$  sending each  $X \in \mathbf{CReg}$  to  $C^*(X)$  and each continuous map  $\varphi : X \rightarrow Y$  to  $\varphi^*$ .

The functor  $C^*$  has a contravariant adjoint which is defined as follows. Let  $Y_A$  be the Yosida space of maximal  $\ell$ -ideals of  $A$ . It is well known that  $Y_A \in \mathbf{KHaus}$ . For a unital  $\ell$ -algebra homomorphism  $\alpha : A \rightarrow B$  let  $\alpha_* : Y_B \rightarrow Y_A$  be given by  $\alpha_*(M) = \alpha^{-1}(M)$ . Then  $\alpha_*$  is continuous, and we have a contravariant functor  $Y : \mathbf{bal} \rightarrow \mathbf{CReg}$  sending each  $A \in \mathbf{bal}$  to  $Y_A$  and each unital  $\ell$ -algebra homomorphism  $\alpha : A \rightarrow B$  to  $\alpha_*$ .

The functors  $C^*$  and  $Y$  yield a contravariant adjunction between **CReg** and **bal**, which restricts to a dual equivalence between **KHaus** and **ubal**. We thus arrive at the following celebrated result:

**Theorem 1** (Gelfand-Naimark-Stone duality). *The categories **KHaus** and **ubal** are dually equivalent, and the dual equivalence is established by the functors  $C^*$  and  $Y$ .*

We recall that  $B \in \mathbf{bal}$  is *Dedekind complete* if every bounded subset of  $B$  has a supremum and infimum. We call  $B$  a *basic algebra* if  $B$  is Dedekind complete and the boolean algebra  $\text{Id}(B)$  of idempotents is atomic. Basic algebras generalize complete and atomic boolean algebras. Let **balg** be the category of basic algebras and unital  $\ell$ -algebra homomorphisms preserving all existing joins and meets.

For any set  $X$ , the  $\ell$ -algebra  $B(X)$  of all bounded real-valued functions is a basic algebra. This defines a contravariant functor  $B$  from the category **Set** of sets to **balg**.

**Theorem 2.** *The functor  $B$  yields a dual equivalence between **Set** and **balg**.*

Theorem 2 generalizes Tarski duality between **Set** and the category of complete and atomic boolean algebras. Theorem 2 is a key ingredient in extending Theorem 1 to completely regular spaces and compactifications. Recall that a compactification of a completely regular space  $X$  is

a topological embedding  $e : X \rightarrow Y$  into a compact Hausdorff space  $Y$  such that  $e[X]$  is dense in  $Y$ .

**Definition 3.** Let  $\mathbf{Comp}$  be the category whose objects are compactifications  $e : X \rightarrow Y$  and whose morphisms are pairs  $(f, g)$  of continuous maps such that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{e} & Y \\ f \downarrow & & \downarrow g \\ X' & \xrightarrow{e'} & Y' \end{array}$$

The composition of two morphisms  $(f_1, g_1)$  and  $(f_2, g_2)$  is defined to be  $(f_2 \circ f_1, g_2 \circ g_1)$ .

If  $e : X \rightarrow Y$  is a compactification, we define  $e^\flat : C^*(Y) \rightarrow B(X)$  by  $e^\flat(f) = f \circ e$ . Then  $e^\flat$  is a monomorphism in  $\mathbf{bal}$  such that  $e^\flat[C^*(Y)]$  is join-meet dense in  $B(X)$ , meaning that each  $b \in B(X)$  is a join of meets from  $e^\flat[C^*(Y)]$ . This motivates the following definition.

**Definition 4.**

1. Let  $A \in \mathbf{bal}$ ,  $B \in \mathbf{balg}$ , and  $\alpha : A \rightarrow B$  be a monomorphism in  $\mathbf{bal}$ . We call  $\alpha : A \rightarrow B$  a *basic extension* if  $\alpha[A]$  is join-meet dense in  $B$ .
2. Let  $\mathbf{basic}$  be the category whose objects are basic extensions and whose morphisms are pairs  $(\rho, \sigma)$  such that  $\rho$  is a morphism in  $\mathbf{bal}$ ,  $\sigma$  is a morphism in  $\mathbf{balg}$ , and  $\sigma \circ \alpha = \alpha' \circ \rho$ . The composition of two morphisms  $(\rho_1, \sigma_1)$  and  $(\rho_2, \sigma_2)$  is defined to be  $(\rho_2 \circ \rho_1, \sigma_2 \circ \sigma_1)$ .
3. Let  $\mathbf{ubasic}$  be the full subcategory of  $\mathbf{basic}$  consisting of the basic extensions  $\alpha : A \rightarrow B$  where  $A \in \mathbf{ubal}$ .

Define a contravariant functor  $\mathbf{E} : \mathbf{Comp} \rightarrow \mathbf{basic}$  as follows. For a compactification  $e : X \rightarrow Y$  let  $\mathbf{E}(e)$  be the basic extension  $e^\flat : C^*(Y) \rightarrow B(X)$ . For a morphism  $(f, g)$  in  $\mathbf{Comp}$ , let  $\mathbf{E}(f, g)$  be the pair  $(g^*, f^+)$ , where  $f^+ : B(X') \rightarrow B(X)$  is given by  $f^+(b) = b \circ f$  for all  $b \in B(X')$ .

**Theorem 5.** *The functor  $\mathbf{E} : \mathbf{Comp} \rightarrow \mathbf{basic}$  yields a dual equivalence between  $\mathbf{Comp}$  and  $\mathbf{ubasic}$ .*

**Definition 6.**

1. We call a basic extension  $\alpha : A \rightarrow B$  *maximal* provided that the only elements of  $B$  that are both a join and meet of elements from  $\alpha[A]$  are those that are in  $\alpha[A]$ .
2. Let  $\mathbf{mbasic}$  be the full subcategory of  $\mathbf{basic}$  consisting of maximal extensions.

The definition is motivated by the fact that the basic extension  $e^\flat : C^*(Y) \rightarrow B(X)$  is maximal if and only if  $e : X \rightarrow Y$  is equivalent to the Stone-Ćech compactification of  $X$ . Let  $\mathbf{SComp}$  be the full subcategory of  $\mathbf{Comp}$  consisting of Stone-Ćech compactifications.

**Theorem 7.** *The restriction of  $\mathbf{E}$  to  $\mathbf{SComp}$  yields a dual equivalence between  $\mathbf{SComp}$  and  $\mathbf{mbasic}$ .*

It is well known that  $\mathbf{CReg}$  and  $\mathbf{SComp}$  are equivalent. Thus, we obtain:

**Theorem 8.** *There is a dual equivalence between  $\mathbf{CReg}$  and  $\mathbf{mbasic}$ .*

Theorem 8 generalizes Gelfand-Naimark-Stone duality to completely regular spaces.

# The van Benthem characterisation theorem for descriptive models

Nick Bezhanishvili and Tim Henke

*Institute for Logic, Language and Computation, University of Amsterdam*

The celebrated van Benthem characterisation theorem states that on Kripke models modal logic is the bisimulation-invariant fragment of first-order logic [1, 3]. The proof of this theorem relies heavily on the compactness theorem of first-order logic. Rosen [10], however, showed that the bisimulation characterisation theorem also holds for the class of finite models on which the compactness theorem fails. Dawar and Otto [4] generalised the bisimulation characterisation theorem to a number of model-classes including rooted finite models, rooted transitive models, etc. Olkhovikov [8, 9] obtained the van Benthem characterisation theorems for intuitionistic logic. The van Benthem theorem for neighbourhood models was proved in [6]. Coalgebraic generalisations can be found in [11].

In this abstract we formulate and sketch a proof of the van Benthem characterisation theorem in the context of descriptive frames. This is an important class of general frames for which every modal logic is complete [3]. These frames can be represented as Stone spaces equipped with a suitable binary relation.

**Definition 1** (Descriptive frames and models). A *descriptive frame* is a pair  $\mathfrak{g} = (W, R)$ , where  $W$  is a Stone space and

1.  $R[x] = \{y \in W : xRy\}$  is a closed set,
2. For every clopen set  $U$  the set  $\diamond_R(U) = \{x \in W : R[x] \cap U \neq \emptyset\}$  is also clopen.

A *descriptive model* is a pair  $(\mathfrak{g}, V)$ , where  $\mathfrak{g}$  is a descriptive frame and  $V$  is a valuation in clopen subsets of  $W$ .

Descriptive models can be seen as topological generalisations of finite models. In particular, the two classes have similar model-theoretic properties. Like finite models the class of descriptive models lacks the compactness property for the language of first-order logic. They also share closure under finite disjoint unions, closure under p-morphic images, and a lack of closure under infinite disjoint unions.

It is well known that Kripke frames are the coalgebras for the powerset functor on **Set** and finite Kripke frames are coalgebras for the powerset functor on finite sets. Descriptive frames are coalgebras for the Vietoris functor on **Stone** [7]. In [2] this representation was used for developing the notion of Vietoris bisimulation for descriptive models. In fact, similarly to finite models (more generally, image-finite models), the class of descriptive models (which are image-compact) also enjoys the Hennessy-Milner property. It was proved in [2] that Vietoris bisimilarity is the same as Kripke bisimilarity since the closure of a Kripke bisimulation is both a Kripke and Vietoris bisimulation. We will refer to this notion as *bisimilarity*.

Our key technique is similar to that of Rosen [10] and Dawar and Otto [4]. Our main tool is the notion of unravelling. However, a standard unravelling of a descriptive model is not necessarily descriptive. Thus, we define a “descriptive unravelling”, which is a kind of a “descriptivisation” of the standard unravelling. Roughly speaking, we add new points to the standard unravelling at “infinite distance” from the points of unravelling, yet guaranteeing that the new model is descriptive.

Our main result states that on descriptive models, modal logic is the bisimulation-invariant fragment of first-order logic (when unary predicates are interpreted as clopen sets).

**Theorem 2** (The van Benthem Characterisation Theorem for Descriptive Models). *Let  $\alpha(x)$  be a first-order formula in one free variable. Then the following are equivalent:*

1. *There exists a modal formula  $\phi$  such that for any pointed, descriptive model  $m, w$  we have*

$$m \models \alpha[w] \text{ if and only if } m, w \Vdash \phi.$$

2. *If two pointed, descriptive models  $m, w$  and  $n, v$  are bisimilar, then  $m \models \alpha[w]$  if and only if  $n \models \alpha[v]$ .*

*Sketch of the proof.* Similarly to [10], the proof relies centrally on the following diagram, for appropriate integers  $n \geq \text{qd}(\alpha)$  (the quantifier depth of  $\alpha$ ),  $\ell \geq 3^n$ , and infinite cardinal  $\kappa \geq \aleph_0 \cdot |m| \cdot |n|$ .

$$\begin{array}{ccccc} m^{\otimes \kappa}, (w, 0) & \longleftrightarrow & m, w & \xleftrightarrow{\ell} & n, v & \longleftrightarrow & n^{\otimes \kappa}, (v, 0) \\ & & & & & & \\ & & \updownarrow & & \updownarrow & & \\ \widetilde{m^{\otimes \kappa}}, \widetilde{w} & \longleftrightarrow & \widetilde{m^{\otimes \kappa}} \uplus \widetilde{n^{\otimes \kappa}}, \widetilde{w} \equiv_n \widetilde{m^{\otimes \kappa}} \uplus \widetilde{n^{\otimes \kappa}}, \widetilde{v} & \longleftrightarrow & \widetilde{n^{\otimes \kappa}}, \widetilde{v} & & \end{array}$$

The operation  $\widetilde{\phantom{x}}$  in the above denotes the previously mentioned descriptive unravelling: the descriptivisation of the unravelling tree. The operation  $\cdot^{\otimes \kappa}$  is a duplication process that makes  $\kappa$  many copies of each point, in such a way that the final result remains descriptive.

Using this diagram, one can see that if  $m \models \alpha[w]$ , it follows from bisimulation-invariance that  $m^{\otimes \kappa} \models \alpha[(w, 0)]$  and thus  $\widetilde{m^{\otimes \kappa}} \models \alpha[\widetilde{w}]$  so that also  $\widetilde{m^{\otimes \kappa}} \uplus \widetilde{n^{\otimes \kappa}} \models \alpha[\widetilde{w}]$ . From there, equivalence up to quantifier depth  $n \geq \text{qd}(\alpha)$  gives that  $\widetilde{n} \models \alpha[\widetilde{v}]$ , where bisimulation-invariance similarly applied provides  $n \models \alpha[v]$ . This means that  $\alpha$  is  $\ell$ -bisimulation-invariant, which means it is equivalent to a modal formula of depth  $\ell$ . It is left to justify the bisimulations and  $n$ -equivalence in the diagram, which is the main challenge of the proof. The  $n$ -equivalence is shown by a variant of Hanf's Lemma [5] and the bisimilarity by the aforementioned structure of the descriptive unravelling.

## References

- [1] J. van Benthem. *Modal correspondence theory*. PhD thesis, University of Amsterdam, 1976.
- [2] N. Bezhanishvili, G. Fontaine, and Y. Venema. Vietoris bisimulations. *Journal of Logic and Computation*, 20(5):1017–1040, 2010.
- [3] P. Blackburn, M. De Rijke, and Y. Venema. *Modal Logic*. Cambridge University Press, 2001.
- [4] A. Dawar and M. Otto. Modal characterisation theorems over special classes of frames. *Annals of Pure and Applied Logic*, 161(1):1–42, 2009.
- [5] W. Hanf. Model-theoretic methods in the study of elementary logic. In *The theory of models*, pages 132–145. Elsevier, 2014.
- [6] H. Hansen, C. Kupke, and E. Pacuit. Neighbourhood Structures: Bisimilarity and Basic Model Theory. *Logical Methods in Computer Science*, 5(2), 2009.
- [7] C. Kupke, A. Kurz, and Y. Venema. Stone coalgebras. *Theor. Comput. Sci.*, 327(1-2):109–134, 2004.
- [8] G. Olkhovikov. Model-theoretic characterization of intuitionistic predicate formulas. *J. Log. Comput.*, 24(4):809–829, 2014.
- [9] G. Olkhovikov. On generalized Van Benthem-type characterizations. *Annals of Pure and Applied Logic*, 168(9):1643–1691, 2017.
- [10] E. Rosen. Modal logic over finite structures. *J. Logic Lang. Inform.*, 6(4):427–439, 1997.
- [11] L. Schröder, D. Pattinson, and T. Litak. A van Benthem/Rosen theorem for coalgebraic predicate logic. *Journal of Logic and Computation*, 27(3):749–773, 2017.

## NEARNESS POSETS

TRISTAN BICE

Nearness spaces generalise uniform spaces, which in turn are a well-known generalisation of metric spaces. These nearness spaces have been studied intermittently since the 50's in various guises, and more recently people have looked at analogous nearness structures in the point-free context, namely on frames. Somewhat surprisingly, a very natural and useful extension of these ideas is possible for general posets, as we endeavour to explain.

In fact, the story really begins 80 years ago in a paper [Wal38] by Wallman. Most attention was focused on the earlier lattice theoretic part of [Wal38] which, together with Stone's famous papers around the same time, gave birth to the field of point-free topology. However, in the latter part, Wallman showed that more general posets arising from abstract simplicial complexes also provide a point-free description of compact  $T_1$  spaces. Here the vertices of the complex correspond to a subbasis of the space, while the simplices in the complex determine the covers of the space. Thus the complex could be considered as “nearness” or “generalised uniformity” on the poset (although these terms did not yet exist - arguably, this is a rare instance where the point-free concept predated the pointed concept). This duality of Wallman's was well ahead of its time and seems to have been largely forgotten in the interim, a situation we hope to rectify by reformulating and extending Wallman's work.

Another important series of papers [Mor51] that took some time to gain the recognition it deserves was due to Morita in the early 50's. This is really the first place nearness spaces were examined in the form of generalised uniformities (the original nearness spaces were introduced later by Herrlich, while Katetov introduced merotopic spaces, both of which turned out to be equivalent to Morita's generalised uniformities). The main idea here was to drop, or at least weaken, the star-refinement axiom for uniformities in order to extend the theory to non-completely regular spaces.

However, Morita's work is also notable for considering basic open covers rather than arbitrary open covers. This is exactly what we are doing, just in the point-free context. Indeed, the covers contain such a wealth of information that there is no need to lean so heavily on the lattice structure, as is usually done in point-free topology. The covers even completely determine the order structure, at least under certain weak “admissibility” conditions. A happy consequence of this observation led us to simplify and generalise an admissibility condition investigated by Herrlich, Picado and Pultr (see [HP00] and [PP12]).

Lastly, we touch on some applications which motivated our work. While nearness frames provide a nice framework for the abstract theory, it is nearness posets that also open the door to actual constructions of interesting spaces, e.g. from continuum theory. For example, in current work with Kubiś, we are using nearness posets arising from Fraïssé sequences in categories of graphs to provide elementary combinatorial constructions of spaces like the pseudoarc and Lelek fan.

---

The author is supported by Cardinal Stefan Wyszyński University in Warsaw, Poland.

TRISTAN BICE

REFERENCES

- [HP00] Horst Herrlich and Ales Pultr. Nearness, subfitness and sequential regularity. *Applied Categorical Structures*, 8(1-2):67–80, 2000. doi:10.1007/978-94-017-2529-3\_5.
- [Mor51] Kiiti Morita. On the simple extension of a space with respect to a uniformity. I-IV. *Proc. Japan Acad.*, 27:65–72,130–137,166–171,632–636, 1951. URL: <http://projecteuclid.org/euclid.pja/1195571517>.
- [PP12] Jorge Picado and Aleš Pultr. *Frames and locales: Topology without points*. Frontiers in Mathematics. Birkhäuser/Springer Basel AG, Basel, 2012. doi:10.1007/978-3-0348-0154-6.
- [Wal38] Henry Wallman. Lattices and topological spaces. *Ann. of Math. (2)*, 39(1):112–126, 1938. doi:10.2307/1968717.

*Email address:* [Tristan.Bice@gmail.com](mailto:Tristan.Bice@gmail.com)

# Towards completeness of logics of information and common belief

Marta Bílková\*

Institute of Computer Science of the Czech Academy of Sciences, Prague  
bilkova@cs.cas.cz

Being at the same time interested in general theory of coalgebraic logic beyond those based on boolean propositional part, and in some rather particular yet well-motivated non-classical epistemic logics, I naturally find myself in the following situation: On one hand, we have been developing general coalgebraic account of logics whose semantics can be based on poset coalgebras [5, 3], on the other, we have quite specific examples of modal logics of information in mind, coming from epistemic or dynamic tradition, in need of a treatment. As a step towards applications of the general coalgebraic theory we pick a common belief logic based on Belnap-Dunn four-valued logic BD [1, 2, 9] as its propositional base.

We consider BD in the language  $\{\wedge, \vee, \neg, \top, \perp\}$  extended with finitely many belief modalities  $\diamond_i$  that are essentially normal diamonds (such epistemic logics of belief-confirmed-by-a-source were considered e.g. in [4]). We then add a common belief operator and understand it as a flat fixed point modality  $b_c$  (cf. [12]), intuitively denoting a greatest fixed point of the form  $\nu x.c(p, x)$  where  $c(p, x)$  is taken to be the scheme  $E(p \wedge x)$  with  $Ea = \bigwedge_{i \in I} \diamond_i a$ . Semantically, we interpret the language over frames of BD (see e.g. [10], cf. de Morgan frames of [8]) i.e., partially order involutive structures  $(X, \leq, *)$ , extended with monotone relations  $S_i : X \multimap X$ , (reading  $yS_i x$  as  $y$  is a trusted source in  $x$  for agent  $i$ ) used to interpret the confirmed-belief modalities as backward looking diamonds. The involution  $* : X \longrightarrow X^{op}$  interprets negation via  $x \Vdash \neg a$  iff  $*x \not\Vdash a$ , and valuations assign to each (atomic) formula an upperset of nodes. The common belief modality is interpreted as the appropriate greatest fixed point, which of course exists in this semantics.

We consider two ways of axiomatizing the common belief over the basic modal logic (i.e. an axiomatization of BD extended by normal diamond modalities): one finitary, which is the standard Kozen's axiomatization of  $b_c$  as the greatest fixed point:

$$b_c(p) \vdash c(p, b_c(p)) \quad \frac{q \vdash c(p, q)}{q \vdash b_c(p)}$$

and the other infinitary, with the following rule replacing the induction rule and using finite approximations of  $b_c(p)$  ( $c^0(p) = \top$  and  $c^{n+1}(p) = c(p, c^n(p))$ ):

$$\{c^n(p) \mid n \in N\} \vdash_\omega b_c(p).$$

We aim at proving completeness of the finitary axiomatization, following closely the method used for classical flat fixed point logics in [12]. For simplicity, we restrict ourselves to a unimodal case at first. We denote  $\mathcal{L}$  the Lindenbaum-Tarski algebra of the finitary system<sup>1</sup>. The frames can be understood as poset coalgebras, using the product of the lowerset and upperset functors: i.e. structures  $\xi : X \longrightarrow (\mathbb{L} \times \mathbb{U})X$ , mapping  $x \mapsto (\{y \mid S(y, x)\}, \{*y \mid S(y, *x)\})$ <sup>2</sup>. We exploit the fact that a coalgebraic language in spirit of [3] is available, namely one based on a cover modality  $\nabla$ , whose arity is given by the (finitary) functor  $\mathbb{U}_\omega \times \mathbb{L}_\omega$  and whose semantics uses the lifting of local satisfaction relation by the same functor (see [3] for precise definitions). The essential properties of this language proved in [3] (namely mutual definability of cover modalities and standard modalities, axiomatization in terms of distributive laws, dual cover modality  $\Delta$ , and availability of normal forms) allow us to prove the following (we consider  $\mathcal{L}_{\nabla}$  to be a Lindenbaum-Tarski algebra of the finitary system in the language based on the cover modality):

\*The work has been supported by the project GA17-04630S of the Czech Science Foundation.

<sup>1</sup>It is the formula algebra factorised by inter-derivability, i.e. the Frege congruence.

<sup>2</sup>We will need to consider also the dual box modalities defined  $\square a \equiv \neg \diamond \neg a$  on their own for normal forms to exist.

- On the algebra  $\mathcal{L}_{\nabla}$ ,  $\nabla_{\mathbb{U} \times \mathbb{L}}$  is a finitary O-adjoint:  $\nabla(\alpha, \beta) \leq b$  iff  $(\alpha, \beta) \overline{\mathbb{U} \times \mathbb{L}}(\leq) G(b)$ , with  $G(b)$  being a pair of finitely generated upper and lower set of formulas, and using the  $\mathbb{U} \times \mathbb{L}$  lifting of relation  $\leq$  (see [3] for precise definitions).

It consequently yields (using in particular the fact that  $\diamond a$  is definable as  $\nabla(\{a\}, \{\top\})$ ) the following:

- The algebra  $\mathcal{L}$  is *residuated*, i.e. each diamond is a finitary O-adjoint, and, using Santocanale's result (Proposition 6.6 in [11]) it further follows:
- The algebra  $\mathcal{L}$  is *constructive*, i.e.  $[b_{\varphi}\alpha] = \bigwedge_{n \in \mathbb{N}} [c^n(\alpha)]$ .

One of the consequences of the last statement is that the infinitary system is conservative over the finitary one (w.r.t. finitary proofs), and the infinitary rule is (globally) sound w.r.t. frame semantics (in contrast to classical common knowledge, it is not locally sound in general).

Turning the attention towards the infinitary system itself, we provide a canonical model construction, based on prime theories closed under the infinitary rule (using a Belnap's pair extension lemma, in an infinitary context proved in [7]). This construction, on one hand provides a strong completeness proof of the infinitary system w.r.t. restricted semantics (namely the class of frames where the rule is locally sound), on the other hand, together with what has been said above, it can be used to prove weak completeness of the finitary axiomatization. The results generalize to a multimodal setting.

This is partially work that has been presented before, and partially an ongoing work, the ongoing part addressing questions on completeness of general flat fixed point logics extending BD (and, further possibly also other logics with poset based semantics — adding an implication being the main challenge). In particular, we are working on generalising the adjointness results of [6] to poset setting.

Frankly speaking, the results offered here are obtained closely following the methods developed before in study of classical (flat) fixed point logics, and, however desirable and not quite trivial, are not surprising. But we find it worth investigating nevertheless, more because theory of fixed point extensions of modal logics with a non-classical base is as yet largely undeveloped.

- [1] BELNAP, N., *A useful four-valued logic*, in J.M. Dunn and G. Epstein (eds.), *Modern Uses of Multiple-Valued Logic*, 1977, pp. 5–37.
- [2] BELNAP, N., *How a computer should think*, in G. Ryle, (ed.), *Contemporary Aspects of Philosophy*, Oriel Press, Boston, 1977, pp. 30–56.
- [3] BÍLKOVÁ, M. AND DOSTÁL, M., *Moss' logic for ordered coalgebras*, *Logical Methods in Computer Science*, accepted, 2017. ArXiv: <https://arxiv.org/abs/1901.06547>.
- [4] BÍLKOVÁ, M., O. MAJER AND M. PELIŠ, *Epistemic logics for sceptical agents*, *Journal of Logic and Computation*, 26(6), 2016, pp. 1815–1841,
- [5] BÍLKOVÁ, M., KURZ, A., PETRISAN, D., AND VELEBIL, J., *Relation liftings on preorders and posets*, *International Conference on Algebra and Coalgebra in Computer Science CALCO 2011*, pp. 115–129.
- [6] BÍLKOVÁ, M., VELEBIL, J., AND VENEMA, Y., *On monotone modalities and adjointness*, *Mathematical Structures in Computer Science*, (21) 2011, pp. 383–416.
- [7] BÍLKOVÁ, M., CINTULA, P., AND LÁVIČKA, T., *Lindenbaum and Pair Extension Lemma in Infinitary Logics*, *Proceedings of WoLLIC 2018*, LNCS 10944, pp. 130–144.
- [8] CORNISH, W.H. AND FOWLER, P.R., *Coproducts of de Morgan algebras*, *Bulletin of the Australian Mathematical Society*, 16 (1977), pp. 1–13.
- [9] DUNN, J. M., *The algebra of intensional logics*, Ph.D. thesis, University of Pittsburgh, Ann Arbor., 1966.
- [10] PŘENOSIL, A., *Reasoning with Inconsistent Information*, Ph.D. Thesis, Charles University, 2018.
- [11] SANTOCANALE L., *Completions of  $\mu$ -algebras*, *Annals of Pure and Applied Logic*, (154), 2008, pp. 27–50.
- [12] SANTOCANALE L. AND VENEMA, Y., *Completeness for flat modal fixpoint logics*, *Annals of Pure and Applied Logic*, (162), 2010, pp. 55–82.



# Counting finite involutive bisemilattices

S. Bonzio<sup>1</sup>, M. Pra Baldi<sup>2</sup>, and Diego Valota<sup>3</sup>

<sup>1</sup> Università Politecnica delle Marche, Ancona, Italy.

<sup>2</sup> Università di Cagliari, Italy.

<sup>3</sup> Università di Milano, Italy

The class of involutive bisemilattices plays the role of the algebraic counterpart among one of the three-valued logics introduced by Kleene in [15], namely paraconsistent weak Kleene logic – PWK for short. PWK, essentially introduced by Halldén [11], can be defined as the logic induced by a matrix given by the weak Kleene tables with  $\{1, n\}$  as truth set:

$\wedge$	0	$n$	1	$\vee$	0	$n$	1	$\neg$	
0	0	$n$	0	0	0	$n$	1	1	0
$n$	$n$	$n$	$n$	$n$	$n$	$n$	$n$	$n$	$n$
1	0	$n$	1	1	1	$n$	1	0	1

Equivalently (see [9, 5]), PWK can be obtained out of (propositional) classical logic (CL) imposing the following syntactical restriction:

$$\Gamma \vdash_{\text{PWK}} \varphi \iff \text{there is } \Delta \subseteq \Gamma \text{ s.t. } \text{Var}(\Delta) \subseteq \text{Var}(\varphi) \text{ and } \Delta \vdash_{\text{CL}} \varphi,$$

where  $\text{Var}(\varphi)$  is the set of variables really occurring in  $\varphi$ .

Involutive bisemilattices consist of a *regular* variety, namely one satisfying identities of the form  $\varepsilon \approx \tau$ , where  $\text{Var}(\varepsilon) = \text{Var}(\tau)$ . More precisely, involutive bisemilattices satisfy only the regular identities holding in Boolean algebras. Due to the general theory of regular varieties, which traces back to the pioneering work of Płonka [17], involutive bisemilattices can be represented as *Płonka sums* of Boolean algebras, that is, a sum over semilattice direct systems of Boolean algebras. Over the years, Płonka sums and (some) regular varieties have been studied in depth both from a purely algebraic perspective [1, 14, 12, 13] and in connection with their topological duals [19, 20, 4]. The machinery of Płonka sums has also found useful applications in the study of the constraint satisfaction problem [2] and in database semantics [16, 18]. Recently, thanks to the extension of this formalism to logical matrices [7, 8], Płonka sums have turned out to play a useful role in the investigation of logics featuring the presence of a non-sensical, infectious truth-value. This family of logics – including PWK and Bochvar logic [3] – provides valuable formal instruments to model computer-programs affected by errors [10].

In this paper we exploit the Płonka sum representation for the purpose of counting the finite members of the class of involutive bisemilattices. In particular, we will start considering a specific subclass of involutive bisemilattices, whose representation consists of a linearly ordered semilattice. In particular, we provide an algorithm offering a solution to the fine spectrum problem [21] for the class of linearly ordered involutive bisemilattices. In order to achieve this goal, we use the categorical apparatus developed in [6]. We believe that the application of the above-mentioned algebraic methods allows us to develop algorithms that are more efficient than “brute-force” procedures. This is confirmed by the computational experiments. In particular, a comparison between the efficiency of the algorithm introduced in this paper and of Mace4 is briefly discussed. The technique will be finally extended to the whole class of involutive bisemilattices.

## References

- [1] R. Balbes. A representation theorem for distributive quasilattices. *Fundamenta Mathematicae*, 68:207–214, 1970.
- [2] C. Bergman and D. Failing. Commutative idempotent groupoids and the constraint satisfaction problem. *Algebra Universalis*, 73(3):391–417, 2015.
- [3] D. Bochvar. On a three-valued calculus and its application in the analysis of the paradoxes of the extended functional calculus. *Matematicheskii Sbornik*, 4:287–308, 1938.
- [4] S. Bonzio. Dualities for Płonka sums. *Logica Universalis*, 12:327–339, 2018.
- [5] S. Bonzio, J. Gil-Férez, F. Paoli, and L. Peruzzi. On Paraconsistent Weak Kleene Logic: axiomatization and algebraic analysis. *Studia Logica*, 105(2):253–297, 2017.
- [6] S. Bonzio, A. Loi, and L. Peruzzi. A duality for involutive bisemilattices. *Studia Logica*, 2018. <https://doi.org/10.1007/s11225-018-9801-0>.
- [7] S. Bonzio, T. Moraschini, and M. Pra Baldi. Logics of left variable inclusion and Płonka sums of matrices. Submitted manuscript, 2018.
- [8] S. Bonzio and M. Pra Baldi. Containment logics and Płonka sums of matrices. Submitted manuscript, 2018.
- [9] R. Ciuni and M. Carrara. Characterizing logical consequence in paraconsistent weak kleene. In L. Fellino, A. Ledda, F. Paoli, and E. Rossanese, editors, *New Developments in Logic and the Philosophy of Science*. College, London, 2016.
- [10] T. Ferguson. A computational interpretation of conceptivism. *Journal of Applied Non-Classical Logics*, 24(4):333–367, 2014.
- [11] S. Halldén. *The Logic of Nonsense*. Lundequista Bokhandeln, Uppsala, 1949.
- [12] J. Harding and A. B. Romanowska. Varieties of Birkhoff systems: part I. *Order*, 34(1):45–68, 2017.
- [13] J. Harding and A. B. Romanowska. Varieties of Birkhoff Systems: Part II. *Order*, 34(1):69–89, 2017.
- [14] J. Kalman. Subdirect decomposition of distributive quasilattices. *Fundamenta Mathematicae*, 2(71):161–163, 1971.
- [15] S. Kleene. *Introduction to Metamathematics*. North Holland, Amsterdam, 1952.
- [16] L. Libkin. *Aspects of Partial Information in Databases*. PhD Thesis, University of Pennsylvania, 1994.
- [17] J. Płonka. On a method of construction of abstract algebras. *Fundamenta Mathematicae*, 61(2):183–189, 1967.
- [18] H. Puhmann. The snack powerdomain for database semantics. In A. M. Borzyszkowski and S. Sokołowski, editors, *Mathematical Foundations of Computer Science 1993*, pages 650–659, Berlin, Heidelberg, 1993. Springer Berlin Heidelberg.
- [19] A. Romanowska and J. Smith. Semilattice-based dualities. *Studia Logica*, 56(1/2):225–261, 1996.
- [20] A. Romanowska and J. Smith. Duality for semilattice representations. *Journal of Pure and Applied Algebra*, 115(3):289–308, 1997.
- [21] W. Taylor. The fine spectrum of a variety. *Algebra Universalis*, 5(1):263–303, 1975.

# Difference hierarchies over lattices\*

Célia Borlido

Laboratoire J. A. Dieudonné, CNRS, Université Côte d'Azur  
cborlido@unice.fr

Hausdorff introduced the notion of a difference hierarchy in his work on set theory [4]. Subsequently, the notion has played an important role in descriptive set theory as well as in complexity theory. More recently, it has seen a number of applications in the theory of regular languages and automata [3, 2]. From a lattice theoretic point of view, the difference hierarchy over a bounded distributive lattice  $D$  stratifies the Booleanization,  $B$ , of the lattice in question. The Booleanization of  $D$  is the (unique up to isomorphism) Boolean algebra containing  $D$  as a bounded sublattice and generated (as a Boolean algebra) by  $D$ . The stratification is made according to the minimum length of difference chains required to describe an element  $b \in B$ :

$$b = a_1 - (a_2 - (\dots (a_{n-1} - a_n)\dots))$$

where  $a_1 \geq a_2 \geq \dots \geq a_{n-1} \geq a_n$  are elements of  $D$ . One difficulty in the study of difference hierarchies is that in general elements  $b \in B$  do not have canonical associated difference chains.

Stone duality [6] represents any bounded distributive lattice as the simultaneously compact and open subsets of an associated topological space known as the Stone dual space of the lattice. Priestley duality [5] is a rephrasing of this duality which uses the Stone space of the Booleanization equipped with a partial order to represent the lattice as the closed and open upsets of the associated Priestley space. Priestley duality provides an elucidating tool for the study of difference hierarchies. For one, the minimum length of difference chains for an element  $b \in B$  has a nice description relative to the Priestley dual space  $X$  of  $D$  as the length of the longest chain of points  $x_1 \leq x_2 \leq \dots \leq x_n$  in  $X$  so that  $x_i$  belongs to the clopen corresponding to  $b$  if and only if  $i$  is odd. Further, if we allow difference chains of closed upsets of the Priestley space, rather than clopen upsets, then every element  $b \in B$  has a canonical difference chain which is of minimum length. In particular, if the lattice  $D$  is a co-Heyting algebra, then the canonical difference chain of closed upsets consists of closed and open upsets and thus every  $b \in B$  has a canonical difference chain in  $D$ . In this talk I will start by explaining how such a canonical difference chain is obtained. Using a compactness argument via canonical extensions, this may be extended to every lattice that is the direct limit of the images of a family of maps  $\{g_i : S_i \rightarrow B\}_{i \in I}$  admitting an upper adjoint, where  $\{S_i\}_{i \in I}$  is a family of semilattices and  $B$  a Boolean algebra. In particular, this provides a topological proof of the well-known fact that every element  $b$  in the Booleanization of a bounded distributive lattice  $D$  may be written as a difference

$$b = a_1 - (a_2 - (\dots (a_{n-1} - a_n)\dots)),$$

where  $a_1 \geq a_2 \geq \dots \geq a_{n-1} \geq a_n$  are elements of  $D$ .

## References

- [1] C. Borlido, M. Gehrke, A. Krebs, and H. Straubing. Difference hierarchies and duality with an application to formal languages. *arXiv e-prints*, page arXiv:1812.01921, Dec 2018.

---

\*This is based on joint work with Mai Gehrke, Andreas Krebs and Howard Straubing [1].

- [2] O. Carton, D. Perrin, and J.-É. Pin. A survey on difference hierarchies of regular languages. *Preprint*, 2017.
- [3] C. Glaßer, H. Schmitz, and V. Selivanov. Efficient algorithms for membership in Boolean hierarchies of regular languages. *Theoret. Comput. Sci.*, 646:86–108, 2016.
- [4] F. Hausdorff. *Set theory*. Chelsea Publishing Company, New York, 1957. Translated by John R. Aumann, et al.
- [5] H. A. Priestley. Representation of distributive lattices by means of ordered stone spaces. *Bull. London Math. Soc.*, 2:186–190, 1970.
- [6] M. H. Stone. Applications of the theory of Boolean rings to general topology. *Trans. Amer. Math. Soc.*, 41(3):375–481, 1937.

# Time-reversal and homotopical properties of concurrent systems

C. Calk

ENS de Lyon

Directed topology was originally introduced as a model, and a tool, for studying and classifying concurrent systems in computer science [13, 7]. In this approach, the possible states of several processes running concurrently are modeled as points in a topological space of configurations, in which executions are described by paths. Restricted areas appear when these processes have to synchronize, to perform a joint task, or to use a shared object that cannot be shared by more than a certain number of processes. Formally, see [8], a *directed space*, or a *dispace* for short, is a pair  $\mathcal{X} = (X, dX)$ , where  $X$  is a topological space and  $dX$  is a set of *paths* in  $X$ , i.e., continuous maps from  $[0, 1]$  to  $X$ , called *directed paths*, of *dipaths*, such that every constant path is directed, and such that  $dX$  is closed under monotonic reparametrization and concatenation.

It is natural to study the homotopical properties of these spaces in order to deduce some interesting properties of the parallel programs they represent, for verification purposes, or for classifying synchronization primitives. In contrast to ordinary algebraic topology, the invariants of interest are invariants under some form of continuous deformation, but which have to respect the flow of time, or direction, given by the dipaths. In short, the only valid homotopies are the ones which never invert the flow of time. For mathematical developments and some applications we refer the reader to the two books [8, 6].

Directed topological invariants, most notably the computationally tractable ones such as homology, have been long in the making (starting with [7]). Most directed homology theories have proven too weak to classify essential features of directed topology, until the proposal [4, 5]. The main idea of [4] is to encode the way in which the homotopy types of the spaces of directed paths vary when we move the end points. The algebraic structure which logs all of the homotopy types of the directed path spaces between each pair of points is that of a natural system. As observed by Fajstrup and Hess, a shortcoming of directed homotopy is that it is invariant under time-reversal. We solve this problem using so-called composition pairings associated to natural systems.

A *natural system on a category  $\mathcal{C}$*  with values in a category  $\mathbf{V}$  is a functor  $D : FC \rightarrow \mathbf{V}$ , where  $FC$  is the factorization category of  $\mathcal{C}$  whose 0-cells are the 1-cells of  $\mathcal{C}$  and the 1-cells correspond to factorizations of 1-cells in  $\mathcal{C}$ . These were introduced in [14] and used as coefficients for cohomology of small categories in [1] and monoids in [11], as well as to define homological finiteness invariants for convergent rewriting systems in [9, 10].

A classic result states that the category of natural systems on a category  $\mathcal{C}$  with values in the category  $\mathbf{Ab}$  of Abelian groups is equivalent to the category of internal abelian groups in the slice category  $\mathbf{Cat}_{\mathcal{C}_0}/\mathcal{C}$  of categories over  $\mathcal{C}$  with the same object set  $\mathcal{C}_0$ . In order to extend such an equivalence to natural systems with values in the category  $\mathbf{Gp}$  of groups, Porter [12] considers natural systems enriched with composition pairings, which can also be interpreted as lax functors. Specifically, given a natural system  $D : FC \rightarrow \mathbf{V}$ , a *composition pairing* associated to  $D$  consists of families

$$\nu_{f,g} : D_f \times D_g \rightarrow D_{fg} \quad \nu_x : \mathbb{T} \rightarrow D_{1_x},$$

of morphisms of  $\mathbf{V}$  indexed by 1-cells  $f, g$  a 0-cell  $x$  in  $\mathcal{C}$ , satisfying coherence conditions. Porter showed that the category of natural systems on a category  $\mathcal{C}$  with values in the category of groups and with composition pairings is equivalent to the category of internal groups in the category of categories over  $\mathcal{C}$ .

We show that the natural systems of directed homotopy as introduced in [5, 3], which give a natural system  $\overrightarrow{P}_n(\mathcal{X})$  for each dispace  $\mathcal{X}$  and every  $n$ , admit composition pairings. We can thereby interpret these functors as internal groups in a certain comma category:

**Theorem 1** Let  $\mathcal{X} = (X, dX)$  be a dispace. For each  $n \geq 2$  there exists an internal group  $C_{\mathcal{X}}^n$  in  $\mathbf{Cat}_X / \overrightarrow{\mathbf{P}}(\mathcal{X})$  such that  $\overrightarrow{\mathbf{P}}_n(\mathcal{X})_f = (C_{\mathcal{X}}^n)_f$ , for all traces  $f$  of  $\mathcal{X}$ , and this assignment is functorial in  $\mathcal{X}$ .

Now, given some dispace  $\mathcal{X}$ , we define its time-reversed, or opposite, dispace  $\mathcal{X}^\# = (X, dX^\#)$ , where

$$dX^\# = \{t \mapsto f(1 - t) \mid f \in dX\}.$$

L. Fajstrup and K. Hess noted that  $\overrightarrow{\mathbf{P}}_n(\mathcal{X})$  and  $\overrightarrow{\mathbf{P}}_n(\mathcal{X}^\#)$  are isomorphic, i.e. that the functor

$$\overrightarrow{\mathbf{P}}_n : \mathbf{dTop} \rightarrow \mathbf{opNat}(\mathbf{Gp})$$

is invariant under time-reversal, where  $\mathbf{dTop}$  (resp.  $\mathbf{opNat}(\mathbf{Gp})$ ) denotes the category of directed spaces (resp. the category of natural systems with values in the category of groups). Interpreting this functor, via the internal group construction, as a functor  $C_-^n : \mathbf{dTop} \rightarrow \mathbf{Cat}$  allows us to capture this reversal via a passage to the opposite category, our main result:

**Theorem 2** For any  $n \geq 0$ , the functor  $C_-^n : \mathbf{dTop} \rightarrow \mathbf{Cat}$  is time-reversal, i.e.

$$C_{\mathcal{X}^\#}^n \simeq (C_{\mathcal{X}}^n)^{\mathbf{op}}$$

The results discussed in this talk are from [2], a joint work with E. Goubault and P. Malbos.

## REFERENCES

- [1] Hans-Joachim Baues and Gunther Wirsching. Cohomology of small categories. *Journal of Pure and Applied Algebra*, 38(2–3):187–211, 1985.
- [2] Cameron Calk, Eric Goubault, and Philippe Malbos. Time-reversal homotopical properties of concurrent systems. *arXiv e-prints:1812.05062*, Dec 2018.
- [3] Jérémy Dubut. *Directed homotopy and homology theories for geometric models of true concurrency*. PhD thesis, 09 2017.
- [4] Jérémy Dubut, Eric Goubault, and Jean Goubault-Larrecq. Natural homology. In *Automata, Languages, and Programming - 42nd International Colloquium, ICALP 2015, Kyoto, Japan, July 6-10, 2015, Proceedings, Part II*, pages 171–183, 2015.
- [5] Jérémy Dubut, Éric Goubault, and Jean Goubault-Larrecq. Directed homology theories and Eilenberg-Steenrod axioms. *Applied Categorical Structures*, pages 1–33, 2016.
- [6] L. Fajstrup, E. Goubault, E. Haucourt, S. Mimram, and M. Raussen. *Directed Algebraic Topology and Concurrency*. Springer, 2016.
- [7] Eric Goubault and Thomas P. Jensen. Homology of higher dimensional automata. In *CONCUR '92, Third International Conference on Concurrency Theory, Stony Brook, NY, USA, August 24-27, 1992, Proceedings*, pages 254–268, 1992.
- [8] Marco Grandis. *Directed Algebraic Topology, Models of non-reversible worlds*. Cambridge University Press, 2009.
- [9] Yves Guiraud and Philippe Malbos. Higher-dimensional normalisation strategies for acyclicity. *Adv. Math.*, 231(3-4):2294–2351, 2012.
- [10] Yves Guiraud and Philippe Malbos. Identities among relations for higher-dimensional rewriting systems. In *OPERADS 2009*, volume 26 of *Sémin. Congr.*, pages 145–161. Soc. Math. France, Paris, 2013.
- [11] Jonathan Leech. Cohomology theory for monoid congruences. *Houston J. Math.*, 11(2):207–223, 1985.
- [12] Timothy Porter. Group objects in  $\mathbf{Cat}_{\Sigma_0} / \mathbf{B}$ , 2012. Preprint.
- [13] Vaughn Pratt. Modeling concurrency with geometry. In *Proceedings of the 18th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, POPL '91, pages 311–322. ACM, 1991.
- [14] Daniel Quillen. Higher algebraic K-theory. I. *Lecture Notes in Mathematics*, 341:85–147, 1973.

# A temporal interpretation of intuitionistic quantifiers

Guram Bezhanishvili and Luca Carai

Department of Mathematical Sciences  
 New Mexico State University  
 Las Cruces NM 88003, USA  
 guram@nmsu.edu  
 lcarai@nmsu.edu

It is well known that the Gödel translation of the intuitionistic propositional calculus IPC into Lewis' modal system S4 is full and faithful, meaning that

$$\text{IPC} \vdash \varphi \text{ iff } \text{S4} \vdash \varphi^t,$$

where we recall that the Gödel translation is defined as follows:

$$\begin{aligned} \perp^t &= \perp \\ p^t &= \Box p && \text{for each propositional letter } p \\ (\varphi \circ \psi)^t &= \varphi^t \circ \psi^t && \circ = \wedge, \vee \\ (\varphi \rightarrow \psi)^t &= \Box(\neg\varphi^t \vee \psi^t). \end{aligned}$$

The translation extends to the predicate setting fully and faithfully by letting

$$\begin{aligned} (\forall x\varphi)^t &= \Box\forall x\varphi^t \\ (\exists x\varphi)^t &= \exists x\varphi^t. \end{aligned}$$

Thus, we have

$$\text{IQC} \vdash \varphi \text{ iff } \text{QS4} \vdash \varphi^t,$$

where IQC is the intuitionistic predicate calculus and QS4 is the predicate S4.

The monadic fragment of IQC, known as MIPC, can be axiomatized by enriching the language of IPC by two “quantifier modalities”  $\forall, \exists$  such that

- $\forall$  satisfies the S4-axioms for  $\Box$ ;
- $\exists$  satisfies the S5-axioms for  $\Diamond$ ;
- $\exists p \rightarrow \forall \exists p$ ;
- $\exists \forall p \rightarrow \forall p$ .

The monadic fragment of QS4, denoted by MS4, is obtained from the fusion  $\text{S4} \otimes \text{S5}$ , where  $\Box$  is the S4-modality and  $\forall$  is the S5-modality, by adding the left commutativity axiom

$$\Box\forall p \rightarrow \forall\Box p.$$

Fischer-Servi [2] proved that the Gödel translation of IQC into QS4 restricts to a full and faithful translation of MIPC into MS4.

Our goal is to develop an alternative temporal interpretation of intuitionistic quantifiers, where  $\forall$  is interpreted as “always in the future” and  $\exists$  is interpreted as “sometime in the past.” This we do by first introducing a new multi-modal temporal system.

Let S4.t be the temporal S4, whose modalities we denote by  $\Box_F$  (always in the future) and  $\Box_P$  (always in the past); see Esakia [1], Wolter [3]. The temporal diamonds  $\Diamond_P$  (sometime in the past) and  $\Diamond_F$  (sometime in the future) are definable from  $\Box_P$  and  $\Box_F$  in the usual way.

The logic TS4 is obtained from the fusion  $\text{S4} \otimes \text{S4.t}$  by adding the axioms

- $\Box_F q \rightarrow \Box \Box_F q$ ;
- $\Diamond_F q \rightarrow \Diamond(\Diamond_F q \wedge \Diamond_P q)$ .

We extend the Gödel translation of IPC into S4 to interpret the monadic quantifiers as follows.

$$\begin{aligned} (\forall \varphi)^t &= \Box_F \varphi^t \\ (\exists \varphi)^t &= \Diamond_P \varphi^t. \end{aligned}$$

Thus, the universal quantifier is interpreted as “always in the future” and the existential quantifier as “sometime in the past.” We prove that this translation is full and faithful, meaning that

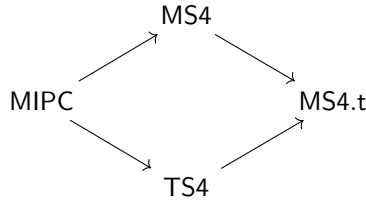
$$\text{MIPC} \vdash \varphi \text{ iff } \text{TS4} \vdash \varphi^t.$$

The proof can be done by either algebraic or frame-theoretic means as the systems involved are canonical.

While the systems MS4 and TS4 are incomparable, they admit a common extension. Let QS4.t be the predicate S4.t and let MS4.t be its monadic fragment. Thus, MS4.t is obtained from the fusion S4.t  $\otimes$  S5 by adding the left commutativity axiom

$$\Box_F \forall p \rightarrow \forall \Box_F p.$$

We prove that both MS4 and TS4 can be translated into MS4.t fully and faithfully. Thus, we arrive at the following commutative diagram of translations, where the commutativity is up to logical equivalence.



Consequently, we have a full and faithful translation of MIPC into MS4.t. We prove that it extends to a full and faithful translation of IQC into QS4.t, where

$$\begin{aligned} (\forall x \varphi)^t &= \Box_F \forall x \varphi^t \\ (\exists x \varphi)^t &= \exists x \Diamond_P \varphi^t. \end{aligned}$$

## References

- [1] L. Esakia. Semantical analysis of bimodal (tense) systems. In *Logic, Semantics and Methodology*, pages 87–99 (Russian). “Metsniereba”, Tbilisi, 1978.
- [2] G. Fischer-Servi. On modal logic with an intuitionistic base. *Studia Logica*, 36:141, 1977.
- [3] F. Wolter. On logics with coimplication. *J. Philos. Logic*, 27(4):353–387, 1998.



# Semantic analysis and proof theory for monotone modal logic

Jinsheng Chen<sup>1</sup>, Giuseppe Greco<sup>2</sup>, Alessandra Palmigiano<sup>1,3</sup>, and Apostolos Tzimoulis<sup>4</sup>

<sup>1</sup> Delft University of Technology, the Netherlands

<sup>2</sup> University of Utrecht, the Netherlands

<sup>3</sup> University of Johannesburg, South Africa

<sup>4</sup> Vrije Universiteit Amsterdam, the Netherlands

*Non normal logics* are understood as those propositional logics algebraically captured by varieties of *Boolean algebra expansions*, i.e. algebras  $\mathbb{A} = (\mathbb{B}, \mathcal{F}^{\mathbb{A}}, \mathcal{G}^{\mathbb{A}})$  such that  $\mathbb{B}$  is a Boolean algebra, and  $\mathcal{F}^{\mathbb{A}}$  and  $\mathcal{G}^{\mathbb{A}}$  are finite, possibly empty families of operations on  $\mathbb{B}$  in which the requirement is dropped that each operation in  $\mathcal{F}^{\mathbb{A}}$  be finitely join-preserving or meet-reversing in each coordinate and each operation in  $\mathcal{G}^{\mathbb{A}}$  be finitely meet-preserving or join-reversing in each coordinate. Very well known examples of non normal logics are *monotone modal logic* which have been intensely investigated, since they capture key aspects of agents' reasoning, such as the epistemic [21], and strategic [19, 18].

Non normal logics have been extensively investigated both with model-theoretic tools [14] and with proof-theoretic tools [17]. Specific to proof theory, the main challenge is to endow non normal logics with analytic calculi which can be modularly expanded with additional rules so as to uniformly capture wide classes of axiomatic extensions of the basic frameworks, while preserving key properties such as cut elimination.

In this talk, we propose a method to achieve this goal. We will illustrate this method for the two specific signatures of monotone modal logic. Our starting point is the very well known observation that, under the interpretation of the modal connective of monotone modal logic in neighbourhood frames  $\mathbb{F} = (W, \nu)$ , the monotone 'box' operation can be understood as the composition of a *normal* (i.e. finitely join-preserving) semantic diamond  $\langle \nu \rangle$  and a *normal* (i.e. finitely meet-preserving) semantic box  $[\exists]$ . The binary relations  $R_\nu$  and  $R_\exists$  corresponding to these two normal operators are not defined on one and the same domain, but span over two domains, namely  $R_\nu \subseteq W \times \mathcal{P}(W)$  is s.t.  $wR_\nu X$  iff  $X \in \nu(w)$  and  $R_\exists \subseteq \mathcal{P}(W) \times W$  is s.t.  $XR_\exists w$  iff  $w \in X$  (cf. [14, Definition 5.7], see also [15, 6]).

We refine and expand these observations so as to: (a) introduce a semantic environment of two-sorted Kripke frames and their heterogeneous algebras; (b) outline a network of discrete dualities and adjunctions among these semantic structures and the algebras and frames for monotone modal logic; (c) based on these semantic relationships, introduce multi-type *normal* logics into which the original non normal logics can embed via suitable translations; (d) retrieve well known dual characterization results for axiomatic extensions of monotone modal logic as instances of general algorithmic correspondence theory for normal (multi-type) LE-logics applied to the translated axioms; (e) extract analytic structural rules from the computations of the first order correspondents of the translated axioms, so that, again by general results on proper display calculi [11] applied to multi-type logical frameworks as done in [3, 4, 5, 2, 8, 13, 20, 16, 10, 9, 12, 1, 7], the resulting calculi are sound, complete, conservative and enjoy cut elimination and subformula property.

## References

- [1] M. Bílková, G. Greco, A. Palmigiano, A. Tzimoulis, and N. Wijnberg. The logic of resources and capabilities. *The Review of Symbolic Logic*, 11(2):371–410, 2018.
- [2] S. Frittella, G. Greco, A. Kurz, and A. Palmigiano. Multi-type display calculus for propositional dynamic logic. *Journal of Logic and Computation*, 26(6):2067–2104, 2016.
- [3] S. Frittella, G. Greco, A. Kurz, A. Palmigiano, and V. Sikimić. Multi-type sequent calculi. *Proc. Trends in Logic XIII, A. Indrzejczak et al. eds*, pages 81–93, 2014.
- [4] S. Frittella, G. Greco, A. Kurz, A. Palmigiano, and V. Sikimić. A proof-theoretic semantic analysis of dynamic epistemic logic. *Journal of Logic and Computation*, 26(6):1961–2015, 2016.
- [5] S. Frittella, G. Greco, A. Palmigiano, and F. Yang. A multi-type calculus for inquisitive logic. In *Proc. WoLLIC 2016*, volume 9803 of *LNCS*, pages 215–233, 2016.
- [6] O. Gasquet and A. Herzig. From classical to normal modal logics. In *Proof theory of modal logic*, pages 293–311. Springer, 1996.
- [7] G. Greco, P. Jipsen, K. Manoorkar, A. Palmigiano, and A. Tzimoulis. Logics for rough concept analysis. In *Proc. ICLA 2019*, volume 11600 of *LNCS*, 2019.
- [8] G. Greco, F. Liang, K. Manoorkar, and A. Palmigiano. Proper multi-type display calculi for rough algebras. *Proc. LSFA 2018, ENTCS*, forthcoming, ArXiv preprint 1808.07278.
- [9] G. Greco, F. Liang, M. Moshier, and A. Palmigiano. Multi-type display calculus for semi De Morgan logic. In *Proc. WoLLIC 2017*, volume 10388 of *LNCS*, pages 199–215, 2017.
- [10] G. Greco, F. Liang, A. Palmigiano, and U. Rivieccio. Bilattice logic properly displayed. *Fuzzy Sets and Systems*, 363:138–155, 2018.
- [11] G. Greco, M. Ma, A. Palmigiano, A. Tzimoulis, and Z. Zhao. Unified correspondence as a proof-theoretic tool. *Journal of Logic and Computation*, 28(7):1367–1442, 2018.
- [12] G. Greco and A. Palmigiano. Linear logic properly displayed. *arXiv preprint: 1611.04184*.
- [13] G. Greco and A. Palmigiano. Lattice logic properly displayed. In *Proc. WoLLIC 2017*, volume 10388 of *LNCS*, pages 153–169, 2017.
- [14] H. H. Hansen. *Monotonic modal logics*. Institute for Logic, Language and Computation (ILLC), University of Amsterdam, 2003.
- [15] M. Kracht and F. Wolter. Normal monomodal logics can simulate all others. *The Journal of Symbolic Logic*, 64(1):99–138, 1999.
- [16] F. Liang. *Multi-type Algebraic Proof Theory*. PhD thesis, TU Delft, 2018.
- [17] S. Negri. Proof theory for non-normal modal logics: The neighbourhood formalism and basic results. *IFCoLog Journal of Logic and its Applications*, 4:1241–1286, 2017.
- [18] M. Pauly. A modal logic for coalitional power in games. *JLC*, 12(1):149–166, 2002.
- [19] M. Pauly and R. Parikh. Game logic—an overview. *Studia Logica*, 75(2):165–182, 2003.
- [20] A. Tzimoulis. *Algebraic and Proof-Theoretic Foundations of the Logics for Social Behaviour*. PhD thesis, TU Delft, 2018.
- [21] J. van Benthem and E. Pacuit. Dynamic logics of evidence-based beliefs. *Studia Logica*, 99(1-3):61, 2011.

# Completeness properties in abstract algebraic logic

Petr Cintula<sup>1</sup> and Carles Noguera<sup>2</sup>

<sup>1</sup> Institute of Computer Science, Czech Academy of Sciences,  
cintula@cs.cas.cz

<sup>2</sup> Institute of Information Theory and Automation, Czech Academy of Sciences,  
noguera@utia.cas.cz

Any (propositional) logic  $L$ , construed as a structural consequence relation, is strongly complete with respect to the class  $\mathbf{Mod}^*(L)$  of its reduced models, i.e.,  $\Gamma \vdash_L \varphi$  if, and only if,  $\Gamma \models_{\mathbf{Mod}^*(L)} \varphi$  (where  $\vdash_L$  is the derivability relation of the logic  $L$  and  $\models_{\mathbf{Mod}^*(L)}$  is the semantical consequence relation with respect to the class  $\mathbf{Mod}^*(L)$ ).

Although  $\mathbf{Mod}^*(L)$  already gives a complete semantics for  $L$ , it is common to consider meaningful subclasses of  $\mathbf{Mod}^*(L)$  which may provide stronger completeness theorems. In particular, if  $L$  is *finitary*, then it is strongly complete w.r.t. the classes  $\mathbf{Mod}^*(L)_{\text{RSI}}$  and  $\mathbf{Mod}^*(L)_{\text{RFSI}}$  of *relatively (finitely) subdirectly irreducible members of  $\mathbf{Mod}^*(L)$* , i.e., matrices which cannot be decomposed as a non-trivial subdirect product of an arbitrary (finite non-empty resp.) family of matrices from  $\mathbf{Mod}^*(L)$ . Recall that in classical logic  $CL$  we have:  $\mathbf{Mod}^*(CL)_{\text{RFSI}} = \mathbf{Mod}^*(CL)_{\text{RSI}} = \{\langle \mathbf{2}, \{1\} \rangle\}$ , where  $\mathbf{2}$  is the two-valued Boolean algebra.

Another interesting example is the Łukasiewicz logic, a prominent many-valued logic, which is known to be complete w.r.t. matrix  $\langle [0, 1]_L, \{1\} \rangle$  where  $[0, 1]_L$  is the so-called standard MV-algebra; but in this case the completeness is weaker, it holds for *finite* sets of premises only.

Finally, in some logics, one can prove that theorems equal tautologies but not more. This observation led us to defining three kinds of completeness for any logic  $L$  and a set of reduced matrices  $\mathbb{K} \subseteq \mathbf{Mod}^*(L)$ :

- *Strong  $\mathbb{K}$ -completeness*,  $S\mathbb{K}C$  for short, if  $L$  and  $\models_{\mathbb{K}}$  coincide, i.e., for every set of formulas  $\Gamma \cup \{\varphi\}$  we have:  $\Gamma \vdash_L \varphi$  if, and only if,  $\Gamma \models_{\mathbb{K}} \varphi$ .
- *Finite strong  $\mathbb{K}$ -completeness*,  $FS\mathbb{K}C$  for short, if finitary companions of  $L$  and  $\models_{\mathbb{K}}$  coincide, i.e., when for every *finite* set of formulas  $\Gamma \cup \{\varphi\}$  we have:  $\Gamma \vdash_L \varphi$  if, and only if,  $\Gamma \models_{\mathbb{K}} \varphi$ .
- *$\mathbb{K}$ -completeness*,  $\mathbb{K}C$  for short, if theorems of  $L$  and  $\models_{\mathbb{K}}$  coincide, i.e., for every formula  $\varphi$  we have:  $\vdash_L \varphi$  if, and only if,  $\models_{\mathbb{K}} \varphi$ .

The aim of this talk is to present characterizations of these properties that will allow, for particular choices of logics and classes of reduced models, either to show or to falsify the corresponding completeness properties, and to prove relationships between completeness properties w.r.t. different matricial semantics. Some of these results are closely related to corresponding results of universal algebra and model theory and several of them are actually already known in the theory of abstract algebraic logic [4].

This abstract is based on a systematic investigation of these properties and corresponding characterizations recently presented in our paper [2]; here in this abstract, as a sample, we showcase some of our results.

The first batch of results is valid in full generality and is based on the localization of the class  $\mathbf{Mod}^*(L)$  using the class of matrices generated from  $\mathbb{K}$  using suitable class operators, such as  $\mathbf{I}$ ,  $\mathbf{H}$ ,  $\mathbf{S}^*$ ,  $\mathbf{P}$ ,  $\mathbf{P}_U$ ,  $\mathbf{P}_\omega$  which stand for isomorphic images, homomorphic images, reductions of submatrices, products, ultraproducts and  $\omega$ -filtered products (products factorized by a filter closed under countable intersections). The subsequent results are proved for logics possessing suitable a generalized equivalence or disjunction connective: for protoalgebraic [4], equivalential [4], or strongly p-disjunctive [2] logics.

**Theorem 1.** *Let  $L$  be a logic and  $\mathbb{K} \subseteq \mathbf{Mod}^*(L)$ .*

1.  *$L$  has the S $\mathbb{K}$ C iff  $\mathbf{Mod}^*(L) \subseteq \mathbf{IS}^*\mathbf{P}_\omega(\mathbb{K})$ .*
2.  *$L$  has the FS $\mathbb{K}$ C iff  $\mathbf{Mod}^*(L) \subseteq \mathbf{IS}^*\mathbf{PP}_U(\mathbb{K})$ .*

*Therefore  $L$  has the FS $\mathbb{K}$ C iff it has the  $\mathbf{SP}_U(\mathbb{K})$ C.*

If  $L$  is protoalgebraic, we could write equality instead of subethood in the first claim, if it is even finitary and finitely equivalential, we could do that in the second claim as well. For protoalgebraic logics we can also prove that  $L$  has the  $\mathbb{K}$ C iff  $\mathbf{H}(\mathbf{Mod}^*(L)) = \mathbf{HS}^*\mathbf{P}(\mathbb{K})$ .

The second kind of results characterize the completeness properties using the localization of the (countable members of the) classes  $\mathbf{Mod}^*(L)_{\text{RSI}}$  and  $\mathbf{Mod}^*(L)_{\text{RFSI}}$ .

**Theorem 2.** *Let  $L$  be a protoalgebraic or strongly  $p$ -disjunctional finitary logic and  $\mathbb{K} \subseteq \mathbf{Mod}^*(L)$ .*

1.  *$L$  has the S $\mathbb{K}$ C iff  $\mathbf{Mod}^*(L)_{\text{RFSI}}^\omega \subseteq \mathbf{IS}^*(\mathbb{K})$ .*
2.  *$L$  has the FS $\mathbb{K}$ C iff  $\mathbf{Mod}^*(L)_{\text{RFSI}} \subseteq \mathbf{IS}^*\mathbf{P}_U(\mathbb{K})$ .*

*If  $L$  is both protoalgebraic and strongly  $p$ -disjunctional logic, then  $L$  has the  $\mathbb{K}$ C iff  $\mathbf{Mod}^*(L)_{\text{RFSI}} \subseteq \mathbf{HS}^*\mathbf{P}_U(\mathbb{K})$ .*

The first claim of the previous theorem can be improved if  $L$  is a finitary protoalgebraic logic with a particularly nice disjunction connective, called *lattice-disjunction* [1]. Then we can prove that  $L$  has the S $\mathbb{K}$ C iff  $\mathbf{Mod}^*(L)_{\text{RFSI}}^\omega \subseteq \mathbf{IS}^*(\mathbb{K}^+)$  (where by  $\mathbb{K}^+$  we denote the class  $\mathbb{K}$  expanded by the trivial reduced matrix), which entails an interesting fact that  $\mathbf{Mod}^*(L)_{\text{RFSI}}^\omega \subseteq \mathbf{IS}^*(\mathbf{Mod}^*(L)_{\text{RSI}}^{\omega,+})$ .

The previous theorem has also an interesting corollary generalizing the above mentioned fact about classical logic.

**Corollary 3.** *Let  $L$  be a protoalgebraic and strongly  $p$ -disjunctional logic which has  $\mathbb{K}$ C w.r.t. a finite set of finite matrices. Then the class  $\mathbf{Mod}^*(L)_{\text{RFSI}}$  consists only of finitely many (up to isomorphism) finite matrices and hence  $\mathbf{Mod}^*(L)_{\text{RSI}} = \mathbf{Mod}^*(L)_{\text{RFSI}}$ .*

As a final illustration we mention a result also speaking about matrices in  $\mathbf{Mod}^*(L)_{\text{RFSI}}$  and their relation to those in  $\mathbb{K}$  but using the notion of *partial* embeddability, which is actually a common technique in the literature on mathematical fuzzy logic for proving completeness theorems.

**Theorem 4.** *Let  $L$  be a finitary equivalential logic in a finite language and  $\mathbb{K} \subseteq \mathbf{Mod}^*(L)$ . Then  $L$  has the FS $\mathbb{K}$ C iff  $\mathbf{Mod}^*(L)_{\text{RFSI}}$  is partially embeddable into  $\mathbb{K}^+$ .*

## References

- [1] CINTULA, PETR, and CARLES NOGUERA, ‘Implicational (semilinear) logics II: additional connectives and characterizations of semilinearity’, *Archive for Mathematical Logic*, 55 (2016), 353–372.
- [2] CINTULA, PETR, and CARLES NOGUERA, ‘Implicational (semilinear) logics III: completeness properties’, *Archive for Mathematical Logic*, 7 (2018), 391–420.
- [3] CINTULA, PETR, and CARLES NOGUERA, ‘The Proof by Cases Property and its Variants in Structural Consequence Relations’, *Studia Logica*, 101 (2013), 713–747.
- [4] CZELAKOWSKI, JANUSZ, *Protoalgebraic Logics*, vol. 10 of *Trends in Logic*, Kluwer, Dordrecht, 2001.

# Orders on Groups: an Approach through Spectral Spaces

Almudena Colacito\*

Mathematisches Institut, Universität Bern  
almudena.colacito@math.unibe.ch

A *right order* on a group  $G$  is a total order  $\leq$  on  $G$  such that  $x \leq y$  implies  $xt \leq yt$ , for all  $x, y, t \in G$ . There are at least two distinct ways that could lead a mathematician to encounter such orders on groups. First, it is standard that a countable group admits a right order if, and only if, it acts faithfully on the real line by orientation-preserving homeomorphisms (see, e.g., [7, Theorem 6.8]). The result indicates that orders on groups play a rôle in topological dynamics. Second, right orders are central to the theory of *lattice-ordered groups* (briefly,  *$\ell$ -groups*), i.e., groups with a lattice structure compatible with the group operation (see, e.g., [9, 8]). For a recent connection between right orders and  $\ell$ -group equations, see [4].

In 2004, Sikora’s paper “Topology on the spaces of orderings of groups” [11] pioneered a different perspective on the study of the interplay between topology and ordered groups, that has led to applications to both orderable groups and algebraic topology. The basic construction in Sikora’s paper is the definition of a topology on the set of right orders  $\mathcal{R}(G)$  on a group  $G$ , thereby associating a natural topological space to any right-orderable group. The space is then proved compact, Hausdorff, and zero-dimensional (see, e.g., [2, Problem 1.38]).

We show in [3] that Sikora’s space arises naturally from the study of  $\ell$ -groups, as the minimal spectrum of the  $\ell$ -group freely generated by the group at hand. The  *$\ell$ -spectrum*  $\text{Spec } H$  of an  $\ell$ -group  $H$  is the root system of all its prime subgroups ordered by inclusion topologised with the hull-kernel topology. Here, a *prime subgroup* of  $H$  is an order-convex sublattice subgroup  $\mathfrak{p}$  of  $H$  with the further property that  $x \wedge y \in \mathfrak{p}$  implies  $x \in \mathfrak{p}$  or  $y \in \mathfrak{p}$ . We write  $\text{Min } H$  for the set of inclusion-minimal prime subgroups of  $H$  with the subspace topology. By an application of Zorn’s Lemma, any prime subgroup of  $H$  contains a minimal prime subgroup. It can be proved that  $\text{Min } H$  is Hausdorff [6, Proposition 49.8], and that compactness of  $\text{Spec } H$  is equivalent to the existence of a *strong order unit*—an element  $u \in H^+ := \{x \in H \mid e \leq x\}$  that generates  $H$  as an order-convex sublattice subgroup, where  $e$  denotes the group identity [5, 1.3]. If  $H$  is finitely generated, every  $\mathfrak{p} \in \text{Spec } H$  is included in a unique maximal prime subgroup, and we write  $\text{Max } H$  for the set of maximal prime subgroups of  $H$  with the subspace topology. We call an  $\ell$ -group *representable* if it is a subdirect product of totally ordered groups.

By replacing right orders with right *pre-orders*—pre-orders that are invariant under group multiplication on the right—we provide a systematic, structural account of the relationship between (total) right pre-orders on a group  $G$  and prime subgroups of the  $\ell$ -group  $F^\ell(G)$  freely generated by  $G$  (or over  $G$ ). This connection is developed in full generality—that is, for any variety of  $\ell$ -groups (see [3, Theorem 2.1]). It follows that the space of right pre-orders on any group  $G$  is homeomorphic to the  $\ell$ -spectrum  $\text{Spec } F^\ell(G)$  and, when  $G$  is right orderable, Sikora’s space  $\mathcal{R}(G)$  is homeomorphic to  $\text{Min } F^\ell(G)$ . Further, when  $G$  admits an *order* (i.e., a right order that is invariant under group multiplication on the left), the minimal spectrum of the free representable  $\ell$ -group  $F_R^\ell(G)$  over  $G$  is homeomorphic to the space  $\mathcal{O}(G)$  of orders on  $G$ . This theoretical framework leads to a few immediate, diverse results, some of which are listed below.

**Example.** The space of right pre-orders on a finitely generated group  $G$  is compact.

This follows immediately from the fact that the finitely many generators of  $G$  induce the existence of a strong order unit on  $F^\ell(G)$ .

---

\*Based on joint work with Vincenzo Marra (Università degli Studi di Milano).

**Example.** For any group  $G$ , the space  $\text{Min } F^\ell(G)$  is compact.

In fact, we show that the space of right pre-orders on  $G$  is either empty or homeomorphic to the space of right pre-orders on a corresponding right-orderable group  $G^*$  [3, Remark 6.2]. The minimal layer of the latter is Sikora's  $\mathcal{R}(G^*)$ , which is indeed compact. This topological property of the minimal spectrum can be reformulated algebraically as follows. For any  $\ell$ -group  $H$ , we say that  $H^+$  is *complemented* if for every  $x \in H^+$  there is a  $y \in H^+$  such that  $x \wedge y = e$ , and  $x \vee y$  is a *weak order unit*—an element  $w \in H^+$  such that  $w \wedge x = e$  implies  $x = e$ .

**Example.** For any group  $G$ , the distributive lattice  $F^\ell(G)^+$  is complemented.

**Example.** The free  $\ell$ -group  $F^\ell(n)$  of rank  $n \geq 2$  acts by homeomorphism on the Cantor space.

For this, it suffices to observe that  $F^\ell(n)$  acts by homeomorphism on  $\text{Min } F^\ell(n)$ . It follows from [10, Corollary 5], [2, §1.5.2], and [3], that  $\text{Min } F^\ell(n)$  is the Cantor space.

These consequences are nothing more than translations of results from the right order setting to the  $\ell$ -group setting, or vice versa, facilitated by the correspondence established in [3]. We mention here two possibilities for further development.

First, we observe that [3] provides a new perspective on the open question whether there exist isolated points in  $\mathcal{O}(F(n))$  for  $n \geq 2$  (this question was raised by McCleary in [1, §4]). More precisely, we get a necessary condition for the existence of such isolated points. In fact, it is possible to argue that  $\text{Max } F_{\mathbb{R}}^\ell(n)$  is the  $(n - 1)$ -sphere  $\mathbb{S}^{n-1}$ , and that there exists a closed continuous map  $\lambda: \text{Min } F_{\mathbb{R}}^\ell(n) \rightarrow \text{Max } F_{\mathbb{R}}^\ell(n)$ . Thus, if  $\lambda$  is irreducible—it sends proper closed subsets to proper closed subsets—then  $\text{Min } F_{\mathbb{R}}^\ell(n)$  has no isolated points. We finally mention the intriguing problem of obtaining a natural representation of  $F_{\mathbb{R}}^\ell(n)$  for  $n \geq 2$  in terms of (possibly, piecewise linear) functions. In tackling this problem, we believe that the connection between topological dynamics and right orders mentioned in the beginning will be a key ingredient.

## References

- [1] Ashok K. Arora and Stephen H. McCleary. Centralizers in free lattice-ordered groups. *Houston J. Math.*, 12, 1986.
- [2] Adam Clay and Dale Rolfsen. *Ordered Groups and Topology*, volume 176 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2016.
- [3] Almudena Colacito and Vincenzo Marra. Orders on groups, and spectral spaces of lattice-groups. arXiv preprint arXiv:1901.07638. 2019.
- [4] Almudena Colacito and George Metcalfe. Ordering groups and validity in lattice-ordered groups. *J. Pure Appl. Algebra*. To appear. 2019.
- [5] Paul Conrad and Jorge Martinez. Complemented lattice-ordered groups. *Indag. Math.*, 1(3):281–297, 1990.
- [6] Michael R. Darnel. *Theory of Lattice-Ordered Groups*, volume 187 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1995.
- [7] Étienne Ghys. Groups acting on the circle. *Enseign. Math. (2)*, 47(3-4):329–407, 2001.
- [8] Andrew M. W. Glass. *Partially Ordered Groups*, volume 7. World Scientific, 1999.
- [9] Valerii M. Kopytov and Nikolai Ya. Medvedev. *The Theory of Lattice-Ordered Groups*, volume 307 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1994.
- [10] Stephen H. McCleary. Free lattice-ordered groups represented as  $o$ -2 transitive  $\ell$ -permutation groups. *Trans. Am. Math. Soc.*, 290(1):69–79, 1985.
- [11] Adam S. Sikora. Topology on the spaces of orderings of groups. *Bull. Lond. Math. Soc.*, 36(4):519–526, 2004.

# The logic of categories and informational entropy

Willem Conradie, Andrew Craig, Alessandra Palmigiano, and Nachoem M. Wijnberg

The contributions discussed in this talk lie at the intersection of several strands of research. They are rooted in the generalized Sahlqvist theory for normal LE-logics [9, 8, 4], i.e. those logics algebraically captured by varieties of normal lattice expansions [16].

Via canonical extensions and discrete duality, basic normal LE-logics of arbitrary signatures and a large class of their axiomatic extensions can be uniformly endowed with complete relational semantics of different kinds. The relational structures on which this talk will focus are based on *formal contexts* [15, 13, 6, 7, 17, 5] and *reflexive graphs* [1, 3]. In a mathematical setting in which the original discrete duality for perfect normal LEs has been relaxed to an adjunction involving complete normal LEs, these semantic structures have yielded uniform theoretical developments in the algebraic proof theory [17] and in the model theory [11] of LE-logics, and also insights on possible interpretations of LE-logics which have generated new opportunities for applications. In particular, via polarity-based semantics, in [6], the basic non-distributive modal logic and some of its axiomatic extensions are interpreted as *epistemic logics of categories and concepts*. Then, in [7], the corresponding ‘common knowledge’-type construction is used to give an epistemic-logical formalization of the notion of *prototype* of a category. In [5, 18], polarity-based semantics for non-distributive modal logic is proposed as an encompassing framework for the integration of rough set theory [19] and formal concept analysis [14]; in this context, the basic non-distributive modal logic is interpreted as the logic of *rough concepts*, and on the basis of this interpretation, polarity-based structures have been used as the semantic framework for a Dempster-Shafer theory of concepts [12]; via its graph-based semantics, in [3], the same logic is interpreted as the logic of *informational entropy*, i.e. an inherent boundary to knowability due e.g. to perceptual, theoretical, evidential or linguistic limits.

In the graphs  $(Z, E)$  on which the relational structures are based, the relation  $E$  is interpreted as the indiscernibility relation induced by informational entropy, much in the same style as Pawlak’s approximation spaces in rough set theory. However, the key difference is that, rather than generating modal operators which associate any subset of  $Z$  with its definable  $E$ -approximations,  $E$  generates a complete lattice (i.e. the lattice of  $E^c$ -concepts). In our approach, concepts are not definable approximations of predicates, but rather they represent ‘all there is to know’, i.e. the theoretical horizon to knowability, given the inherent boundary encoded into  $E$  (in their turn,  $E^c$ -concepts are approximated by means of the additional relations of the graph-based relational structures, from which the semantic modal operators arise). Interestingly,  $E$  is required to be reflexive but in general neither transitive nor symmetric, which is in line with proposals in rough set theory [20, 21] that indiscernibility does not need to give rise to equivalence relations.

Time permitting, we will discuss the *many-valued* [2, 5] and many-valued and *multi-type* [10] versions of the polarity-based and graph-based semantics of basic normal non-distributive modal logic, and in particular their potential for modelling situations in which categories and concepts are vague, and informational entropy is graded.

## References

- [1] W. Conradie and A. Craig. Relational semantics via TiRS graphs. *Proc. TACL 2015*, long abstract.



- [2] W. Conradie, A. Craig, A. Palmigiano, and N. Wijnberg. Modelling competing theories. In *Proc. EUSFLAT 2019*, Atlantis Studies in Uncertainty Modelling, 2019, accepted.
- [3] W. Conradie, A. Craig, A. Palmigiano, and N. Wijnberg. Modelling informational entropy. In *Proc. WoLLIC 2019*, Lecture Notes in Computer Science. Springer, 2019, accepted.
- [4] W. Conradie, A. Craig, A. Palmigiano, and Z. Zhao. Constructive canonicity for lattice-based fixed point logics. In *Proc. WoLLIC 2017*, volume 10388 of *Lecture Notes in Computer Science*, pages 92–109. Springer, 2017. ArXiv preprint arXiv:1603.06547.
- [5] W. Conradie, S. Frittella, K. Manoorkar, S. Nazari, A. Palmigiano, A. Tzimoulis, and N. Wijnberg. Rough concepts. *Submitted*, 2019.
- [6] W. Conradie, S. Frittella, A. Palmigiano, M. Piazzai, A. Tzimoulis, and N. Wijnberg. Categories: How I Learned to Stop Worrying and Love Two Sorts. In *Proc. WoLLIC 2016*, volume 9803 of *LNCS*, pages 145–164, 2016.
- [7] W. Conradie, S. Frittella, A. Palmigiano, M. Piazzai, A. Tzimoulis, and N. Wijnberg. Toward an epistemic-logical theory of categorization. In *Proc. TARK 2017*, volume 251 of *EPTCS*, pages 167–186, 2017.
- [8] W. Conradie and A. Palmigiano. Constructive canonicity of inductive inequalities. *arXiv preprint arXiv:1603.08341*, 2016.
- [9] W. Conradie and A. Palmigiano. Algorithmic correspondence and canonicity for non-distributive logics. *Annals of Pure and Applied Logic*, DOI: 10.1016/j.apal.2019.04.003, 2019.
- [10] W. Conradie, A. Palmigiano, C. Robinson, A. Tzimoulis, and N. Wijnberg. Modelling socio-political competition. *Submitted*, 2019.
- [11] W. Conradie, A. Palmigiano, and A. Tzimoulis. Goldblatt-Thomason for LE-logics. *arXiv preprint arXiv:1809.08225*, 2018.
- [12] S. Frittella, K. Manoorkar, A. Palmigiano, A. Tzimoulis, and N. Wijnberg. Towards a Dempster-Shafer theory of concepts. *Submitted*, 2019.
- [13] N. Galatos and P. Jipsen. Residuated frames with applications to decidability. *Transactions of the American Mathematical Society*, 365(3):1219–1249, 2013.
- [14] B. Ganter and R. Wille. *Formal concept analysis: mathematical foundations*. Springer, 2012.
- [15] M. Gehrke. Generalized Kripke frames. *Studia Logica*, 84(2):241–275, 2006.
- [16] M. Gehrke and J. Harding. Bounded lattice expansions. *Journal of Algebra*, 238(1):345–371, 2001.
- [17] G. Greco, P. Jipsen, F. Liang, A. Palmigiano, and A. Tzimoulis. Algebraic proof theory for LE-logics. *arXiv preprint arXiv:1808.04642*, 2018.
- [18] G. Greco, P. Jipsen, K. Manoorkar, A. Palmigiano, and A. Tzimoulis. Logics for rough concept analysis. In *Proc. ICLA 2019*, volume 11600 of *LNCS*, pages 144–159, 2019.
- [19] Z. Pawlak. Rough set theory and its applications to data analysis. *Cybernetics & Systems*, 29(7):661–688, 1998.
- [20] Y. Yao and T. Y. Lin. Generalization of rough sets using modal logics. *Intelligent Automation & Soft Computing*, 2(2):103–119, 1996.
- [21] Y. Yao and P. Lingras. Interpretations of belief functions in the theory of rough sets. *Information Sciences*, 104(1-2):81–106, 1998.



# Goldblatt-Thomason for LE-logics

Willem Conradie, Alessandra Palmigiano, and Apostolos Tzimoulis

We present the results of [6], in which we state and prove a version of the Goldblatt-Thomason theorem which applies uniformly to normal LE-logics in arbitrary signatures, i.e. classes of logics algebraically captured by varieties of normal lattice expansions. This class of logics includes well known logics such as the full Lambek calculus and its axiomatic extensions [20, 8], orthologic [13], and the Lambek-Grishin calculus [17]. The theorem is formulated as usual in terms of four model-theoretic constructions (coproduct, bounded morphic image, generated subframe, filter-ideal frame) on polarity-based structures, defined and justified on duality-theoretic grounds.

This result contributes to the theory of *polarity-based semantics* for normal LE-logics. Building on the theory of canonical extensions [10, 7], polarity-based semantics was introduced in [9] for the multiplicative fragment of the Lambek calculus, based on RS-polarities (i.e. those polarities that dually correspond to perfect lattices). With the same method, a polarity-based semantics for arbitrary LE-languages is introduced [15] in which the ‘RS’ restriction is dropped.

Thanks to its generality and uniformity, the polarity-based semantics for LE-logics lends itself to support a rich mathematical theory, uniformly developed for the whole class of LE-logics or large subclasses thereof: examples of such results are the generalized Sahlqvist theory [5], and the uniform proof of semantic cut elimination and finite model property for certain classes of LE-logics [15], paving the way to a research program aimed at extending also other results in algebraic proof theory (e.g. decidability via finite embeddability property, disjunction property, Craig interpolation) from substructural logics to LE-logics.

Interestingly, the polarity-based semantics has also proved suitable to support a number of interpretations of the meaning of (some) LE-languages, in the same way in which Kripke semantics captures the essentials of various independent conceptual frameworks of reference for modal logic. Specifically, in [4, 3], a poly-modal lattice-based logic was given a natural interpretation as an epistemic logic of formal concepts, under which,  $\Box_i\phi$  intuitively denotes ‘the concept  $\phi$  according to agent  $i$ ’. This interpretation is also consistent with the epistemic interpretation of well known (Sahlqvist) modal principles such as factivity and positive introspection. In [21, 16], the polarity-based semantics of the LE-logic in the language  $\wedge, \vee, \top, \perp, \Box, \Diamond$  is used as a natural framework for *rough concepts* which unifies Formal Concept Analysis and Rough Set Theory [22] under which,  $\Box\phi$  denotes the category of the certified members of  $\phi$ . Also this interpretation is consistent with the interpretation of well known (Sahlqvist) modal principles such as seriality.

Precisely the availability of these and other interpretations makes it interesting to study the expressivity of LE-logics in regard to their polarity-based semantics, and further motivates the contributions discussed in the present talk. Besides its centrality in the build-up of a uniform mathematical theory of the polarity-based semantics of LE-logics, the Goldblatt-Thomason theorem provides a useful strategy to determine whether a certain elementary class of polarity-based structures can be captured by an LE-axiomatic principle. It is enough to show that the given class fails to reflect/be closed under one of the usual constructions to establish that no such axiomatic principle exists.

Goldblatt-Thomason theorem [14] has been extended to Positive Modal Logic [2], coalgebraic logic [19], graded modal logic [23], distributive substructural logics [1], Łukasiewicz logic [24], and possibility semantics for modal logic [18]. Recently, Goldblatt himself gave a version of it for the logic of general lattices [12] and developed two different versions of it for lattice-based modal logic [11], one for classes closed under canonical extensions of polarity structures, and another for classes closed under ultrapowers.

## References

- [1] M. Bilková, R. Horčík, and J. Velebil. Distributive substructural logics as coalgebraic logics over posets. *Advances in Modal Logic*, 9:119–142, 2012.
- [2] S. Celani and R. Jansana. Priestley duality, a Sahlqvist theorem and a Goldblatt-Thomason theorem for positive modal logic. *Logic Journal of IGPL*, 7(6):683–715, 1999.
- [3] W. Conradie, S. Frittella, A. Palmigiano, M. Piazzai, A. Tzimoulis, and N. Wijnberg. Toward an epistemic-logical theory of categorization. In Proc. TARK 2017, *EPTCS* 251 (2017), 170–189.
- [4] W. Conradie, S. Frittella, A. Palmigiano, M. Piazzai, A. Tzimoulis, and N. M. Wijnberg. Categories: how I learned to stop worrying and love two sorts. In Proc. Wollic 2016, *LNCS* 10388 (2016), 145–164.
- [5] W. Conradie and A. Palmigiano. Algorithmic correspondence and canonicity for non-distributive logics. *Annals of Pure and Applied Logic*, accepted, 2019. arXiv:1603.08515.
- [6] W. Conradie, A. Palmigiano, and A. Tzimoulis. Goldblatt-Thomason for LE-logics. *arXiv preprint arXiv:1809.08225*, submitted, 2018.
- [7] J. M. Dunn, M. Gehrke, and A. Palmigiano. Canonical extensions and relational completeness of some substructural logics. *The Journal of Symbolic Logic*, 70(3):713–740, 2005.
- [8] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono. *Residuated lattices: an algebraic glimpse at substructural logics*, volume 151. Elsevier, 2007.
- [9] M. Gehrke. Generalized Kripke frames. *Studia Logica*, 84(2):241–275, 2006.
- [10] M. Gehrke and J. Harding. Bounded lattice expansions. *Journal of Algebra*, 238(1):345–371, 2001.
- [11] R. Goldblatt. Morphisms and duality for polarities and lattices with operators. *arXiv preprint, arXiv:1902.09783*, 2019.
- [12] R. Goldblatt. Canonical extensions and ultraproducts of polarities. *Algebra Universalis*, 2018.
- [13] R. I. Goldblatt. Semantic analysis of orthologic. *Journal of Philosophical logic*, 3(1-2):19–35, 1974.
- [14] R. I. Goldblatt and S. K. Thomason. Axiomatic classes in propositional modal logic. In *Algebra and logic*, pages 163–173. Springer, 1975.
- [15] G. Greco, P. Jipsen, F. Liang, A. Palmigiano, and A. Tzimoulis. Algebraic proof theory for LE-logics. *submitted, arXiv preprint arXiv:1808.04642*, 2018.
- [16] G. Greco, P. Jipsen, K. Manoorkar, A. Palmigiano, and A. Tzimoulis. Logics for rough concept analysis. *Proc. ICLA 2019, LNCS 11600*, pp. 144–159. *ArXiv preprint 1811.07149*, 2019.
- [17] V. N. Grishin. On a generalization of the Ajdukiewicz-Lambek system. *Studies in nonclassical logics and formal systems*, pages 315–334, 1983.
- [18] W. H. Holliday. Possibility frames and forcing for modal logic. 2016.
- [19] A. Kurz and J. Rosický. The Goldblatt-Thomason theorem for coalgebras. In *International Conference on Algebra and Coalgebra in Computer Science*, pages 342–355. Springer, 2007.
- [20] J. Lambek. The mathematics of sentence structure. *The American Mathematical Monthly*, 65(3):154–170, 1958.
- [21] W. Conradie, S. Frittella, K. Manoorkar, S. Nazari, A. Palmigiano, A. Tzimoulis and N. M. Wijnberg. Rough concepts. 2018.
- [22] Z. Pawlak. Rough set theory and its applications to data analysis. *Cybernetics & Systems*, 29(7):661–688, 1998.
- [23] K. Sano and M. Ma. Goldblatt-Thomason-style theorems for graded modal language. *Advances in Modal Logic*, 2010:330–349, 2010.
- [24] B. Teheux. Goldblatt-Thomason theorem for Łukasiewicz finitely-valued modal language. *Graded Logical Approaches and their Applications-Abstracts*, pages 122–125, 2014.

# Intuitionistic and classical non-normal modal logics: An embedding\*

Tiziano Dalmonte, Charles Grellois, and Nicola Olivetti

Aix Marseille Univ, Université de Toulon, CNRS, LIS, Marseille, France

The motivations for classical non-normal modal logics (CNNML) (see [3]) come from the analysis of epistemic and deontic modalities, whereas intuitionistic modal logics (see [9]) have been studied with the purpose of giving a constructive account of modal logics, in particular for type theoretic application. We claim that it is interesting to combine the two logical traditions, so that in [4] and [5] we have defined a class of intuitionistic non-normal modal logics (INNML) that can be seen as intuitionistic counterparts of the *classical cube* of CNNMLs. In particular, we have defined a family of 24 systems containing two non-interdefinable modalities  $\Box$  and  $\Diamond$ . Each system contains some of the modal axioms characterising the classical cube:

$$\begin{array}{l} \text{RE}_{\Box} \frac{A \supset B \quad B \supset A}{\Box A \supset \Box B} \quad \text{M}_{\Box} \Box(A \wedge B) \supset \Box A \quad \text{N}_{\Box} \Box \top \quad \text{C}_{\Box} \Box A \wedge \Box B \supset \Box(A \wedge B) \\ \text{RE}_{\Diamond} \frac{A \supset B \quad B \supset A}{\Diamond A \supset \Diamond B} \quad \text{M}_{\Diamond} \Diamond A \supset \Diamond(A \vee B) \quad \text{N}_{\Diamond} \neg \Diamond \perp \quad \text{C}_{\Diamond} \Diamond(A \vee B) \supset \Diamond A \vee \Diamond B \end{array}$$

In addition, each system contains some of the following interaction axioms, that express under which conditions two formulas  $\Box A$  and  $\Diamond B$  are jointly inconsistent:

$$\begin{array}{lll} \text{weak}_a \neg(\Box \top \wedge \Diamond \perp) & \text{weak}_b \neg(\Diamond \top \wedge \Box \perp) & \text{str} \frac{\neg(A \wedge B)}{\neg(\Box A \wedge \Diamond B)} \\ \text{neg}_a \neg(\Box A \wedge \Diamond \neg A) & \text{neg}_b \neg(\Box \neg A \wedge \Diamond A) & \end{array}$$

In [5] we have given cut-free calculi and a proof of decidability for all systems. We have also provided a modular semantic characterisation of the logics in terms of neighbourhood models:

**Definition 0.1.** A *coupled intuitionistic neighbourhood model* is a tuple  $\mathcal{M} = \langle \mathcal{W}, \preceq, \mathcal{N}_{\Box}, \mathcal{N}_{\Diamond}, \mathcal{V} \rangle$ , where  $\mathcal{W}$  is a non-empty set,  $\preceq$  is a preorder over  $\mathcal{W}$ ,  $\mathcal{V}$  is a hereditary valuation function  $\mathcal{W} \rightarrow \mathcal{P}(\text{Atm})$  ( $w \preceq v$  implies  $\mathcal{V}(w) \subseteq \mathcal{V}(v)$ ), and  $\mathcal{N}_{\Box}, \mathcal{N}_{\Diamond}$  are two neighbourhood functions  $\mathcal{W} \rightarrow \mathcal{P}(\mathcal{P}(\mathcal{W}))$  such that  $w \preceq v$  implies  $\mathcal{N}_{\Box}(w) \subseteq \mathcal{N}_{\Box}(v)$  and  $\mathcal{N}_{\Diamond}(w) \supseteq \mathcal{N}_{\Diamond}(v)$ . The functions  $\mathcal{N}_{\Box}$  and  $\mathcal{N}_{\Diamond}$  can be *supplemented*, *closed under intersection*, or *contain the unit*. Moreover, letting  $\neg\alpha$  denote the set  $\{w \in \mathcal{W} \mid \text{for all } v \succeq w, v \notin \alpha\}$ ,  $\mathcal{N}_{\Box}$  and  $\mathcal{N}_{\Diamond}$  can be related as follows:

- (i) For all  $w \in \mathcal{W}$ ,  $\mathcal{N}_{\Box}(w) \subseteq \mathcal{N}_{\Diamond}(w)$ ;
- (ii) If  $\alpha \in \mathcal{N}_{\Box}(w)$ , then  $\mathcal{W} \setminus \neg\alpha \in \mathcal{N}_{\Diamond}(w)$ ;
- (iii) If  $\neg\alpha \in \mathcal{N}_{\Box}(w)$ , then  $\mathcal{W} \setminus \alpha \in \mathcal{N}_{\Diamond}(w)$ ;
- (iv) If  $\alpha \in \mathcal{N}_{\Box}(w)$  and  $\alpha \subseteq \beta$ , then  $\beta \in \mathcal{N}_{\Diamond}(w)$ .

The associated forcing relation  $w \Vdash A$  is the usual one for  $p, \perp, B \wedge C, B \vee C$ , whereas for  $B \supset C, \Box B$  and  $\Diamond B$  it is as follows:  $w \Vdash B \supset C$  iff for all  $v \succeq w, v \Vdash B$  implies  $v \Vdash C$ ;  $w \Vdash \Box B$  iff  $[B] \in \mathcal{N}_{\Box}(w)$ ; and  $w \Vdash \Diamond B$  iff  $\mathcal{W} \setminus [B] \notin \mathcal{N}_{\Diamond}(w)$ .

Translation or embedding of intuitionistic modal logics into classical bimodal logics have been studied, for instance, in [8], [11], and [6]. We present an embedding of our INNMLs into classical multimodal logics of the form (S4, cL<sub>2</sub>, cL<sub>3</sub>) – where cL<sub>2</sub> and cL<sub>3</sub> range over CNNMLs. Logics (S4, cL<sub>2</sub>, cL<sub>3</sub>) are defined on a propositional modal language  $\mathcal{L}_3$  containing three modalities  $\Box, \Box, \Box$  (by duality we can define  $\Diamond, \Diamond, \Diamond$ ).

**Definition 0.2.** Given two CNNMLs cL<sub>2</sub> and cL<sub>3</sub>, the classical multimodal logics (S4, cL<sub>2</sub>, cL<sub>3</sub>) are defined by taking S4 axioms and rules on  $\Box$ ; cL<sub>2</sub> axioms and rules on  $\Box$ ; cL<sub>3</sub> axioms

\*This work was partially supported by the Project TICAMORE ANR-16-CE91-0002-01.

and rules on  $\boxtimes$ ; and the connecting axioms  $\boxtimes A \rightarrow \boxtimes \boxtimes A$  and  $\diamond A \rightarrow \boxtimes \diamond A$ . In addition, each multimodal logics contains either (a) axiom  $\boxtimes A \rightarrow \boxtimes A$ , or (b) both axioms  $\boxtimes A \rightarrow \boxtimes \diamond A$  and  $\boxtimes \boxtimes A \rightarrow \boxtimes A$ , or (c) the rule  $\frac{A \rightarrow B}{\boxtimes A \rightarrow \boxtimes B}$ .

For each INNML  $L$ , the classical multimodal logic ( $S4, cL_2, cL_3$ ) associated to  $L$  (denoted as  $emb(L)$ ) is determined as follows: (i)  $cL_2$  is the CNNML defined by the  $\square$ -axioms of  $L$ . (ii)  $cL_3$  is the CNNML defined by the  $\diamond$ -axioms of  $L$ . (iii) The axioms connecting  $\boxtimes$  and  $\boxtimes$  depend on the interaction axioms of  $L$ :  $emb(L)$  contains (a) if  $L$  has axioms  $weak_a$  and  $weak_b$ , it contains (b) if  $L$  has axioms  $neg_a$  and  $neg_b$ , and it contains (c) if  $L$  has rule  $str$ . For logics containing axioms  $M_\square$  and  $M_\diamond$  we always consider axiom (a).

We consider the following translation of formulas of INNMLs into formulas of  $\mathcal{L}_3$ :  $\dagger(p) = \boxtimes p$ ,  $\dagger(\perp) = \perp$ ,  $\dagger(A \wedge B) = \dagger(A) \wedge \dagger(B)$ ,  $\dagger(A \vee B) = \dagger(A) \vee \dagger(B)$ ,  $\dagger(A \supset B) = \boxtimes(\dagger(A) \rightarrow \dagger(B))$ ,  $\dagger(\square A) = \boxtimes \boxtimes \dagger(A)$ ,  $\dagger(\diamond A) = \boxtimes \diamond \dagger(A)$ . For each INNML  $L$ , we show that  $L$  is embedded into the corresponding classical multimodal logic  $emb(L)$  by proving the following:

**Theorem 0.1.**  $\vdash_L A$  if and only if  $\vdash_{emb(L)} \dagger(A)$ .

In a similar way we can prove analogous embeddings for other INNMLs studied in the literature, such as Constructive K by Bellin *et al.* [2] and the propositional fragment of Wijesekera's Constructive Concurrent Dynamic Logic [10]. Both logics can be included in our semantic framework by considering a simple additional property. Their embedding into classical non-normal multimodal logics can be proved by considering the connecting axiom  $\boxtimes A \wedge \diamond B \rightarrow \diamond(A \wedge B)$ . This embedding shows the usefulness of studying multimodal non-normal modal logics, a topic explored *e.g.* in [1] and [7], and that we plan to treat in future research.

## References

- [1] Baader, F., S. Ghilardi, and C. Tinelli, A new combination procedure for the word problem that generalizes fusion decidability results in modal logics, *Information and Computation*, 204(10) (2006), pp. 1413–1452.
- [2] Bellin, G., V. de Paiva, and E. Ritter, Extended Curry-Howard Correspondence for a Basic Constructive Modal Logic, in: *Proceedings of Methods for Modalities*, 2001.
- [3] Chellas, B. F., *Modal Logic: An Introduction*, Cambridge University Press, 1980.
- [4] Dalmonte, T., C. Grellois, and N. Olivetti, Towards intuitionistic non-normal modal logic and its calculi, [https://members.loria.fr/DGalmiche/files/=papers/EICNCL2018/EICNCL\\_2018\\_paper\\_3.pdf](https://members.loria.fr/DGalmiche/files/=papers/EICNCL2018/EICNCL_2018_paper_3.pdf)
- [5] Dalmonte, T., C. Grellois, and N. Olivetti, Intuitionistic non-normal modal logics: A general framework, <https://arxiv.org/pdf/1901.09812.pdf>
- [6] Fairtlough, M., and M. Mendler, Propositional Lax Logic, *Information and Computation*, 137(1) (1997), pp. 1–33.
- [7] Fajardo, R., and M. Finger, How Not to Combine Modal Logics, In: *IICA 2005*, pp. 1629–1647.
- [8] Fischer Servi, G., Semantics for a class of intuitionistic modal calculi, in: *Italian studies in the philosophy of science*, Springer, 1980, pp. 59–72.
- [9] Simpson, A. K., *The Proof Theory and Semantics of Intuitionistic Modal Logic*. PhD thesis, School of Informatics, University of Edinburgh, 1994.
- [10] Wijesekera, D., Constructive modal logics I, *Annals of Pure and Applied Logic*, 50 (1990), pp. 271–301.
- [11] Wolter, F., and M. Zakharyashev, Intuitionistic modal logics as fragment of classical bimodal logics, in: E. Orłowska (ed.), *Logic at Work*, Springer, 1999, pp. 168–186.

# Lifting Functors from Pos to Pries

Jim de Groot

The Australian National University, Canberra, Australia  
 jim.degroot@anu.edu.au

Analogously to classical logic, coalgebraic methods can be used to introduce modal operators to *positive* logic. In particular, **Pre**- and **Pos**-functors yield frames for interpreting positive logic with modalities corresponding to *positive predicate liftings*. Coalgebras for endofunctors on **Pries** can be seen as *descriptive* frames for modal positive logic, because they are dual to the algebraic semantics. We observe that some functors are closely related: We give and compare two ways to lift a **Pos**-functor to a **Pries**-functor.

The framework is as follows: Let  $\mathbf{P}$  be a category with a forgetful functor  $U : \mathbf{P} \rightarrow \mathbf{Set}$ . Let  $P : \mathbf{P} \rightarrow \mathbf{DL}$  be a contravariant functor to the category of bounded distributive lattices and  $T$  an endofunctor on  $\mathbf{P}$ . For  $n \in \mathbb{N}$ , an *n-ary positive predicate lifting* for  $T$  is a natural transformation  $\lambda : UP^n \rightarrow UPT$ . A set  $\Lambda$  of predicate liftings yields a language which can be interpreted in  $T$ -coalgebras in the usual way.

**Examples.** All items (a) below yield frame semantics for Dunn’s modal positive logic [4].

1. Let  $\mathbf{P} = \mathbf{Pre}$ , the category of preorders and monotone functions, and  $P = Up$ , the functor taking upsets and inverse images.
  - (a) For a preorder  $X$  let  $\mathcal{P}_{\sqsubseteq}X$  be the powerset of  $X$  ordered by:  $a \sqsubseteq b$  iff  $[\forall x \in a \exists y \in b$  s.t.  $x \leq_X y$  and  $\forall y \in b \exists x \in a$  s.t.  $x \leq_X y]$ . For a monotone function  $f$  let  $\mathcal{P}_{\sqsubseteq}f$  be the direct image of  $f$ . Then  $\mathcal{P}_{\sqsubseteq}$  is a functor whose coalgebras are precisely PML-frames [3]. The modalities  $\square$  and  $\diamond$  correspond to the positive predicate liftings given by  $\lambda_X^{\square}(a) = \{b \in \mathcal{P}_{\sqsubseteq}X \mid b \subseteq a\}$  and  $\lambda_X^{\diamond}(a) = \{b \in \mathcal{P}_{\sqsubseteq}X \mid b \cap a \neq \emptyset\}$ .
  - (b) Although used for intuitionistic logic,  $H\square$ -frames from [2] are coalgebras for the functor  $\mathcal{P}_2 : \mathbf{Pre} \rightarrow \mathbf{Pre}$ , which sends a preorder  $X$  to the powerset of  $X$  ordered  $a \sqsubseteq_2 b$  iff every element of  $b$  has a  $\leq_X$ -predecessor in  $a$ , and a morphism  $f$  to its direct image.
2. Let  $\mathbf{P} = \mathbf{Pos}$ , the category of posets and monotone functions, and  $P = Up$ .
  - (a) Closely related to  $\mathcal{P}_{\sqsubseteq}$  is the convex powerset functor  $\mathcal{P}_c$  on **Pos**, which sends a poset to the collection of convex sets ordered by  $\sqsubseteq$ , and a monotone functor  $f : X \rightarrow X'$  to  $\mathcal{P}_c f$  defined by  $\mathcal{P}_c f(a) = \{x' \in X' \mid \exists x, y \in X$  s.t.  $f(x) \leq x' \leq f(y)\}$ .
  - (b) Also used for intuitionistic logic are  $\square$ -frames [8]. These are coalgebras for the upper powerset functor  $\mathcal{P}_{up}$ , which sends a poset  $X$  to the set of upsets of  $X$  ordered by reverse inclusion, and a morphism  $f : X \rightarrow X'$  to  $\mathcal{P}_{up}f : a \mapsto \uparrow a = \{x' \in X' \mid \exists y \in a$  s.t.  $f(y) \leq_{X'} x'\}$ .
3. Let  $\mathbf{P} = \mathbf{Pries}$ , the category of Priestley spaces and morphisms, and  $P = ClpUp$  [7].
  - (a) Coalgebras for the convex Vietoris functor  $\mathcal{V}_c$  are equivalent to  $K^+$ -spaces [6], which can be used to interpret positive modal logic [4], using the obvious predicate liftings.

For all these examples there are straightforward analogs of  $\lambda^{\square}$  yielding  $\square$ -like modalities, and for all items (a) the same goes for  $\lambda^{\diamond}$ . The relation between **Pre**-functors and **Pos**-functors is studied in [1, Sec. 4]. In particular  $\mathcal{P}_{\sqsubseteq}$  and  $\mathcal{P}_2$  lift to  $\mathcal{P}_c$  and  $\mathcal{P}_{up}$  respectively, as do corresponding predicate liftings. Since this is understood, we focus on the connection between **Pos**-functors and **Pries**-functors. More precisely, we give two ways of lifting **Pos**-functors to **Pries**-functors.

Our first lifting method resembles [5]. Let  $W : \text{Pries} \rightarrow \text{Pos}$  be the forgetful functor which sends a Priestley space to the underlying poset, and  $pf : \text{DL} \rightarrow \text{Pries}$  the contravariant functor taking a distributive lattice to its Priestley space of prime filters [7]. Let  $T : \text{Pos} \rightarrow \text{Pos}$  be any functor and  $\Lambda$  a set of positive predicate liftings for  $T$ . For a Priestley space  $\mathcal{X}$  let  $\widehat{D}_\Lambda \mathcal{X}$  be the sub-distributive lattice of  $UpTW\mathcal{X}$  generated by  $\{\lambda(a_1, \dots, a_n) \mid \lambda \in \Lambda, a_i \in ClpUp\mathcal{X}\}$ , and for a morphism  $f : \mathcal{X} \rightarrow \mathcal{X}'$  let  $\widehat{D}_\Lambda f : \widehat{D}_\Lambda \mathcal{X}' \rightarrow \widehat{D}_\Lambda \mathcal{X}$  be restriction of  $UpTWf$  to  $\widehat{D}_\Lambda \mathcal{X}'$ . Then  $\widehat{D}_\Lambda \mathcal{X}$  is a contravariant functor  $\text{Pries} \rightarrow \text{DL}$ . Define

$$\widehat{T}_\Lambda = pf \circ \widehat{D}_\Lambda : \text{Pries} \rightarrow \text{Pries}.$$

The set  $\Lambda$  lifts to a set  $\widehat{\Lambda}$  of predicate liftings for  $\widehat{T}$  yielding the same language (with different semantics). For example:  $\widehat{\mathcal{P}}_{c, \{\lambda^\square, \lambda^\diamond\}} = \mathcal{V}_c$ , and the lift of the functor  $\mathcal{P}_{up}$  with respect to  $\lambda^\square$  is the so-called *upper Vietoris functor*, whose coalgebras are descriptive  $\square$ -frames [8].

A second way of lifting functors is as follows: Suppose  $T : \text{Pos} \rightarrow \text{Pos}$  restricts to the category  $\text{Pos}_f$  of finite posets. Denote by  $\text{Pries}_f$  the full subcategory of  $\text{Pries}$  whose objects are finite Priestley spaces. For a Priestley space  $\mathcal{X}$ , let  $U_{\mathcal{X}} : (\mathcal{X} \downarrow \text{Pries}_f) \rightarrow \text{Pries}$  be the obvious forgetful functor from the coslice category to  $\text{Pries}$ . Then the (cofiltered) limit of the diagram  $U_{\mathcal{X}}$  is  $\mathcal{X}$  itself. Since  $\text{Pos}_f \cong \text{Pries}_f$ , we can apply  $T$  to the diagram  $U_{\mathcal{X}}$  and define

$$\overline{T}\mathcal{X} = \lim(TU_{\mathcal{X}}).$$

A morphism  $f : \mathcal{X} \rightarrow \mathcal{X}'$  entails that  $\overline{T}\mathcal{X}$  is a cone of  $TU_{\mathcal{X}'}$ , hence gives a unique mediating morphism  $\overline{T}\mathcal{X} \rightarrow \overline{T}\mathcal{X}'$ , which we take to be  $\overline{T}f$ .

**Theorem.** *Let  $T$  be an endofunctor on  $\text{Pos}$  which restricts to  $\text{Pos}_f$  and preserves epis and cofiltered limits. Let  $\Lambda$  be the set of all positive predicate liftings for  $T$ . Then there is a natural isomorphism  $\overline{T} \rightarrow \widehat{T}$ .*

Both  $\mathcal{P}_c$  and  $\mathcal{P}_{up}$  satisfy the preconditions of the Theorem. The connection between the two lifts appears to be an instance of a more general phenomenon, where instead of  $\text{DL}$  we have a variety of algebras. I am still investigating this more general setting.

## References

- [1] A. Balan and A. Kurz. Finitary functors: From set to preord and poset. In *CALCO*, volume 6859 of *Lecture Notes in Computer Science*, pages 85–99. Springer, 2011.
- [2] M. Božić and K. Došen. Models for normal intuitionistic modal logics. *Studia Logica*, 43:217–245, 1984.
- [3] S. Celani and R. Jansana. Priestley duality, a Sahlqvist theorem and a Goldblatt-Thomason theorem for positive modal logic. *Logic Journal of the IGPL*, 7:683–715, 12 1999.
- [4] J.M. Dunn. Positive modal logic. *Studia Logica*, 55:301–317, 1995.
- [5] C. Kupke, A. Kurz, and D. Pattinson. Ultrafilter extensions for coalgebras. In J. Luiz Fiadeiro, N. Harman, M. Roggenbach, and J. Rutten, editors, *Algebra and Coalgebra in Computer Science*, pages 263–277, Berlin, Heidelberg, 2005. Springer Berlin Heidelberg.
- [6] Alessandra Palmigiano. A coalgebraic view on positive modal logic. *Theoretical Computer Science*, 327:175–195, 10 2004.
- [7] H. A. Priestley. Representation of distributive lattices by means of ordered Stone spaces. *Bulletin of the London Mathematical Society*, 2(2):186–190, 1970.
- [8] F. Wolter and M. Zakharyashev. Intuitionistic modal logic. In A. Cantini, E. Casari, and P. Minari, editors, *Logic and Foundations of Mathematics: Selected Contributed Papers of the Tenth International Congress of Logic, Methodology and Philosophy of Science, Florence, August 1995*, pages 227–238, Dordrecht, 1999. Springer Netherlands.

# A Sahlqvist theorem for subordination algebras

Laurent De Rudder and Georges Hansoul

Department of Mathematics, University of Liege  
l.derudder@uliege.be

Subordination algebras have been studied under several different names: precontact algebras ([6]), proximity algebras ([7]), strict implications algebras ([3]) or even quasi-modal algebras ([4]). The equivalences between all these definitions are well discussed in [2] and [5]. Modal algebras and subordination algebras share a common characteristic: their duals, in the sense of the Stone Duality, are Boolean topological spaces with a closed relation and, in particular, Kripke frames. Therefore the latter denomination, quasi-modal, is not a surprise. Moreover, since subordination algebras form a more general class than modal algebra, we propose an investigation of modal logic through subordination algebras.

While this problem has already been studied under other perspectives, for instance in [1] or in [3], our approach is slightly different as we are interested in validity of modal formulas in subordinations algebras instead of validity of subordination formulas.

**Definition 1** (see [2]). A subordination algebra (or quasimodal algebra) is a pair  $\mathcal{B} = (B, \prec)$  where  $B$  is a Boolean algebra and  $\prec$  is a binary relation on  $B$ , called subordination, verifying the following properties :

1.  $0 \prec 0$  and  $1 \prec 1$ ,
2.  $a \prec b, c$  implies  $a \prec b \wedge c$ ,
3.  $a, b \prec c$  implies  $a \vee b \prec c$ ,
4.  $a \leq b \prec c \leq d$  implies  $a \prec d$ .

In [2], it is stated that modal algebras are particular subordination algebras. The authors also provide a sufficient condition for subordination algebras to be modal algebras.

**Definition 2.** A subordination space is a pair  $\mathcal{X} = (X, R)$  where  $X$  is a Stone space and  $R$  is a binary closed relation on  $X$ .

**Theorem 3** (see [4]). *The category **Sub**, whose objects are subordination algebras and whose morphisms are the  $q$ -homomorphisms defined in [4], and the category **SubS**, whose objects are subordination spaces and whose morphisms are the  $q$ -morphisms defined in [4], are dually equivalent.*

To be seen as models for modal logic, we need to define valuation and validity on subordination algebras. The problem is that we cannot extend freely the valuation for variables to modal formulas, as for instance  $\Box p$  may fail to be a clopen set of the dual. In order to resolve this issue, we will focus on the canonical extension of a subordination algebra.

**Theorem 4.** *If  $\mathcal{B} = (B, \prec)$  is a subordination algebra, then its canonical extension  $\mathcal{B}^\delta = (\mathcal{P}(X_{\mathcal{B}}), \prec_R)$ , where  $(X_{\mathcal{B}}, R)$  is the subordination space dual to  $\mathcal{B}$  and  $\prec_R$  is defined by*

$$E \prec_R F \Leftrightarrow R(-, E) \subseteq F, \tag{1}$$

*is a complete tense bimodal algebra with  $\Diamond E = R(-, E)$  and  $\blacklozenge E = R(E, -)$ .*



**Definition 5.** Let  $\mathcal{B}$  be a subordination algebra. A valuation on  $\mathcal{B}$  is a map  $v : \text{Var} \rightarrow B$ , where  $\text{Var}$  is the set of variables. In particular, this map can be considered as a map  $v : \text{Var} \rightarrow B^\delta$  and, as such, extend to a bimodal morphism between the set of all bimodal formulas and  $B^\delta$ . As usual, we will say that a formula  $\varphi$  is valid in  $\mathcal{B}$  under the valuation  $v$ , which will be denoted by  $\mathcal{B} \models_v \varphi$ , if  $v(\varphi) = 1$ , where 1 is the top element of both  $\mathcal{B}$  and  $B^\delta$ . The formula  $\varphi$  is valid in  $\mathcal{B}$  if  $\mathcal{B} \models_v \varphi$  for all valuation  $v$ , this is denoted by  $\mathcal{B} \models \varphi$ .

**Definition 6.** Let  $\varphi$  be a bimodal formula. It is closed (resp. open) if it is obtained from propositional variables, negation of propositional variables,  $\top$  and  $\perp$  by applying  $\wedge$ ,  $\vee$ ,  $\diamond$  and  $\blacklozenge$  (resp.  $\Box$  and  $\blacksquare$ ).

It is positive (resp. negative) if it is obtained from propositional variables (resp. negation of propositional variables)  $\top$  and  $\perp$  by applying  $\wedge$ ,  $\vee$ ,  $\diamond$ ,  $\Box$ ,  $\blacklozenge$  and  $\blacksquare$ .

It is strongly positive if it is a conjunction of formulas of the form

$$\Box^{(k)} p = \Box^{k_1} \blacksquare^{k_2} \dots \Box^{k_n} p$$

where  $p$  is a propositional variable,  $n \in \mathbb{N}$  and  $k \in \mathbb{N}^n$ .

It is  $s$ -positive (resp.  $s$ -negative) if it is obtained from closed positive formulas (resp. open negative formulas) by applying  $\wedge$ ,  $\vee$ ,  $\Box$  and  $\blacksquare$  (resp.  $\diamond$  and  $\blacklozenge$ ).

It is  $s$ -untied if it is obtained from strongly positive and  $s$ -negative formulas by applying  $\wedge$ ,  $\vee$ ,  $\diamond$  and  $\blacklozenge$ .

**Theorem 7.** Let  $\varphi = \Box^{(k)}(\varphi_1 \rightarrow \varphi_2)$  be a bimodal formula where  $\varphi_1$  is  $s$ -untied and  $\varphi_2$   $s$ -positive. Then, there exists a first order formula  $\mathfrak{f}$  in the language of the accessibility relation such that for a subordination algebra  $(B, \prec)$  and its dual  $(X_B, R)$  we have

$$(B, \prec) \models \varphi \text{ if and only if } (X_B, R) \models \mathfrak{f}.$$

This theorem is relatively similar to the one obtained in [8] with the particularity that in the formula  $\varphi$  there is no  $\Box$  within the scope of a  $\diamond$ .

## References

- [1] Balbiani, P., Kikot, S.: Sahlqvist theorems for precontact logics. *Advances in Modal Logic* **9**, 55–70 (2012)
- [2] Bezhanishvili, G., Bezhanishvili, N., Sourabh, S. and Venema, Y.: Irreducible Equivalence Relations, Gleason Spaces, and de Vries Duality. *Appl Categ Struct* **25**, 381–401 (2017)
- [3] Bezhanishvili, G., Bezhanishvili, N., Santoli, T. and Venema, Y.: A simple propositional calculus for compact Hausdorff spaces.
- [4] Celani, S.: Quasi-modal algebras. *Mathematica Bohemica* **126**(4), 721–736 (2001)
- [5] Celani, S.: Precontact relations and quasi-modal operators in Boolean algebras. *Actas del XIII congreso Dr. Antonio A.R. Monteiro*, 63-79 (2016)
- [6] Dimov, G., Vakarelov, D.: Topological representation of precontact algebras. MacCaull, Winter, and Düntsch, editors, *Relational Methods in Computer Science*, Lecture notes in in Computer Science **3929**, 1–16 (2006)
- [7] Düntsch, I., Vakarelov, D.: Region based theory of discrete spaces: A proximity approach. *Ann. Math. Artif. Intell.* **49**, 5–14 (2007)
- [8] Sambin, G., Vaccaro, V.: A new proof of Sahlqvist’s theorem on modal definability and completeness. *The Journal of Symbolic Logic* **54**(3), 992–999 (1989)



# Extensions of the Stone Duality to the category of zero-dimensional Hausdorff spaces

Georgi Dimov<sup>1</sup> and Elza Ivanova-Dimova<sup>2\*</sup>

<sup>1</sup> Faculty of Math. and Informatics, Sofia University, 5 J. Bourchier Blvd., 1164 Sofia, Bulgaria  
`gdimov@fmi.uni-sofia.bg`

<sup>2</sup> Faculty of Math. and Informatics, Sofia University, 5 J. Bourchier Blvd., 1164 Sofia, Bulgaria  
`elza@fmi.uni-sofia.bg`

In 1937, M. Stone [5] proved that there exists a dual equivalence  $S^t : \mathbf{Stone} \rightarrow \mathbf{Bool}$  between the category **Stone** of compact zero-dimensional Hausdorff spaces and continuous maps and the category **Bool** of Boolean algebras and Boolean homomorphisms. In 1964, H. P. Doctor [4] extended the Stone duality to a duality between the category **PZHLC** of all locally compact zero-dimensional Hausdorff spaces and all perfect maps between them and the category **GBPL** of all generalized Boolean algebras and suitable morphisms between them. Later on, G. Dimov [1, 2] extended the Stone Duality to the category **ZHLC** of zero-dimensional locally compact Hausdorff spaces and continuous maps.

In this talk, extending the Stone Duality Theorem, we will describe two categories which are dually equivalent to the category **ZHaus** of zero-dimensional Hausdorff spaces and continuous maps. We will find as well two categories which are dually equivalent to the category **ZComp** of all zero-dimensional Hausdorff compactifications of zero-dimensional Hausdorff spaces. The details are given in [3].

## References

- [1] G. Dimov. A de Vries-type duality theorem for locally compact spaces – II. arXiv:0903.2593v4, 1–37, 2009.
- [2] G. Dimov. Some generalizations of the Stone Duality Theorem. *Publicationes Mathematicae Debrecen*, 80, 255–293, 2012.
- [3] G. Dimov and E. Ivanova-Dimova. Extensions of the Stone Duality to the category of zero-dimensional Hausdorff spaces. arXiv:1901.04537, 1–15, 2019.
- [4] H. Doctor. The categories of Boolean lattices, Boolean rings and Boolean spaces. *Canad. Math. Bulletin*, 7, 245–252, 1964.
- [5] M. H. Stone. The theory of representations for Boolean algebras. *Trans. Amer. Math. Soc.*, 40, 37–111, 1936.

---

The research of these authors was supported by Bulgarian National Science Fund, contract no. DN02/15/19.12.2016.

\*Presenter at TACL, Nice 2019

# Semi-Reflective Extensions of Dualities and a New Approach to the Fedorchuk Duality

Georgi Dimov<sup>1</sup>, Elza Ivanova-Dimova<sup>2</sup>, and Walter Tholen<sup>3\*</sup>

<sup>1</sup> Faculty of Mathematics and Informatics, Sofia University, Sofia, Bulgaria  
gdimov@fmi.uni-sofia.bg

<sup>2</sup> Faculty of Mathematics and Informatics, Sofia University, Sofia, Bulgaria  
elza@fmi.uni-sofia.bg

<sup>3</sup> Department of Mathematics and Statistics, York University, Toronto, Canada  
tholen@mathstat.yorku.ca

A dual equivalence often arises as the restriction of a dual adjunction to its fixed subcategories, given by those objects for which the adjunction units and co-units are isomorphisms. This work is about the converse procedure: given a dual equivalence, furnished by contravariant functors

$$T : \mathcal{A} \rightarrow \mathcal{B} \quad \text{and} \quad S : \mathcal{B} \rightarrow \mathcal{A},$$

where  $\mathcal{B}$  is a full subcategory of an ambient category  $\mathcal{C}$ , find a *good* description of a category  $\mathcal{D}$  containing  $\mathcal{A}$  as a full subcategory, as well as of a pair of dual equivalences

$$\tilde{T} : \mathcal{D} \rightarrow \mathcal{C} \quad \text{and} \quad \tilde{S} : \mathcal{C} \rightarrow \mathcal{D}$$

extending  $T$  and  $S$ . As the mere existence for such  $\mathcal{D}, \tilde{T}, \tilde{S}$  may be quite trivially verified, the important qualifier for this task lies in the word *good*.

We are guided by our role model, the *Fedorchuk duality*, which arises as an extension of a restriction of the Stone duality, as follows. As noted by Dimov [2, 3], the restriction of the Stone duality to the category  $\mathcal{A}$  of complete Boolean algebras and their sup-preserving maps produces a dual equivalence between  $\mathcal{A}$  and the category  $\mathcal{B}$  of extremally disconnected compact Hausdorff spaces and their open continuous maps. Considering  $\mathcal{B}$  as a full subcategory of the category  $\mathcal{C}$  of all compact Hausdorff spaces and their quasi-open continuous maps, the goal is then to provide a categorical framework for the construction of Fedorchuk's category CNCA of so-called *complete normal contact algebras* and their suitably chosen morphisms, which he had identified as dually equivalent to  $\mathcal{C}$  in his paper [4].

Key to our categorical construction is the equivalent description of the CNCA-objects as pairs  $(A, p)$ , with  $A$  a complete Boolean algebra and  $p : TA \rightarrow C$  an irreducible map of its Stone dual space  $TA$  onto a compact Hausdorff space  $C$  based on a theorem of Bezhanishvili [1]. We therefore consider the class  $\mathcal{P}$  of all irreducible continuous maps  $B \rightarrow C$  with domain in  $\mathcal{B}$ , which are well-known to be quasi-open. The challenge in the general categorical context is then to find a set of suitable conditions on an arbitrary class  $\mathcal{P}$  of morphisms in any category  $\mathcal{C}$  which, in the abstract context, allows for the construction of a category  $\mathcal{D}$  dually equivalent to  $\mathcal{C}$  and, in the concrete context, reproduces the Fedorchuk duality.

Given a dual equivalence  $T : \mathcal{A} \leftarrow \mathcal{B} : S$ , where  $\mathcal{B}$  is a full subcategory of a category  $\mathcal{C}$ , we call a class  $\mathcal{P}$  of morphisms in  $\mathcal{C}$  a  $(\mathcal{B}, \mathcal{C})$ -covering class if

$$(P1) \quad \forall (p : B \rightarrow C) \in \mathcal{P} : B \in |\mathcal{B}|;$$

$$(P2) \quad \forall B \in |\mathcal{B}| : 1_B \in \mathcal{P};$$

$$(P3) \quad \mathcal{P} \circ \text{Iso}(\mathcal{B}) \subseteq \mathcal{P};$$

---

The research of the first two authors was supported by the Bulgarian National Science Fund, contract no. DN02/15/19.12.2016, and that of the third author by the Natural Sciences and Engineering Council of Canada under the Discovery Grants Program.

\*Presenter at TACL, Nice 2019

(P4)  $\forall C \in |\mathcal{C}| \exists (p : B \rightarrow C) \in \mathcal{P}$ ;

(P5) for morphisms in  $\mathcal{C}$ , there is a functorial assignment

$$((p : B \rightarrow C) \in \mathcal{P}, v : C \rightarrow C', (p' : B' \rightarrow C') \in \mathcal{P}) \mapsto (\hat{v} : B \rightarrow B' \text{ with } v \circ p = p' \circ \hat{v});$$

we emphasize that, in this assignment,  $\hat{v}$  depends not only on  $v$ , but also on  $p$  and  $p'$ . In condition (P4) we tacitly assume that, for every  $C \in |\mathcal{C}|$ , there is a *chosen* morphism  $p \in \mathcal{P}$  with codomain  $C$ . In the presence of (P2), that morphism may be taken to be an identity morphism whenever  $C \in |\mathcal{B}|$ . To highlight the choice, we may reformulate (P4) as

(P4')  $\forall C \in |\mathcal{C}| \exists (\pi_C : EC \rightarrow C) \in \mathcal{P}$  (with  $\pi_C = 1_C$  when  $C \in |\mathcal{B}|$ ).

It is then clear that (P5) enables us to make  $E$  a functor  $\mathcal{C} \rightarrow \mathcal{B}$  and  $\pi$  a natural transformation  $IE \rightarrow \text{Id}_{\mathcal{C}}$ .

In our role model,  $EC$  is the absolute of the compact Hausdorff space  $C$ , and  $\pi_C$  serves as a projective cover. Actually,  $E$  turns out to be a coreflector of the category  $\mathcal{C}$  onto  $\mathcal{B}$ , as we show in extension of results by Henriksen and Jerison [5] and Bereznitskij (as cited in [7]). In general, the existence of a  $(\mathcal{B}, \mathcal{C})$ -covering class is weaker than having a right adjoint for the full embedding  $I : \mathcal{B} \hookrightarrow \mathcal{C}$ ; rather, it is equivalent to  $I$  being part of a *semi-adjunction* in the sense of Medvedev [6]; this means that the embedding has *almost* a right adjoint  $E$ , except that one of the triangular identities required for an adjunction is missing. This clarifies the status of conditions (P1-5) in standard categorical terms. They permit us to prove:

**Theorem.** *With the category  $\mathcal{D}$  constructed as suggested by the Fedorchuk duality,  $\mathcal{D}$  contains  $\mathcal{A}$  as a full subcategory and admits a dual equivalence  $\tilde{T} : \mathcal{D} \leftarrow \mathcal{C} : \tilde{S}$ , extending the given dual equivalence  $T : \mathcal{A} \leftarrow \mathcal{B} : S$ , in the sense that*

$$\tilde{T}J = IT \text{ and } \tilde{S}I \cong JS.$$

One can actually establish a rather complete array of identities which show the smooth interaction of the two dual adjunctions with each other, as well as with the semi-reflective embedding  $\mathcal{A} \hookrightarrow \mathcal{D}$  and the semi-coreflective embedding  $\mathcal{B} \hookrightarrow \mathcal{C}$ . (Some of these identities require a slight strengthening of condition (P4).) The identities allow us to go back and forth between  $\mathcal{D}$  and  $\mathcal{C}$  as efficiently as between  $\mathcal{A}$  and  $\mathcal{B}$ .

Armed with this categorical extension theorem, we sketch a new proof of Fedorchuk's duality theorem which, unlike its original proof, does *not* make use of the de Vries duality theorem. Time permitting, we will also comment on further applications of the categorical framework.

## References

- [1] G. Bezhanishvili. Stone duality and Gleason covers through de Vries duality. *Topology and its Applications*, 157:1064–1080, 2010.
- [2] G. Dimov. A de Vries-type duality theorem for locally compact spaces – II. arXiv:0903.2593v4, 1–37, 2009.
- [3] G. Dimov. Some generalizations of the Stone Duality Theorem. *Publicationes Mathematicae Debrecen*, 80:255–293, 2012.
- [4] V. V. Fedorchuk. Boolean  $\delta$ -algebras and quasi-open mappings. *Sibirsk. Mat. Ž.*, 14:1088–1099, 1973.
- [5] M. Henriksen and M. Jerison. Minimal projective extensions of compact spaces. *Duke Mathematical Journal*, 32:291–295, 1965.
- [6] M.Ya. Medvedev. Semiadjoint functors and Kan extensions. *Siberian Mathematical Journal*, 15:674–676, 1975.
- [7] V.I. Ponomarev and L.B. Šapiro. Absolutes of topological spaces and their continuous mappings. *Usbekhi Mat. Nauk*, 31:121–136, 1976.

# Extensions of dualities and a new approach to de Vries' Duality Theorem

Georgi Dimov<sup>1\*</sup>, Elza Ivanova-Dimova<sup>2</sup>, and Walter Tholen<sup>3</sup>

<sup>1</sup> Faculty of Math. and Informatics, Sofia University, 5 J. Bourchier Blvd., 1164 Sofia, Bulgaria  
gdimov@fmi.uni-sofia.bg

<sup>2</sup> Faculty of Math. and Informatics, Sofia University, 5 J. Bourchier Blvd., 1164 Sofia, Bulgaria  
elza@fmi.uni-sofia.bg

<sup>3</sup> Department of Mathematics and Statistics, York University, Toronto, Ontario, M3J 1P3, Canada  
tholen@mathstat.yorku.ca

In [2], we proved a general categorical theorem for extensions of dualities (briefly, ED-Theorem) and derived from it the Fedorchuk Duality Theorem [4] which says that there exists a dual equivalence between the category  $\mathbf{CHaus}_{\text{qop}}$  of compact Hausdorff spaces and their quasi-open maps and the category  $\mathbf{Fed}$  of complete normal contact algebras and suprema-preserving Boolean homomorphisms which reflect the contact relation. In this talk, we present the results from the continuation [3] of [2]. They concern the de Vries dual equivalence between the category  $\mathbf{CHaus}$  of compact Hausdorff spaces and their continuous maps and the category  $\mathbf{DeV}$  of complete normal contact algebras and de Vries' morphisms between them. We start by deriving from our ED-Theorem the recent Duality Theorem of Bezhanishvili-Morandi-Olberding [1] which extends de Vries' duality from the category  $\mathbf{CHaus}$  to the category  $\mathbf{Tych}$  of Tychonoff spaces and their continuous maps. We do this by obtaining first a new category  $\mathcal{C}'$  and a dual equivalence between it and the category  $\mathbf{Tych}$  which extends de Vries' duality. Then, using the Tarski Duality Theorem, we show that the category  $\mathcal{C}'$  is equivalent to the category  $\mathbf{BMO}$  obtained in [1] as a dual to the category  $\mathbf{Tych}$ . Further, we present a new general categorical theorem for extensions of dualities and derive from it de Vries' Duality Theorem. Moreover, in the process of doing this, we obtain a new duality theorem for the category  $\mathbf{CHaus}$ .

## References

- [1] G. Bezhanishvili, P. J. Morandi, and B. Olberding. De Vries duality for compactifications and completely regular spaces. *Topology and its Applications*, 257, 85–105, 2019.
- [2] G. Dimov, E. Ivanova-Dimova, and W. Tholen. Extensions of dualities and a new approach to the Fedorchuk duality. arXiv:1808.06168v2, 1–24, 2018.
- [3] G. Dimov, E. Ivanova-Dimova, and W. Tholen. Extensions of dualities and a new approach to de Vries' Duality Theorem, 2019, In preparation.
- [4] V. V. Fedorchuk. Boolean  $\delta$ -algebras and quasi-open mappings. *Sibirsk. Mat. Ž.* 14, 1088–1099, 1973.

---

The research of the first two authors was supported by Bulgarian National Science Fund, contract no. DN02/15/19.12.2016.

\*Presenter at TACL, Nice 2019

# Resource Reasoning in Duality-theoretic Form: Stone-type Dualities for Bunched and Separation Logics

Simon Docherty and David Pym

University College London, London, United Kingdom  
simon.docherty@ucl.ac.uk and d.pym@ucl.ac.uk

Bunched logics, beginning with O’Hearn and Pym’s **BI** [9, 10], have proved to be exceptionally useful tools in modelling and reasoning about computational and information-theoretic phenomena such as resources, the structure of complex systems, and access control. Perhaps the most striking example is Separation Logic [11] a specific theory of predicate **BI** with primitives for mutable data structures. Separation Logic has heralded a paradigm shift in deployable program correctness proving, key examples of which being the static analysis tool Infer ([www.fbinfer.com](http://www.fbinfer.com)) — now part of the code review production line at Facebook, with millions of lines of code automatically checked for memory bugs to date — and the Coq-implemented Concurrent Separation Logic framework Iris, which has been used to give machine-checked safety proofs for the systems programming language Rust [8].

Bunched logics provide an alternative to the resource-sensitive reasoning facilitated by linear logic. In linear logic, the structural rules of weakening and contraction are dropped, leading to a splitting of conjunction and disjunction into *additive* and *multiplicative* forms. These structural rules are reintroduced in a controlled manner via the *exponentials* ! and ?. This leads to an operational *number-of-uses* interpretation of formulae: a formula  $\varphi$  is a resource that may be used once; however,  $!\varphi$  denotes a duplicable resource  $\varphi$  that can be used as many times as one needs. In bunched logics, the control of structural rules is implemented very differently: in bunched sequent calculi, contexts are tree-shaped structures — *bunches* — built from *two* context formers to which different structural rules apply: one in which all apply, and another in which weakening and contraction (and possibly more) are dropped. Such systems can safely be seen as the free combination of intuitionistic propositional logic with multiplicative fragments of linear logics. The upshot of this is the existence (in contrast with linear logic) of a simple Kripke semantics of abstract resource: formulae have a declarative *separation* interpretation, describing properties a resource may satisfy, and, in particular, the manner in which resources must be (de)composed into components in order to meet a specification.

In the characteristic case of **BI**, Kripke resource models are given by ordered partial commutative monoids, in which worlds are seen to be resources that can be compared via an order  $\leq$  and, when compatible, composed by a partial composition  $\circ$ . For example, in the standard model of Separation Logic the resources are heaps (chunks of dynamically allocated computer memory) which can be compared (when one heap contains another) and, when compatible (when the memory addresses assigned by each heap are disjoint), composed by disjoint union. The Kripke semantics then extends that for intuitionistic logic with clauses for the multiplicative connectives. In particular, the multiplicative conjunction,  $*$ , is interpreted as follows:

$$x \vDash \varphi * \psi \text{ iff there exists resources } y, z \text{ such that } y \circ z \leq x \text{ and } y \vDash \varphi \text{ and } z \vDash \psi,$$

to be read as “the resource  $x$  is sufficient for  $\varphi * \psi$  iff part of  $x$  can be split into separate resources,  $y$  and  $z$ , with  $y$  sufficient for  $\varphi$  and  $z$  sufficient for  $\psi$ ”. Further multiplicative connectives—corresponding to implications, negation, disjunction, verum and falsum—are similarly given a straightforward Kripke semantics via operations on resources.

Resource semantics has been hugely influential; in particular, in its instantiations in Separation Logic and its descendents, with a huge body of literature and automated reasoning

tools successfully applying the idea to a range of computational phenomena. In contrast, the alternative *algebraic* view on bunched logics — as Heyting algebras extended with additional residuated monoidal operations — has seen little attention, with recent work by Galatos & Jipsen [5] and Litak & Jipsen [7] rare exceptions. This is quite an usual situation for a family of systems closely related to intuitionistic, modal and substructural logics.

In this talk, we give a systematic account of resource semantics via a family of Stone-type duality theorems between categories of bunched logic algebras and categories of ordered topological spaces. This framework encompasses the full range of systems: from the weakest bunched logics to those involving multiplicative variants of all of the standard propositional connectives, as well as those featuring (separating) modalities. By considering the category theoretic structures of bunched logic hyperdoctrines and indexed topological spaces, the duality theorems are extended to the predicate case, thus additionally capturing Separation Logic. As corollaries we retrieve soundness and completeness for the standard Kripke semantics found in the literature as well as new results for logics that previously lacked a semantic formulation.

To do so, we synthesise a variety of related work from modal [6], relevant [12], substructural [1] and categorical logic [2]. Much of the theory these areas enjoy is produced by way of algebraic and topological techniques. We argue that by recontextualizing the resource semantics of bunched logics in this way, similar theory can be given for both Separation Logic and its underlying systems. As examples, we prove a range of metatheory, including: decidability of weak bunched logics, the failure of interpolation, and a Goldblatt-Thomason-style characterisation of the definable classes of resource models. Further, we indicate a range of future directions building on our framework, including the *natural duality* generalisation of our results, extensions with semantics of program execution, and the development of Sahlqvist-style correspondence theory for bunched logics. This talk is based on material from the first author’s PhD thesis [3], some of which will appear in a forthcoming journal article [4].

## References

- [1] K. Bimbó and J.M. Dunn. *Generalized Galois Logics. Relational Semantics of Nonclassical Logical Calculi*. CSLI Publications, 2008.
- [2] D. Coumans. Duality for first-order logic. <http://www.math.ru.nl/~coumans/talkAC.pdf>.
- [3] S. Docherty. Bunched logics: a uniform approach. PhD thesis, University College London, 2019.
- [4] S. Docherty and D. Pym. Stone-type dualities for separation logics. *Log. Meth. Comp. Sci.*, to appear.
- [5] N. Galatos and P. Jipsen. Distributive residuated frames and generalized bunched implication algebras. *Algebr. Univ.*, 78(3): 303–336, 2017.
- [6] R. Goldblatt. Varieties of complex algebras. *Ann. Pure Appl. Logic*, 44(3):173–242, 1989.
- [7] P. Jipsen and T. Litak. An algebraic glimpse at bunched implications and separation logic. In *Hiroakira Ono on Residuated Lattices and Substructural Logics*, arXiv:1709.07063v2, to appear.
- [8] R. Jung, J.-H. Jourdan, R. Krebbers and D. Dreyer. RustBelt: Securing the foundations of the Rust programming language. *POPL 2018*, Article 66, 2018.
- [9] P. O’Hearn and D. Pym. The logic of bunched implications. *Bull. Symb. Log.*, 5(2):215–244, 1999.
- [10] D. Pym. Resource semantics: logic as a modelling technology. *ACM SIGLOG News*, April 2019, Vol. 6, No. 2, 5–41.
- [11] J. Reynolds. Separation logic: a logic for shared mutable data structures. In *LICS 2002*, 55–74, 2002.
- [12] Alasdair Urquhart. Duality for algebras of relevant logics. *Studia Logica*, 56(1/2): pp. 263–276, 1996.

## AMALGAMATING POSET EXTENSIONS

ROB EGROT

*Faculty of ICT, Mahidol University, 999 Phutthamonthon Sai 4 Rd, Salaya, Nakhon Pathom 73170, Thailand*

Given a poset  $P$  define  $\mathcal{F}$  and  $\mathcal{I}$  to be, respectively, the sets of all filters and ideals of  $P$ . A standard method for constructing a canonical extension  $e : P \rightarrow C$  is to first define an antitone Galois connection between  $\wp(\mathcal{F})$  and  $\wp(\mathcal{I})$  based on the relation of ‘non-empty intersection’ between ideals and filters, and then to define  $C$  to be set of all subsets of  $\mathcal{F}$  that are closed with respect to the induced closure operator. The embedding  $e$  is then defined by  $e : p \mapsto \{F \in \mathcal{F} : p \in F\}$  [3, 2]. When working with canonical extensions, it is often convenient to define the so-called ‘intermediate structure’ as a subposet of  $C$  generated from the embedded image  $e[P]$  by including certain joins and meets. Pulling back the natural embeddings of  $\mathcal{F}$  and  $\mathcal{I}$  into  $C$  we obtain orderings of  $\mathcal{F}$  and  $\mathcal{I}$  which agree with the ‘intrinsic’ orderings of these sets by reverse inclusion and inclusion respectively.

Generalizing, we can relax the requirements on  $\mathcal{F}$  and  $\mathcal{I}$  to allow for alternative definitions of ‘filter’ and ‘ideal’, and also to allow for situations where  $\mathcal{F}$  and  $\mathcal{I}$  include some, but not necessarily all, filters and ideals (as they are defined) respectively. Provided some consistency properties are satisfied, the ‘completion via Galois connection’ process described above with respect to the resulting *polarization* produces another class of completions [8, 6].

Generalizing further, we can extend to relations between filters and ideals other than non-empty intersection, using *polarities*  $(\mathcal{F}, \mathcal{I}, R)$ . For example, this method can be used to produce  $\Delta_1$ -completions, of which canonical extensions are one example [4]. As with canonical extensions, we can define the intermediate structure for a  $\Delta_1$ -completion  $d : P \rightarrow D$ . This is a subposet,  $I_d$ , of  $D$  generated from  $d[P]$  in a canonical way.

For a polarity  $(\mathcal{F}, \mathcal{I}, R)$  not producing a  $\Delta_1$ -completion, the natural maps from  $\mathcal{F}$  and  $\mathcal{I}$  into the lattice of Galois closed sets  $C$  need not be order embeddings, so the intrinsic orders on  $\mathcal{F}$  and  $\mathcal{I}$  and the orders induced by  $C$  are not necessarily the same. It is natural to ask what must be true of  $R$  for the intrinsic and induced orders to match, and this question is easy enough to answer.

We can also approach the situation in another way, by defining an *extension polarity* to be a triple  $(e_X, e_Y, R)$  such that  $e_X : P \rightarrow X$  and  $e_Y : P \rightarrow Y$  are extensions of  $P$ , and  $R$  is a relation between  $X$  and  $Y$ . Given such an extension polarity, we may ask (1) “*When is it possible to define a pre-order on  $X \cup Y$  that agrees with the orders on  $X$  and  $Y$ , and also with  $R$ ?*”, and (2) “*If  $e_X$  and  $e_Y$  are meet- and join-extensions respectively, when can such a pre-order be defined so that the natural embeddings  $\iota_X : X \rightarrow X \cup Y$  and  $\iota_Y : Y \rightarrow X \cup Y$  additionally have strong preservation properties as in the case of canonical extensions?*”. These questions are also easy enough to answer. More interestingly, it turns out that when a pre-order does exist for  $(e_X, e_Y, R)$  in affirmative answer to (2), it is necessarily unique. We use  $X \uplus Y$  to denote the poset over  $X \cup Y$  induced by this unique pre-order.

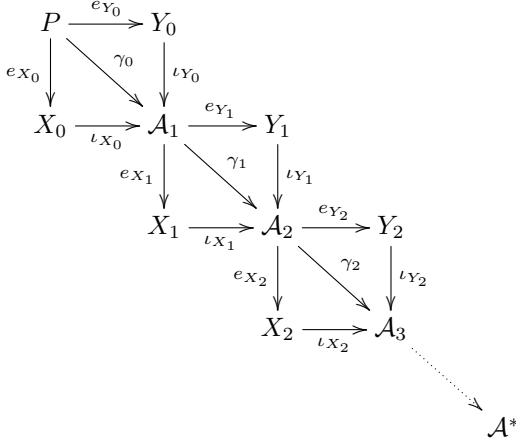


FIGURE 1.

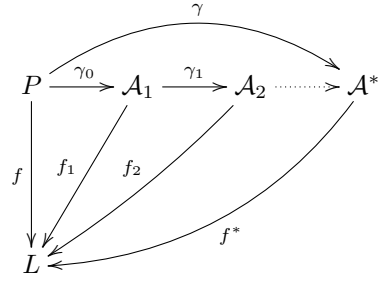


FIGURE 2.

Given  $e_X$  and  $e_Y$ , the relation  $R_l$  of ‘non-empty intersection’ defined by  $x R_l y \iff e_X^{-1}(x^\uparrow) \cap e_Y^{-1}(y^\downarrow) \neq \emptyset$  is the minimal relation such that there is an affirmative answer to (2). In the case where  $R = R_l$ , we show that  $X \uplus Y$  has a universal property, and based on this we define the **canonical amalgamation of  $e_X$  and  $e_Y$**  to be a triple  $(\mathcal{A}_{(e_X, e_Y)}, \pi_X, \pi_Y)$ , where  $\mathcal{A}_{(e_X, e_Y)}$  is a poset and  $\pi_X$  and  $\pi_Y$  are maps satisfying some additional properties.  $(\mathcal{A}_{(e_X, e_Y)}, \pi_X, \pi_Y)$  is unique up to a suitable notion of isomorphism, and always exists as  $(X \uplus Y, \iota_X, \iota_Y)$  provides a concrete construction.

Following [4], by combining  $(\mathcal{A}_{(e_X, e_Y)}, \pi_X, \pi_Y)$  with taking MacNeille completions we obtain a construction for canonical extension like completions. Moreover, we can in some circumstances extend the universal properties of meet- and join-completions from [7, Theorem 2] to canonical amalgamations. As an application of this, we present a construction of the free lattice generated by a poset  $P$  and preserving selected bounds (as in [1, 5]) as a colimit. This construction and the proof it has the required universal property are illustrated by the diagrams in Figures 1 and 2 respectively.

REFERENCES

- [1] R. A. Dean. Free lattices generated by partially ordered sets and preserving bounds. *Canad. J. Math.*, 16:136–148, 1964.
- [2] J. Dunn, M. Gehrke, and A. Palmigiano. Canonical extensions and relational completeness of some substructural logics. *J. Symb. Logic*, 70:713–740, 2005.
- [3] M. Gehrke and J. Harding. Bounded lattice expansions. *J. Algebra*, 238(1):345–371, 2001.
- [4] M. Gehrke, R. Jansana, and A. Palmigiano.  $\Delta_1$ -completions of a poset. *Order*, 30(1):39–64, 2013.
- [5] H. Lakser. Lattices freely generated by an order and preserving certain bounds. *Algebra Universalis*, 67(2):113–120, 2012.
- [6] W. Morton and C. J. van Alten. Distributive and completely distributive lattice extensions of ordered sets. *Internat. J. Algebra Comput.*, 28(3):521–541, 2018.
- [7] J. Schmidt. Each join-completion of a partially ordered set is the solution of a universal problem. *J. Austral. Math. Soc.*, 17:406–413, 1974.
- [8] W. R. Tunnichliffe. The completion of a partially ordered set with respect to a polarization. *Proc. London Math. Soc. (3)*, 28:13–27, 1974.



# Artin glueings as semidirect products

Peter Faul<sup>1</sup> and Graham Manuell<sup>2</sup>

<sup>1</sup> University of Cambridge, Cambridge, United Kingdom  
peter@faul.io

<sup>2</sup> University of Edinburgh, Edinburgh, United Kingdom  
graham@manuell.me

## Abstract

An Artin glueing [2] of two frames  $H$  and  $N$  is a frame  $G$  in which  $H$  and  $N$  are included as sublocales, with  $H$  open and  $N$  its closed complement. Artin glueings are not unique, but are determined by meet preserving maps  $f: H \rightarrow N$ .

Compare this to a semidirect product  $G$  of two groups  $N$  and  $H$ . Both  $N$  and  $H$  are subgroups of  $G$ , with  $N$  being normal. They satisfy that  $N \cap H = \{e\}$  and  $NH = G$ , which if thought of in terms of the lattice of subobjects of  $G$ , says that  $N$  and  $H$  are complements. Furthermore, just as with the Artin glueing, semidirect products of groups are not unique and are similarly determined by a map  $f: H \rightarrow \text{Aut}(N)$ .

In order to show that this analogy has substance we examine a link to extension problems. It is well known that the split extensions between any two groups  $N$  and  $H$  are precisely the semidirect products of  $N$  and  $H$ . We show that Artin glueings are precisely the solutions to a natural extension problem in the category **RFrm** of frames with meet-preserving maps.

We consider a chain  $N \xrightarrow{m} G \xrightarrow{e} H$  to be an extension if  $m$  is the kernel of  $e$  and  $e$  is the cokernel of  $m$ . We are really interested in split extensions. In the case of groups, split extensions satisfy the property that if  $s$  is a splitting of  $e$ , then the images of  $m$  and  $s$  together generate  $G$ . This is not so in **RFrm** and it will only occur when  $s$  is the right adjoint of  $e$ . For our purposes, this generation property is crucial and so we restrict to the generating split extensions. We briefly discuss some connections with the theory of  $\mathcal{S}$ -protomodularity [1].

We show that there is a natural way to view an Artin glueing as an extension in **RFrm** and that every extension  $N \xrightarrow{m} G \xrightarrow{e} H$  can be thought of as the glueing of  $H$  and  $N$  along  $m^*e_*$ .

We develop these ideas further by constructing the corresponding Ext functor, which takes in two frames and returns the set of Artin glueings. In the case of groups we have that  $\text{Ext}(H, N) \simeq \text{Hom}(H, \text{Aut}(N))$  and we find analogously that for frames we have  $\text{Ext}(H, N) \simeq \text{Hom}(H, N)$ .

## References

- [1] D. Bourn. Mal'tsev reflection, S-Mal'tsev and S-protomodular categories. *preprint Cahiers LMPA*, (497), 2014.
- [2] G. Wraith. Artin glueing. *Journal of Pure and Applied Algebra*, 4(3):345–348, 1974.

# Generic Models for Topological Evidence Logics

A. Baltag<sup>1</sup>, N. Bezhanishvili<sup>1</sup>, and S. Fernández González<sup>2</sup>

<sup>1</sup> ILLC, Universiteit van Amsterdam

<sup>2</sup> IRIT, Université de Toulouse

## 1 Introduction

In [BBÖS16] a topological semantics for evidence-based belief and knowledge is introduced, where knowledge amounts to not only having evidence but having *indefeasible evidence*, a notion encoded by the dense open sets<sup>1</sup>.

**Definition 1** (The dense interior semantics). Our language is the propositional language enriched with a knowledge modality  $K$ . A *topological evidence model* (topo-e-model) is a tuple  $(X, \tau, V)$ , where  $(X, \tau)$  is a topological space, and  $V : \mathbf{Prop} \rightarrow 2^X$  is a valuation. The semantics of a formula  $\phi$  is defined as follows:  $\|p\| = V(p)$ ;  $\|\phi \wedge \psi\| = \|\phi\| \cap \|\psi\|$ ;  $\|\neg\phi\| = X \setminus \|\phi\|$ ;  $x \in \|K\phi\|$  iff  $x \in \text{Int } \|\phi\|$  and  $\text{Int } \|\phi\|$  is dense, where  $\text{Int}$  is the interior operator.

The framework introduced in [BBÖS16] is single-agent and the logic of topo-e-models is S4.2. In this abstract we present a multi-agent generalisation, along with some “generic models”. Our proposal differs conceptually from previous multi-agent approaches to the dense interior semantics in that we build it on the notion of *local density*.

## 2 One-Agent Generic Models

Following the spirit of the McKinsey-Tarski theorem [MT44], one of our aims is finding *generic models* for this logic, i.e. single topological spaces whose logic under the dense interior semantics is S4.2. The proof of the next two theorems can be found in [BBG19].

**Theorem 2.** *S4.2 is sound and complete in the dense interior semantics with respect to any dense-in-itself metrisable space such as  $\mathbb{R}$ ,  $\mathbb{Q}$ , etc.*

In [BBÖS16] also an expansion of this logic is considered with the universal modality  $\forall$  and topological interior modality  $\square$ , which encodes “having evidence”. We denote this logic by  $\text{Logic}_{\forall\square}$ .

**Theorem 3.** *Logic <sub>$\forall\square$</sub>  is sound and complete in the dense interior semantics with respect to any dense-in-itself, metrisable and idempotent space such as  $\mathbb{Q}$ .*

## 3 Going Multi-Agent

For simplicity of presentation we work in a two-agent system (it is rather straightforward to extend these results to  $n > 2$  agents). Our language now contains modalities  $K_1$  and  $K_2$ , each encoding the same notion as in the single-agent system.

---

<sup>1</sup>A set  $U \subseteq X$  is dense whenever  $\text{Cl}U = X$ , or equivalently when it has nonempty intersection with every nonempty open set.

**The Problem of Density. Topological-partitional models.** A first (naive) approach to multi-agent topo-e-models would be to simply consider two topologies  $\tau_1$  and  $\tau_2$  on the same space and have agent  $i$  know  $\phi$  whenever  $\text{Int}_{\tau_i} \|\phi\|$  is dense in  $\tau_i$ . This is undesirable: this global notion of density does not account for cases in which two agents do not consider the same set of worlds possible. Instead, we want to make explicit, at each world  $x \in X$ , which subsets of worlds in  $X$  are compatible with each agent’s information. We will do this via partitions.

**Definition 4.** A *topological-partitional model* is a tuple  $(X, \tau_1, \tau_2, \Pi_1, \Pi_2, V)$  where  $X$  is a set,  $\tau_1$  and  $\tau_2$  are topologies defined on  $X$ ,  $\Pi_1$  and  $\Pi_2$  are partitions and  $V$  is a valuation.

For  $U \subseteq X$  we write  $\Pi_i[U] := \{\pi \in \Pi_i : U \cap \pi \neq \emptyset\}$ . For  $i = 1, 2$  and  $\pi \in \Pi_i[U]$  we say that  $U$  is  *$i$ -locally dense in  $\pi$*  whenever  $U \cap \pi$  is dense in the subspace topology  $(\pi, \tau_i|_\pi)$ . We simply say  $U$  is  *$i$ -locally dense* if it is locally dense in every  $\pi \in \Pi_i[U]$ .

For the remainder of this abstract, we limit ourselves to the fragment of the language including the  $K_1$  and  $K_2$  modalities.

**Definition 5 (Semantics).** We read  $x \in \|\!|K_i\phi\|\!$  iff there exists an  $i$ -locally dense  $\tau_i$ -open set  $U$  with  $x \in U \subseteq \|\phi\|$ .

This definition generalises one-agent models, appears to be more suitable conceptually and, moreover, gives us the logic one would extrapolate from the one-agent case:

**Theorem 6.** *The  $\mathcal{L}_{K_1K_2}$ -logic of topological-partitional models is  $\text{S4.2}_{K_1} + \text{S4.2}_{K_2}$ , the least normal modal logic containing the  $\text{S4.2}$  axioms for each  $K_i$ .*

## 4 Multi-Agent Generic Models

**The Quaternary Tree  $\mathcal{T}_{2,2}$ .** The *quaternary tree*  $\mathcal{T}_{2,2}$  is the full infinite tree with two relations  $R_1$  and  $R_2$  where every node has exactly four successors: a left  $R_i$ -successor and a right  $R_i$ -successor for  $i = 1, 2$ . We can define two topologies  $\tau_i$  and two partitions  $\Pi_i$  on  $\mathcal{T}_{2,2}$  by taking, respectively, the set of  $R_i$ -upsets and the set of  $R_i$ -connected components.

**Theorem 7.**  *$\text{S4.2}_{K_1} + \text{S4.2}_{K_2}$  is sound and complete with respect to  $(\mathcal{T}_{2,2}, \tau_{1,2}, \Pi_{1,2})$ .*

**The rational plane  $\mathbb{Q} \times \mathbb{Q}$ .** We can define two topologies on  $\mathbb{Q} \times \mathbb{Q}$  by “lifting” the open sets in the rational line horizontally or vertically [BBtCS06]. Formally, the *horizontal topology*  $\tau_H$  is the topology generated by  $\{U \times \{y\} : U \text{ is open, } y \in \mathbb{Q}\}$  and likewise for the vertical topology  $\tau_V$ .

**Proposition 8.** *There exist partitions  $\Pi_H$  and  $\Pi_V$  such that  $(\mathbb{Q} \times \mathbb{Q}, \tau_{H,V}, \Pi_{H,V})$  is a topological-partitional model whose logic is  $\text{S4.2}_{K_1} + \text{S4.2}_{K_2}$ .*

## References

- [BBG19] A. Baltag, N. Bezhanishvili, and S. Fernández González. The McKinsey-Tarski Theorem for Topological Evidence Logics. *Submitted. Available as ILLC preprint PP-2019-08*, 2019.
- [BBÖS16] A. Baltag, N. Bezhanishvili, A. Özgün, and S. Smets. Justified belief and the topology of evidence. In *WoLLIC 2016*, pages 83–103. Springer, 2016.
- [BBtCS06] J. van Benthem, G. Bezhanishvili, B. ten Cate, and D. Sarenac. Multimodal logics of products of topologies. *Studia Logica*, 84(3):369–392, 2006.
- [MT44] J.C.C. McKinsey and A. Tarski. The algebra of topology. *Annals of mathematics*, pages 141–191, 1944.

# On non-distributive lattices with involution \*

Wesley Fussner

Laboratoire J.A. Dieudonné, CNRS, and Université Côte d'Azur  
 Nice, France  
 wfussner@unice.fr

A *lattice with involution* (also known as an *i-lattice*) is an expansion of a lattice  $(L, \wedge, \vee)$  by an anti-isomorphism, i.e., a function  $\neg: L \rightarrow L$  such that for all  $x, y \in L$ ,

$$\neg\neg x = x,$$

$$\neg(x \wedge y) = \neg x \vee \neg y, \text{ and}$$

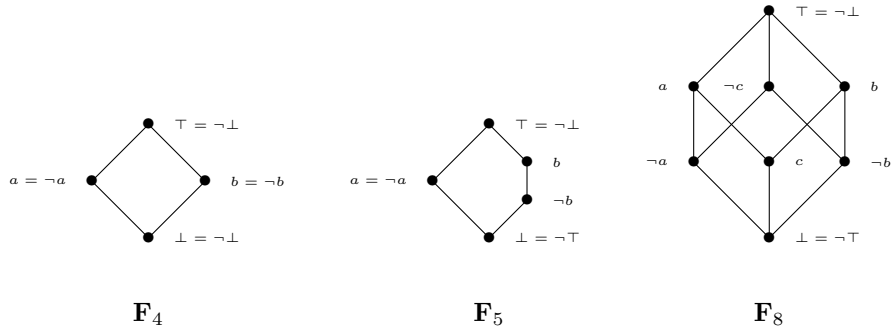
$$\neg(x \vee y) = \neg x \wedge \neg y.$$

Distributive lattices with involution have received extensive attention under the name *De Morgan lattices* (or *De Morgan algebras* when equipped with distinguished lattice bounds). For De Morgan lattices, there is a complete description of the lattice of subvarieties [4] and even lattice of subquasivarieties [6], as well as duality-theoretic analyses [2, 3]. In all of these studies, De Morgan lattices satisfying the normality condition,

$$x \wedge \neg x \leq y \vee \neg y,$$

play a crucial role, and comprise the subvariety of *Kleene lattices*. Non-distributive lattices with involution have received much less attention and are not as well-understood (but see, e.g., [5] and [1] for some recent studies).

The purpose of the present work is to contribute to our understanding of the normality condition in the context of non-distributive lattices with involution. For this, three finite lattices with involution play a decisive role, and have labeled Hasse diagrams as follows:



Each of  $\mathbf{F}_4$ ,  $\mathbf{F}_5$ , and  $\mathbf{F}_8$  can readily be seen to refute the normality condition. The *i-lattice*  $\mathbf{F}_4$  is shown in [4] to generate the variety of distributive *i-lattices* as a quasivariety, and  $\mathbf{F}_8$  is shown in [6] to generate the quasivariety of distributive *i-lattices* whose non-trivial members lack a  $\neg$ -fixed element. The following theorem shows that these finite structures play a decisive role among non-distributive *i-lattices* as well.

---

\*This project received funding from the European Research Council (ERC) under the European Unions Horizon 2020 research and innovation program (grant agreement No. 670624).

**Theorem 1.** *Let  $\mathbf{L} = (L, \wedge, \vee, \neg)$  be a lattice with involution. Then the following hold:*

1. *If  $\mathbf{L}$  has a modular lattice reduct, then  $\mathbf{L}$  refutes  $x \wedge \neg x \leq y \vee \neg y$  if and only if one of  $\mathbf{F}_4$  or  $\mathbf{F}_8$  embeds in  $\mathbf{L}$ .*
2. *If  $\mathbf{L}$  has a  $\neg$ -fixed element, then  $\mathbf{L}$  refutes  $x \wedge \neg x \leq y \vee \neg y$  if and only if one of  $\mathbf{F}_4$  or  $\mathbf{F}_5$  embeds in  $\mathbf{L}$ .*

The preceding theorem offers an account of normality in the non-distributive setting in the same spirit as the celebrated result of lattice theory that a lattice  $\mathbf{L}$  is distributive if and only if neither the five-element non-modular lattice nor the five-element modular, non-distributive lattice embeds into  $\mathbf{L}$ .

## References

- [1] Chajda, I.: A note on pseudo-Kleene algebras. *Acta Universitatis Palackianae Olomucensis, Facultas Rerum Naturalium, Mathematica*. 55(1):39–45, 2016.
- [2] Davey, B.A. and Werner, H.: Dualities and equivalences for varieties of algebras. In A.P. Huhn and E.T. Schmidt, editors, *Contributions to Lattice Theory*, pages 101–275. North-Holland, Amsterdam, New York, 1983.
- [3] Fussner, W. and Galatos, N.: Categories of models of R-mingle. Preprint, arXiv:1710.04256, 2017.
- [4] Kalman, J.: Lattices with involution. *Trans. Amer. Math. Soc.* 87(2):485–491, 1958.
- [5] Mureşan, C.: Some properties of lattice congruences preserving involutions and their largest numbers in the finite case. Preprint, arXiv:1802.05344, 2018.
- [6] Pynko, A.: Implicational classes of De Morgan lattices. *Discrete Mathematics* 205:171–181, 1999.

# Characterization of flat polygonal logics

David Gabelaia, Mamuka Jibladze, Evgeny Kuznetsov, and Levan Uridia

TSU A. Razmadze Mathematical Institute

Let  $\mathfrak{B}^n$  be the Boolean subalgebra generated by halfspaces inside the powerset  $\wp(\mathbb{R}^n)$ . We call elements of  $\mathfrak{B}^n$  *polyhedra*.  $\mathfrak{B}^n$  is in fact an **S4** modal subalgebra of  $\wp(\mathbb{R}^n)$  with respect to the Euclidean closure operator  $\diamond$ . Moreover,  $\mathfrak{B}^n$  turns out to be an **S4.Grz**-algebra.

Take a polyhedron  $P \in \mathfrak{B}^n$  and denote the relativization of  $\mathfrak{B}^n$  to  $P$  by  $P^+$ . Then  $P^+$  is a **S4.Grz**-algebra consisting of all subpolyhedra of  $P$ , endowed with the subspace closure operator. Inside  $P^+$  there also sits a Heyting algebra of all *open* subpolyhedra of  $P$ . These structures give rise to the corresponding formalisms – modal logics above **S4.Grz** if we concentrate on the modal algebras  $P^+$  and intermediate logics if we instead focus on the Heyting algebras sitting inside. Here we consider only the modal case, but the results easily translate to intermediate logics. The systematic study of the polyhedral semantics for modal and intuitionistic logics has been initiated in [4], [2] and [5].

We consider *polyhedral modal logics* – the logics  $\text{Log}\{P_i^+ \mid i \in I\}$ , generated by some family  $(P_i)_{i \in I}$  of polyhedra  $P_i \in \mathfrak{B}^{n_i}$ . Each  $P^+$  is of finite *height* and hence, locally finite [3]. This has to do with the geometric dimension of  $P$  being finite. It follows that polyhedral logics enjoy the finite model property and their study can be reduced to the study of the corresponding finite posets.

Each polyhedral logic  $L$  has well-defined *dimension*  $\dim L$ : it is either the smallest  $d$  for which  $L$  contains the Jankov-Fine axiom of the  $(d + 1)$ -element chain, or infinity, if such a  $d$  does not exist. This happens to coincide with the maximum of the geometric dimensions of the polyhedra  $P$  which validate  $L$ . The polyhedral logics of finite dimension are of finite height and hence, locally finite.

The smallest extension of **S4.Grz** of height  $n$  is **S4.Grz<sub>n</sub>** (the logic of posets of height  $\leq n$ ), while the largest is **S4.Grz<sub>n</sub>.3** (the logic of the  $n$ -element chain). It follows from the results of [2] interpreted modally that the smallest polyhedral logic of  $\dim n$  is **S4.Grz<sub>n+1</sub>** (the modal logic of all polyhedra of  $\dim \leq n$ ). On the other hand, **S4.Grz<sub>n</sub>.3** is polyhedrally *incomplete* for any  $n > 1$ . The largest polyhedral logic of  $\dim n$  turns out to be  $\mathbf{PL}_n = \text{Log}(\mathfrak{B}^n)$ .

**Theorem 1.** *Let  $L$  be a polyhedral modal logic of  $\dim n$ . Then  $\mathbf{S4.Grz}_{n+1} \subseteq L \subseteq \mathbf{PL}_n$ .*

There is a single polyhedral logic of  $\dim 0$  – the trivial logic  $\mathbf{Triv} = \mathbf{S4.Grz}_1 = \mathbf{PL}_0$ . Let us denote by  $\mathfrak{F}_n$  the *n-fan* – the rooted poset of height 2 with  $n$ -many maximal points.

**Theorem 2.** *The modal logic  $L$  is a polyhedral logic of dimension 1 iff  $L = \mathbf{S4.Grz}_2$  or  $L$  coincides with the modal logic of an  $n$ -fan  $\mathfrak{F}_n$  for some  $n > 1$ . These logics form a descending countable chain (under inclusion) between  $\mathbf{S4.Grz}_2$  and  $\mathbf{PL}_1$  where the latter is the modal logic of the 2-fan.*

We now turn to *flat polyhedra* – those  $\dim n$  polyhedra that are embedded into the ambient Euclidean space  $\mathbb{R}^n$  of the *same dimension*. The relevant algebraic notion is that of *relativization*, while the relevant modal notion is that of *downward subframization* [6], [1]. Call the polyhedral logic  $L$  of  $\dim n$  *flat* iff  $L$  is complete w.r.t. to some class  $(P_i^+)_{i \in I}$  of polyhedral algebras such that  $P_i \in \mathfrak{B}^n$  are polyhedra of  $\dim n$  inside  $\mathbb{R}^n$  for all  $i \in I$ .

**Theorem 3.** *The least flat polyhedral logic  $\mathbf{Flat}_n$  of  $\dim n$  is the downward subframization of  $\mathbf{PL}_n$ .*

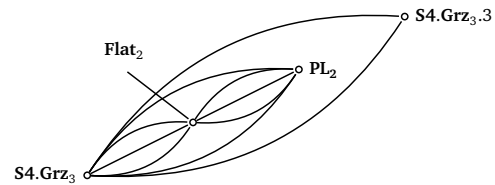


Figure 1: The (flat) polygonal logics inside the lattice of all height 3 extensions of **S4.Grz**

Our main results concern the flat logics of dim 2 – we call them *Flat Polygonal Logics*. By definition, such logics are generated by some family of relativizations of  $\mathfrak{B}^2$ . In other words, flat polygonal logics are complete w.r.t. some class  $(P_i^+)_{i \in I}$  where each  $P_i$  is a flat polygon – a polygonal subset of the Euclidean plane  $\mathbb{R}^2$ . We will give a full characterization of flat polygonal logics, using an explicit collection of Jankov-Fine and subframe formulas for certain finite posets. It turns out that  $\mathbf{Flat}_2$  is the logic of finite height-3 posets without a subframe isomorphic to the 3-fork poset  $\begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \end{array}$ .

**Theorem 4.**  $\mathbf{Flat}_2 = \mathbf{S4.Grz}_3 + \sigma \left( \begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \end{array} \right)$  where  $\sigma \left( \begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \end{array} \right)$  is the subframe formula forbidding the 3-fork. Moreover, flat polygonal logics are all in the interval  $[\mathbf{Flat}_2, \mathbf{PL}_2]$ .

To describe the flat polygonal logics occurring between  $\mathbf{Flat}_2$  and  $\mathbf{PL}_2$ , we introduce posets  $\mathfrak{F}_{m,n}$  depicted below that are ordered by *reducibility* –  $\mathfrak{F}$  is reducible to  $\mathfrak{G}$  if there exists an onto p-morphism from  $\mathfrak{F}$  to  $\mathfrak{G}$ . The poset of these frames is depicted on Figure 2.

The reducibility among  $\mathfrak{F}_{m,n}$  can be described as follows:  $\mathfrak{F}_{m,n}$  reduces to  $\mathfrak{F}_{m',n'}$  iff  $m+n \geq m'+n'$  and  $m \geq m'$ . Denote the poset of these frames by  $Q$ .

**Lemma 5.** The dual poset of  $Q$  is a well partial order, i.e.  $Q$  contains neither infinite strictly ascending chains, nor infinite antichains.

For every antichain  $\alpha$  in  $Q$  the corresponding logic  $L_\alpha$  is obtained by adding to  $\mathbf{Flat}_2$  the Jankov-Fine axioms  $\chi(\mathfrak{F}_{m,n})$  for each  $\mathfrak{F}_{m,n} \in \alpha$ . It is not difficult to see, that  $L_\alpha \subseteq L_\beta$  iff  $\alpha \subseteq \downarrow\beta$ . Moreover:

**Theorem 6.** The logics  $L_\alpha$ , for  $\alpha \subset Q$  an antichain, are all different, and exhaust all flat polygonal logics, that is all polygonal logics between  $\mathbf{Flat}_2$  and  $\mathbf{PL}_2$ .

It follows that there are only countably many flat polygonal logics, each one of them being finitely axiomatizable and decidable. In the talk we will also present a way to describe the Kripke frames for each flat polygonal logic  $L$  based on the upset of  $L$ -frames inside  $Q$  and a certain operation on rooted posets defining  $\mathbf{PL}_2$  and  $\mathbf{PL}_1$  – crown frames [4] and  $n$ -fans.

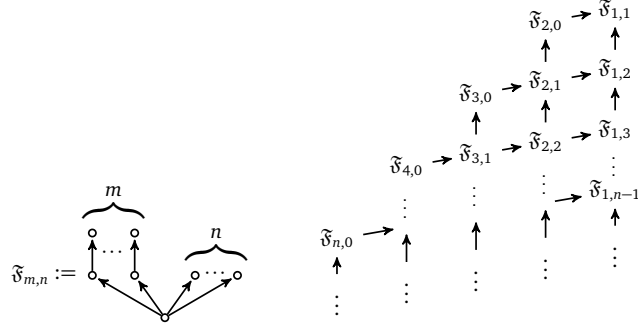


Figure 2: Poset  $Q$  of the frames  $\mathfrak{F}_{m,n}$  ordered by reducibility

## References

- [1] G. Bezhanishvili, N. Bezhanishvili and J. Ilin *Subframization and stabilization for superintuitionistic logics* Journal of Logic and Computation, 29(1):1–35, 2019
- [2] N. Bezhanishvili, V. Marra, D. Mcneill, and A. Pedrini. Tarski’s theorem on intuitionistic logic, for polyhedra. *Annals of Pure and Applied Logic*, 169(5):373–391, 2018.
- [3] A. Chagrov and M. Zakharyashev. *Modal logic*, volume 35 of *Oxford Logic Guides*. The Clarendon Press, Oxford University Press, New York, 1997. Oxford Science Publications.
- [4] D. Gabelaia, K. Gogoladze, M. Jibladze, E. Kuznetsov, and M. Marx. Modal logic of planar polygons. Preprint available at <https://arxiv.org/abs/1807.02868>.
- [5] D. Gabelaia, K. Gogoladze, M. Jibladze, E. Kuznetsov and L. Uridia An Axiomatization of the d-logic of Planar Polygons Accepted for publication in the proceedings of TbiLLC-2017, Springer, 2019.
- [6] F. Wolter The structure of lattices of subframe logics, *Annals of Pure and Applied Logic*, 86(1):47–100, 1997.

# Undecidability methods for residuated lattices

Nikolaos Galatos<sup>1</sup> and Gavin St. John<sup>2</sup>

<sup>1</sup> University of Denver, Denver, Colorado, USA  
ngalatos@du.edu

<sup>2</sup> University of Denver, Denver, Colorado, USA  
Gavin.StJohn@du.edu

## Abstract

We describe various encodings of counter machines and interpretations of semigroups into varieties of residuated lattices. We obtain the undecidability of the word problem and thus of the quasi-equational theory for these varieties.

Proofs of undecidability in substructural logics employ encodings of counter machines or interpretations of semigroups.

The hardware of a *counter machine* consists of a finite number of *registers*, which can be thought of as empty boxes labeled by the name of the register, and *tokens* each of which can be in some register, as well a final set of internal *states* in which the machine can be in, with designated initial state  $q_I$  and final state  $q_F$ . We write  $R = \{r_1, \dots, r_k\}$  for the set of registers,  $Q$  for the set of states, and we label each token within register  $r_i$  by  $r_i$ . Therefore the configuration of a machine can be represented by the monoid term  $qS_0r_1^{n_1}S_1 \cdots S_{k-1}r_k^{n_k}S_k$ . The auxiliary letters  $S_0, \dots, S_k$  are called *stoppers*. The software consists of a finite set of instructions taken from three different types. Increment instructions: when in state  $q$ , increment register  $r_i$  by one token and change the internal state to  $q'$ . Analogously we can decrement a non-zero counter (and do nothing if it is empty). Finally zero-test instructions: when in state  $q$ , check the contents of register  $r_i$  and if they are empty then move to state  $q'$ . We represent these instructions by the following inequalities:  $qS_i \leq q'r_iS_i$ ,  $qr_iS_i \leq q'S_i$  and  $S_{i-1}qS_i \leq S_{i-1}q'S_i$ . We also assume that for every letter  $x$  and every state  $q$ , we have the ambient instructions  $xq \leq qx$  and  $qx \leq xq$ .

The computation relation  $\leq$  of a machine is defined as the reflexive-transitive closure of the smallest compatible relation containing the instructions. We say that a configuration  $C$  is accepted if  $C \leq q_F S_0 S_1 \cdots S_k$ . For example, consider the machine that has set of states  $Q = \{q_1, q_F\}$ , with initial and final state  $q_F$ , set of registers  $R = \{r_1, r_2\}$  and set of instructions  $P = \{q_F r_1 S_1 \leq q_1 S_1, q_1 r_2 S_2 \leq q_F S_2\}$ , then we have

$$\begin{aligned} q_F S_0 r_1 S_1 r_2 S_2 &\leq S_0 q_F r_1 S_1 r_2 S_2 \\ &\leq S_0 q_1 S_1 r_2 S_2 \\ &\leq S_0 S_1 q_1 r_2 S_2 \\ &\leq S_0 S_1 q_F S_2 \\ &\leq S_0 q_F S_1 S_2 \\ &\leq q_F S_0 S_1 S_2. \end{aligned}$$

The only initial configurations that are accepted are of the form  $q_F S_0 r_1^n S_1 r_2^n S_2$ , where  $n$  is a natural number, so the machine checks if we have an equal number of  $r_1$ -tokens as  $r_2$ -tokens.

It is well known that there is a counter machine with an undecidable set of accepted configurations. This fact can be used to prove that the quasiequational theory of residuated lattices is undecidable. Horčík proves that the quasiequational theory of square-increasing ( $x \leq x^2$ )



residuated lattices is also undecidable, via a different encoding that is resilient to applications of square increasingness. The need for a different encoding is explained by the fact that in the standard encoding tokens can be doubled at will by the use of the ambient instruction of square increasingness. Unfortunately, this result does not cover more involved inequalities such as  $xy \leq xyx \vee x \vee y$ . We describe a new encoding (using powers of a fixed appropriate integer) that is resilient to such ambient instructions, thus obtaining undecidability results for the corresponding varieties.

For the implementation and correct encoding of the zero test, it is imperative that we do not have ambient instructions of the form  $xy \leq yx$  for all letters, as then the tokens  $r_i$  are not constrained within the designated stoppers  $S_{i-1}$  and  $S_i$  as they are expected to be for the correct application of zero tests; the stoppers fail to stop them. Therefore, all of the above encodings provide no help in proving the undecidability of the quasiequational theory of any variety of commutative residuated lattices. (We discuss some innocent generalizations of commutativity for which the encoding still works, but for commutativity itself and many of its variants the above encodings fail.) We present variants of the above encoding that implement the zero test in an indirect way: we can simulate parallel computation, one strand of which will make sure that any zero tests used were applied correctly and the other strand will perform the main computation. Acceptance is defined when all strands terminate successfully. Parallel computation is performed with the use of the join operation. (We note that join, when it appears on the left-hand side of an inequality, has a conjunctive effect.) By further combining two of the above ideas, we can modify the encoding to include parallel computation and powers of an integer at the same time. This results in encodings that are resilient to a plethora of inequalities in the signature of  $\{\vee, \cdot, 1\}$ , even in the presence of commutativity. All of the above results can be strengthened to obtain the undecidability of the word problem for these varieties.

We also briefly explore the method of proving undecidability via interpreting semigroups. The argument relies on identifying a term that is definable in the existing language, usually with the aid of a few constants (hence a polynomial term). These constants are needed for importing a geometric intuition analogous to that of a coordinate frame in (projective) geometry. A subset (a line) is identified and an associative operation is defined on it. The proof concludes by defining a specific finitely presented algebra in the variety, in which decidability of the word problem is equivalent to decidability of the word problem for a specific semigroup, which is known to have an undecidable word problem. This method can be used to study varieties of distributive residuated lattices, as the argument relies on the distributive residuated lattice of all subsets of an infinitely-dimensional vector space.

# A Glivenko theorem for lattice-ordered groups\*

José Gil-Férez<sup>1</sup>, Frederik Möllerström Lauridsen<sup>2</sup>, and George Metcalfe<sup>1</sup>

<sup>1</sup> Mathematisches Institut, University of Bern, Switzerland.  
gilferez@gmail.com, george.metcalfe@math.unibe.ch

<sup>2</sup> Institute for Logic, Language and Computation, University of Amsterdam, The Netherlands.  
f.m.lauridsen@uva.nl

Glivenko’s theorem, proved by Valery Glivenko in 1929, may be formulated algebraically as the statement that an equation  $\neg\neg s \leq \neg\neg t$  is valid in all Heyting algebras if and only if  $s \leq t$  is valid in all Boolean algebras [8]. In this work [7], we show that “integrally closed” residuated lattices, enjoying close connections to (pseudo) BCI-algebras [10, 12, 3], Dubreil-Jacotin semigroups [1, Chap. 12–13], and algebras for Casari’s comparative logic [2, 11], admit an analogous Glivenko property with respect to lattice-ordered groups ( $\ell$ -groups) and indeed form the largest variety of residuated lattices admitting such a property. We also use this Glivenko property for  $\ell$ -groups to give a sequent calculus for the variety of integrally closed residuated lattices and establish the decidability (indeed PSPACE-completeness) of its equational theory.

A residuated lattice is called *integrally closed* (cf., [4, Chap. XII.3]) if it satisfies the equation  $x \setminus x \approx e$  or, equivalently, the equation  $x/x \approx e$ . It is easily shown that any integral, cancellative, or divisible residuated lattice is integrally closed. Also, it is not hard to prove that any integrally closed residuated lattice  $\mathbf{A}$  is *e-cyclic*, i.e., satisfies the equation  $x \setminus e \approx e/x$ .

For any e-cyclic residuated lattice  $\mathbf{A}$  and  $a \in A$ , let  $\sim a$  denote the common result  $a \setminus e = e/a$ . The map  $\alpha: A \rightarrow A$ ;  $a \mapsto \sim \sim a$  is always a nucleus on  $\mathbf{A}$  and induces a residuated lattice structure on its image which we denote by  $\mathbf{A}_{\sim \sim}$  (see, e.g., [6, Lem. 5.2–5.3]). The relationship between  $\mathbf{A}$  and  $\mathbf{A}_{\sim \sim}$  is particularly interesting when  $\mathbf{A}$  is integrally closed.

**Proposition 1.** *Let  $\mathbf{A}$  be an e-cyclic residuated lattice.*

- (a)  $\mathbf{A}$  is integrally closed if, and only if,  $\mathbf{A}_{\sim \sim}$  is an  $\ell$ -group.
- (b) If  $\mathbf{A}$  is integrally closed, then  $\alpha: \mathbf{A} \rightarrow \mathbf{A}_{\sim \sim}$  is a homomorphism and  $\alpha^{-1}(\downarrow e) = \downarrow e$ .

Indeed, any residuated lattice  $\mathbf{A}$  admitting a homomorphism  $h: \mathbf{A} \rightarrow \mathbf{G}$  such that  $\mathbf{G}$  is an  $\ell$ -group and  $h^{-1}(\downarrow e) = \downarrow e$  must be integrally closed, in which case  $\mathbf{A}_{\sim \sim}$  is the unique (up to isomorphism) homomorphic image of  $\mathbf{A}$  with these two properties.

Let us denote the variety of integrally closed residuated lattices by  $\mathcal{ICRL}$  and the variety of  $\ell$ -groups by  $\mathcal{LG}$ . Given any class  $\mathcal{K} \subseteq \mathcal{ICRL}$ , we define the class  $\mathcal{K}_{\sim \sim} := \{\mathbf{A}_{\sim \sim} \mid \mathbf{A} \in \mathcal{K}\} \subseteq \mathcal{LG}$ . The following result is then a straightforward consequence of Proposition 1.

**Proposition 2.** *The map  $\mathcal{V} \mapsto \mathcal{V}_{\sim \sim}$  is an interior operator on the lattice of subvarieties of  $\mathcal{ICRL}$  whose image is the lattice of subvarieties of  $\mathcal{LG}$ .*

Following [6], we say that a variety  $\mathcal{V}$  of residuated lattices admits the (*equational*) *Glivenko property* with respect to another variety  $\mathcal{W}$  of residuated lattices if for all terms  $s, t$  in the language of residuated lattices,

$$\mathcal{V} \models e/(s \setminus e) \leq e/(t \setminus e) \iff \mathcal{W} \models s \leq t \iff \mathcal{V} \models (e/s) \setminus e \leq (e/t) \setminus e,$$

noting that if  $\mathcal{V}$  is a variety of e-cyclic residuated lattices, this simplifies to

$$\mathcal{V} \models \sim \sim s \leq \sim \sim t \iff \mathcal{W} \models s \leq t.$$

We can now state the following Glivenko theorem for (varieties of)  $\ell$ -groups.

---

\*The research reported here was supported by Swiss National Science Foundation (SNF) grant 200021\_165850.

**Theorem 3.** *Any variety  $\mathcal{V}$  of integrally closed residuated lattices admits the Glivenko property with respect to the variety  $\mathcal{V}_{\sim\sim}$ . Moreover,  $\mathcal{IcRL}$  is the largest variety of residuated lattices that admits the Glivenko property with respect to  $\mathcal{LG}$ .*

Let us now define a *sequent* to be an expression of the form  $\Gamma \Rightarrow t$  where  $\Gamma$  is a finite sequence of terms and  $t$  a term in the language of residuated lattices. We say that a sequent  $s_1, \dots, s_n \Rightarrow t$  is *valid* in a class  $\mathcal{K}$  of residuated lattices, denoted by  $\models_{\mathcal{K}} \Gamma \Rightarrow t$ , if  $\mathcal{K} \models s_1 \cdots s_n \leq t$ , where the empty product is understood as  $e$ . Theorem 3 may then be used to establish the soundness, with respect to validity in  $\mathcal{IcRL}$ , of the following “non-standard” weakening rule:

$$\frac{\Gamma, \Pi \Rightarrow t \quad \models_{\mathcal{LG}} \Delta \Rightarrow e}{\Gamma, \Delta, \Pi \Rightarrow t} \text{ (LG-w)}$$

where  $\models_{\mathcal{LG}} \Delta \Rightarrow e$  may be understood as a side-condition for weakening that is decidable [9], indeed co-NP-complete [5]. This condition can also be understood proof-theoretically as asking for a derivation of the sequent  $\Delta \Rightarrow e$  in some calculus for  $\ell$ -groups, such as the hypersequent calculus admitting cut-elimination provided in [5]. Adding the rule (LG-w) to the standard sequent calculus for residuated lattices we obtain a sequent calculus for  $\mathcal{IcRL}$  that admits cut-elimination. This allows us to establish the following result.

**Theorem 4.** *The equational theory of  $\mathcal{IcRL}$  is decidable, indeed PSPACE-complete.*

Let us mention finally that, by considering the appropriate reducts and expansions of the language of residuated lattices, these results can be related to previous work on (pseudo) BCI-algebras [10, 12, 3], Dubreil-Jacotin semigroups [1, Chap. 12–13], and algebras for Casari’s comparative logic [2, 11].

## References

- [1] T. S. Blyth. *Lattices and Ordered Algebraic Structures*. Springer, 2005.
- [2] E. Casari. Comparative logics and abelian  $\ell$ -groups. In C. Bonotto, R. Ferro, S. Valentini, and A. Zanardo, editors, *Logic Colloquium ’88*, pages 161–190. Elsevier, 1989.
- [3] P. Emanovský and J. Kühn. Some properties of pseudo-BCK- and pseudo-BCI-algebras. *Fuzzy Sets and Systems*, 339:1–16, 2018.
- [4] L. Fuchs. *Partially Ordered Algebraic Systems*. Pergamon Press, 1963.
- [5] N. Galatos and G. Metcalfe. Proof theory for lattice-ordered groups. *Ann. Pure Appl. Logic*, 8(167):707–724, 2016.
- [6] N. Galatos and H. Ono. Glivenko theorems for substructural logics over FL. *J. Symbolic Logic*, 71(4):1353–1384, 2006.
- [7] J. Gil-Férez, F. M. Lauridsen, and G. Metcalfe. Self-cancellative residuated lattices. Manuscript, available to download at <http://arxiv.org/abs/1902.08144>, 2019.
- [8] V. Glivenko. Sur quelques points de la logique de M. Brouwer. *Bulletins de la classe des sciences, Académie Royale de Belgique*, 15:183–188, 1929.
- [9] W. C. Holland and S. H. McCleary. Solvability of the word problem in free lattice-ordered groups. *Houston Journal of Mathematics*, 5(1):99–105, 1979.
- [10] R. Kashima and Y. Komori. The word problem for free BCI-algebras is decidable. *Math. Japon.*, 37(6):1025–1029, 1992.
- [11] G. Metcalfe. Proof calculi for Casari’s comparative logics. *J. Logic Comput.*, 16(4):405–422, 2006.
- [12] J. G. Raftery and C. J. van Alten. Residuation in commutative ordered monoids with minimal zero. *Rep. Math. Logic*, 34:23–57, 2000.

# Nonclassical first order logics: semantics and proof theory

Giuseppe Greco, Peter Jipsen, Alexander Kurz, M. Andrew Moshier,  
Alessandra Palmigiano, and Apostolos Tzimoulis

The existential and universal quantifiers in first order logic have a clear intuitive meaning and a very well understood behaviour from their interpretation in classical models. However, many challenges arise when trying to interpret quantifiers in a non-classical setting. Already in intuitionistic logic the semantics of the universal quantifier cannot be defined locally (i.e. its intuitionistic interpretation requires to look across possible worlds in models and across the individuals inhabiting those worlds) [7], and unless the constant domain axiom  $\forall x(A \vee B(x)) \leftrightarrow A \vee \forall xB(x)$  is assumed, the domains of the models might vary. Likewise for modal logic, even within a classical framework, different semantics have led to different axiomatizations [8, 11]. For weaker logics than the intuitionistic, it is unclear how to properly axiomatize the quantifiers and how to interpret them. In [12, 14, 13], a general approach on quantification is given based on the theory of hyperdoctrines. In [18], semantics with a constant domain are given for distributive modal logic. In [17, 3, 15], an algebraic approach is explored which covers a class of logics, based on the algebraic interpretation of quantifiers as suprema and infima [16]. For classical modal logic, a very general complete axiomatization is given in [4].

Our proposal builds on [19, 1], where a proper display calculus is introduced for classical first order logic, based on a well known semantic analysis that represents classical first order models as hyperdoctrines. As was the case of other logical frameworks (cf. e.g. [5, 6, 9]), this semantic analysis makes it possible to define a suitable multi-type calculus for first order logic in which the side conditions of introduction rules for the quantifiers are encoded into analytic structural rules involving different types. Wansing’s insight [21, 20, 2] that quantifiers can be treated proof-theoretically as modal operators is incorporated into this approach by simply regarding  $(\forall x)$  and  $(\exists x)$  as modal operators bridging different types (i.e. as *heterogeneous* operators). Following Lawvere [12, 14, 13] and Halmos [10], this requires to consider as many types as there are finite sets of free variables; that is, two first order formulas have the same type if and only if they have exactly the same free variables.

In this environment, both substitutions and quantifiers are explicitly represented as logical and structural connectives, which allows to encode the axioms capturing their interaction as analytic (heterogeneous) modal reduction principles, and hence as analytic structural rules. Thanks to this, we are now in a position to explore systematically the space of properties of substitutions and quantifiers and their possible interactions, and more importantly, to conduct a finer-grained analysis of fundamental interactions between quantifiers and intensional connectives. For instance, certain rules encode the fact that the cylindrification maps are Boolean algebra homomorphisms, which in turn captures the fact that classical propositional connectives are all extensional.

This talk reports on preliminary results about multi-type algebraic semantics and analytic multi-type display calculi for first order logics based on varieties of lattice expansions (normal LE-logics). Using duality theory and the theory of canonical extensions we investigate the corresponding relational models. We recast the completeness proofs for classical and intuitionistic first order logic and discuss the challenges that arise in the interpretation of quantifiers in first order logic when the propositional base is weaker than classical or intuitionistic. Time permitting, we discuss relational semantics for first order co-intuitionistic and distributive propositional based logic as case studies, and show that the expected meaning of the quantifiers drastically changes unless extra assumptions are added which are hard to justify. We also

discuss the modal logics obtained from this process and compare them with those of [4, 11]. Finally we present general polarity-based semantics for first order LE-logics and discuss possible interpretations.

## References

- [1] S. Balco, G. Greco, A. Kurz, M. A. Moshier, A. Palmigiano, and A. Tzimoulis. First order logic properly displayed. *in preparation*.
- [2] A. Ciabattoni, R. Ramanayake, and H. Wansing. Hypersequent and display calculi—a unified perspective. *Studia Logica*, 102(6):1245–1294, 2014.
- [3] P. Cintula and C. Noguera. A Henkin-style proof of completeness for first-order algebraizable logics. *The Journal of Symbolic Logic*, 80(1):341–358, 2015.
- [4] G. Corsi. A unified completeness theorem for quantified modal logics. *The Journal of Symbolic Logic*, 67(4):1483–1510, 2002.
- [5] S. Frittella, G. Greco, A. Kurz, A. Palmigiano, and V. Sikimić. Multi-type display calculus for dynamic epistemic logic. *Journal of Logic and Computation*, 26(6):2017–2065, 2016.
- [6] S. Frittella, G. Greco, A. Palmigiano, and F. Yang. A multi-type calculus for inquisitive logic. In J. Väänänen, Å. Hirvonen, and R. de Queiroz, editors, *Logic, Language, Information, and Computation*, LNCS 9803, pages 215–233. Springer, 2016.
- [7] D. Gabbay, V. Shehtman, and D. Skvortsov. *Quantification in Nonclassical Logic*. Studies in Logic and the Foundations of Mathematics. Elsevier, 2009.
- [8] J. W. Garson. Quantification in modal logic. In *Handbook of philosophical logic*, pages 267–323. Springer, 2001.
- [9] G. Greco and A. Palmigiano. Linear logic properly displayed. *Submitted. ArXiv preprint 1611.04181*.
- [10] P. R. Halmos. Polyadic Boolean algebras. *Proceedings of the National Academy of Sciences*, 40(5):296–301, 1954.
- [11] K. Kishida. Neighborhood-sheaf semantics for first-order modal logic. *Electronic Notes in Theoretical Computer Science*, 278:129–143, 2011.
- [12] F. W. Lawvere. Functorial semantics of elementary theories. In *Journal of symbolic logic*, volume 31, page 294, 1966.
- [13] F. W. Lawvere. Adjointness in foundations. *Dialectica*, 1969.
- [14] F. W. Lawvere. Equality in hyperdoctrines and comprehension schema as an adjoint functor. *Proceedings of the AMS Symposium on Pure Mathematics XVII*, pages 1–14, 1970.
- [15] F. Montagna. Storage operators and multiplicative quantifiers in many-valued logics. *J. Log. Comput.*, 14(2):299–322, 2004.
- [16] A. Mostowski. Axiomatizability of some many valued predicate calculi. *Fundamenta mathematicae*, 50:165–190, 1961.
- [17] H. Ono. Algebraic semantics for predicate logics and their completeness. *Logic at work. Stud. Fuzziness Soft Comput*, 24:637–650, 1999.
- [18] G. Restall. Constant domain quantified modal logics without Boolean negation. 2005.
- [19] A. Tzimoulis. *Algebraic and Proof-Theoretic Foundations of the Logics for Social Behaviour*. PhD thesis, Delft University of Technology, 2018.
- [20] H. Wansing. *Displaying modal logic*, volume 3. Springer, Trends in Logic, 1998.
- [21] H. Wansing. Predicate logics on display. *Studia Logica*, 62(1):49–75, 1999.

# Algebraic proof theory for LE-logics

Giuseppe Greco, Peter Jipsen, Fei Liang, Alessandra Palmigiano, and  
Apostolos Tzimoulis

This presentation reports on the results of [9] where we extend the research programme in algebraic proof theory from axiomatic extensions of the full Lambek calculus to logics algebraically captured by certain varieties of normal lattice expansions (normal LE-logics). Specifically, we generalise the *residuated frames* in [7] to arbitrary signatures of normal lattice expansions. This generalization is a valuable tool for proving key properties of LE-logics in full uniformity. We prove semantic cut elimination for the display calculi D.LE associated with the basic normal LE-logics and their axiomatic extensions with analytic inductive axioms. We also prove the finite model property for each such calculus D.LE, as well as for its extensions with analytic structural rules satisfying certain additional properties.

Algebraic proof theory [3] is a research area aimed at establishing systematic connections between results in structural proof theory (such as cut elimination theorems) and in algebraic logic (such as representation theorems for classes of algebras). While results of each type have been traditionally formulated and developed independently from the other type, algebraic proof theory aims at realizing a deep integration of these fields. The main results in algebraic proof theory have been obtained for axiomatic extensions of the full Lambek calculus, and, building on the work of many authors [1, 4, 12, 8, 3, 7], establish a systematic connection between a strong form of cut elimination for certain substructural logics (on the proof-theoretic side) and the closure of their corresponding varieties of algebras under MacNeille completions (on the algebraic side). Specifically, given a cut eliminable sequent calculus for a basic logic (e.g. the full Lambek calculus), a core question in structural proof theory concerns the identification of axioms which can be added to the given basic logic so that the resulting axiomatic extensions can be captured by calculi which are again cut eliminable. This question is very hard, since the cut elimination theorem is notoriously a very fragile result. However, in [2, 3] a very satisfactory answer is given to this question for substructural logics, by identifying a hierarchy  $(\mathcal{N}_n, \mathcal{P}_n)$  of axioms in the language of the full Lambek calculus, referred to as the *substructural hierarchy*, and guaranteeing that, up to the level  $\mathcal{N}_2$ , these axioms can be effectively transformed into special structural rules (called *analytic*) which can be safely added to a cut eliminable calculus without destroying cut elimination. Algebraically, this transformation corresponds to the possibility of transforming equations into equivalent quasiequations, and remarkably, such a transformation (which we will expand on shortly) is also key to proving preservation under MacNeille completions and canonical extensions.

The second major contribution of algebraic proof theory is the identification of the algebraic essence of cut elimination (for substructural logics) in the relationship between a certain polarity-based relational structure (residuated frame)  $\mathbb{W}$  arising from the given sequent calculus, and a certain ordered algebra  $\mathbb{W}^+$  which can be viewed as the complex algebra of  $\mathbb{W}$  by analogy with modal logic. Specifically, the fact that the calculus is cut-free is captured semantically by  $\mathbb{W}$  being an *intransitive* structure, while  $\mathbb{W}^+$  is by construction an ordered algebra, on which the cut rule is sound. Hence, in this context, cut elimination is encoded in the preservation of validity from  $\mathbb{W}$  to  $\mathbb{W}^+$ . For instance, the validity of analytic structural rules/quasiequations is preserved from  $\mathbb{W}$  to  $\mathbb{W}^+$  (cf. [3]), which shows that analytic structural rules can indeed be safely added to the basic Lambek calculus in a way which preserves its cut elimination.

In [7], residuated frames are introduced. Much in the same way as Kripke frames for modal logic, residuated frames provide relational semantics for substructural logics and underlie

the representation theory for the algebraic semantics of substructural logics. The algebraic proof theory program is developed in [7] by showing the existence of a connection between Gentzen-style sequent calculi for substructural logics and residuated frames, which translates into a connection between a cut-free proof system and the finite model property and the finite embeddability property for the corresponding variety of algebras.

We generalize the residuated frames of [7] and their connection with proof calculi. Specifically: we introduce *LE-frames* as the counterparts of residuated frames for arbitrary logical signatures of normal lattice expansions (LE-signatures); in particular, arbitrary signatures do not need to be closed under the residuals of each connective. We introduce *functional D-frames* as the LE-frames associated with any proper display calculus in any LE-signature; this generalization involves moving from structural rules of so-called simple shape to the more general class of analytic structural rules (cf. [10], Definition 4) in any LE-signature. Our results include the proof of semantic cut elimination for the display calculus D.LE associated with the basic normal LE-logic in any signature, the transfer of this cut elimination to extensions of D.LE with analytic structural rules, and the finite model property for D.LE and for extensions of D.LE with analytic structural rules satisfying certain additional properties. We also discuss how these results recapture the semantic cut elimination results in [3] and apply in a modular way to a range of logics which includes (analytic extensions of) the basic epistemic logic of categories [5, 6], and the Lambek-Grishin calculus [11].

## References

- [1] F. Belardinelli, P. Jipsen, and H. Ono. Algebraic aspects of cut elimination. *Studia Logica*, 77(2):209–240, 2004.
- [2] A. Ciabattoni, N. Galatos, and K. Terui. From axioms to analytic rules in nonclassical logics. In *LICS’08*, pages 229–240. IEEE, 2008.
- [3] A. Ciabattoni, N. Galatos, and K. Terui. Algebraic proof theory for substructural logics: cut-elimination and completions. *Annals of Pure and Applied Logic*, 163(3):266–290, 2012.
- [4] A. Ciabattoni and K. Terui. Towards a semantic characterization of cut-elimination. *Studia Logica*, 82(1):95–119, 2006.
- [5] W. Conradie, S. Frittella, A. Palmigiano, M. Piazzai, A. Tzimoulis, and N. Wijnberg. Categories: How I Learned to Stop Worrying and Love Two Sorts. In *Proc. WoLLIC 2016*, volume 9803 of *LNCS*, pages 145–164, 2016.
- [6] W. Conradie, S. Frittella, A. Palmigiano, M. Piazzai, A. Tzimoulis, and N. Wijnberg. Toward an epistemic-logical theory of categorization. In *Proc. TARK 2017*, volume 251 of *EPTCS*, pages 167–186, 2017.
- [7] N. Galatos and P. Jipsen. Residuated frames with applications to decidability. *Transactions of the American Mathematical Society*, 365(3):1219–1249, 2013.
- [8] N. Galatos and H. Ono. Cut elimination and strong separation for substructural logics: an algebraic approach. *Annals of Pure and Applied Logic*, 161(9):1097–1133, 2010.
- [9] G. Greco, P. Jipsen, F. Liang, A. Palmigiano, and A. Tzimoulis. Algebraic proof theory for LE-logics. *submitted, arXiv preprint arXiv:1808.04642*, 2018.
- [10] G. Greco, M. Ma, A. Palmigiano, A. Tzimoulis, and Z. Zhao. Unified correspondence as a proof-theoretic tool. *Journal of Logic and Computation*, 28(7):1367–1442, 2018.
- [11] V. N. Grishin. On a generalization of the Ajdukiewicz-Lambek system. *Studies in nonclassical logics and formal systems*, pages 315–334, 1983.
- [12] K. Terui. Which structural rules admit cut elimination? An algebraic criterion. *The Journal of Symbolic Logic*, 72(3):738–754, 2007.



# Proof theory and semantics for structural control

Giuseppe Greco<sup>1</sup>, Michael Moortgat<sup>1</sup>, and Apostolos Tzimoulis<sup>2</sup>

<sup>1</sup> University of Utrecht, the Netherlands

<sup>2</sup> Vrije Universiteit Amsterdam, the Netherlands

In the tradition of ‘parsing as deduction’, various logical calculi have been considered for applications in formal linguistics. In recent years a line of research has emerged focusing on the analysis of logical systems specifically designed to model a controlled linguistic resource management [13, 12, 14, 15, 10, 1, 18]. Research on structural control and substructural logics is also motivated by applications in other domains and has given rise to a rich literature in logic (see [6, 9, 11, 4, 17]). In particular, [16] introduces a language expansion of the non-associative Lambek calculus NL with a set of unary adjoint operators and a further generalization with a set of residuated  $n$ -ary connectives (see also [2]). The general strategy is to define a multi-modal logic where linguistic composition is relativized to specific resource management *modes* via a language expansion of the basic logic, and the extra expressivity needed for linguistics applications is obtained in a controlled fashion via the addition of *interaction postulates* between modes. Notably, the extra expressivity can be used to *licence* or to *block* the access to different regimes of resource management (see also [12]).

In the present proposal we introduce the class of *multi-type* logics for explicit structural control management (mSCLs) together with their algebraic and relational semantics, and provide proper display calculi for the basic mSCLs and their analytic axiomatic extensions in a modular fashion (e.g. preserving completeness, subformula property and cut elimination) according to the general methodology emerged in the field of structural and algebraic proof theory [3, 5, 7, 8]; in particular, we show how all the logics considered in [16] and related work, when recast as mSCLs, can profit from the pleasant proof-theoretic and model-theoretic benefits that the multi-type approach brings with it.

For each  $i \in I$ , a *heterogeneous structural control algebra* is a structure

$$\mathbb{H} := (G, S_i, B, \diamond_i, \blacksquare_i, \square_i, \blacklozenge_i, \blacklozenge_i, \blacklozenge_i, \blacklozenge_i, \mathcal{F}, \mathcal{G}, \mathcal{F}_B, \mathcal{G}_B, \leq_G, \leq_{S_i}, \leq_B)$$

such that

- $\mathbb{G} := (G, \leq_G, \mathcal{F}, \mathcal{G})$  is a partially ordered algebra,  $\mathcal{F}$  (resp.  $\mathcal{G}$ ) is a set of maps from  $\mathbb{G}^n$  to  $\mathbb{G}$  for some natural number  $n$ , and for each map in  $\mathcal{F} \cup \mathcal{G}$  the corresponding adjoint/residual is also in  $\mathcal{F} \cup \mathcal{G}$ ;
- $(S_i, \leq_{S_i})$  is a partial order;  $\blacksquare_i : G \rightarrow S_i$ ,  $\diamond_i : S_i \hookrightarrow G$ , and  $\diamond_i \dashv \blacksquare_i$ ;  $\blacklozenge_i : G \rightarrow S_i$ ,  $\square_i : S_i \hookrightarrow G$ , and  $\blacklozenge_i \dashv \square_i$ ;
- the composition  $\diamond_i \blacksquare_i : G \rightarrow G$  (resp.  $\square_i \blacklozenge_i : G \rightarrow G$ ) defines an interior operator (resp. a closure operator); the compositions  $\blacksquare_i \diamond_i : S_i \rightarrow S_i$  and  $\blacklozenge_i \square_i : S_i \rightarrow S_i$  define identity on  $S_i$ ;
- $(B, \leq_B)$  is a partial order;  $\blacksquare : G \hookrightarrow B$ ,  $\diamond : B \hookrightarrow G$ , and  $\diamond \dashv \blacksquare$ ;  $\square : G \hookrightarrow B$ ,  $\blacklozenge : B \hookrightarrow G$ , and  $\blacklozenge \dashv \square$ ; moreover, for each map  $f \in \mathcal{F}$  (resp.  $g \in \mathcal{G}$ ) with domain  $\mathbb{G}^n$ , there exists a map  $\mathcal{F}_B \ni f_B : \mathbb{B} \times \mathbb{G}^{n-1} \rightarrow \mathbb{G}$  (resp.  $\mathcal{G}_B \ni g_B : \mathbb{B} \times \mathbb{G}^{n-1} \rightarrow \mathbb{G}$ ), and for each map in  $\mathcal{F}_B \cup \mathcal{G}_B$  the corresponding adjoint/residual is also in  $\mathcal{F}_B \cup \mathcal{G}_B$ ;
- the composition  $\diamond \blacksquare : G \rightarrow G$  and  $\square \blacklozenge : G \rightarrow G$  define identity on  $G$ ; the compositions  $\blacksquare \diamond : B \rightarrow B$  and  $\blacklozenge \square : B \rightarrow B$  define identity on  $B$ .



$G$  is the set of *general* elements, the elements that inhabit the more restrictive regime. The sort  $(S_i, \leq_{S_i})$  is a set of elements that witness the licence of a *special* (more liberal) regime. The indexed unary heterogeneous operators are the licensing modalities and therefore they play a rôle in each interaction postulate: they identify special elements in the general regime/type modulo the composition of adjoint pairs. E.g. in the expanded signature of the Lambek calculus the postulate (A)  $(x \otimes y) \otimes \diamond_a \alpha \leq_G x \otimes (y \otimes \diamond_a \alpha)$  represents a controlled form of left-to-right associativity. The  $x, y$  here are general elements and  $\diamond_a \alpha$  is the image of a special element  $\alpha$  which then *licenses* the restructuring. Complementary to this licensing use of the control operators, the sort  $(B, \leq_B)$  is a set of elements that provide the room to *block* structural transformations that would otherwise be allowed. The non-indexed unary operators  $\square$  and  $\diamond$  plus the  $n$ -ary heterogeneous operators in  $\mathcal{F}_B \cup \mathcal{G}_B$  guarantee the necessary expressivity to make the needed type distinctions. E.g we could impose the interaction postulate (E)  $x \otimes \diamond_e \alpha \leq_G \diamond_e \alpha \otimes x$  (where  $x$  is general and  $\alpha$  is special) without imposing  $x \otimes_{B_1} \beta \leq_G \beta \otimes_{B_2} x$  (where  $x$  is general,  $\beta$  is a blocking element,  $\otimes_{B_1} : G \times B \rightarrow G$ , and  $\otimes_{B_2} : B \times G \rightarrow G$ ).

In the full paper, we show how the multi-type approach leads to elegant analyses of a number of motivating examples for structural control in grammatical type logics.

## References

- [1] G. Barry and G. Morrill, editors. *Studies in Categorical Grammar*, volume 5 of CCS. Edinburgh Working Papers in Cognitive Science, Edinburgh, 1990.
- [2] W. Buszkowski. Interpolation and FEP for logics of residuated algebras. *L.J. of IGPL*, 19(3):437–454, 2010.
- [3] A. Ciabattoni, N. Galatos, and K. Terui. Algebraic proof theory for substructural logics: cut-elimination and completions. *Annals of Pure and Applied Logic*, 163(3):266–290, 2012.
- [4] V. de Paiva and H. Eades. Dialectica categories for the Lambek calculus. In *International Symposium on Logical Foundations of Computer Science*, pages 256–272. Springer, 2018.
- [5] N. Galatos and P. Jipsen. Residuated frames with applications to decidability. *Transactions of the American Mathematical Society*, 365(3):1219–1249, 2013.
- [6] J.Y. Girard. Linear logic. *Theoretical Computer Science*, 50(1):1–101, 1987.
- [7] G. Greco, P. Jipsen, F. Liang, A. Palmigiano, and A. Tzimoulis. Algebraic proof theory for LE-logics. *submitted, arXiv preprint arXiv:1808.04642*, 2018.
- [8] G. Greco, M. Ma, A. Palmigiano, A. Tzimoulis, and Z. Zhao. Unified correspondence as a proof-theoretic tool. *Journal of Logic and Computation*, page exw022, 2016.
- [9] G. Greco and A. Palmigiano. Linear logic properly displayed. Submitted. ArXiv: 1611.04184.
- [10] M. Hepple. Labelled deduction and discontinuous constituency. In M. Abrusci, C. Casadio, and M. Moortgat, editors, *Linear Logic and Lambek Calculus*, Proceedings 1993 Rome Workshop, pages 123–150. ILLC, Amsterdam, 1993.
- [11] B. Jacobs. Semantics of weakening and contraction. *Annals of Pure and Applied Logic*, 69(1):73–106, 1994.
- [12] N. Kurtonina and M. Moortgat. Structural control. *Specifying syntactic structures*, pages 75–113, 1997.
- [13] M. Moortgat. Categorical type logics. In J. van Benthem, editor, *Handbook of logic and language*, chapter 2. Elsevier, 1997.
- [14] M. Moortgat and G. Morrill. Heads and phrases. Type calculus for dependency and constituent structures. *Ms OTS Utrecht*, 1991.
- [15] M. Moortgat and R.T. Oehrle. Adjacency, dependency and order. In P. Dekker and M. Stokhof, editors, *Proceedings Ninth Amsterdam Colloquium*, pages 447–466. ILLC, 1994.
- [16] Michael Moortgat. Multimodal linguistic inference. *Journal of Logic, Language and Information*, 5(3-4):349–385, 1996.
- [17] Y. Venema. Meeting strength in substructural logics. *Studia Logica* 54, 54:3–32, 1995.
- [18] K. Versmissen. Categorical grammar, modalities and algebraic semantics. *Proceedings EACL93*, pages 377–383, 1996.

# On the variety of Gödel-MV-algebras

Antonio Di Nola<sup>1</sup>, Revaz Grigolia<sup>2</sup>, and Gaetano Vitale<sup>1</sup>

<sup>1</sup>University of Salerno, Salerno, Italy  
[adinola@unisa.it](mailto:adinola@unisa.it), [gvitale@unisa.it](mailto:gvitale@unisa.it)

<sup>2</sup>Tbilisi State University, Tbilisi, Georgia  
[revaz.grigolia@tsu.ge](mailto:revaz.grigolia@tsu.ge)

A "symmetric" formulation of intuitionistic propositional calculus, suggested by various authors (G. Moisil, A. Kuznetsov, C. Rauszer), presupposes that any connective  $\&$ ,  $\vee$ ,  $\rightarrow_G$ ,  $\top$ ,  $\perp$  has its dual  $\vee$ ,  $\&$ ,  $\rightarrow_{Br}$ ,  $\perp$ ,  $\top$ , and the duality principle of the classical logic is restored. The notion of double-Brouwerian algebras was introduced by J. McKinsey and A. Tarski in [MT], based on the idea considered by T. Skolem in 1919.

MV-algebras were introduced by Chang in [Ch] as an algebraic model for infinitely valued Łukasiewicz logic. An MV-algebra is an algebra  $(A, \otimes, \oplus, *, 0, 1)$  where  $(A, \oplus, 0)$  is an abelian monoid, and the following identities hold for all  $x, y \in A$ :

$$x \oplus 1 = 1, x^{**} = x, 0^* = 1, x \oplus x^* = 1, x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x, x \otimes y = (x^* \oplus y^*)^*.$$

Every MV-algebra has an underlying ordered structure defined by  $x \leq y$  iff  $x^* \oplus y = 1$  which is a lattice order on  $A$ . The unit interval of real numbers  $[0, 1]$  endowed with the following operations:  $x \oplus y = \min(1, x + y)$ ,  $x \otimes y = \max(0, x + y - 1)$ ,  $x^* = 1 - x$ , becomes an MV-algebra. From these operations are defined the lattice operations  $x \vee y = \max(x, y) = (x \otimes y^*) \oplus y$  and  $x \wedge y = \min(x, y) = (x^* \oplus y) \otimes x$ . It is well known that the MV-algebra  $S = ([0, 1], \otimes, \oplus, *, 0, 1)$  generates the variety  $\mathbf{MV}$  of all MV-algebras, i. e.  $V(S) = \mathbf{MV}$ . The algebra  $S_n = (\{0, 1/n, \dots, n-1/n, 1\}, \otimes, \oplus, *, 0, 1)$  generates the subvariety  $\mathbf{MV}_n$ , the algebras of which is called MV<sub>n</sub>-algebras [Gr], i. e.  $V(S_n) = \mathbf{MV}_n$ . Notice that  $\mathbf{MV} = V(\bigcup_n \mathbf{MV}_n)$ .

A Heyting algebra  $(A, \vee, \wedge, \rightarrow_H, 0, 1)$  is a bounded distributive lattice where the implication  $\rightarrow_H$  is adjoint to the lattice operation infimum  $\wedge$ . A Brouwerian algebra  $(A, \vee, \wedge, \rightarrow_{Br}, 0, 1)$  is a bounded distributive lattice where the co-implication (relatively pseudo-difference)  $\rightarrow_{Br}$  is adjoint to the lattice operation supremum  $\vee$ . If for bounded distributive lattice  $(A, \vee, \wedge, 0, 1)$  there exist both implication  $\rightarrow_H$  and relatively pseudo-difference  $\rightarrow_{Br}$  then it is named *bi-Heyting (Heyting-Brouwerian)* algebra. A Gödel algebra  $(A, \vee, \wedge, \rightarrow_G, 0, 1)$  is a Heyting algebra satisfying the identity  $(x \rightarrow_G y) \vee (y \rightarrow_G x) = 1$ .

We introduce a new algebra  $(A, \otimes, \oplus, *, \vee, \wedge, \rightarrow_G, 0, 1)$  called *Gödel-MV-algebra (GMV-algebra)* if  $(A, \otimes, \oplus, *, 0, 1)$  is MV-algebra and  $(A, \vee, \wedge, \rightarrow_G, 0, 1)$  is a Gödel algebra. In other words we have symbiosis of two algebras – MV-algebra and Gödel algebra.

Let  $\mathcal{S}_n = (\{0, 1/n, \dots, n-1/n, 1\}, \otimes, \oplus, *, \vee, \wedge, \rightarrow_G, 0, 1)$  be GMV<sub>n</sub>-algebra where  $(\{0, 1/n, \dots, n-1/n, 1\}, \otimes, \oplus, *, 0, 1) = S_n$  and  $(\{0, 1/n, \dots, n-1/n, 1\}, \vee, \wedge, \rightarrow_G, 0, 1)$  is a Gödel algebra. Let  $\mathbf{GMV}_n$  be the variety generated by the family  $\{\mathcal{S}_k : 1 \leq k \leq n\}$ .

Taking into account that Łukasiewicz implication  $x \Rightarrow y = x^* \oplus y$  is expressible in MV-algebra, we have two distinct residuations. Moreover, the relatively pseudo-difference  $b \rightarrow_{Br} a = (a^* \rightarrow_G b^*)^*$  and Brouwerian negation  $\neg a = (-a^*)^* = 1 \rightarrow_{Br} a$ .

Let  $A = (A, \otimes, \oplus, *, \vee, \wedge, \rightarrow_G, 0, 1)$  be *GMV*-algebra. Denote by  $A^\circ = (A, \otimes, \oplus, *, 0, 1)$  the *MV*-reduct of the *GMV*-algebra  $A$  and by  $A^\wedge = (A, \vee, \wedge, \rightarrow_G, 0, 1)$  the Heyting-reduct of the *GMV*-algebra  $A$ .

**Theorem 1.** *Let  $(A, \otimes, \oplus, *, \vee, \wedge, \rightarrow_G, 0, 1)$  be a *GMV*-algebra. Then  $A^\wedge$  is a bi-Heyting (Heyting-Brouwerian) algebra, where the relatively pseudo-difference  $b \rightarrow_{Br} a = (a^* \rightarrow_G b^*)^*$  and Brouwerian negation  $\neg a = (\neg a^*)^* = 1 \rightarrow_{Br} a$ .*

Let  $(A, \otimes, \oplus, *, \vee, \wedge, \rightarrow_G, 0, 1)$  be a *GMV*-algebra. A subset  $F \subset A$  is said to be a *Skolem MV-filter*, if  $F$  is a *MV*-filter (i. e.  $1 \in F$ , if  $x \in F$  and  $x \leq y$ , then  $y \in F$ , if  $x, y \in F$ , then  $x \otimes y \in F$ ) and if  $x \in F$ , then  $\neg \neg x \in F$ .

**Theorem 2.** *Let  $F$  be a Skolem *MV*-filter of the *GMV*-algebra  $A$ . The equivalence relation  $x \equiv y \Leftrightarrow (x^* \oplus y) \wedge (y^* \oplus x) \in F$  is a congruence relation for the *GMV*-algebra  $A$ .*

**Theorem 3.** (i) *Any chain *GMV*-algebra  $A$  is simple.*  
(ii) *Let  $\{F_i\}_{i \in I}$  be the family of all maximal Skolem *MV*-filters of the *GMV*-algebra  $A$ . Then  $A$  is isomorphic to the subdirect product of the algebras of  $A/F_i$  ( $i \in I$ ).*  
(iii) *Any *GMV*-algebra is subdirect product of chain *GMV*-algebras.*  
(iv) *Any *GMV*-algebra is semi-simple.*

**Theorem 4.**  *$m$ -generated free *GMV*-algebra  $F_{GMV}(m)$  is isomorphic to a subalgebra of an inverse limit  $F_\infty(m)$  of a chain of order type  $\omega^*$  of finite algebras, for  $m \in \omega$ , and the finite algebras are isomorphic to the  $m$ -generated free algebra  $F_{GMV_n}(m)$  of the variety  $\mathbf{GMV}_n$ .*

**Theorem 5.** *A *GMV*-algebra  $A$  is finitely presented iff  $A \cong F_{GMV}(m)/[u]$ , where  $[u]$  is a principal Skolem *MV*-filter generated by some element  $u \in F_{GMV}(m)$ .*

## References

- [Ch] C. C. Chang, *Algebraic Analysis of Many-Valued Logics*, Trans. Amer. Math. Soc., 88(1958), 467-490.  
[Gr] R. Grigolia, *Algebraic analysis of Lukasiewicz-Tarski  $n$ -valued logical systems*, Selected papers on Lukasiewicz Sentential Calculi, Wroclaw, 81-91 (1977)  
[MT] McKinsey J. and Tarski A. (1946). *On closed elements in Closure algebras*, Ann. Math, v. 47 (122-162).

# Some theorems concerning Grzegorzcyk contact lattices

Rafał Gruszczynski and Andrzej Pietruszczak

Chair of Logic  
Nicolaus Copernicus University in Toruń  
Poland  
{gruszka,pietrusz}@umk.pl

A pair  $\mathfrak{L} = \langle R, \leq \rangle$  is a *Grzegorzcyk lattice* iff it is a lattice with zero element and satisfies the following strong polarization condition<sup>1</sup>:

$$x \not\leq y \implies \exists z \in R (z \leq x \wedge z \perp y \wedge \forall u \in R (u \leq x \wedge u \perp y) \implies u \leq z), \quad (\text{P})$$

where  $x \perp y \stackrel{\text{df}}{\iff} x \sqcap y = 0$  (with  $\sqcap$  being the standard *meet* operation). All elements from the class have the relative complement operation in  $R \times R$ :

$$x - y := \max\{z \in R \mid z \leq x \wedge z \perp y\}, \quad (\text{df } -)$$

which is well-defined thanks to (P).

A triple  $\mathfrak{C} = \langle R, \leq, \mathsf{C} \rangle$ , where  $\langle R, \leq \rangle$  is a Grzegorzcyk lattice and  $\mathsf{C} \subseteq R \times R$  satisfies:

$$0 \not\mathsf{C} x, \quad (\text{C0})$$

$$x \leq y \implies x \mathsf{C} y, \quad (\text{C1})$$

$$x \mathsf{C} y \implies y \mathsf{C} x, \quad (\text{C2})$$

$$x \leq y \wedge x \mathsf{C} z \implies y \mathsf{C} z, \quad (\text{C3})$$

will be called a *quasi-contact lattice*.<sup>2</sup> Elements of  $R$  are called regions and  $\mathsf{C}$  is a *contact* (*connection*) relation. In  $\mathfrak{C}$  we define *non-tangential inclusion* and *overlap* relations by means of the following two conditions (respectively):

$$x \ll y \stackrel{\text{df}}{\iff} \forall z \in R (z \perp y \implies z \not\mathsf{C} y), \quad x \circ y \stackrel{\text{df}}{\iff} x \sqcap y \neq 0.$$

A *pre-point* of  $\mathfrak{C}$  is a non-empty set  $X$  of regions such that:

$$0 \notin X, \quad (\text{r0})$$

$$\forall u, v \in X (u = v \vee u \ll v \vee v \ll u), \quad (\text{r1})$$

$$\bigwedge \forall u \in X \exists v \in X v \ll u, \quad (\text{r2})$$

$$\forall x, y \in R \forall u \in X (u \circ x \wedge u \circ y) \implies x \mathsf{C} y. \quad (\text{r3})$$

The purpose of this definition is to formally grasp the intuition of point as the system of diminishing regions determining the unique location in space.

<sup>1</sup>In the sequel,  $\wedge, \vee, \implies$  and  $\iff$  are standard logical connectives of conjunction, disjunction, implication and equivalence, respectively.

<sup>2</sup>The prefix ‘quasi’ is due to absence of the standard contact relation axiom:  $x \mathsf{C} y \sqcup z \implies x \mathsf{C} y \vee x \mathsf{C} z$  (where  $\sqcup$  is the standard join operation).

Let  $\mathbf{Q}$  be the set of all pre-points of  $\mathfrak{C}$ . We extend the set of axioms for  $\mathfrak{C}$  with the following postulates:

$$x \circ y \implies \exists Q \in \mathbf{Q} \exists z \in Q z \leq x \sqcap y, \quad (\text{G1})$$

$$x \mathbf{C} y \wedge x \perp y \implies \exists Q \in \mathbf{Q} \forall z \in Q (z \circ x \wedge z \circ y). \quad (\text{G2})$$

called *Grzegorzcyk axioms*, introduced in [5]. Any  $\mathfrak{C}$  which satisfies all the aforementioned axioms is called *Grzegorzcyk contact lattice* (GCL for short). Every such lattice satisfies the standard contact relation axioms (see e.g. [1]).

A *point* of GCL is any filter generated by a pre-point:

$$\mathfrak{p} \text{ is a point iff } \exists Q \in \mathbf{Q} \mathfrak{p} = \{x \in R \mid \exists q \in Q q \leq x\}.$$

In every GCL we can introduce a topology in the set of all points, first by defining the set of all internal points of a region  $x$ :

$$\mathbf{Irl}(x) := \{\mathfrak{p} \mid x \in \mathfrak{p}\},$$

and second, taking all  $\mathbf{Irl}(x)$  as a basis. As a result we obtain a concentric topological space (see. [2, 4]).

The following set-theoretical representation theorem holds for GCLs:

**Theorem 1.** *If  $\mathfrak{C}$  is a GCL, then  $\mathbf{Irl}[\mathfrak{C}]$  is a GCL and:*

1.  $\mathbf{Irl}$  is an isomorphism of  $\mathfrak{C}$  onto  $\mathbf{Irl}[\mathfrak{C}]$ ,
2. the operation  $\mathbf{Irl}$  is a reduced and perfect representation of  $\mathfrak{C}$ ,
3.  $\mathfrak{C}$  is complete iff  $\mathbf{Irl}[\mathfrak{C}]$  is complete.

In our talk we would like to sketch a proof of the above theorem and give a full characterization of finite Grzegorzcyk contact lattices, in particular we would like to show that:

**Theorem 2.** *A Grzegorzcyk contact lattice  $L$  is finite iff it is finite as a lattice and  $\mathbf{C} = \circ$ .*

**Theorem 3.** *A Grzegorzcyk contact lattice  $L$  is finite iff it is complete and the set of Grzegorzcyk points coincides with the set of maximal filters.*

All results from the talk can be found in [2, 3, 4].

The presentation is a continuation of presentations given at TACL 2009, 2015 and SYSMICS 2019 conferences.

## References

- [1] B. Bennett and I. Düntsch. Axioms, algebras and topology. In J. Van Benthem M. Aiello, I. Pratt-Hartmann, editor, *Handbook of Spatial Logics*, chapter 3, pages 99–159. Springer, 2007.
- [2] R. Gruszczyński. *Non-standard theories of space*. Nicolaus Copernicus University Scientific House, Toruń, 2016.
- [3] R. Gruszczyński and A. Pietruszczak. A study in Grzegorzcyk point-free topology. Part I: Separation and Grzegorzcyk structures. *Studia Logica*, 106(6):1197–1238, 2018.
- [4] R. Gruszczyński and A. Pietruszczak. A study in Grzegorzcyk point-free topology. Part II: Spaces of points. *Studia Logica*, 2018.
- [5] A. Grzegorzcyk. Axiomatizability of geometry without points. *Synthese*, 12(2–3):228–235, 1960.

# Computing the validity degree in Łukasiewicz logic

Zuzana Haniková

Institute of Computer Science of the Czech Academy of Sciences

hanikova@cs.cas.cz

Łukasiewicz logic ( $\mathbf{L}$ ) is an important and familiar example of a many-valued logic; a vast amount of knowledge has been accumulated on its algebraic semantics, provided by the class of MV-algebras. Semantic investigations allowed a fairly good understanding of the computational aspects of the logic, with a range of results. The propositional logic (tautologousness and provability from finite theories) is known to be coNP complete, a result that follows Mundici’s approach to determining the complexity of the SAT problem for  $[0, 1]_{\mathbf{L}}$ , the standard MV-algebra [10]. First-order standard tautologies turned out to be complete for the class  $\Pi_2$  of the arithmetical hierarchy [13], and also the monadic fragment of first-order logic (both general and standard semantics) is undecidable, a result due to Bou. Admissible rules of Łukasiewicz logic are decidable in polynomial space, and complete therein [7, 8]. One can further show, e.g., that the set of formulas that are positively, but not fully, satisfiable in the standard semantics is complete for the class DP [6], or that the set of prenex formulas with one existential propositional quantifier valid in the standard semantics belongs to the class  $\Pi_2^P$  of the polynomial hierarchy [2].

The present work looks at an optimization problem in (propositional) Łukasiewicz logic. To introduce the problem, we consider a conservative expansion of Łukasiewicz logic with constants for the rational elements of the standard semantics and the bookkeeping axioms; this logic is sometimes called Rational Pavelka logic (RPL) [4, 3, 12]. Pavelka completeness theorem states that for  $T$  a theory and  $\varphi$  a formula in RPL, the provability degree of  $\varphi$  in  $T$  equals its validity degree, i.e., one has

$$|\varphi|_T = \sup\{r \mid T \vdash_{\text{RPL}} r \rightarrow \varphi\} = \inf\{v(\varphi) \mid v \text{ model of } T\} = \|\varphi\|_T$$

where, in the definition of the provability degree, the  $r$ ’s range in the rationals in  $[0, 1]$ , while the assignments  $v$  used to define the validity degree are taken in the standard MV-algebra. Computing the validity degree appears easier than directly obtaining results on the provability degree, because for the former, one can appeal to the standard semantics. In fact [4] shows that for a finite  $T$ , the validity degree is attained on a rational assignment (and is therefore rational); it follows also that the denominators are of polynomial size in  $|T|$  and  $|\varphi|$ . Another reason for working with the validity degree is that the definition applies easily even in Łukasiewicz logic without any new rational constants. For a finite  $T$ , computing the validity degree is a natural optimization problem.

## ValDegree

Instance: a finite theory  $T$  and a formula  $\varphi$  of RPL (with or without rational constants).

Output:  $\|\varphi\|_T$ .

It is shown in [5] that provability from finite theories in RPL, taken as a decision problem, is coNP complete (so expansion with rational constants does not affect the complexity of Łukasiewicz logic). In particular, the problem “given  $T$ ,  $\varphi$ , and a rational  $r$ , does  $T \vdash_{\text{RPL}} r \rightarrow \varphi$ ?” is coNP complete.

We first establish an expectable result: on input  $T$  and  $\varphi$ , one can compute, in time polynomial in the input size, the validity degree  $\|\varphi\|_T$  if one has as oracle a subroutine solving the decision version of the problem; this is achieved by binary search, taking into account the above observation about small denominators. In other words, there is a Turing reduction of the optimization to the

decision version of the problem, analogously to other optimization problems such as TSP (cf. [11]). This puts **ValDegree** in  $\text{FP}^{\text{NP}}$ . We note [1] provides tight upper bounds for denominators needed to demonstrate that an L-formula is not a tautology of the standard semantics.

A lower bound for **ValDegree** can be established by reducing another optimization problem to it, via a metric reduction (a many-one polynomial-time reduction suitable for optimization problems). We provide a metric reduction from the **Weighted-MaxSAT** problem: “for a Boolean CNF formula  $\varphi(x_1, \dots, x_k)$  with (binary) weights on clauses  $w_1, \dots, w_n$ , maximize the sum of weights of true clauses over all assignments to variables in  $\varphi$ ”, to **ValDegree**. The problem **Weighted-MaxSAT** has been proved complete for the classes  $\text{FP}^{\text{NP}}$  and OptP in [9].

## References

- [1] Stefano Aguzzoli. An asymptotically tight bound on countermodels for Łukasiewicz logic. *International Journal of Approximate Reasoning*, 43(1):76–89, 2006.
- [2] Stefano Aguzzoli and Daniele Mundici. Weirstrass approximation theorem and Łukasiewicz formulas. In Melvin Chris Fitting and Ewa Orłowska, editors, *Beyond Two: Theory and Applications of Multiple-Valued Logic*, volume 114 of *Studies in Fuzziness and Soft Computing*, pages 251–272, Heidelberg, 2003. Physica-Verlag.
- [3] Joseph Amadee Goguen. The logic of inexact concepts. *Synthese*, 19(3–4):325–373, 1969.
- [4] Petr Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer, Dordrecht, 1998.
- [5] Petr Hájek. Computational complexity of t-norm based propositional fuzzy logics with rational truth constants. *Fuzzy Sets and Systems*, 157(5):677–682, 2006.
- [6] Zuzana Haniková and Petr Savický. Term satisfiability in  $\text{FL}_{ew}$ -algebras. *Theoretical Computer Science*, 631:1–15, 2016.
- [7] Emil Jeřábek. Admissible rules of Łukasiewicz logic. *Journal of Logic and Computation*, 20(2):425–447, 2010.
- [8] Emil Jeřábek. The complexity of admissible rules of Łukasiewicz logic. *Journal of Logic and Computation*, 23(3):693–705, 2013.
- [9] Mark W. Krentel. The complexity of optimization problems. *Journal of Computer and System Sciences*, 36:490–509, 1988.
- [10] Daniele Mundici. Satisfiability in many-valued sentential logic is NP-complete. *Theoretical Computer Science*, 52(1–2):145–153, 1987.
- [11] Christos H. Papadimitriou. *Computational Complexity*. Theoretical Computer Science. Addison Wesley, 1993.
- [12] Jan Pavelka. On fuzzy logic I, II, III. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 25:45–52, 119–134, 447–464, 1979.
- [13] Matthias Emil Ragaz. *Arithmetische Klassifikation von Formelmengen der unendlichwertigen Logik*. PhD thesis, Swiss Federal Institute of Technology, Zürich, 1981.

# Remarks on the Topos Approach to Quantum Mechanics

John Harding<sup>1</sup> and Chris Heunen<sup>2</sup>

<sup>1</sup> New Mexico State University, Las Cruces, New Mexico, USA  
`jharding@nmsu.edu`

<sup>2</sup> University of Edinburgh, Edinburgh, Scotland, U.K.  
`chris.heunen@ed.ac.uk`

Our central theme of study in this note is the topos approach to quantum mechanics that was introduced by Isham and Butterfield [14], and continued by others including Döring and Flori, see [2, 3, 4]. The two books by Flori [5, 6] are the most current source. This approach has also spurred a related topos approach [11, 12, 13, 17].

The idea of this topos approach is to take the von Neumann algebra  $\mathcal{N}$  used to represent a quantum system, form its poset of abelian von Neumann subalgebras  $\mathcal{V}(\mathcal{N})$ , and then take the topos of presheaves on this poset as the fundamental ingredient. The philosophy is that quantum mechanics behaves classically if one restricts attention “locally”, that is to pieces consisting of observables one can measure at the same time, the abelian subalgebras. The topos approach is a machinery to paste together these local “snapshots” to form the whole picture. For instance, the spectral presheaf  $\Sigma$  associates to each  $V \in \mathcal{V}(\mathcal{N})$  its Gelfand spectrum  $\Sigma_V$ . This spectral presheaf is the analog of a classical state space. Events are classically certain subsets of the state space, and in the topos approach these correspond to clopen subobjects of the spectral presheaf  $\Sigma$ , that is, subobjects of  $\Sigma$  where each component is a clopen subset of the Gelfand spectrum  $\Sigma_V$ . Many aspects of quantum theory are developed in this approach such as events, observables, states, and automorphisms. The Kochen-Specker Theorem [15] that it is impossible to assign simultaneous truth values to all questions of a quantum system is equivalent to the spectral presheaf  $\Sigma$  failing to have a global element.

Several results are of interest in looking at this topos approach from afar. First, the structure of the poset  $\mathcal{V}(\mathcal{N})$  determines the von Neumann algebra up to Jordan isomorphism [1], a result that has various extensions [8, 9, 10, 16]. Most pertinent here is the result of [9] where it is shown that the poset  $\mathcal{V}(\mathcal{N})$  is determined by its subposet  $\mathcal{V}(\mathcal{N})^*$  of elements of height at most two, and that these posets  $\mathcal{V}(\mathcal{N})^*$  have representations reminiscent of projective geometries consisting of points and lines. These results are extended in [9] to provide a near categorical equivalence between orthomodular posets and certain structures that closely resemble projective geometries and the morphisms between them. This can be viewed as a version of Greechie diagrams [7] that treats arbitrary orthomodular posets and the morphisms between them.

Our purpose here, is to turn these results back to the original focus, that of the topos treatment of quantum mechanics. Rather than consider the topos of presheaves over  $\mathcal{V}(\mathcal{N})$ , we consider the topos of presheaves over  $\mathcal{V}(\mathcal{N})^*$ . We show that the ingredients of [2, 5, 6] hold in this altered setting. The point is that the topos approach embeds quantum mechanics into the topos of presheaves, but presheaves without physical importance abound. The same is true whether one works in the topos over  $\mathcal{V}(\mathcal{N})$  or its short version  $\mathcal{V}(\mathcal{N})^*$ .

Not only can one achieve the same result working with presheaves over the short poset  $\mathcal{V}(\mathcal{N})^*$ , it provides considerable simplification. For example, each  $V \in \mathcal{V}(\mathcal{N})^*$  has a Boolean algebra of projections with at most 8 elements. So the spectral presheaf  $\Sigma^*$  over  $\mathcal{V}(\mathcal{N})^*$  associates to such  $V$  its spectrum which is a set with at most 3 elements. The Stone topology vanishes into discreteness. The result that the clopen subobjects of  $\Sigma^*$  form a Heyting algebra is then a triviality from topos theory, they are all subobjects. Similar comments apply even moreso to various other aspects of the topos approach such as observables, states, and automorphisms.



This also points to a question related to the view expressed in [2] that the internal logic of the topos of presheaves over  $\mathcal{V}(\mathcal{N})$  captures a tangible aspect of quantum mechanics. If this is the case, why this logic, rather than that of the topos of presheaves over  $\mathcal{V}(\mathcal{N})^*$ ?

## References

- [1] A. Döring and J. Harding, *Abelian subalgebras and the Jordan structure of a von Neumann algebra*, Houston J. Math. 42(2):559–568, 2016.
- [2] A. Döring and C. J. Isham, ‘*What is a thing?*’: *topos theory in the foundations of physics*, New structures for physics, 753?937, Lecture Notes in Phys., 813, Springer, Heidelberg, 2011.
- [3] A. Döring, A. and C. J. Isham, *Classical and quantum probabilities as truth values*, J. Math. Phys. 53 (3), 2012.
- [4] A. Döring, *Flows on generalized Gelfand spectra of nonabelian unital  $C^*$ -algebras and time evolution of quantum systems*, arXiv:1212.4882v2, 2013.
- [5] C. Flori, *A first course in topos quantum theory*, Lecture Notes in Physics 868, Springer-Verlag, 2013.
- [6] C. Flori, *A second course in topos quantum theory*, Lecture Notes in Physics 944, Springer-Verlag, 2018.
- [7] R. J. Greechie, *On the structure of orthomodular lattices satisfying the chain condition*, J. Combinatorial Theory 4:210-218, 1968.
- [8] J. Hamhalter, *Isomorphisms of ordered structures of abelian  $C^*$ -subalgebras of  $C^*$ -algebras*, J. Math. Anal. Appl. 383(2):391–399, 2011.
- [9] J. Harding, C. Heunen, A. J. Lindenhovius, and M. Navara, *Boolean subalgebras of orthoalgebras*, arXiv:1711.03748, 2018.
- [10] J. Harding and M. Navara, *Subalgebras of orthomodular lattices*, Order 28(3):549–563, 2011.
- [11] C. Heunen, N. P. Landsman, and B. Spitters, *A Topos for Algebraic Quantum Theory*, Communications in Mathematical Physics 291. 63-110. 2009.
- [12] C. Heunen, N. P. Landsman, and B. Spitters, *Bohrification of Operator Algebras and Quantum Logic*, Synthese. <http://dx.doi.org/10.1007/s11229-011-9918-4>. 2011.
- [13] C. Heunen, N. P. Landsman and B. Spitters, *Bohrification. Deep Beauty: Mathematical Innovation and Research for Underlying Intelligibility in the Quantum World*, ed. Hans Halvorson. Cambridge University Press. 2011.
- [14] C. J. Isham and J. Butterfield, *A Topos perspective on the Kochen-Specker theorem. I – IV*, Internat. J. Theoret. Phys. 37:2669–2733, 1998; 38:827–859, 1999; 39:1413–1436, 2000; 41:613–639, 2002.
- [15] S. Kochen and E. P. Specker, *Logical structures arising in quantum theory*, The theory of models, ed. J. W. Addison, L. Henkin and A. Tarski, North-Holland, Amsterdam, 1965.
- [16] B. Lindenhovius,  *$\mathcal{C}(A)$* , Radboud University Nijmegen, PhD thesis, [hdl.handle.net/2066/158429](http://hdl.handle.net/2066/158429), 2016.
- [17] S. Wolters, *A comparison of two topos-theoretic approaches to quantum theory*, arXiv:1010.2031, 2011.

# Hyper-MacNeille completions of Heyting algebras\*

John Harding<sup>1</sup> and Frederik Möllerström Lauridsen<sup>2</sup>

<sup>1</sup> Department of Mathematical Sciences, New Mexico State University, Las Cruces, NM 88003, U.S.A.  
[hardingj@math.nmsu.edu](mailto:hardingj@math.nmsu.edu)

<sup>2</sup> Institute for Logic, Language and Computation, University of Amsterdam, The Netherlands.  
[f.m.lauridsen@uva.nl](mailto:f.m.lauridsen@uva.nl)

Given any Heyting algebra  $\mathbf{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$  we obtain monoids  $(A, \vee, 0)$  and  $(A, \wedge, 1)$ . The direct product of these two monoids determines a monoidal structure on  $A^2$  which lifts to the power set  $\wp(A^2)$  in the evident way. The set  $N \subseteq A^2 \times A^2$  defined as,

$$(s, a)N(t, b) \iff s \vee t \vee (a \rightarrow b) = 1,$$

is a so-called *nuclear relation* on the lattice-ordered monoid  $\wp(A^2)$  and as such gives rise to a complete lattice, see, e.g., [4], which we denote by  $\mathbf{A}^+$ . It can be shown that  $\mathbf{A}^+$  is in fact a Heyting algebra and that  $\mathbf{A}$  embeds into it, making  $\mathbf{A}^+$  a completion of  $\mathbf{A}$ . This completion is (isomorphic) to the so-called *hyper-MacNeille completion* introduced by Ciabattini, Galatos, and Terui [2] in connection with their algebraic proof of cut-admissibility for a certain class of hyper-sequent calculi. In the context of Heyting algebras<sup>1</sup> they showed that any variety of Heyting algebras axiomatised by so-called  $\mathcal{P}_3$ -equations is closed under hyper-MacNeille completions [2, Thm. 6.8]. They also showed that for any subdirectly irreducible Heyting algebra  $\mathbf{A}$  the hyper-MacNeille completion  $\mathbf{A}^+$  and the MacNeille completion  $\overline{\mathbf{A}}$  coincide [2, Prop. 6.6]. Furthermore, they provided sufficient conditions for the embedding  $\mathbf{A} \hookrightarrow \mathbf{A}^+$  to be *regular*, i.e., to preserve all existing meets and joins in  $\mathbf{A}$ , [2, Prop. 6.11].

We report on work to understand the hyper-MacNeille completion of Heyting algebras from a more algebraic point of view allowing us to generalize some of the results from [2].

A bounded distributive lattice  $\mathbf{D}$  is *supplemented* if for each  $a \in D$  the equation  $a \vee x \approx 1$  has a least solution which we denote  $\sim a$ . Thus a bounded distributive lattice is supplemented iff its order dual is pseudo-complemented, cf., e.g., [1, Chap. VIII]. We call a supplemented bounded distributive lattice *De Morgan supplemented* if it satisfies the equation  $\sim(x \vee y) \approx \sim x \wedge \sim y$ , noting that the dual equation is satisfied in any supplemented bounded distributive lattice.

Examples of De Morgan supplemented Heyting algebras include all Boolean algebras and all Heyting algebras with a join irreducible top element, viz., the finitely subdirectly irreducible (fsi) Heyting algebras. In fact, a Heyting algebra is De Morgan supplemented iff it is a Boolean product of fsi Heyting algebras, cf., [7, Thm. 9.5]. For De Morgan supplemented Heyting algebras the hyper-MacNeille completion is easy to understand.

**Proposition 1.** *Let  $\mathbf{A}$  be a De Morgan supplemented Heyting algebra. Then,  $\mathbf{A}^+$  is isomorphic to  $\overline{\mathbf{A}}$ .*

In particular, we obtain that the MacNeille and the hyper-MacNeille completions coincide for finitely subdirectly irreducible Heyting algebras and for Boolean algebras.

Each prime filter  $x$  on a Heyting algebra  $\mathbf{A}$  gives rise to a congruence  $\theta_x$  on  $\mathbf{A}$  such that the quotient  $\mathbf{A}/\theta_x$  is finitely subdirectly irreducible. Letting  $M$  denote the set of prime filters of

---

\*This project has received funding from the European Unions Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 689176.

<sup>1</sup>The results from [2] applies in one form or other, *mutatis mutandis*, in the much more general setting of pointed residuated lattices or FL-algebras.

$\mathbf{A}$  which are minimal with respect to set-theoretic inclusion, it is not difficult to see that  $\mathbf{A}$  is a subdirect product of the family  $\{\mathbf{A}/\theta_x\}_{x \in M}$ . One may topologize the (disjoint) union of this family to obtain a sheaf of Heyting algebras with base space  $M$  and stalks  $\{\mathbf{A}/\theta_x\}_{x \in M}$ . By identifying *dense open sections* of this sheaf when they agree on a dense open subset of  $M$  we obtain a Heyting algebra  $Q(\mathbf{A})$  which belongs to the variety generated by  $\mathbf{A}$ . The algebra  $Q(\mathbf{A})$  is always De Morgan supplemented and completely determines the hyper-MacNeille completion of  $\mathbf{A}$  in the following sense.

**Theorem 2.** *The hyper-MacNeille completion  $\mathbf{A}^+$  of a Heyting algebra  $\mathbf{A}$  is isomorphic to the MacNeille completion of  $Q(\mathbf{A})$ .*

**Remark 3.** We note that algebras of dense open sections also play a crucial role in the study of MacNeille completions of (weak) Boolean products of lattice-based algebras, cf., [5, 3].

From Theorem 2 and Proposition 1 we obtain the following.

**Corollary 4.** *A variety  $\mathcal{V}$  of Heyting algebras is closed under hyper-MacNeille completions iff the class of De Morgan supplemented members of  $\mathcal{V}$  is closed under MacNeille completions.*

Even for supplemented Heyting algebras MacNeille completions are relatively easier to work with than for arbitrary Heyting algebras. Consequently, using Corollary 4 it is not difficult to show that many varieties of Heyting algebras are closed under hyper-MacNeille completions, e.g., the well-known varieties  $\mathcal{LC}$ ,  $\mathcal{KC}$  and  $\mathcal{BD}_2$ . Note that unlike the first two, the variety  $\mathcal{BD}_2$  cannot be axiomatized by  $\mathcal{P}_3$ -equations, see [6, Prop. 3.24].

One may show that when the size of the algebras  $\{\mathbf{A}/\theta_x\}_{x \in M}$  is uniformly bounded on a dense open subset of  $M$  then  $Q(\mathbf{A})$  is complete. Jónsson's Lemma then implies the following.

**Theorem 5.** *Any finitely generated variety of Heyting algebras is closed under hyper-MacNeille completions.*

Theorem 5 in conjunction with the characterization of varieties of Heyting algebras determined by  $\mathcal{P}_3$ -equations [6] allows us to give more examples of varieties of Heyting algebras closed under hyper-MacNeille completions but not determined by  $\mathcal{P}_3$ -equations.

Finally, from Theorem 2 it also follows that the hyper-MacNeille completion of a Heyting algebra is always De Morgan supplemented. Using this it can be shown that, at least for Heyting algebras, the sufficient condition for the hyper-MacNeille completion to be regular given by Ciabattini et al. is in fact also necessary.

## References

- [1] R. Balbes and Ph. Dwinger. *Distributive lattices*. University of Missouri Press, Columbia, Mo., 1974.
- [2] A. Ciabattini, N. Galatos, and K. Terui. Algebraic proof theory: hypersequents and hypercompletions. *Ann. Pure Appl. Logic*, 168(3):693–737, 2017.
- [3] G. D. Crown, J. Harding, and M. F. Janowitz. Boolean products of lattices. *Order*, 13(2):175–205, 1996.
- [4] N. Galatos and P. Jipsen. Residuated frames with applications to decidability. *Trans. Amer. Math. Soc.*, 365(3):1219–1249, 2013.
- [5] J. Harding. Completions of orthomodular lattices. II. *Order*, 10(3):283–294, 1993.
- [6] F. M. Lauridsen. Intermediate logics admitting a structural hypersequent calculus. *Studia Logica*, 2018. To appear.
- [7] D. J. Vaggione. Locally Boolean spectra. *Algebra Universalis*, 33(3):319–354, 1995.

## Fibrations of toposes

SINA HAZRATPOUR

(JOINT WORK WITH STEVE VICKERS)

SCHOOL OF COMPUTER SCIENCE, UNIVERSITY OF BIRMINGHAM, UK.

Categorifying the comprehension scheme of ZF naturally leads to the notion of the subobject classifier, namely the “true” morphism from the terminal object to the truth object of the category is the universal subobject and any subobject is obtained as a pullback of the universal subobject. It also yields the epi-mono factorization system in the category.

Going one level higher, it was first realized by Street and Walters ([1]) that category of pointed sets discretely opfibred over the category of sets plays the same role as the subobject classifier. This was internalized in suitably structured 2-categories. Similarly the comprehension construction yields that discrete opfibrations and initial functors form a factorization system. There is a dual comprehensive factorization system, namely that of discrete fibrations and final functors.

Going yet another dimension higher, we introduce the notion of *comprehension constructions for bicategories* ([10]) (that is internal to the tricategory  $\text{Hom}$  of bicategories, pseudo functors, pseudo natural transformations, and modifications). This will rely on earlier work on fibrations of bicategories ([4], [5], [6]).

For us, the main example is the bicategory of *generalized spaces* (that is Grothendieck toposes defined over a varying elementary base) fibred over the bicategory of elementary toposes and geometric morphisms. We use the structure of comprehension to prove results about fibrations and opfibrations of toposes from fibrational extension of generalized geometric theories ([9]). As we shall see the notions of (op)fibration of toposes have close connections to topological properties. For instance, every local homeomorphism is an opfibration while every fibrewise Stone space is a fibration.

To study fibrations of toposes, Johnstone defined fibrations internal to 2-categories ([3]). If toposes are taken to be bounded over some fixed base  $\mathcal{S}$ , the analysis of fibrations and opfibrations in the 2-category  $\mathcal{B}\mathcal{T}\text{op}/\mathcal{S}$  of bounded toposes over base  $\mathcal{S}$  is much easier than the general case where there is no canonical choice of base topos and one has to work in the 2-category  $\mathcal{B}\mathcal{T}\text{op}$ . Indeed, Johnstone proved several important (op)fibrational results in  $\mathcal{B}\mathcal{T}\text{op}$ .

I will introduce the 2-category  $\mathcal{C}\text{on}$  of Arithmetic Universe (AU) contexts developed by Vickers ([7], [8]). It provides a language to reason about geometric construction within the predicative fragment of internal language of toposes, that is within the language of Arithmetic Universes.

Borrowing from work of Street ([2]), we introduce a syntactic notion of (op)fibration in  $\mathcal{C}\text{on}$  which is based on Chevalley’s internal characterization of fibrations obtained as a theorem in there. Note that Johnstone’s definition of internal (op)fibrations is more general than Chevalley’s definition: neither strictness of 2-categories nor the existence of the structure of strict pullbacks and comma objects are assumed.

I shall explain our result that gives a recipe for obtaining (op)fibrations of toposes from the finitary syntactic (op)fibrations of contexts ([9]). The scaffolding of the proof of this result is based on a certain comprehension bicategory involving fibred bicategory of generalized spaces over elementary toposes.

As an applications of this result, I will discuss the construction of the colimits of Grothendieck topos from generalized point-free *bag contexts*. Hopefully, this sheds some light on the relationship between AUs and traditional Grothendieck topos theory.

## References

- 1 Ross Street and Robert Walters, “The comprehensive factorization of a functor”, *Bulletin of the American Mathematical Society*, vol. **79** 1973, 936–941.
- 2 Ross Street, “Fibrations and Yoneda’s lemma in a 2-category”, *Lecture Notes in Math.*, Springer, Berlin, vol. **420**, 1974, pp. 104–133.
- 3 Peter Johnstone. “Fibrations and partial products in a 2-category”, *Applied Categorical Structures*, vol. **1**, 1993, pp. 141–179.
- 4 Claudio Hermida. “Some properties of Fib as a fibred 2-category”, *Journal of Pure and Applied Algebra*, vol. **134**, 1999, pp. 83–109
- 5 Igor Bakovic. “Some properties of Fib as a fibred 2-category”, *93rd Peripatetic Seminar on Sheaves and Logic*, University of Cambridge, 2012
- 6 Mitchell Buckley. “Fibred 2-categories and bicategories”, *Journal of Pure and Applied Algebra* 218, 2014, pp. 1034–1074.
- 7 Steven Vickers. “Sketches for arithmetic universes”, 2016, *Journal of Logic and Analysis Accepted for publication June 2018*.  
URL: <https://arxiv.org/abs/1608.01559>
- 8 Steven Vickers. “Arithmetic universes and classifying toposes”, *Cahiers de Topologie et Géométrie Différentielle Catégoriques* 58(4):213–248, 2017.  
<https://arxiv.org/abs/1701.04611>
- 9 Sina Hazratpour and Steven Vickers. “Fibrations of AU-contexts beget fibrations of toposes” *Submitted to Theory and Application of Categories (TAC) 2018*.  
URL: <https://arxiv.org/abs/1808.08291>
- 10 Sina Hazratpour. “A logical study of some 2-categorical aspects of toposes”, *PhD thesis*, University of Birmingham, 2019

# Twist products arising from residuated bimodules

Pengfei He<sup>1</sup>, Adam Přenosil<sup>2</sup>, and Constantine Tsinakis<sup>2</sup>

<sup>1</sup> Shaanxi Normal University, School of Mathematics and Information Science  
pfhe@snnu.edu.cn

<sup>2</sup> Vanderbilt University, Department of Mathematics  
adam.prenosil@vanderbilt.edu

<sup>3</sup> Vanderbilt University, Department of Mathematics  
constantine.tsinakis@vanderbilt.edu

In this contribution we investigate and slightly generalize a certain construction due to Tsinakis and Wille [5] of “residuated lattices of fractions” based on bimodules of residuated lattices. In particular, we investigate to what extent residuated bimodules (two-sorted algebras consisting of a residuated lattice acting on a lattice) can be presented as one-sorted algebras consisting of a residuated lattice equipped with an interior and a closure operator satisfying certain conditions, and vice versa. In doing so, we generalize the results of Busaniche and Cignoli [1] for integral commutative residuated lattices to arbitrary residuated lattices.

A *residuated  $\ell$ -bimodule* is a two-sorted algebra consisting of a residuated lattice  $\mathbf{L}$ , a lattice  $\mathbf{M}$ , a residuated left action  $*$  of  $\mathbf{L}$  on  $\mathbf{M}$  with residuals  $\backslash_*$  and  $*/$ , and a residuated right action  $*$  of  $\mathbf{L}$  on  $\mathbf{M}$  with residuals  $\backslash^*$  and  $/_*$ . The preceding operations satisfy the following conditions for all  $a, b \in \mathbf{L}$  and  $x \in \mathbf{M}$ :

$$\begin{aligned} e * x &= x, & x * e &= x, \\ ab * x &= a * (b * x), & (a * x) * b &= a * (x * b), & x * ab &= (x * a) * b, \\ x \leq a \backslash_* y &\iff a * x \leq y \iff a \leq y^*/x, \\ x \leq y /_* a &\iff x * a \leq y \iff a \leq x \backslash^* y. \end{aligned}$$

In particular, each residuated lattice  $\mathbf{L}$  acts on its own order dual  $\mathbf{L}^\partial$  with the residuated action  $a * x = x/a$  and  $x * a = a \backslash x$ . We call this the *canonical action* of  $\mathbf{L}$  on  $\mathbf{L}^\partial$ .

Residuated  $\ell$ -bimodules arise whenever a residuated lattice  $\mathbf{L}$  is equipped with a conucleus  $\sigma$  (see [2, 3] for the definition of a conucleus  $\sigma$  on  $\mathbf{L}$  and the conuclear image  $\mathbf{L}_\sigma$ ) and a closure operator  $\gamma$  satisfying the compatibility conditions  $\sigma a \cdot \gamma x \leq \gamma(a \cdot x)$  and  $\gamma x \cdot \sigma a \leq \gamma(x \cdot a)$ . Then  $\mathbf{L}_\sigma$  acts on the lattice of  $\gamma$ -closed elements  $\mathbf{L}_\gamma$  via the residuated action  $a * x = \gamma(a \cdot x)$  and  $x * a = \gamma(x \cdot a)$ . In fact, we show that each residuated  $\ell$ -bimodule equipped with an element  $0 \in \mathbf{M}$  such that  $a * 0 = 0 * a$  for each  $a \in \mathbf{L}$  arises in this way by modifying the twist product construction in [5] slightly. (Such bimodules will be called *cyclic-pointed*. Accordingly, a *cyclic-pointed residuated lattice* is a residuated lattice  $\mathbf{L}$  equipped with a constant  $0$  such that  $a \backslash 0 = 0/a$  for each  $a \in \mathbf{L}$ .)

This is because each cyclic-pointed residuated  $\ell$ -bimodule yields a residuated lattice  $\mathbf{L} \rtimes_0 \mathbf{M}$  which consists of the set  $\{\langle a, x \rangle \mid a \in \mathbf{L} \ \& \ x \in \mathbf{M} \ \& \ 0 * a \leq x\}$  equipped with the operations

$$\begin{aligned} e_{\mathbf{L} \rtimes_0 \mathbf{M}} &= \langle e, 0 \rangle, & \langle a, x \rangle \cdot \langle b, y \rangle &= \langle ab, (x * b) \vee (a * y) \rangle, \\ \langle a, x \rangle \wedge \langle b, y \rangle &= \langle a \wedge b, x \vee y \rangle, & \langle a, x \rangle \backslash \langle b, y \rangle &= \langle a \backslash b \wedge x \backslash^* y, a \backslash_* y \rangle, \\ \langle a, x \rangle \vee \langle b, y \rangle &= \langle a \vee b, x \wedge y \rangle, & \langle a, x \rangle / \langle b, y \rangle &= \langle a/b \wedge x^*/y, x /_* b \rangle. \end{aligned}$$

In the case of  $\mathbf{L}$  acting canonically on its order dual  $\mathbf{L}^\partial$ , the resulting algebra will be denoted  $\mathbf{L}^\boxtimes$ . The residuated lattice  $\mathbf{L}$  can be recovered as the image of the conucleus  $\sigma: \langle a, x \rangle \mapsto \langle a, a * 0 \rangle$ , and  $\mathbf{M}$  can be recovered as the image of the closure operator  $\gamma: \langle a, x \rangle \mapsto \langle x \backslash^* 0, x \rangle$ . This yields the following improvement of a result proved in [3] for bounded residuated lattices.

**Theorem 1.** *The conuclear images of (commutative) involutive residuated lattices are exactly (commutative) cyclic-pointed residuated lattices. More precisely, if  $0$  is a cyclic element of a (commutative) residuated lattice  $\mathbf{L}$ , then there is a (commutative) involutive residuated lattice  $\mathbf{K}$  and a conucleus  $\sigma$  on  $\mathbf{K}$  which preserves joins and products such that  $(\mathbf{L}, 0) \cong (\mathbf{K}_\sigma, \sigma(0))$ .*

An algebra of the form  $\mathbf{L} \rtimes_0 \mathbf{M}$  (of the form  $\mathbf{L}^\boxtimes$ ) will be called a *bimodule (square) twist product*. Square twist products are in fact *involutive* residuated lattices (in the sense of e.g. [5]).

**Theorem 2.** *A residuated lattice  $\mathbf{L}$  with a conucleus  $\sigma$ , a closure operator  $\gamma$ , and a constant  $0 = \gamma 0$  embeds into a bimodule twist product if and only if it satisfies the following equations:*

$$\begin{array}{lll} \sigma x \setminus y \wedge x \setminus \gamma y = x \setminus y & & \sigma(x \cdot y) = \sigma x \cdot \sigma y \\ x / \sigma y \wedge \gamma x / y = x / y & & \sigma(x \vee y) = \sigma x \vee \sigma y \\ \sigma x \cdot y \vee x \cdot \sigma y = x \cdot y & & \gamma(x \wedge y) = \gamma x \wedge \gamma y \\ \\ \sigma x \setminus \gamma y = \gamma(x \setminus y) & \gamma(0 \cdot \sigma x) = \gamma \sigma x & \sigma(0 \setminus \gamma x) = \sigma \gamma x \\ \sigma y / \gamma x = \gamma(y / x) & \gamma(\sigma x \cdot 0) = \gamma \sigma x & \sigma(\gamma x / 0) = \sigma \gamma x \end{array}$$

In that case the embedding is given by the map  $\eta: x \mapsto \langle \sigma x, \gamma x \rangle$ .

**Theorem 3.** *An involutive residuated lattice  $\mathbf{L}$  with a conucleus  $\sigma$  embeds into a square twist product if and only if it embeds into a bimodule twist product when expanded by the closure operator  $\gamma x = (\sigma(x \setminus 0)) \setminus 0$ .*

These embeddings in fact yield categorical adjunctions. Restricting to square twist products of integral commutative residuated lattices corresponds to imposing the equations  $0 = e$ ,  $\sigma x = e \wedge x$ , and  $x \cdot y = y \cdot x$  on the square twist products. In particular, the above equations then reduce to the equational description of the so-called K-lattices of Busaniche and Cignoli [1]. The  $\mathcal{T}$ -lattices of Ono and Riviuccio [4] are also related, although distinct, structures.

Finally, we remark that the preceding construction can be used to prove that any Brouwerian algebra is isomorphic to the negative cone of an idempotent involutive residuated lattice.

**Theorem 4.** *Brouwerian algebras are precisely the negative cones of idempotent involutive residuated lattices. In particular, each Brouwerian algebra  $\mathbf{L}$  is isomorphic to the negative cone of an idempotent involutive nucleus image of a conucleus image of  $\mathbf{L}^\boxtimes$ .*

## References

- [1] Manuela Busaniche and Roberto Cignoli. The subvariety of commutative residuated lattices represented by twist-products. *Algebra universalis*, 71(1):5–22, 2014.
- [2] Nikolaos Galatos and Constantine Tsinakis. Generalized MV-algebras. *Journal of Algebra*, 283:254–291, 2005.
- [3] Franco Montagna and Constantine Tsinakis. Ordered groups with a conucleus. *Journal of Pure and Applied Algebra*, 214(1):71–88, 2010.
- [4] Hiroakira Ono and Umberto Riviuccio. Modal twist-structures over residuated lattices. *Logic Journal of the IGPL*, 22(3):440–457, 2013.
- [5] Constantine Tsinakis and Annika M. Wille. Minimal varieties of involutive residuated lattices. *Studia Logica*, 83:407–423, 2006.



# Relational semantics for extended contact algebras

Tatyana Ivanova

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria  
tatyana.ivanova@math.bas.bg

The notion of contact algebra is one of the main tools in the region based theory of space. It is an extension of Boolean algebra with an additional relation  $C$  called contact. The elements of the algebra are called regions and are considered as analogs of physical bodies. A ternary predicate of extended contact  $\vdash$  has been introduced in [2]. Extended contact gives the possibility to define the unary predicate of internal connectedness  $c^\circ$  which cannot be defined in the language of contact algebras ( $c^\circ(a)$  iff  $\forall b \forall c (b \neq 0 \wedge c \neq 0 \wedge a = b + c \rightarrow b, c \not\vdash a^*)$ ).

**Definition 1.** [2] *Extended contact algebra (ExtCA, for short) is a system  $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, \vdash, C, c^\circ)$ , where  $(B, \leq, 0, 1, \cdot, +, *)$  is a nondegenerate Boolean algebra,  $\vdash$  is a ternary relation in  $B$  such that the following axioms are true:*

- (1)  $a, b \vdash c \rightarrow b, a \vdash c$ ,
- (2)  $a \leq c \rightarrow a, b \vdash c$ ,
- (3)  $a, b \vdash x, a, b \vdash y, x, y \vdash c \rightarrow a, b \vdash c$ ,
- (4)  $a, b \vdash c \rightarrow a \cdot b \leq c$ ,
- (5)  $a, b \vdash c \rightarrow a + x, b \vdash c + x$ ,

$C$  is a binary relation in  $B$  such that

- (6)  $aCb \leftrightarrow a, b \not\vdash 0$ ,

$c^\circ$  is a unary predicate in  $B$  such that

- (7)  $c^\circ(a) \leftrightarrow \forall b \forall c (b \neq 0 \wedge c \neq 0 \wedge a = b + c \rightarrow b, c \not\vdash a^*)$ .

Primary semantics for ExtCAs is topological. Let  $X$  be a topological space and  $a$  be its subset. We say that  $a$  is regular closed if  $a = Cl Int a$ . A topological ExtCA over  $X$  is the structure with universe the set  $RC(X)$  of all regular closed subsets together with the following interpretations:  $a \leq b$  iff  $a \subseteq b$ ,  $0 = \emptyset$ ,  $1 = X$ ,  $a \cdot b = Cl Int (a \cap b)$ ,  $a + b = a \cup b$ ,  $a^* = Cl (X \setminus a)$ ,  $a, b \vdash c$  iff  $a \cap b \subseteq c$ ,  $aCb$  iff  $a, b \not\subseteq \emptyset$ ,  $c^\circ(a)$  iff  $Int a$  is a connected subspace of  $X$ .

It is interesting also to consider a relational semantics for ExtCAs. An *equivalence frame of type 2* is a relational structure of the form  $(W, R_1, R_2)$ , where  $W$  is a nonempty set and  $R_1$  and  $R_2$  are equivalence relations on  $W$ .

We relate to any equivalence frame of type 2 a relational ExtCA with underlying structure the Boolean algebra of all subsets of  $W$ :  $\underline{B} = (2^W, \subseteq, \emptyset, W, \cap, \cup, *, \vdash, C, c^\circ)$ , where  $*$  denotes the set theoretical complement and for any subsets of  $W$   $a, b$ , and  $c$ :

- $a, b \vdash c$  iff  $\forall A, A_1, B, B_1 \left( AR_1 A_1 \in a, BR_1 B_1 \in b, AR_2 B \rightarrow (\exists C, C_1)(CR_1 C_1 \in c, AR_2 C) \right)$   
and  $a \cap b \subseteq c$ ,
- $aCb$  iff  $a, b \not\subseteq \emptyset$ ,
- $c^\circ(a)$  iff  $(\forall b, c \subseteq W)(b \neq \emptyset, c \neq \emptyset, a = b \cup c \rightarrow b, c \not\subseteq (W \setminus a))$ .

We say that a formula is true in  $(W, R_1, R_2)$  if it is true in the ExtCA related to  $(W, R_1, R_2)$ .

**Theorem 1.** [1] *Let  $\underline{B}$  be a finite ExtCA. Then  $\underline{B}$  is isomorphically embedded in the ExtCA related to some equivalence frame of type 2  $(W, R_1, R_2)$ .*



We consider a quantifier-free first-order logic  $\mathbb{L}$  for ExtCAs which has the following:

• axioms:

- the axioms of the classical propositional logic
- the axioms of Boolean algebra
- the axioms of ExtCA concerning the relations extended contact and contact
- the axiom schemes:

$$(\text{Ax } c^o) \quad c^o(p) \wedge q \neq 0 \wedge r \neq 0 \wedge p = q + r \rightarrow q, r \not\vdash p^*$$

$$(\text{Ax } c^o \ 1) \quad c^o(0)$$

$$(\text{Ax } c^o \ 2) \quad \neg c^o(p + q) \rightarrow \neg c^o(p) \vee \neg c^o(q)$$

$$(\text{Ax } c^o \ 3) \quad c^o(p + q) \rightarrow c^o(p) \wedge c^o(q)$$

• rules:

- MP

This logic is decidable and we have the following

**Theorem 2.** *For every quantifier-free formula  $\alpha$  the following conditions are equivalent:*

- i)  $\alpha$  is a theorem of  $\mathbb{L}$ ;
- ii)  $\alpha$  is true in all equivalence frames of type 2.

Extended contact gives also the possibility to define the relation of contact ( $aCb$  iff  $a, b \not\vdash 0$ ) and the binary relation  $RC_{\cap}$  meaning that the intersection of two regular closed sets is a regular closed set ( $RC_{\cap}(a, b)$  iff  $a, b \vdash a \cdot b$ ). It is worth to consider also a quantifier-free first-order language without the predicate of internal connectedness i.e.  $\mathcal{L}(0, 1; \cdot, +, *, \leq, \vdash, C)$ . In this weaker language one equivalence relation is enough - we consider *equivalence frames of type 1*. They are relational structures  $(W, R)$ , where  $W$  is a nonempty set and  $R$  is an equivalence relation on  $W$ .

We relate to any equivalence frame of type 1 a relational ExtCA in  $\mathcal{L} \underline{B} = (2^W, \subseteq, \emptyset, W, \cap, \cup, *, \vdash, C)$ , where  $*$  denotes the set theoretical complement and for any subsets of  $W$   $a, b$ , and  $c$ :

- $a, b \vdash c$  iff  $((\exists A \in a)(\exists B \in b)ARB \rightarrow (\exists C \in c)ARC)$  and  $a \cap b \subseteq c$ ,
- $aCb$  iff  $a, b \not\vdash \emptyset$

**Theorem 3.** [1] *Let  $\underline{B}$  be a finite ExtCA. Then in  $\mathcal{L} \underline{B}$  is isomorphically embedded in the ExtCA related to some equivalence frame of type 1  $(W, R)$ .*

Let  $\mathbb{L}_1$  be the logic obtained from  $\mathbb{L}$  by removing axioms  $(\text{Ax } c^o)$ ,  $(\text{Ax } c^o \ 1)$ ,  $(\text{Ax } c^o \ 2)$  and  $(\text{Ax } c^o \ 3)$ . This logic is called extended contact logic. We have the following

**Theorem 4.** *For every formula  $\alpha$  in  $\mathcal{L}$  the following conditions are equivalent:*

- i)  $\alpha$  is a theorem of  $\mathbb{L}_1$ ;
- ii)  $\alpha$  is true in all equivalence frames of type 1.

**Thanks.** Thanks to the anonymous referee for the remarks. This abstract is supported by contract DN02/15/19.12.2016 "Space, Time and Modality: Relational, Algebraic and Topological Models" with Bulgarian NSF.

## References

- [1] P. Balbiani and T. Ivanova. Representation theorems for extended contact algebras based on equivalence relations. 2019. arXiv:1901.10367, Submitted.
- [2] T. Ivanova. Extended contact algebras and internal connectedness. *Studia Logica*, 2019. <https://doi.org/10.1007/s11225-019-09845-6>.

# Colimits of effect algebras via a reflection \*

Gejza Jenča

Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak  
University of Technology, Bratislava, Slovak Republic, [gejza.jenca@stuba.sk](mailto:gejza.jenca@stuba.sk)

Effect algebras [1, 5, 2] are positive, cancellative, unital partial abelian monoids. The category of effect algebras is denoted by **EA**.

It was proved by Jacobs and Mandemaker in [4] that the category **EA** is cocomplete. Coproducts of effect algebras are very easy to describe, this construction is called *0,1-pasting*. It is much more difficult to describe coequalizers. Let us note that a regular epimorphism of effect algebras need not be surjective, see [4] for an example. This is probably the reason why previous attempts to give a sufficiently general theory of congruences of effect algebras were not entirely satisfactory.

In [4], colimits are transferred along a coreflection from a pseudovariety of *barred commutative monoids*, which has all colimits. In the present paper, we use a similar but different method. We fully embed the category of effect algebras into a category of *finite multiset covers* **FinMulCov**. The object of this category consist of pairs  $(X, \mathcal{J}(X))$ , where  $X$  is a set and  $\mathcal{J}(X)$  is a system of finite  $X$ -based multisets such that every element of  $X$  belongs to at least one multiset in  $\mathcal{J}(X)$ . The notion of a set equipped with a multiset cover is a generalization of the notion of an  $E$ -test space [3].

Morphisms in **FinMulCov** are given by pushforwards, as follows; if  $\mathbf{t}: X \rightarrow \mathbb{N}$  is a finite  $X$ -based multiset and  $f: X \rightarrow Y$  is a mapping, then a *pushforward of  $\mathbf{t}$  along  $f$*  is a finite  $Y$ -based multiset  $f_*(\mathbf{t}): Y \rightarrow \mathbb{N}$  is given by the rule

$$f_*(\mathbf{t})(y) = \sum_{x \in f^{-1}(y)} \mathbf{t}(x)$$

If  $(X, \mathcal{J}(X))$  and  $(Y, \mathcal{J}(Y))$  are sets equipped with a finite multiset cover, then a *morphism* from  $(X, \mathcal{J}(X))$  to  $(Y, \mathcal{J}(Y))$  is a mapping  $f: X \rightarrow Y$  such that for every  $\mathbf{t} \in \mathcal{J}(X)$ ,  $f_*(\mathbf{t}) \in \mathcal{J}(Y)$ . It is relatively straightforward to prove that **FinMulCov** has all colimits.

From every effect algebra  $(E, +, 0, 1)$  one can construct a finite multiset cover  $(E, \mathcal{J}(E))$  such that  $\mathcal{J}(E)$  is a collection of all finite multisets  $\mathbf{t}: E \rightarrow \mathbb{N}$  satisfying

$$\sum_{x \in \text{supp}(\mathbf{t})} \mathbf{t}(x) \cdot x = 1.$$

This construction is a functor from the category of effect algebras **EA** into the category **FinMulCov**. Moreover, it is a right adjoint and the counit of the adjunction is an isomorphism. Therefore, we obtain the following theorem:

**Theorem 1.** ***EA** is equivalent to a reflective subcategory of **FinMulCov**.*

We may then use this result to give an explicit construction of colimits of effect algebras: a colimit of a diagram in **EA** can be computed in **FinMulCov** and then reflected to **EA**.

---

\*This research is supported by grants VEGA 2/0069/16, 1/0420/15, Slovakia and by the Slovak Research and Development Agency under the contracts APVV-14-0013, APVV-16-0073.

## References

- [1] D.J. Foulis and M.K. Bennett. Effect algebras and unsharp quantum logics. *Found. Phys.*, 24:1325–1346, 1994.
- [2] R. Giuntini and H. Greuling. Toward a formal language for unsharp properties. *Found. Phys.*, 19:931–945, 1989.
- [3] S. Gudder. Effect test spaces. *Intern. J. Theor. Phys.*, 36:2681–2705, 1997.
- [4] Bart Jacobs and Jorik Mandemaker. Coreflections in algebraic quantum logic. *Foundations of physics*, 42(7):932–958, 2012.
- [5] F. Kópka and F. Chovanec. D-posets. *Math. Slovaca*, 44:21–34, 1994.

# Partially ordered varieties of involutive residuated posets

Peter Jipsen

Chapman University, Orange, California  
jipsen@chapman.edu

An *involutive residuated poset* is of the form  $(A, \leq, \cdot, \sim, -, 0)$  such that  $(A, \leq)$  is a poset,  $\cdot$  is associative, and for all  $x, y \in A$ ,

$$x \leq y \iff x \cdot \sim y \leq 0 \iff -y \cdot x \leq 0.$$

The element  $-0$  is denoted by  $1$ , and  $x \cdot y$  is usually written  $xy$ .

**Lemma 1.** *Involutive residuated posets satisfy the following identities and quasi-inequalities.*

1.  $\sim -x = x = -\sim x$
2.  $x \leq y \iff \sim y \leq \sim x \iff -y \leq -x$
3.  $1x = x = x1$
4.  $1 = \sim 0, \quad -1 = \sim 1 = 0$
5.  $xy \leq z \iff y \leq \sim(-z \cdot x) \iff x \leq -(y \cdot \sim z)$

Hence they are residuated po-monoids with residuals  $x \setminus y = \sim(-y \cdot x)$  and  $x / y = -(y \cdot \sim x)$ , and  $\cdot$  is order-preserving in both arguments.

The class of involutive residuated posets is denoted by InRP. Since all operations are order-preserving or order-reversing in each argument this class forms a *partially ordered quasivariety* (Pigozzi [4]). It is in fact a partially ordered *variety* (or po-variety) defined by the (in)equations  $(xy)z = x(yz)$ ,  $\sim -x = x = -\sim x$ ,  $\sim 0 = -0$ ,  $-0 \cdot x = x$ ,  $-x \cdot x \leq 1$ ,  $x \cdot \sim(yx) \leq \sim y$  together with the order-preservation of  $\cdot$  and the order-reversal of  $\sim, -$ .

The po-subvarieties of *commutative* ( $xy = yx$ ), *cyclic* ( $\sim x = -x$ ), *integral* ( $x \leq 1$ ) and *idempotent* ( $xx = x$ ) InRLs are denoted by CInRL, CyInRL, IInRL and IdInRL respectively.

InRP contains several well-known (term-equivalent) subclasses of (po-)algebras:

- The variety of *pointed groups* is axiomatized by adding  $x \leq y \implies x = y$  to InRP.
- The variety of *groups* is axiomatized by adding  $0 = 1$  to pointed groups. Hence involutive residuated posets may be considered the analogue of (pointed) groups over the category of posets.
- The po-subvariety of *pregroups* (Lambek [2]) is obtained by adding the identities  $0 = 1$  and  $xy = \sim(-y \cdot -x)$  to InRP.
- The po-subvariety of *partially ordered groups* (Glass [1]) is obtained by adding  $\sim x = -x$  to pregroups.
- *Involutive pocrimms* (Raftery [5]) are defined as involutive partially ordered commutative residuated integral monoids, hence they are the subvariety CIInRP. They are a class of algebras since  $x \leq y \iff -y \cdot x = 0$ . Involutive pocrimms include the subvarieties of IMTL-algebras, (pseudo)-MV-algebras and Boolean algebras.

- The variety of *involutive residuated lattices* is the expansion of InRP with a semilattice operation  $\vee$  such that  $x \leq y \iff x \vee y = y$ , and this class includes the subvarieties of lattice-ordered groups, classical linear logic algebras (without exponentials), De Morgan monoids and Sugihara algebras from relevance logic.

For the po-variety of idempotent involutive residuated posets we have the following result, which is from joint work with José Gil-Ferez.

**Theorem 2.** 1. *Cyclic idempotent involutive residuated posets are commutative.*

2. *Finite idempotent involutive residuated chains are commutative.*

3. *There exists an infinite noncyclic idempotent involutive residuated chain.*

We conjecture that all finite idempotent involutive residuated posets are commutative. The following partial result has been obtained with the help of Prover9 [3].

**Theorem 3.** *The po-subvariety of IdInRP determined by the identity  $\sim\sim x = --x$  satisfies cyclicity and commutativity.*

The smallest idempotent involutive residuated poset that is not a lattice has 10 elements and is depicted in Figure 1. The  $\cdot$  operation defines a semilattice since it is associative, commutative and idempotent. This semilattice is displayed in the same figure as a meet-semilattice with top element 1. Idempotence implies that  $0 \leq 1$  and that  $([0, 1], \cdot, +, -, 0, 1)$  is a Boolean algebra, where  $x + y = \sim(-y \cdot -x)$ .

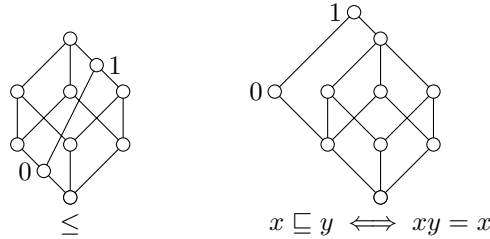


Figure 1: The smallest idempotent involutive residuated poset that is not a lattice.

For a po-algebra  $A$  in CIIdInRP, define the terms  $0_x = -x \cdot x$  and  $1_x = -(-x \cdot x)$ , and let  $\llbracket a, b \rrbracket = \{c \in A : ac = a, bc = c\}$ . Then the semilattice intervals  $(\llbracket 0_x, 1_x \rrbracket, \cdot, +, -, 0_x, 1_x)$  are also Boolean algebras and they partition  $A$ .

It is an open problem to characterize the posets that are reducts of involutive residuated posets, even for the idempotent and/or finite members of InRP.

## References

- [1] A. M. W. Glass: Partially ordered groups. World Scientific, 1999.
- [2] J. Lambek: *Type grammar revisited*, In A. Lecomte, F. Lamarche and G. Perrier, editors, Logical Aspects of Computational Linguistics, Springer LNAI 1582, 1999, 1–27.
- [3] W. McCune: Prover9 and Mace4. <http://www.cs.unm.edu/~mccune/prover9/>, 2005–2010.
- [4] D. Pigozzi: Partially ordered varieties and quasivarieties. Unpublished lecture notes at [https://orion.math.iastate.edu/dpigozzi/notes/santiago\\_notes.pdf](https://orion.math.iastate.edu/dpigozzi/notes/santiago_notes.pdf), 2004, 1–26.
- [5] J. Raftery: On the variety generated by involutive pocrimms. Reports on Mathematical Logic, 42, (2007), 71–86.

# On the structure of finite (commutative) idempotent involutive residuated lattices

Peter Jipsen<sup>1</sup>, Olim Tuyt<sup>2</sup>, and Diego Valota<sup>3</sup>

<sup>1</sup> Chapman University, Orange, USA  
jipsen@chapman.edu

<sup>2</sup> Mathematical Institute, University of Bern, Switzerland  
olim.tuyt@math.unibe.ch

<sup>3</sup> Department of Computer Science, Università degli Studi di Milano,  
valota@di.unimi.it

A (*pointed*) *residuated lattice* is an algebraic structure  $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1, 0 \rangle$  such that  $\langle A, \wedge, \vee \rangle$  is a lattice,  $\langle A, \cdot, 1 \rangle$  is a monoid and  $\cdot$  is residuated in the underlying lattice order with residuals  $\backslash$  and  $/$ , i.e. for  $a, b, c \in L$ ,  $ab \leq c \iff a \leq c/b \iff b \leq a \backslash c$  [3]. A residuated lattice  $\mathbf{A}$  is called *idempotent* if  $aa = a$  for all  $a \in A$  and *commutative* if  $ab = ba$  for all  $a, b \in A$ . The linearly ordered members of the variety of commutative idempotent residuated lattices have been studied in e.g. [6].

We define two *linear negations* on  $\mathbf{A}$  by  $\sim x := x \backslash 0$  and  $-x := 0/x$ . A residuated lattice where  $-\sim a = a = \sim -a$  for all  $a \in A$  is called *involutive*. Let  $\mathbf{CIdInRL}$  denote the variety of commutative idempotent involutive residuated lattices. In this setting, both residuals and both negations coincide. We work in the signature  $\langle A, \wedge, \vee, \cdot, /, -, 1, 0 \rangle$ . Interesting subvarieties of  $\mathbf{CIdInRL}$  include Sugihara monoids, the algebraic semantics of relevance logic  $\mathbf{RM}^t$  [1].

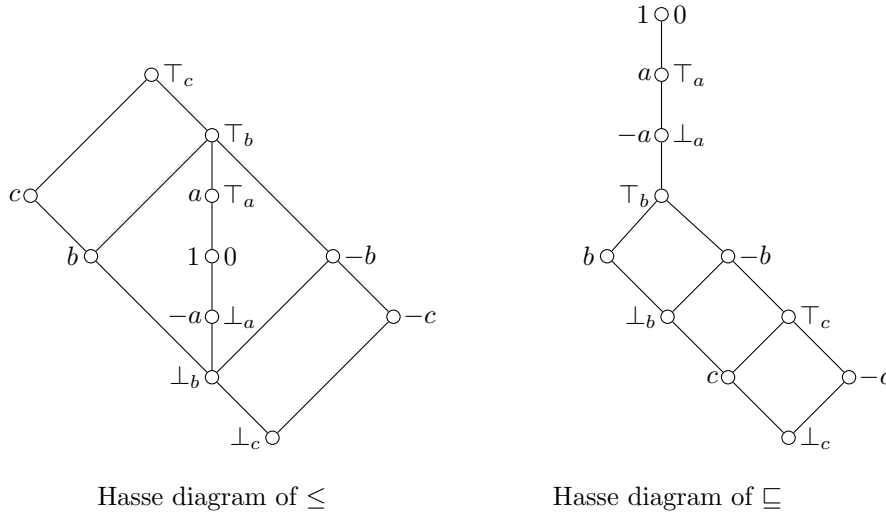
The algebras  $\mathbf{A} \in \mathbf{CIdInRL}$  are studied by considering their monoidal reduct. For  $a, b \in A$ , consider the *monoidal order*  $\sqsubseteq$ , where  $a \sqsubseteq b$  if and only if  $a \cdot b = a$  ([4]). By the properties of  $\mathbf{A}$ ,  $\sqsubseteq$  is a meet-semilattice order with  $\cdot$  as the meet operator and 1 as its maximum.

For each  $a \in A$ , let  $\perp_a$  and  $\top_a$  denote the terms  $a \wedge -a$  and  $a \vee -a$  respectively. We write  $[a, b]_{\sqsubseteq}$  and  $[a, b]_{\leq}$  for the sets  $\{c \in A \mid a \sqsubseteq c \sqsubseteq b\}$  and  $\{c \in A \mid a \leq c \leq b\}$  respectively. The following theorem summarizes a number of interesting properties of the structure of members of  $\mathbf{CIdInRL}$ , relating its monoidal and lattice order. These properties are illustrated by an example in Figure 1.

**Theorem.** *Let  $\mathbf{A} \in \mathbf{CIdInRL}$ .*

- *For each  $a \in A$ ,  $\langle [\perp_a, \top_a]_{\sqsubseteq}, \wedge, \vee, -, \perp_a, \top_a \rangle$  is a Boolean algebra.*
- *The intervals  $[\perp_a, \top_a]_{\sqsubseteq}$  partition  $A$ .*
- *The algebra  $\langle \{\perp_a \mid a \in A\}, \cdot, \vee \rangle$  is a distributive lattice. Moreover,  $\{\perp_a \mid a \in A\} = \{a \in A \mid a \leq 0\}$ .*
- *For each  $a \in A$ ,  $[\perp_a, 1]_{\sqsubseteq} = [\perp_a, \top_a]_{\leq}$ .*

Aiming for a structural characterization of the finite members of the variety  $\mathbf{CIdInRL}$ , we exploit the properties listed above to construct new members of the variety. The methods to build such new members involve the doubling of a filter in the monoidal order generated by an element  $a \leq 1$ . When  $a = \perp_a$  this construction generalizes Day's doubling for lattices [2]. We conjecture that these constructions suffice to construct all finite members of the variety.

Figure 1: The Hasse diagrams for  $\leq$  and  $\sqsubseteq$  of a residuated lattice  $\mathbf{A} \in \text{CIdInRL}$ .

### Acknowledgements

Some of the computations leading to our results have been obtained with the help of Prover9 and Mace4 [5]. The authors acknowledge the support of funding from the European Unions Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 689176.

### References

- [1] A. R. Anderson and N. D. Belnap, Jr. *Entailment. Volume I: The logic of relevance and necessity*. Princeton University Press, Princeton, N. J.-London, 1975.
- [2] A. Day. A simple solution to the word problem for lattices. *Canadian Mathematical Bulletin*, 13(2):253254, 1970.
- [3] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*. Elsevier, 2007.
- [4] J. Gil-Férez, P. Jipsen, and G. Metcalfe. Structure theorems for idempotent residuated lattices. *preprint*.
- [5] W. McCune. Prover9 and Mace4. <http://www.cs.unm.edu/~mccune/prover9/>, 2005–2010.
- [6] J. G. Raftery. Representable idempotent commutative residuated lattices. *Trans. Amer. Math. Soc.*, 359(9):4405–4427, 2007.

# From MV-semirings to Involutive semirings – Abstract

Peter Jipsen and Sara Vannucci\*

Chapman University, Orange, California, USA  
jipsen@chapman.edu  
University of Salerno, Italy  
svannucci@unisa.it

We first establish a term equivalence between involutive residuated lattices [3] and a special class of semirings, called *involutive 0-free semirings*. The semiring perspective helps us find a necessary and sufficient condition for  $[0, 1]$  to be a subalgebra of an involutive residuated lattice. We also import some results and techniques of semimodule theory in the study of this class of semirings. Following what was already done with MV-semirings ([1], [4]), we generalize some results about injective and projective semimodules [2]. Indeed, we note that the involution plays a crucial role and that the results for MV-semirings are still true for involutive semirings whenever the Mundici functor [6] is not involved. Using the Brzozowski derivative [5], we find a new characterization of injective semimodules over additively idempotent and commutative semirings.

An *involutive residuated lattice* is an algebra  $(A, \vee, \wedge, \cdot, 1, \sim, -)$  of type  $(2, 2, 2, 0, 1, 1)$  such that  $(A, \vee, \wedge)$  is a lattice,  $(A, \cdot, 1)$  is a monoid and for all  $x, y, z \in A$

$$x \cdot y \leq z \iff x \leq -(y \cdot \sim z) \iff y \leq \sim(-z \cdot x).$$

It follows that the identity  $\sim -x = -\sim x = x$  holds. A *semiring* is an algebra  $(S, +, \cdot, 0, 1)$  of type  $(2, 2, 0, 0)$  such that  $(S, +, 0)$  is a commutative monoid,  $(S, \cdot, 1)$  is a monoid and for all  $x, y, z \in S$   $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ ,  $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$  and  $0 \cdot x = x \cdot 0 = 0$ . If the semiring  $S$  is idempotent (i. e.  $x + x = x$  for all  $x \in S$ ), then  $(S, +, 0)$  is a join-semilattice. A semiring  $S$  is *zero-free* if it doesn't have 0 (i. e.  $(S, +)$  is a commutative semigroup and the last axiom of semirings isn't assumed to hold). An *involutive zero-free idempotent semiring* is an algebra  $(A, \vee, \cdot, 1, \sim, -)$  such that

- $(A, \vee, \cdot, 1)$  is a *zero-free idempotent semiring* and
- $x \leq y \iff x \cdot \sim y \leq -1 \iff -y \cdot x \leq -1$  for all  $x, y \in A$ .

**Proposition 1.** 1. *Involutive residuated lattices and involutive zero-free semirings are term-equivalent.*

2. *Denoting the element  $-1$  in an involutive residuated lattice by  $0$ , the interval  $[0, 1]$  is a subalgebra if and only if  $0 \cdot 0 = 0$ .*

An *involutive semiring* is an algebra  $(A, \vee, \cdot, 0, 1, \sim, -)$  of type  $(2, 2, 0, 0, 1, 1)$  such that

- $(A, \vee, \cdot, 0, 1)$  is an idempotent semiring and
- $x \leq y \iff x \cdot \sim y = 0 \iff -y \cdot x = 0$  for all  $x, y \in A$ .

Let  $S$  be an idempotent semiring. A *(left)  $S$ -semimodule* is a join semilattice  $(M, \vee, 0)$  with a scalar multiplication  $\cdot : A \times M \rightarrow M$ , denoted  $a \cdot x$ , such that the following conditions hold for all  $a, b \in A$  and  $x, y \in M$ :



1.  $(a \cdot b) \cdot x = a \cdot (b \cdot x)$ ;
2.  $a \cdot (x \vee y) = (a \cdot x) \vee (a \cdot y)$ ;
3.  $(a \vee b) \cdot x = (a \cdot x) \vee (b \cdot x)$ ;
4.  $0_A \cdot x = 0_M = a \cdot 0_M$ ;
5.  $1 \cdot x = x$ .

It is known that, in every variety of algebras, the projective objects are the retracts of the free ones. In particular, in the category of  $S$ -semimodule over a fixed semiring  $S$ , the free objects are  $S^{(X)}$ , for some set  $X$ , ([1]).

**Proposition 2.** *Let  $A$  be a finite commutative involutive semiring and  $M$  a finitely generated  $A$ -semimodule. Then,  $M$  is injective if and only if it is projective.*

Let  $Id(A)$  denote the set of join-semilattice ideals of an involutive semiring  $A$ . With the Brzozowski derivative [5] as action,  $Id(A)$  is an  $A$ -semimodule.

**Proposition 3.** *Let  $A$  be a commutative idempotent semiring and  $M$  an  $A$ -semimodule. Then,  $M$  is injective if and only if  $M$  is a retract of  $Id(A)^X$  for some set  $X$ .*

A semiring  $S$  is called *left (right) self-injective* if the regular left (right)  $S$ -semimodule  $S$  is injective. If the semiring is commutative and has this property then it is called *self-injective*. It is clear from the two previous results that every finite commutative involutive semiring is self-injective.

**Lemma 4.** ([4]) *Let  $S = \prod_{i \in I} S_i$  be a direct product of semirings. Then  $S$  is left self-injective if and only if each  $S_i$  is left self-injective.*

Thanks to the preceding lemma we have the following result.

**Proposition 5.** *Every direct product of finite commutative involutive semirings is self-injective.*

**Acknowledgement.** The second author is very grateful for support from the SYSMICS project and for the opportunity to spend a month doing research at Chapman University.

## References

- [1] A. Di Nola and C. Russo, Semiring and semimodule issues in MV-algebras, *Communications in Algebra*, **41** (2013), 1017–1048.
- [2] J.S. Golan, *Semirings and their Applications*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1999.
- [3] P. Jipsen, Relation algebras, idempotent semirings and generalized bunched implication algebras, in proceedings of the 16th International Conference on Relational and Algebraic Methods in Computer Science (RAMiCS), May 15 - 19, 2017, ENS Lyon, France, *Lecture Notes in Computer Science*, Vol. 10226, Springer (2017), 144–158
- [4] A. Di Nola, G. Lenzi, T.G. Nam, S. Vannucci, On injectivity of semimodules over additively idempotent division semirings and chain MV-semirings, preprint.
- [5] Janusz A. Brzozowski, Derivatives of regular expressions, *Journal of the ACM* **11**: 481–494.
- [6] R. L. O. Cignoli, I. M. L. D'Ottaviano and D. Mundici, *Algebraic foundations of many-valued reasoning*. Trends in Logic. Vol. 7. Dordrecht: Kluwer Academic Publishers, 2000.

# Presenting de Groot duality of stably compact spaces by entailment relations

Tatsuji Kawai

Japan Advanced Institute of Science and Technology  
tatsuji.kawai@jaist.ac.jp

Goubault-Larrecq [2] showed that the de Groot duality of stably compact spaces induces a family of dualities on various powerdomain constructions. For example, he showed that the dual of the Smyth powerdomain of a stably compact space  $X$  is the Hoare powerdomain of the dual  $X^d$  and that the Plotkin and probabilistic powerdomain constructions commute with the duality  $(-)^d$ .

Our aim is to give a simple account of the above phenomena for stably compact locales, the pointfree side of stably compact spaces. To this end, we introduce the following structure.

**Definition 1.** An *entailment relation* [1] on a set  $S$  is a binary relation  $\vdash$  on the finite subsets of  $S$  such that

$$a \vdash a \qquad \frac{A \vdash B}{A', A \vdash B, B'} \qquad \frac{A \vdash B, a \quad a, A \vdash B}{A \vdash B}$$

where  $a \in S$  and  $A, B, A', B'$  are finite subsets of  $S$ , and “ $\cup$ ” denotes a union. An entailment relation  $(S, \vdash)$  is *continuous* if it is equipped with an idempotent relation  $\prec$  on  $S$  such that

$$\exists C (A \prec_U C \vdash B) \leftrightarrow \exists D (A \vdash D \prec_L B),$$

where  $A \prec_U B \stackrel{\text{def}}{\iff} \forall b \in B \exists a \in A (a \prec b)$ , and  $\prec_L$  is defined dually.

Every continuous entailment relation  $(S, \vdash, \prec)$  presents a stably compact locale by the set  $S$  of generators and relations  $a = \bigvee_{b \prec a} b$  and  $\bigwedge A \leq \bigvee B$  for each  $A \vdash B$ . Conversely, any stably compact locale can be represented by such a structure.

The notion of continuous entailment relation is related to a well-known representation of a stably compact locale, called *strong proximity lattice* [3], which is a pair  $(S, \prec)$  of a distributive lattice  $(S, 0, \vee, 1, \wedge)$  and an idempotent relation  $\prec$  on  $S$  such that  $\leq \circ \prec \circ \leq = \prec$ , and satisfying

$$0 \prec 0, \qquad a \prec c \ \& \ b \prec c \rightarrow a \vee b \prec c, \qquad a \prec b \vee c \rightarrow \exists b' \prec b \exists c' \prec c (a \prec b' \vee c'),$$

and the dual properties for  $1$  and  $\wedge$ . It can be shown that the category of strong proximity lattices and approximable relations is equivalent to that of continuous entailment relations with a suitable notion of morphism.

The equivalence between the above two structures provides us with a simple method for analysing the de Groot duals of various constructions on a stably compact locale presented by a strong proximity lattice. Specifically, the method rests on an observation that the dual  $S^d = (S, \dashv, \succ)$  of a continuous entailment relation  $S = (S, \vdash, \prec)$  is a continuous entailment relation, which presents the de Groot dual of the locale presented by  $S$ . Moreover, if the relation  $\vdash$  is generated from a set  $R$  of initial entailments (i.e., axioms), then the dual  $\dashv$  is generated from axioms  $R^{\text{op}} := \{A \dashv B \mid A \vdash B \in R\}$ .

In what follows, we take up the construction of a probabilistic powerdomain in the pointfree setting to illustrate the point just mentioned.

**Definition 2.** A *probabilistic valuation* on a locale  $X$  is a Scott continuous function  $\mu: X \rightarrow \overrightarrow{[0, 1]}$  from  $X$  to the lower reals  $\overrightarrow{[0, 1]}$  satisfying  $\mu(0) = 0$ ,  $\mu(1) = 1$ , and the modular law:  $\mu(x) + \mu(y) = \mu(x \wedge y) + \mu(x \vee y)$ . The dual of a probabilistic valuation, called *covaluation*, is a Scott continuous function  $\nu: X \rightarrow \overleftarrow{[0, 1]}$  from  $X$  to the upper reals  $\overleftarrow{[0, 1]}$  satisfying  $\nu(1) = 0$ ,  $\nu(0) = 1$ , and the modular law. Let  $\mathfrak{V}(X)$  and  $\mathfrak{C}(X)$  be locales whose points are probabilistic valuations on  $X$  and covaluations on  $X$ , respectively (cf. Vickers [5]).

**Proposition 3.** *Let  $X$  be a stably compact locale represented by a strong proximity lattice  $(S, \prec)$ . Then, the locale  $\mathfrak{V}(X)$  is presented by a continuous entailment relation on  $\{\langle p, a \rangle \mid p \in \mathbb{Q}, a \in S\}$  generated by the axioms*

$$\begin{aligned} \emptyset \vdash \langle p, a \rangle \quad (p < 0) & \quad \langle p, a \rangle \vdash \emptyset \quad (1 < p) & \quad \langle p, 0 \rangle \vdash \emptyset \quad (0 < p) & \quad \emptyset \vdash \langle p, 1 \rangle \quad (p < 1) \\ & \quad \langle p, a \rangle \vdash \langle q, b \rangle & \quad (q \leq p \ \& \ a \leq b) \\ & \quad \langle p, a \rangle, \langle q, b \rangle \dashv\vdash \langle r, a \wedge b \rangle, \langle s, a \vee b \rangle & \quad (p + q = r + s) \end{aligned}$$

together with an idempotent relation  $\langle p, a \rangle \prec_{\mathfrak{V}} \langle q, b \rangle \stackrel{\text{def}}{\iff} q < p \ \& \ a \prec b$ .

Note that each generator  $\langle p, a \rangle$  of  $\mathfrak{V}(X)$  represents a basic open set of all probabilistic valuations  $\mu$  on  $X$  such that  $p < \mu(a)$ . A simple inspection shows that the locale  $\mathfrak{C}(X)$  of covaluations is presented by the dual of the continuous entailment relation given above, where each generator  $\langle p, a \rangle$  represents a basic open set of all covaluations  $\nu$  on  $X$  such that  $\nu(a) < p$ . This immediately yields the following.

**Theorem 4.** *If  $X$  is a stably compact locale, then  $\mathfrak{V}(X)^{\text{d}} \cong \mathfrak{C}(X^{\text{d}})$ .*

The observation by Vickers [5, Proposition 6.3] implies  $\mathfrak{V}(X) \cong \mathfrak{C}(X)$  and hence the following.

**Theorem 5.** *If  $X$  is a stably compact locale, then  $\mathfrak{V}(X)^{\text{d}} \cong \mathfrak{V}(X^{\text{d}})$ .*

The de Groot duals of the other powerlocale constructions (which correspond to Smyth, Hoare, and Plotkin powerdomains) can be analysed in a similar manner [4], which yields the pointfree analogues of the results by Goubault-Larrecq [2].

## References

- [1] J. Cederquist and T. Coquand. Entailment relations and distributive lattices. In S. Buss, P. Hajek, and P. Pudlak, editors, *Logic Colloquium '98*, Lecture Notes in Logic 13, pages 127–139. Association of Symbolic Logic, 2000.
- [2] J. Goubault-Larrecq. De Groot duality and models of choice: angels, demons and nature. *Math. Structures Comput. Sci.*, 20(2):169–237, 2010.
- [3] A. Jung and P. Sünderhauf. On the duality of compact vs. open. *Ann. N. Y. Acad. Sci.*, 806(1):214–230, 1996.
- [4] T. Kawai. Presenting de Groot duality of stably compact spaces. *ArXiv e-prints*, 1812.06480.
- [5] S. Vickers. A localic theory of lower and upper integrals. *Math. Log. Q.*, 54(1):109–123, 2008.

# Projective unification in $NExt(\mathbf{K4})$

Slawomir Kost

University of Opole, Opole, Poland  
 skost@uni.opole.pl

We study the problem of (projective) unification in normal modal logics (modal logic, for short) extending  $\mathbf{K4}$ . A substitution  $\sigma$  is a *unifier* for a formula  $A$  (in a given modal logic  $\mathbf{L}$ ) if  $\vdash_{\mathbf{L}} A[\sigma]$ . The set of constants in a modal logic  $\mathbf{L}$  is denoted by  $Cons(\mathbf{L})$ . It can be shown (see e.g. [3]) that the modal algebra  $\langle Cons(\mathbf{K4T}^{\square}), \wedge, \neg, \top, \square \rangle$  is isomorphic to the product of modal algebras  $\mathbb{A} = (\{0, 1\}, \cap, -, 1, \square_1)$  and  $\mathbb{B} = (\{0, 1\}, \cap, -, 1, \square_2)$  in which  $\square_1 0 = 0$  and  $\square_2 0 = 1$ . Together with some facts about the logic  $\mathbf{K4G}$  this yields:

**Lemma 1.** *Let  $\mathbf{L}$  be a modal logic extending  $\mathbf{K4GT}^{\square}$  such that  $Cons(\mathbf{L}) = \{\top, \perp, \diamond\top, \square\perp\}$ . If a formula  $A$  is not unifiable (there is no unifier for  $A$ ) in  $\mathbf{L}$ , then*

$$A \vdash_{\mathbf{L}} \diamond\top$$

or there exists a formula  $B$  such that

$$A \vdash_{\mathbf{L}} \square\perp \vee (\diamond B \wedge \diamond\neg B).$$

An inference rule  $A/B$  is said to be *passive* (in  $\mathbf{L}$ ) if  $A$  is not unifiable (in  $\mathbf{L}$ ). As a consequence of Lemma 1 we obtain the following:

**Lemma 2.** *Let  $\mathbf{L}$  be a modal logic extending  $\mathbf{K4GT}^{\square}$  such that  $Cons(\mathbf{L}) = \{\top, \perp, \diamond\top, \square\perp\}$ . The set consisting of the rules*

$$\frac{\diamond\top}{\perp} \quad \text{and} \quad \frac{\square\perp \vee (\diamond A \wedge \diamond\neg A)}{\perp}$$

is a basis for the set of all passive rules in  $\mathbf{L}$ .

Let  $\mathbf{L} \in NExt(\mathbf{K4GT}^{\square})$  be a modal logic with two constants such that  $\vdash_{\mathbf{L}} \diamond\top$ . In this case the second rule from Lemma 2 could be replaced with

$$P_2: \frac{\diamond A \wedge \diamond\neg A}{\perp}.$$

Moreover, the rule  $P_2$  forms a basis for all passive rules in  $\mathbf{L}$ .

The only consistent modal logic  $\mathbf{L} \in NExt(\mathbf{K4GT}^{\square})$  with two constants such that  $\vdash_{\mathbf{L}} \diamond\top \leftrightarrow \perp$  is  $\mathbf{Ver} = \mathbf{K} \oplus \square\perp$ . The only passive formula in this logic is  $\perp$ . Moreover, it is known that the modal logic  $\mathbf{Ver}$  is structurally complete.

A unifier  $\sigma$  for a formula  $A$  is said to be *projective* (in a modal logic  $\mathbf{L}$ ) if

$$A \vdash_{\mathbf{L}} x \leftrightarrow x[\sigma]$$

for each variable  $x$ . A formula is said to be *projective* (in  $\mathbf{L}$ ) iff there exists a projective unifier for the formula. We say that  $\mathbf{L}$  has *projective unification* if there exists a projective unifier for each unifiable formula (in  $\mathbf{L}$ ).

To show the main result on projective unification in  $NExt(\mathbf{K4})$  we use Gilardi's characterization of projective formulas through the so-called extension property, see [2]. Theorem 2.2 from [2] implies the following:

**Theorem 1** (Ghilardi). *Let  $L$  be a modal logic in  $NExt(K4)$  characterized by a class  $\mathcal{C}$  of finite rooted frames. A formula  $A$  is projective in  $L$  if and only if the class*

$$\{\langle \mathfrak{F}, v \rangle : \mathfrak{F} \in \mathcal{C} \text{ and } \langle \mathfrak{F}, v \rangle \models A\}$$

*has the extension property.*

Below is the main result.

**Theorem 2.** *A modal logic  $L \in NExt(K4)$  has projective unification if and only if  $K4D1 \subseteq L$ .*

By Theorem 2 we obtain.

**Corollary 1.** *Every modal logic containing  $K4D1$  is almost structurally complete.*

**Corollary 2.** *A modal logic  $L$  extending  $K4D1$  is structurally complete if and only if either  $L = Ver$  or  $K4D1M \subseteq L$ .*

As an immediate consequence of Theorem 2, we obtain the following result proved by Dzik and Wojtylak [1] using another method.

**Theorem 3** (Dzik and Wojtylak). *A modal logic in  $NExt(S4)$  has projective unification if and only if it contains  $S4.3$ .*

List of axioms

- K  $\Box(x \rightarrow y) \rightarrow (\Box x \rightarrow \Box y)$ ,
- T  $\Box x \rightarrow x$ ,
- T $^\Box$   $\Box(\Box x \rightarrow x)$ ,
- 4  $\Box x \rightarrow \Box \Box x$ ,
- M  $\Box \Diamond x \rightarrow \Diamond \Box x$ ,
- G  $\Diamond \Box x \rightarrow \Box \Diamond x$ ,
- .3  $\Box(\Box^+ x \rightarrow y) \vee \Box(\Box^+ y \rightarrow x)$ ,
- D1  $\Box(\Box x \rightarrow y) \vee \Box(\Box y \rightarrow x)$ .

## References

- [1] Wojciech Dzik and Piotr Wojtylak. Projective unification in modal logic. *Logic Journal of the IGPL*, 20:121–153, 2012.
- [2] Silvio Ghilardi. Best solving modal equations. *Annals of Pure and Applied Logic*, 102:183–198, 2000.
- [3] Sawomir Kost. Projective unification in transitive modal logic. *Logic Journal of the IGPL*, 26:548–566, 2018.

# From intuitionism to Brouwer's modal logic

Zofia Kostrzycka<sup>1</sup>

Opole University of Technology, Opole, Poland  
z.kostrzycka@po.opole.pl

The Brouwer modal logic **KTB** is defined as the following normal extension of the minimal normal modal logic **K**:  $\mathbf{KTB} := \mathbf{K} \oplus T \oplus B$  where the new axioms are the following:  $T := \Box p \rightarrow p$  and  $B := p \rightarrow \Box \Diamond p$ .

The set of rules consists of the modus ponens, the rule of uniform substitution and the rule of necessitation.

The axiom  $T$  is called the axiom of necessity, whereas the axiom  $B$  is known as the Brouwerian axiom. We paraphrase here the following justification of the second name given by G.E. Hughes and M.J. Cresswell in [2], p. 57. As it is known, L. Brouwer is the founder of the intuitionist school of mathematics. The law of double negation does not hold in intuitionistic logic. Exactly it holds that (i)  $\vdash_{INT} p \rightarrow \neg\neg p$  but (ii)  $\not\vdash_{INT} \neg\neg p \rightarrow p$ . Suppose that negation has a stronger meaning – necessarily negative. Hence  $\neg p$  may be translated as  $\Box\neg p$ . The corresponding modal formula to (i) is  $p \rightarrow \Box\neg\Box\neg p$ , which gives us  $p \rightarrow \Box\Diamond p$  and obviously  $\vdash_{KTB} p \rightarrow \Box\Diamond p$ . If we translate (ii) in this way, we obtain:  $\Box\Diamond p \rightarrow p$ , which is not a thesis even of the system **S5** defined below. Hence  $\not\vdash_{KTB} \Box\Diamond p \rightarrow p$ . Further, G.E. Hughes and M.J. Cresswell write: ‘Thus although the connection with Brouwer is somewhat tenuous, historical usage has continued to associate his name with this formula.’

This motivation combining Brouwer's axiom with the intuitionistic logic will be the starting point in our research. Of course, we have to find out a translation other than the Gödel-McKinsey-Tarski one. The Gödel-McKinsey-Tarski translation leads to **S4** logic and its normal extensions. They are known as the modal companions of intuitionistic logic (and intermediate ones) and are well described in literature, (see [1], [3]).

We propose some naive translation which will work within the language of formulas written in one variable. Let

$$\begin{aligned} \alpha^0 &= \perp, & \alpha^1 &= p, & \alpha^2 &= p \rightarrow \perp, \\ \alpha^{2n+1} &= \alpha^{2n} \vee \alpha^{2n-1}, & \alpha^{2n+2} &= \alpha^{2n} \rightarrow \alpha^{2n-1} & \text{for } n \geq 1 \\ \alpha^\omega &= p \rightarrow p & \text{for } \omega \notin \mathbb{N}. \end{aligned}$$

In the set of all formulas written in one variable we introduce an equivalence relation  $\equiv$  in the standard way:  $\varphi \equiv \psi$  if both  $\varphi \rightarrow \psi$  and  $\psi \rightarrow \varphi$  are intuitionistic tautologies.

Every formula from our language falls into one of the equivalence classes  $A^m = [\alpha^m]_{\equiv}$ . Therefore, up to this equivalence relation on the classes of formulas  $A^m$ , the quotient algebra rises to the so-called Rieger - Nishimura lattice  $\mathcal{R}$ , which is a single-generated free Heyting algebra (see Figure 1).

We define the translation in the following way:

$$\begin{aligned} t(\perp) &= \perp, & t(p) &= p, & t(\alpha \rightarrow \beta) &= \Box(t(\alpha) \rightarrow t(\beta)), & (1) \\ t(\alpha \wedge \beta) &= t(\alpha) \wedge t(\beta), & t(\alpha \vee \beta) &= t(\alpha) \vee t(\beta). & (2) \end{aligned}$$

Then we get:  $t(\neg p) = \Box\neg p$  (because  $\neg p = p \rightarrow \perp$ ) and further  $t(\neg\neg p) = \Box\neg\Box\neg p = \Box\Diamond p$ . One may notice that the Gödel-McKinsey-Tarski translation (symb.  $T$ ) differs from our translation because  $T(\perp) = \Box\perp$  and  $T(p) = \Box p$ .

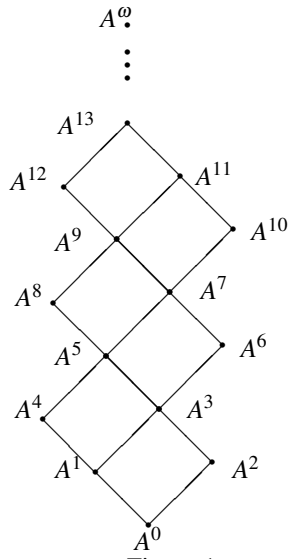


Figure 1.

The bottom of the Rieger-Nishimura lattice after translation  $t$  is presented in Figure 2.

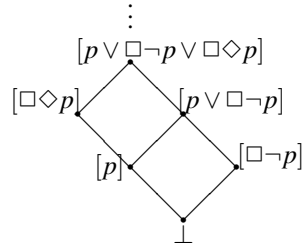


Figure 2.

We shall build up the modal equivalent of the Rieger-Nishimura lattice. It will not be possible to interpret the whole lattice, however, we will be able to obtain an infinite upper sublattice. From this translation we obtain many theorems combining intuitionistic logic of one variable with the same fragment of the modal Brouwer logic.

Further, we shall find the connection between the height of the upper sublattice and the degree of branching the considered KTB-frames.

Our next task is to generalize this result for formulas written in two variables.

## References

- [1] A. Chagrov, M. Zakharyashev, *Modal Logic*, Oxford Logic Guides 35, (1997).
- [2] G.E. Hughes, M.J. Cresswell, *An Introduction to Modal Logic*, Methuen and Co Ltd, London, (1968).
- [3] J. C. C. McKinsey and Alfred Tarski, *Some Theorems About the Sentential Calculi of Lewis and Heyting*, J. Symbolic Logic Volume 13, Issue 1 (1948), pp. 1–15.

# Stone dualities between étale categories and restriction semigroups

Ganna Kudryavtseva\*

Faculty of Civil and Geodetic Engineering,  
University of Ljubljana,  
Jamova cesta 2, SI-1000 Ljubljana, SLOVENIA  
`ganna.kudryavtseva@fgg.uni-lj.si`

In this talk, we present the results of the paper [3] by Mark V. Lawson and the speaker. Our work connects, unifies and extends the two approaches [4, 5] in relating inverse semigroups with étale localic or topological groupoids. The paper [5] achieves this by showing how to construct étale localic groupoids from pseudogroups by means of a class of quantales, whereas the paper [4] achieves this by relating distributive inverse semigroups and pseudogroups to étale topological groupoids making use of prime and completely prime filters. Both of these approaches were motivated by the theory of  $C^*$ -algebras but from slightly different traditions.

In [3] we replace étale localic or topological *groupoids* of [4, 5] by étale localic or topological *categories*. Although our work is a generalization of both [4] and [5], we argue that working at this level of generality actually clarifies and simplifies the theory. The use of localic categories generalized from [5] greatly sharpens some of results of [4], whereas the use of involutions in [5], which we avoid in our generalization, renders the theory superficially more complex. One very important additional feature of our theory is that our results are fully functorial.

A class of semigroups, called *restriction semigroups*, plays the role in our theory similar to that inverse semigroups play in [4] and [5]. Restriction semigroups are non-regular generalizations of inverse semigroups. They are equipped with two unary operations,  $*$  and  $+$ , that generalize the operations  $a \mapsto a^{-1}a$  and  $a \mapsto aa^{-1}$ , respectively, in an inverse semigroup. Such semigroups and their one-sided analogues arise naturally from various sources and have been widely studied by many authors, see the survey article [2].

Projecting down from localic to topological categories, we extend the classical adjunction between locales and topological spaces [1]. This yields extensions of the classical Stone duality, the role of Boolean algebras being played by so-called Boolean restriction semigroups (resp. Boolean restriction  $\wedge$ -semigroups), and the role of Boolean spaces by étale topological categories  $(C_1, C_0)$  where the space  $C_0$  is a Boolean space (resp. both of the spaces  $C_1$  and  $C_0$  are Boolean spaces).

## References

- [1] P. T. Johnstone, *Stone spaces*, CUP, 1986.
- [2] C. Hollings, From right *PP* monoids to restriction semigroups: a survey, *European J. Pure Appl. Math.* **2** (2009), 21–57.
- [3] G. Kudryavtseva, M. V. Lawson, A perspective on non-commutative frame theory, *Adv. Math.* **311** (2017), 378–468.
- [4] M. V. Lawson, D. H. Lenz, Pseudogroups and their étale groupoids, *Adv. Math.* **244** (2013), 117–170.
- [5] P. Resende, Étale groupoids and their quantales, *Adv. Math.* **208** (2007), 147–209.

---

\*The author was partially supported by ARRS grant P1-0288.



# The spectrum of a localic semiring

Graham Manuell

University of Edinburgh, Edinburgh, United Kingdom  
graham@manuell.me

## Abstract

A number of ways have been devised to associate topological spaces to various algebraic structures, starting with Stone's spectrum of a Boolean algebra [6] (or even a general bounded distributive lattice [7]) and Gelfand's spectrum of a commutative Banach algebra [2]. Later Grothendieck defined the Zariski spectrum of a commutative ring and Hofmann and Lawson described the spectrum of a distributive continuous lattice [4].

These spectra have a lot in common. For instance, the points of Stone's and Grothendieck's spectra correspond to prime ideals of the semirings in question. Gelfand's spectrum is usually described in terms of maximal ideals, but at least in the case of  $C^*$ -algebras, it can equivalently be phrased in terms of prime ideals which are closed in the norm topology. Finally, the points of the Hofmann-Lawson spectrum can be thought of as closed prime ideals with respect to the Scott topology. Furthermore, there are obvious similarities in the topologies on these sets in the different cases. This suggests that there might be a single construction encompassing all of these examples.

Constructively, it is better to treat spectra as locales instead of as topological spaces and then it is reasonable to view the lattices, rings and  $C^*$ -algebras as localic commutative semirings (as in [3]). We define the spectrum of a localic semiring as a classifying locale of closed prime ideals (or equivalently, open prime anti-ideals). For a general localic semiring such a spectrum might fail to exist, but we provide conditions under which it does. Under these conditions it is isomorphic to the localic reflection of the quantale of overt weakly closed ideals.

Let us describe this construction in more detail. Recall from [1] that the overt weakly closed sublocales of a frame  $L$  correspond to suplattice homomorphisms from  $L$  to the frame  $\Omega$  of truth values. When  $L$  is a localic semiring, the suplattice inherits operations from  $L$  and we may take a quotient to obtain a quantale of overt weakly closed ideals. In the case of a discrete ring, the overt weakly closed ideals are just the usual set-theoretic ideals and the universal localic quotient of this quantale gives the frame of radical ideals. But the frame of radical ideals is known to be the frame of the Zariski spectrum (see [5]) and, in fact, this construction yields the usual spectra in all of our core examples.

## References

- [1] M. Bunge and J. Funk. Constructive theory of the lower power locale. *Math. Structures Comput. Sci.*, 6(1):69–83, 1996.
- [2] I. Gelfand. Normierte Ringe. *Mat. Sb.*, 9(1):3–24, 1941.
- [3] S. Henry. Localic metric spaces and the localic Gelfand duality. *Adv. Math.*, 294:634–688, 2016.
- [4] K. H. Hofmann and J. D. Lawson. The spectral theory of distributive continuous lattices. *Trans. Amer. Math. Soc.*, 246:285–310, 1978.
- [5] P. T. Johnstone. *Stone spaces*, volume 3 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1982.
- [6] M. H. Stone. Applications of the theory of Boolean rings to general topology. *Trans. Amer. Math. Soc.*, 41(3):375–481, 1937.
- [7] M. H. Stone. Topological representations of distributive lattices and Brouwerian logics. *Časopis Pěst. Mat. Fys.*, 67(1):1–25, 1938.

# A topos for piecewise-linear geometry, and its logic

Vincenzo Marra<sup>1\*</sup> and Matías Menni<sup>2</sup>

<sup>1</sup> Dipartimento di Matematica *Federigo Enriques*  
Università degli Studi di Milano, Italy

<sup>2</sup> CONICET, Argentina

Let us write  $\mathbb{A}^n$ ,  $n \in \mathbb{N} := \{0, 1, 2, \dots\}$ , for  $n$ -dimensional real affine space. A subset of  $\mathbb{A}^n$  is a *polytope* if it is the convex hull of a finite set of points. Affine maps between polytopes are restrictions of affine maps between the ambient affine spaces. Let us then write  $\mathbf{C}$  for the category of nonempty polytopes and affine maps, and  $\widehat{\mathbf{C}}$  for the topos of presheaves of sets on  $\mathbf{C}$ . While  $\widehat{\mathbf{C}}$  receives a fully faithful functor from  $\mathbf{C}$  (Yoneda embedding), sums—more generally, colimits—in  $\mathbf{C}$  are not preserved by the embedding. To remedy this, one equips  $\mathbf{C}$  with a Grothendieck topology  $J$  that captures some of the content of the affine geometry of polytopes, and then replaces presheaves by sheaves. We define  $J$  to be the topology on  $\mathbf{C}$  generated by all finite families of injective affine maps  $C_i \xrightarrow{f_i} D$  such that their joint image covers  $D$ ; in symbols,  $\bigcup_{i=1}^n f_i[C_i] = D$ . That is, we take the smallest topology generated by “covers of a polytope  $D$  by finite families of subpolytopes  $C_1, \dots, C_n$ ”. It turns out this topology is not subcanonical—i.e., representable presheaves need not be sheaves—which, it will soon transpire, detracts nothing from its usefulness and mathematical naturality.

Let now  $\mathcal{P}$  be the topos of sheaves on the site  $(\mathbf{C}, J)$ . We call  $\mathcal{P}$  the *PL* (=Piecewise-Linear) *topos*, and think of it as a spatial setting wherein to do PL geometry in ways that, conjecturally, are less affected by the contingent peculiarities of the classical compact PL category  $\mathbf{P}$ . The latter, we recall, has as objects the not necessarily convex subsets of affine spaces known as the *compact polyhedra*—namely, the finite unions of polytopes—and as morphisms the *PL maps*, that is, functions  $f: P \rightarrow Q$  between polyhedra such that  $f$  is continuous with respect to the Euclidean topologies, and there are finitely many affine maps  $a_1, \dots, a_m$  such that for each  $p \in P$  there is  $i_p \in \{1, \dots, m\}$  with  $f(p) = a_{i_p}$ . As a preliminary step towards vindicating the conjecture just stated, one can prove that  $\mathcal{P}$  embeds  $\mathbf{P}$  fully faithfully so as to preserve “enough finite colimits” to perform the fundamental constructions of polyhedral geometry. The PL topos  $\mathcal{P}$  automatically takes care of gluing the affine maps in  $\mathbf{C}$  into PL maps, thanks to the choice of the topology  $J$ .

In fact, more is true. Consider on  $\mathbf{P}$  the Grothendieck topology  $J^*$  generated by “covers of a polyhedron  $D$  by finite families of subpolyhedra  $C_1, \dots, C_n$ ”. Theorem: The topos of sheaves on the site  $(\mathbf{P}, J^*)$  is again just the PL topos  $\mathcal{P}$  (and  $J^*$ , in contrast to  $J$ , is subcanonical). Although the affine site of definition  $(\mathbf{C}, J)$  is conceptually and mathematically much more economical than  $(\mathbf{P}, J^*)$ —note that the latter, unlike the former, *presumes* knowledge of PL geometry—we can now use the larger, subcanonical PL site to prove results about  $\mathcal{P}$ , whenever convenient; an example of this strategy is given below.

A fuller defence of the conjecture requires considerably deeper developments guided by the theory of Axiomatic Cohesion (see [5] for the initial paper on the subject). We must gloss over these geometric aspects, even if they motivate much of the present research. We focus instead on what we know so far of the connections of  $\mathcal{P}$  with logic.

Every topos classifies a theory in geometric logic. Question: What is the theory classified by the PL topos  $\mathcal{P}$ ? Answer (Theorem): The theory of non-trivial linearly ordered *Riesz MV-algebras*. Here, MV-algebras are well known and have a prominent rôle in substructural logics;

---

\*Presenting author.

and Riesz MV-algebras are what one obtains upon equipping an MV-algebra  $A$  with the action of  $[0, 1] \subseteq \mathbb{R}$  given by “multiplication of elements of  $A$  by scalars in  $[0, 1]$ ”. The Chang-Mundici equivalence between MV-algebras and Abelian lattice-groups with a (strong order) unit has an adaptation to vector lattices (=lattice-ordered real linear spaces) with a unit, yielding an equivalence with Riesz MV-algebras. Thus, the latter are precisely unit intervals of vector lattices with a unit. A fundamental result of Beynon [1], inspired by previous work by Baker, establishes that  $\mathbf{P}$  is dually equivalent to  $\mathbf{V}_{\text{fp}}$ , where the latter is the category of finitely presented vector lattices with a unit. Translating,  $\mathbf{P}$  is equivalent to the opposite of the category of finitely presented Riesz MV-algebras. As these form a finitary variety—whereas, note,  $\mathbf{V}$  is not even an elementary class—a standard result entails that the presheaf topos  $\widehat{\mathbf{V}}_{\text{fp}}^{\text{op}} = \widehat{\mathbf{P}}$  classifies the theory of Riesz MV-algebras. The theorem is now established by proving that the axioms for total order and non-triviality induce on  $\mathbf{P}$  precisely the Grothendieck topology  $J^*$ .

As a further step in understanding the relation between  $\mathscr{P}$  and Riesz MV-algebras one can contrast polyhedra in  $\mathbf{P}$ , seen in  $\mathscr{P}$  as representable sheaves, and the spectral spaces of finitely presented Riesz MV-algebras. For such an algebra  $A$  let us write  $\text{Spec } A$  for the set of prime ideals of  $A$ . It is known that  $\text{Spec } A$  coincides with the spectrum of the distributive lattice of compact congruences on  $A$ ; thus,  $\text{Spec } A$  can be equipped with one amongst the Priestley, Stone, or dual Stone topologies. For our purposes here we choose the dual Stone topology. Then  $A$  can be represented as the algebra of global sections of a sheaf of Riesz MV-algebras over the base space  $\text{Spec } A$  with *totally ordered* stalks, as first proved in [3] and reproved by Priestley duality in [4]. Heyting algebras of subobjects of a sheafified representable in a Grothendieck topos may have a complex structure, one that can be hard to relate to a construction in the site that goes beyond the standard description by closed sieves. Perhaps surprisingly, though, the PL topos begs to differ. Theorem: Pick any polyhedron  $P$  in  $\mathbf{P}$ , and write  $A$  for its Beynon-dual Riesz MV-algebra. Let  $\mathcal{O}(\text{Spec } A)$  denote the frame of open sets of the spectrum of  $A$ , and  $\text{Sub } P$  for the Heyting algebra of subobjects in  $\mathscr{P}$  of (the sheaf represented by)  $P$ . Then  $\mathcal{O}(\text{Spec } A)$  and  $\text{Sub } P$  are isomorphic Heyting algebras. Open problem: Characterise in a useful manner the extension of intuitionistic propositional logic jointly determined by the Heyting algebras  $\text{Sub } P$ , as  $P$  ranges over all polyhedra. At the time of writing it is not known whether the extension at hand is proper or coincides with intuitionistic logic. Let us note that this is a different problem from the one recently solved in [2], where it was shown that the logic determined by the Heyting algebras of open subpolyhedra of  $P$ , as  $P$  ranges over all polyhedra, is intuitionistic logic.

## References

- [1] W. M. Beynon. Duality theorems for finitely generated vector lattices. *Proc. London Math. Soc.* (3), 31:114–128, 1975 part 1.
- [2] N. Bezhanishvili, V. Marra, D. McNeill, and A. Pedrini. Tarski’s theorem on intuitionistic logic, for polyhedra. *Ann. Pure Appl. Logic*, 169(5):373–391, 2018.
- [3] E. J. Dubuc and Y. A. Poveda. Representation theory of MV-algebras. *Ann. Pure Appl. Logic*, 161(8):1024–1046, 2010.
- [4] M. Gehrke, S. J. van Gool, and V. Marra. Sheaf representations of MV-algebras and lattice-ordered abelian groups via duality. *J. Algebra*, 417:290–332, 2014.
- [5] F. W. Lawvere. Axiomatic cohesion. *Theory Appl. Categ.*, 19:No. 3, 41–49, 2007.

# Frege's Basic Law V via Partial Orders

Giovanni M. Martino<sup>1</sup>

Vita-Salute San Raffaele University, Milan  
giovanni.martino3@outlook.it

My aim is to present a consistent model-theoretic representation of Frege's Basic Law V (BLV):  $\forall F \forall G [\varepsilon F(x) = \varepsilon G(x) \leftrightarrow \forall x (F(x) \leftrightarrow G(x))]$  with a full-impredicative comprehension schema (CA):  $\exists X \forall x (Xx \leftrightarrow \varphi(x))$  employing a model based on some properties of posets. The resulting theory manages to block the Russell's Paradox recovering at least Second-Order Peano-Dedekind Frege's style axioms. Despite any syntactical predicative restriction, HECK [5], FERREIRA and WEHMEIER [2] and an impredicative approach based on concepts predicatively defined, FERREIRA [3], my purpose is to employ a semantical approach where both BLV and CA won't be syntactically restricted.

In order to retain the consistency, I shall build-up a model for the theory  $\mathcal{T}_K$  based on a poset  $\mathcal{M} = \langle \mathcal{D}, \subseteq \rangle$ , wherein  $\mathcal{D} = \mathcal{P}(\omega)$  and  $\subseteq$  is a relation, reflexive, antisymmetric, and transitive over  $\mathcal{D}$ . Subsequently, I shall define over  $\mathcal{M}$  a monotone unary function  $\phi$  order-preserving. According to MOSCHOVAKIS [6],  $\phi$  has least fixed point property. Thus, in agreement with Frege's definition, I shall apply  $\phi$  to  $\mathbb{N}(x)$ , the concept of natural number, and I will show that  $\mathbb{N}(x)$  is in the least fixed point, namely,  $\mathcal{T}_K$  manages to recover Second-Order Peano-Dedekind axioms.

In order to carry-out the former challenge, firstly, I have to fix over  $\mathcal{M}$  an interpretation for the syntax of  $\mathcal{T}_K$ : let  $M_1$  be the first-order domain;  $M_2$  the second-order domain and  $V$  the Universe, the pair  $(\mathcal{E}, \mathcal{A})$  interprets any second-order variable  $\vartheta(x)$ , with at most one free first-order variable, where  $\mathcal{E}(\vartheta(x)) \subseteq M_1$  is the extension of the concept;  $\mathcal{A}(\vartheta(x)) := V - \mathcal{E}$  is anti-extension of the concept where,  $\mathcal{E} \cap \mathcal{A} = \emptyset$  and  $\mathcal{E} \cup \mathcal{A} = V$ . Moreover,  $\mathcal{A}^- \subseteq \mathcal{A}(\vartheta(x))$  when  $\mathcal{E} \cap \mathcal{A} \neq \emptyset$ .

The function  $\pi : M_2 \rightarrow M_1$  interprets the abstraction operator  $\varepsilon$ . However, BLV does not delivers always *admissible* extensions. Given a characteristic function  $\chi(x)$  any time defined for a particular concept  $\vartheta(x)$ , I call *admissible* extensions those objects that  $\chi(x) = 1$  for an  $x \in \mathcal{E}(\vartheta)$ ; I call *unadmissible* extensions those objects that  $\chi(x) = 0$  for an  $x \in \mathcal{A}(\vartheta)$ . In agreement with the full-impredicative view, the interpretation of the quantifier is given in standard SOL definition.

Secondly, I have to construe a hierarchy  $\mathcal{S}$  of interpretations of  $\vartheta(x)$  based on  $(\mathcal{E})$ . Only at the limit stage of this hierarchy, the extension and the anti-extension of  $\vartheta$  will be fixed, namely,  $\vartheta$  has the corresponding and admissible extension.

It is now clear that  $\mathcal{T}_K$  avoids the Russell's Paradox. Let me assume that the function  $\pi : R \rightarrow \{R\}$ , delivers to  $R$  his extension  $\{R\}, \{R\} \in \{\mathcal{E}(R) \cup \mathcal{A}(R)\}$ : if  $\{R\}$  belongs to  $\mathcal{A}^-$  and then the value of the characteristic function  $\chi(x) = 0$ , namely,  $\varepsilon R$  is an inadmissible extension.

Finally,  $\mathcal{T}_K$  results both consistent and strong enough to recover Second-Order Peano-Dedekind axioms. A poset  $\mathcal{M} = \langle \mathcal{D}, \subseteq \rangle$  is a model for the former structure: indeed, according to MOSCHOVAKIS [6], the hierarchy  $\mathcal{S}$  is a chain, the concept of everything,  $x = x$ , namely  $U = \max D$ , is the unique maximal element.

Thus, I may form the concept  $\mathbb{N}(x) =_{def} \text{Pred}^+(0, x)$  because only with a predicative fragment I have at least Dedekind-infinitely many  $M_1$  individuals that fall under it. If  $\text{Pred}^+(y, x) = \exists F \exists u (Fu \wedge y = \#F \wedge x = \#[\lambda z. Fz \wedge z \neq u])$ , it easy to show that applying  $\phi$  to  $F$ ,  $F$  is in the least fixed point of  $\phi$ . Thus, by using some Partial Orders properties, I have a model-theoretic representation of Frege's BLV.

## References

- [1] BURGESS, J. P., *Fixing Frege*, Princeton: Princeton University Press, 2005.
- [2] FERREIRA, F. and WEHMEIER, K. F., *On the consistency of the  $\Delta_1^1$ -CA fragment of Frege's Grundgesetze*, *Journal of Philosophical Logic*, 31 (2002) 4, pp. 301-311.
- [3] FERREIRA, F., *Zig Zag and Frege Arithmetic*, <http://webpages.fc.ul.pt/~fjferreira/Zigzag.pdf>
- [4] FREGE, G., *Grundgesetze der Arithmetik. Begriffsschriftlich abgeleitet*, vol. I-II, Jena: H. Pohle, 1893-1903 (trans. by EBERT, P. A. and ROSSBERG, M., *The Basic Laws of Arithmetic*, Oxford: Oxford University Press, 2013).
- [5] HECK, R. K., *The consistency of predicative fragments of Frege's Grundgesetze der Arithmetik*, *History and Philosophical Logic*, 17 (1996) 4, pp. 209-220 (originally published under the name "Richard G. Heck, Jr").
- [6] MOSCHOVAKIS, Y., *Notes on Set Theory*, New York: Springer (2nd edition), 2006.

# B-frame Representations for Complete Lattices

Guillaume Massas

University of California, Berkeley  
gmassas@berkeley.edu

It is a basic fact about bi-topological spaces that for any bi-topological space  $(X, \tau_1, \tau_2)$ , the map  $U \mapsto I_1 C_2(U)$ , sending every open set in  $\tau_1$  to the  $\tau_1$ -interior of its  $\tau_2$ -closure, is a closure operator on the lattice of open sets in  $\tau_1$ . As a consequence, the fixpoints  $\text{RO}_{12}(X)$  of this map, called *generalized regular opens*, always form a complete lattice. This result can be seen of a generalization of a celebrated result, attributed to Tarski [8], that the regular open sets of any topological space always form a complete Boolean algebra. It has been used, particularly in [9] and [5], to present a duality between bounded lattices and a subcategory of the category of bi-topological spaces. Tarski's result also plays an essential role in the choice-free Stone duality recently presented in [4].

A *b-frame* is a bi-preordered set  $(X, \leq_1, \leq_2)$ . Any b-frame induces a bi-topological space  $(X, \tau_1, \tau_2)$  where each topology is the upset topology induced by the corresponding preordering. Several representations of complete lattices as generalized regular opens of some b-frame exist in the literature. In particular, Allwein in [1] and [2] observes that every complete lattice  $L$  is isomorphic to the generalized regular opens of its dual Allwein b-frame  $(P_L, \leq_1, \leq_2)$  defined as:

- $P_L = \{(a, b) \in L \times L ; a \not\leq_L b\}$ ,
- $(a, b) \leq_1 (c, d)$  iff  $a \geq_L c$ , and
- $(a, b) \leq_2 (c, d)$  iff  $b \leq_L d$ .

In the case of Heyting algebras, it has been shown in [3] and [6] that any complete Heyting algebra  $A$  is isomorphic to  $\text{RO}_{12}(X)$  for some b-frame  $(X, \leq_1, \leq_2)$  such that  $\leq_1 \subseteq \leq_2$ . Given a Heyting algebra  $A$ , a standard way of constructing  $(X, \leq_1, \leq_2)$  is to define it such that:

- $X = \{(a, a \rightarrow b) \in L \times L ; a \not\leq_A b\}$ ,
- $(a, b) \leq_1 (c, d)$  iff  $a \geq_A c$ , and
- $(a, b) \leq_2 (c, d)$  iff  $a \geq_A c$  and  $b \leq_A d$ .

It is also well-known, particularly in the forcing literature (see for example [7]), that any complete Boolean algebra is isomorphic to the regular open sets of some Alexandroff space  $(X, \tau)$ , which can be regarded as the generalized regular opens of the b-frame  $(X, \leq_\tau, \leq_\tau)$ , where  $\leq_\tau$  is the specialization order induced by  $\tau$ . In this case, the dual space of a Boolean algebra  $B$  is simply  $(B \setminus \mathbf{0}_B, \geq_B)$ .

In this talk, I will bring some uniformity to the various representations listed above. First, I will define the category **bFrm** of bframes and b-morphisms, and present Allwein's result as an idempotent adjunction between **bFrm** and the category of **cLat** of complete lattices and complete lattice morphisms. This adjunction restricts to a duality between its fixpoints, complete lattices and normal b-frames, i.e. b-frames which are the Allwein dual of some complete lattice. I will give a characterization of normal b-frames in a language with two relational predicates and monadic second-order quantifiers.

Finally, I will show how this equivalence restricts to several subcategories of **cLat**, including **cDL**, **cHA**, **cBA** and the categories of spatial locales and Kripke frames. In each case, I will also present first-order characterizations of the dual subcategories of normal b-frames. This can be seen as a first step towards a correspondence theory between lattice-theoretic notions and relational properties of b-frames.

## References

- [1] Gerard Thomas Allwein. “The duality of algebraic and Kripke models for linear logic.” In: (1993).
- [2] Gerard Allwein and Wendy MacCaull. “A Kripke semantics for the logic of Gelfand quantales”. In: *Studia Logica* 68.2 (2001), pp. 173–228.
- [3] Guram Bezhanishvili and Wesley Halcrow Holliday. “Locales, nuclei, and Dragalin frames”. In: *Advances in Modal Logic* 11 (2016).
- [4] Nick Bezhanishvili and Wesley Holliday. “Choice-free Stone Duality”. In: *Journal of Symbolic Logic* (forthcoming).
- [5] Chrysafis Hartonas. “Duality for lattice-ordered algebras and for normal algebraizable logics”. In: *Studia Logica* 58.3 (1997), pp. 403–450.
- [6] Guillaume Massas. “Possibility spaces, Q-completions and Rasiowa-Sikorski lemmas for non-classical logics”. In: *ILLC Master of Logic Thesis* (2016).
- [7] Gaisi Takeuti and Wilson M Zaring. *Axiomatic set theory*. Vol. 8. Springer Science & Business Media, 2013.
- [8] Alfred Tarski. “Der Aussagenkalkül und die Topologie”. In: *Fundamenta Mathematicae* (1938).
- [9] Alasdair Urquhart. “A topological representation theory for lattices”. In: *Algebra Universalis* 8.1 (1978), pp. 45–58.

# Free Kleene algebras with domain

Brett McLean

Université Nice Sophia Antipolis, Nice, France  
brett.mclean@unice.fr

We identify the finitely generated free algebras of the variety generated by algebras of binary relations equipped with the ‘domain’ operation in addition to the Kleene algebra operations. Elements of the free algebras are ‘regular’ sets of pointed labelled trees.

Let  $\text{Rel}(\cdot, +, *, 0, 1)$  denote the isomorphic closure of the class of all algebras of binary relations, with  $\cdot$  interpreted as relational composition,  $+$  as union,  $*$  as reflexive, transitive closure,  $0$  as the empty relation, and  $1$  as the universal relation  $X \times X$  on the base set  $X$ . It is easy to see that  $\text{Rel}(\cdot, +, *, 0, 1)$  is not a first-order axiomatisable class, not even closed under elementary equivalence, by a simple argument showing that  $\text{Rel}(\cdot, +, *, 0, 1)$  is not closed under ultrapowers. However, if we let  $\mathbf{V} = \text{HSP Rel}(\cdot, +, *, 0, 1)$ —the variety generated by  $\text{Rel}(\cdot, +, *, 0, 1)$ —it is well known that the free  $\mathbf{V}$ -algebra over the finite set  $\Sigma$  is the set of all regular languages over the alphabet  $\Sigma$  (with the operations of language concatenation, union, and so on). Although  $\mathbf{V}$  has no finite equational axiomatisation we do however have Kozen’s quasivariety of *Kleene algebras*, defined by a finite number of equations/quasiequations and generating the same variety [5].

Various unary ‘test’ operations can be defined on binary relations. Here is a selection.

- The unary operation  $D$  is the operation of taking the diagonal of the **domain** of a relation:

$$D(R) = \{(x, x) \in X^2 \mid \exists y \in X : (x, y) \in R\}.$$

- The unary operation  $R$  is the operation of taking the diagonal of the **range** of a relation:

$$R(R) = \{(y, y) \in X^2 \mid \exists x \in X : (x, y) \in R\}.$$

- The unary operation  $A$  is the operation of taking the diagonal of the **antidomain** of a relation—those points of  $X$  at which the image of the relation is empty:

$$A(R) = \{(x, x) \in X^2 \mid \nexists y \in X : (x, y) \in R\}.$$

The term *Kleene algebra with domain* refers to a certain algebraic theory extending Kozen’s Kleene algebra with a domain operation and some associated algebraic laws [2]. One intended model for this theory is algebras of binary relations in the signature  $(\cdot, +, *, 0, 1, D)$ , and it is hoped that the theory will prove useful for reasoning about the actions of nondeterministic computer programs [1].

Indeed, one can vary the operations from those of  $\text{Rel}(\cdot, +, *, 0, 1)$  and/or restrict the binary relations to some particular form and the resulting class will generate a variety whose free algebras should be identified.

Restricting the binary relations to be some type of function (total functions, partial functions, or injective partial functions, for example) tends to yield free algebras whose elements are a ‘single object’, rather than a ‘set of objects’. The class of semigroups, for example, is the variety generated by  $\text{Tot}(\cdot)$ —algebras of total functions with composition—and an element of



a free semigroup is a single string. Similarly, elements of free groups are also strings, groups forming the variety generated by *bijective* functions, with the familiar operations.

There is also an observable pattern when tests operations are added to the signature: strings are replaced by (labelled) trees. The following results are known.

1. The variety generated by  $\text{Inj}(\cdot, \cdot^{-1})$ , algebras of injective partial functions with composition and inverse, is the variety generated by the inverse semigroups [10, 7]. Elements of free inverse semigroups are certain trees, so-called *Munn trees* [6].
2. The isomorphic closure of the class  $\text{Par}(\cdot, D)$ —partial functions with composition and domain—is a variety [9], most commonly known as the *restriction semigroups*. A description of the free algebras has been given, and again, elements can be viewed as trees [3].
3. The isomorphic closure of the class  $\text{Par}(\cdot, D, R)$ —partial functions with composition, domain, and range—is a proper quasivariety; a finite quasiequational axiomatisation was given by Schein [8]. Once more, a description of the free algebras has been given, and elements can be viewed as trees [4].

Having noted that *binary relations*  $\rightsquigarrow$  *sets*, *functions*  $\rightsquigarrow$  *singletons*, and *tests*  $\rightsquigarrow$  *trees*, one can anticipate that when tests are added to the case  $\text{Rel}(\cdot, +, *, 0, 1)$ , elements of free algebras will be sets of labelled trees.

This talk is focused on the free algebras for the Kleene algebra with domain signature. Let  $\mathbf{W} = \text{HSP Rel}(\cdot, +, *, 0, 1, D)$  and let  $\Sigma$  be a finite alphabet. Let  $\mathcal{R}_\Sigma$  be the set of reduced pointed  $\Sigma$ -labelled rooted trees, a pointed tree being one with a distinguished point (in addition to the root). We will explain how the free  $\mathbf{W}$ -algebra consists of certain ‘regular’ subsets of  $\mathcal{R}_\Sigma$  and describe the proof of this.

## References

- [1] Jules Desharnais, Bernhard Möller, and Georg Struth. Kleene algebra with domain. *ACM Transactions on Computational Logic*, 7(4):798–833, 2006.
- [2] Jules Desharnais and Georg Struth. Internal axioms for domain semirings. *Science of Computer Programming*, 76(3):181–203, 2011.
- [3] John Fountain. Free right type A semigroups. *Glasgow Mathematical Journal*, 33(2):135–148, 1991.
- [4] John Fountain, Gracinda M. S. Gomes, and Victoria Gould. The free ample monoid. *International Journal of Algebra and Computation*, 19(04):527–554, 2009.
- [5] Dexter Kozen. A completeness theorem for Kleene algebras and the algebra of regular events. *Information and Computation*, 110(2):366–390, 1994.
- [6] Walter D. Munn. Free inverse semigroups. *Proceedings of the London Mathematical Society*, 3(3):385–404, 1974.
- [7] Gordon B. Preston. Inverse semi-groups. *Journal of the London Mathematical Society*, s1-29(4):396–403, 1954.
- [8] Boris M. Schein. Restrictively multiplicative algebras of transformations. *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, 95(4):91–102, 1970.
- [9] Valentin S. Trokhimenko. Menger’s function systems. *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, pages 71–78, 1973.
- [10] Viktor V. Wagner. Generalised groups. *Proceedings of the USSR Academy of Sciences*, 84:1119–1122, 1952.

# Singly generated quasivarieties and residuated structures

T. Moraschini<sup>1,\*</sup>, J.G. Raftery<sup>2</sup>, and J.J. Wannenburg<sup>2,3</sup>

<sup>1</sup> Institute of Computer Science, Academy of Sciences of the Czech Republic,  
Pod Vodárenskou věží 2, 182 07 Prague 8, Czech Republic.  
`moraschini@cs.cas.cz`

<sup>2</sup> Department of Mathematics and Applied Mathematics, University of Pretoria, Private Bag X20,  
Hatfield, Pretoria 0028, South Africa  
`jamie.wannenburg@up.ac.za`

<sup>3</sup> DST-NRF Centre of Excellence in Mathematical and Statistical Sciences (CoE-MaSS), South Africa

Familiar logics often have an algebraic counterpart that is a quasivariety  $\mathbf{K}$  of algebras; in many cases it is a variety. In this situation, the derivable inference rules of the logic may or may not be determined by a single set of ‘truth tables’, i.e., by the operation tables of a single algebra  $\mathbf{A} \in \mathbf{K}$ . It turns out that if some member of  $\mathbf{K}$  determines the *finite* rules of the logic, then another member determines *all* of the rules, so what is needed is only that  $\mathbf{K}$  be generated by a single algebra—briefly,  $\mathbf{K} = \mathbb{Q}(\mathbf{A})$  for some  $\mathbf{A} \in \mathbf{K}$ . Even when  $\mathbf{K}$  is a variety, it must be generated *as a quasivariety* by one of its members, if the generator is to determine rules (as opposed to theorems only).

Obviously, classical propositional logic has this property: its algebraic counterpart—the variety of Boolean algebras—is generated as a quasivariety by its unique two-element member. More surprisingly, the same holds for intuitionistic propositional logic (though not with a finite algebra), and for the relevance logic  $\mathbf{R}$  [7], but not for its conservative expansion  $\mathbf{R}^{\mathbf{t}}$  (with the so-called Ackermann constants of [1]). In the intuitionistic case, the algebra determining the (possibly infinite) rules cannot be countable [8].

Maltsev [5] proved that a quasivariety  $\mathbf{K}$  is generated by a single algebra iff it has the *joint embedding property* (JEP), i.e., any two nontrivial members of  $\mathbf{K}$  can both be embedded into some third member. Los and Suszko [4] characterized this demand by a syntactic ‘relevance principle’. Various strengthenings of the JEP have received attention in the literature. One of these, called *structural completeness*, asks (in effect) that a quasivariety be generated by its free  $\aleph_0$ -generated member (see [2]). A weaker variant, now called *passive structural completeness* (PSC), amounts to the demand that any two nontrivial members of  $\mathbf{K}$  have the same existential positive theory [9]. Note that this property is hereditary, unlike structural completeness and the JEP.

Our original goal was to investigate these completeness properties for classes of De Morgan monoids (i.e., the models of  $\mathbf{R}^{\mathbf{t}}$ ). It became clear, however, that in many of our results, large parts of the proofs had a general universal algebraic (or even model-theoretic) character, so the first half of this talk concerns such generalities. We call  $\mathbf{K}$  a *Kollár quasivariety* (after [3]) if its nontrivial members lack trivial subalgebras. We prove the following.

**Theorem 1.** *If a quasivariety is PSC, then it has the JEP.*

**Theorem 2.** *If a Kollár quasivariety  $\mathbf{K}$  has the JEP, then its relatively simple members all belong to the universal class generated by one of them. If, in addition,  $\mathbf{K}$  is relatively semisimple, then it is generated (as a quasivariety) by one  $\mathbf{K}$ -simple algebra.*

**Theorem 3.** *A quasivariety of finite type with a finite nontrivial member is PSC iff its nontrivial members have a common retract.*

---

\*Presenter

The second half of the talk deals with (quasi)varieties of De Morgan monoids. A *De Morgan monoid* is a distributive lattice-ordered commutative monoid with a compatible involution  $\neg$ , satisfying  $x \leq x^2 := x \cdot x$ . It is called an *odd Sugihara monoid* if its neutral element  $e$  is a fixed point of  $\neg$ , in which case it satisfies  $x = x^2$ . The varieties of odd Sugihara monoids form a transparent chain of order type  $\omega + 1$ . There are just two simple 0-generated four-element De Morgan monoids,  $\mathbf{C}_4$  and  $\mathbf{D}_4$ . The former is a chain in which  $e < f := \neg e$ . In the latter,  $e$  and  $f$  are incomparable. There is a largest variety  $\mathbf{M}$  of De Morgan monoids such that each member of  $\mathbf{M}$  has  $\mathbf{C}_4$  as a retract or is trivial. This  $\mathbf{M}$  is axiomatized, relative to De Morgan monoids, by  $e \leq f$  and  $x \leq f^2$  and  $f^2 \cdot \neg((f \cdot x) \wedge (f \cdot \neg x)) = f^2$  (see [6]).

Among other results, we describe completely the varieties of De Morgan monoids that are PSC, and we characterize those with the JEP. Specifically:

**Theorem 4.** *A variety  $\mathbf{K}$  of De Morgan monoids is PSC iff it is the variety of Boolean algebras or the variety generated by  $\mathbf{D}_4$  (briefly,  $\mathbb{V}(\mathbf{D}_4)$ ) or a variety of odd Sugihara monoids or a subvariety of  $\mathbf{M}$ . (In the first three cases,  $\mathbf{K}$  is structurally complete.)*

**Theorem 5.** *A variety  $\mathbf{K}$  of De Morgan monoids has the JEP iff one of the following (mutually exclusive) conditions holds: (1)  $\mathbf{K}$  is PSC; (2)  $\mathbf{K} = \mathbb{V}(\mathbf{A})$  for some simple De Morgan monoid  $\mathbf{A}$  such that  $\mathbf{D}_4$  is a proper subalgebra of  $\mathbf{A}$ ; (3) there exist  $\mathbf{A}, \mathbf{B}$  such that  $\mathbf{K} = \mathbb{Q}(\mathbf{B})$ ,  $\mathbf{A}$  is a simple subalgebra of  $\mathbf{B}$ , and  $\mathbf{C}_4$  is a proper subalgebra of  $\mathbf{A}$ .*

It follows from Theorem 4 that the structurally complete varieties of De Morgan monoids fall into two classes—a denumerable family that is fully transparent and a more opaque collection of subvarieties of  $\mathbf{M}$ . In the subvariety lattice of  $\mathbf{M}$ , the variety  $\mathbb{V}(\mathbf{C}_4)$  has just six covers [6]. In the join of these six covers, every subquasivariety  $\mathbf{K}$  is a variety (whence every such  $\mathbf{K}$  is structurally complete). We prove that  $\mathbf{M}$  has uncountably many structurally incomplete subvarieties as well, by exhibiting  $2^{\aleph_0}$  structurally incomplete varieties of Brouwerian algebras (of depth 3) and applying a suitable ‘reflection’ construction.

## References

- [1] A.R. Anderson, N.D. Belnap, Jr., ‘Entailment: The Logic of Relevance and Necessity, Vol. 1’, Princeton University Press, 1975.
- [2] C. Bergman, *Structural completeness in algebra and logic*, in H. Andréka, J.D. Monk and I. Nemeti (eds.), ‘Algebraic Logic’, Colloquia Mathematica Societatis János Bolyai Vol. 54, North-Holland, Amsterdam, 1991, pp. 59–73.
- [3] J. Kollár, *Congruences and one-element subalgebras*, Algebra Universalis **9** (1979), 266–267.
- [4] J. Łoś, R. Suszko, *Remarks on sentential logics*, Proc. Kon. Nederl. Akad. van Wetenschappen, Series A **61** (1958), 177–183.
- [5] A.I. Maltsev, *Several remarks on quasivarieties of algebraic systems*, Algebra i Logika **5** (1966), 3–9 (Russian).
- [6] T. Moraschini, J.G. Raftery, J.J. Wannenburg, *Varieties of De Morgan monoids: covers of atoms*, Rev. Symb. Log., to appear.
- [7] M. Tokarz, *The existence of matrices strongly adequate for  $E$ ,  $R$  and their fragments*, Studia Logica **38** (1979), 75–85.
- [8] A. Wroński, *On cardinalities of matrices strongly adequate for the intuitionistic propositional logic*, Rep. Math. Logic **2** (1974), 55–62.
- [9] A. Wroński, *Overflow rules and a weakening of structural completeness*, in J. Sytnik-Czetwertyński (ed.), ‘Rozważania o Filozofii Prawdziwej. Jerzemu Perzanowskiemu w Darze’, Wydawnictwo Uniwersytetu Jagiellońskiego, Kraków, 2009, pp. 67–71.

# Epimorphisms in Varieties of Heyting Algebras\*

T. Moraschini<sup>1</sup> and J.J. Wannenburg<sup>2,3†</sup>

<sup>1</sup> Institute of Computer Science, Academy of Sciences of the Czech Republic,  
Pod Vodárenskou věží 2, 182 07 Prague 8, Czech Republic.  
`moraschini@cs.cas.cz`

<sup>2</sup> Department of Mathematics and Applied Mathematics, University of Pretoria, Private Bag X20,  
Hatfield, Pretoria 0028, South Africa  
`jamie.wannenburg@up.ac.za`

<sup>3</sup> DST-NRF Centre of Excellence in Mathematical and Statistical Sciences (CoE-MaSS), South Africa

A homomorphism  $f: \mathbf{A} \rightarrow \mathbf{B}$  between algebras in a variety  $\mathbf{K}$  is an *epimorphism* provided it is right-cancellative, i.e., for every pair of homomorphisms  $g, h: \mathbf{B} \rightarrow \mathbf{C}$  with  $\mathbf{C} \in \mathbf{K}$ ,

$$\text{if } g \circ f = h \circ f, \text{ then } g = h.$$

Clearly any surjective homomorphism is an epimorphism. We say that  $\mathbf{K}$  has the *epimorphism surjectivity (ES) property* if the converse holds.

When a variety  $\mathbf{K}$  algebraizes a logic  $\vdash$ , then  $\mathbf{K}$  has the ES property if and only if  $\vdash$  has the *infinite (deductive) Beth (definability) property*, i.e., whenever an arbitrarily large set  $Z$  of variables is defined *implicitly* in terms of other variables by means of some formulas over  $\vdash$ , then it can also be defined *explicitly* [4]. When this is demanded only for *finite*  $Z$ , then  $\vdash$  is said to have the *finite Beth property*, which corresponds similarly to the so-called *weak ES property*. This invites us to question which varieties of Heyting algebras have surjective epimorphisms, or equivalently which intermediate logics have the infinite Beth property.

Classic results of Kreisel and Maksimova, respectively, state that *all* varieties of Heyting algebras have the weak ES property [6], while only finitely many of them have a certain stronger version of it [3, 8, 9]. No simple characterization of the (unqualified) ES property for Heyting algebra varieties is known, however. One of the few general positive results on the topic yields a continuum of varieties *with* the ES property:

**Theorem 1** ([2, Thm. 5.3]). *If a variety of Heyting algebras has finite depth, then it has surjective epimorphisms.*

In fact, until now, only one variety of Heyting algebras *lacking* the ES property was identified in the literature [2, Cor. 6.2]. This variety is generated by an algebra  $\mathbf{D}$ , whose lattice reduct is depicted below. Actually,  $\mathbb{V}(\mathbf{D})$  was the first example showing that the weak ES property is indeed strictly weaker than the ES property, as conjectured by Blok and Hoogland in [4]. This raises the question: is it rare for varieties of Heyting algebras to have non-surjective epimorphisms? We demonstrate that this is not the case by disproving the ES property for a wide range of well-known varieties. Let  $\mathbf{RN}$  denote the Rieger-Nishimura lattice, which is depicted below. We prove:

---

\*This work received funding from the European Unions Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 689176 (project “Syntax Meets Semantics: Methods, Interactions, and Connections in Substructural logics”). The first author was also supported by project CZ.02.2.69/0.0/0.0/17 050/0008361, OPVVV MŠMT, MSCA-IF Lidské zdroje v teoretické informatice. The second author was supported by the DST-NRF Centre of Excellence in Mathematical and Statistical Sciences (CoE-MaSS), South Africa. Opinions expressed and conclusions arrived at are those of the authors and are not necessarily to be attributed to the CoE-MaSS

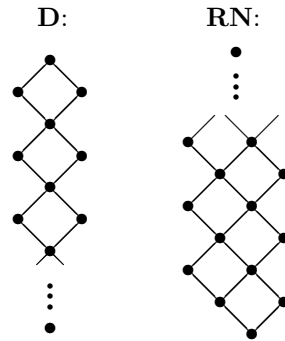
†Presenter

**Theorem 2.** *The ES property fails for all the varieties in the interval  $[\mathbb{V}(\mathbf{D}), \mathbb{V}(\mathbf{RN})]$ . Moreover, this interval contains a continuum of locally finite varieties.*

In contrast with Theorem 1, we also prove:

**Theorem 3.** *For every integer  $n \geq 2$ , the variety of all Heyting algebras with width at most  $n$  does not have the ES property.*

Recall that the Kuznetsov-Gerčiu variety  $\mathbf{KG}$  is generated by all finite linear sums of one-generated Heyting algebras [1, 5, 7]. We describe exactly the subvarieties of  $\mathbf{KG}$  that have surjective epimorphisms, and those that do not. Although the ES property is not generally hereditary, our description implies that, when a variety  $\mathbf{K} \subseteq \mathbf{KG}$  has the ES property, then so do all subvarieties of  $\mathbf{K}$ . Another consequence is that all subvarieties of  $\mathbf{KG}$  with surjective epimorphisms are locally finite. Finally, our description yields an alternative proof of the well-known fact that every variety of Gödel algebras has surjective epimorphisms.



## References

- [1] G. Bezhanishvili, N. Bezhanishvili, D. de Jongh. *The Kuznetsov-Gerčiu and Rieger-Nishimura logics: the boundaries of the finite model property*, Logic and Logical Philosophy, **17** (2008), 73–110.
- [2] G. Bezhanishvili, T. Moraschini, J.G. Raftery, *Epimorphisms in varieties of residuated structures*, J. Algebra **492** (2017), 185–211.
- [3] D.M. Gabbay, L. Maksimova, ‘Interpolation and Definability: Modal and Intuitionistic Logics’, Oxford Logic Guides 46, Clarendon Press, Oxford, 2005.
- [4] W.J. Blok, E. Hoogland, *The Beth property in algebraic logic*, Studia Logica **83** (2006), 49–90.
- [5] V. J. Gerčiu, A. V. Kuznetsov, *The finitely axiomatizable superintuitionistic logics*, Soviet Mathematics Doklady, **11** (1970), 1654–1658.
- [6] G. Kreisel, *Explicit definability in intuitionistic logic*, J. Symbolic Logic **25** (1960), 389–390.
- [7] A. V. Kuznetsov, V. J. Gerčiu, *Superintuitionistic logics and finite approximability*, Soviet Mathematics Doklady, **11** (1970), 1614–1619.
- [8] L.L. Maksimova, *Intuitionistic logic and implicit definability*, Ann. Pure Appl. Logic **105** (2000), 83–102.
- [9] L.L. Maksimova, *Implicit definability and positive logics*, Algebra and Logic **42** (2003), 37–53.

# Exact and Fitted Sublocales

Rick Ball<sup>1</sup>, M. Andrew Moshier<sup>2</sup>, and Aleš Pultr<sup>3</sup>

<sup>1</sup> University of Denver  
Denver, Colorado

<sup>2</sup> Chapman University  
Orange, California

<sup>3</sup> Charles University  
Prague, Cech Republic

In point-free topology, the sublocales of a frame are the point-free analogues of subspaces. Ordered to comport with this view, the sublocales of a frame do not constitute a frame, but rather a coframe  $\mathbf{S}(L)$ . Within  $\mathbf{S}(L)$ , a sublocale is called *fitted* if it is a meet of open sublocales. The analogous, seemingly dual, joins of closed sublocales are also important but do not seem to have a convenient name. In this work, we characterize both sorts of sublocales as arising from certain filters on the underlying frame. In particular, the filters corresponding to joins of closed sublocales are themselves the exact filters (defined below). Thus we propose to call these sublocales *exact*. On the other hand, fitted, being quite a natural name for meets of opens, is now also a suitable name for those corresponding filters.

Let  $\mathbf{S}_C(L)$  and  $\mathbf{S}_O(L)$  denote the collections consisting of joins of closed sublocales (exact sublocales) and meets of open sublocales (fitted sublocales), respectively. Quite a lot is already known about these (see for example [2, 4, 7]). For example,  $\mathbf{S}_C(L)$  is a frame (appearing as a join sublattice of  $\mathbf{S}(L)$ ). Under the rather mild condition of subfitness [5] of  $L$ , this frame is in fact Boolean. Also when  $L$  is the topology of a  $T_1$  space,  $\mathbf{S}_C(L)$  is precisely the frame of sublocales induced by sets of points. This and other considerations lead some researchers [6], to take  $\mathbf{S}_C(L)$  to be the natural point-free analogue of a discrete topology over  $L$ , so that frame maps  $M \rightarrow \mathbf{S}_C(L)$  represent general (non-continuous) point-free maps from  $L$ .

Define the following functions  $\mathcal{P}(L) \rightarrow \mathbf{S}(L)$ :

$$M(A) = \bigwedge_{a \in A} \mathfrak{o}(a)$$

$$J(A) = \bigvee_{a \in A} \mathfrak{c}(a)$$

where  $\mathfrak{o}(a)$  denotes the open sublocale and  $\mathfrak{c}(a)$ , the closed sublocale determined by  $a$ . Evidently,  $M$  is antitonic and  $J$  is monotonic. By definition, the image of  $M$  is  $\mathbf{S}_O(L)$  and the image of  $J(A)$  is  $\mathbf{S}_C(L)$ . Moreover, clearly  $M(\bigcup_i A_i) = \bigwedge_i M(A_i)$ ;  $J(\bigcup_i A_i) = \bigvee_i J(A_i)$ . Consequently,  $M$  possesses a dual right adjoint and  $J$  possesses a right adjoint defined by

$$M_*(S) = \{a \in L \mid S \subseteq \mathfrak{o}(a)\}$$

$$J^*(S) = \{a \in L \mid \mathfrak{c}(a) \subseteq S\}$$

So  $A \subseteq M_*(S)$  if and only if  $S \subseteq M(A)$ , and  $A \subseteq J^*(S)$  if and only if  $J(A) \subseteq S$ .

In this work, we investigate the kernels of these two adjunctions — that is, the subsets of  $L$  satisfying  $M_*(M(A)) = A$  or  $J^*(J(A)) = A$ . To start the analysis rather trivially, open sublocales are closed under finite meets, and closed sublocales are closed under finite joins. Thus on the face of things, the relevant subsets are always filters in  $L$ . This is good news. In addition to filters being generally well-understood, the filters on any distributive lattice

constitute a frame. So by characterizing those filters  $F$  for which  $M(F) \subseteq \mathfrak{o}(a)$  implies  $a \in F$ , and separately those for which  $\mathfrak{c}(a) \subset J(F)$  implies  $a \in F$ , and we can shed light how  $S_C(L)$  and  $S_O(L)$  relate to the frame of general filters.

In a lattice, an *exact meet* is a meet  $\bigwedge_i b_i$  for which  $a \vee \bigwedge_i b_i = \bigwedge_i a \vee b_i$  for all  $a$ . Evidently, a lattice is distributive if and only if every finite meet is exact, and a lattice is a co-frame if and only if every meet is exact, and the dual notion of exact join characterizes frames. In the context of distributive lattices, we can regard exact meets as a generalization of finite ones, leading naturally to the concept of an exact filter, closed under all the exact meets. The notion of exactness appeared first (under another name and in its dual formulation) in [3] in the study of injective hulls of semilattices. Exactness has also proved to be useful for other purposes, for example in [1, 2].

Our first main result is that the filters of the form  $J^*(S)$  are precisely the exact filters, thus justifying the name *exact sublocales*. Consequently,  $S_C(L)$  is not only a frame, but is precisely a sublocale of the frame of filters, and specifically is the injective hull of the meet semilattice reduct of  $L^{\text{op}}$ . We will discuss some of the applications of this observation.

The second part of our investigation is to characterize the *fitted* filters, corresponding to fitted sublocales. The situation here is more complicated. Although meets of sublocales are merely intersections, a criterion for an open sublocale to contain an intersection of open sublocales is complicated. We discuss a suitable criterion that leads to characterizing the fitted filters.

## References

- [1] R.N. Ball, *Distributive Cauchy lattices*, Algebra Univ. **18** (1984), 134-174
- [2] R.N. Ball, J. Picado and A. Pultr, *Notes on exact meets and joins*, Appl.Categor.Struct. **22** (2014), 699-714
- [3] G. Bruns and H. Lakser, *Injective hulls of semilattices*, Canad.Math.Bull. **13** (1970), 115-118
- [4] M.M. Clementino, J. Picado, and A. Pultr, *The Other Closure and Complete Sublocales*, Appl.Categor.Struct. **26** (2018), 892-906, corr. 907-908
- [5] J. R. Isbell, *Atomless parts of spaces*, Math. Scand. **31** (1972), 5-32.
- [6] J. Picado and A. Pultr, *A Boolean extension of a frame and a representation of discontinuity*, Quaestiones Math. **40,8** (2017), 1111-1125
- [7] J. Picado, A. Pultr and A. Tozzi, *Joins of closed sublocales*, Houston J. of Math. **45,1** (2019), 21-38



# The undecidability of profiniteness

Anvar M. Nurakunov<sup>1</sup> and Michał M. Stronkowski<sup>2</sup>

<sup>1</sup> Institute of Mathematics, National Academy of Sciences, Bishkek, Kyrgyzstan  
a.nurakunov@gmail.com

<sup>2</sup> Warsaw University of Technology, Poland  
m.stronkowski@mini.pw.edu.pl

A topological algebra is *profinite* if it is isomorphic to an inverse limit of finite algebras endowed with the discrete topology. Their topologies are always Boolean, i.e., Hausdorff, compact and totally disconnected. However, not all Boolean topological algebras are profinite. As an example, one may take any Boolean topological algebras whose algebraic reduct is subdirectly irreducible (has a least nontrivial congruence).

Let  $\mathcal{V}$  be a variety (an equationally defined class of algebras). We consider the class  $\mathcal{V}_{Bt}$  of Boolean topological algebras with the algebraic reducts in  $\mathcal{V}$ , and the class  $\mathcal{V}_{Bc}$  of profinite algebras with the algebraic reducts in  $\mathcal{V}$ . The class  $\mathcal{V}_{Bc}$  is called the *Boolean core* of  $\mathcal{V}$ . A general problem is the axiomatization of  $\mathcal{V}_{Bc}$  relative to  $\mathcal{V}_{Bt}$ . Or, more precisely, when we can axiomatize  $\mathcal{V}_{Bc}$  relative to  $\mathcal{V}_{Bt}$  without referring to topology. (Indeed, in [4] a general scheme for axiomatizations of even more general classes of topological algebras with the use of topology was given.)

The basic question is simply when  $\mathcal{V}_{Bc} = \mathcal{V}_{Bt}$ ? If it is the case we say that  $\mathcal{V}$  is *standard*. It appears that it is true for many varieties of classical algebras like varieties of groups, rings, semigroups, distributive lattices or Heyting algebras. A property which implies standardness and is shared by the listed varieties was discovered in [2]. Let  $\mathbf{A}$  be an algebra and  $\theta$  be an equivalence relation on the carrier of  $\mathbf{A}$ . Let  $\text{Syn}(\theta)$  be the largest congruence of  $\mathbf{A}$  contained in  $\theta$ . Note that

$$\text{Syn}(\theta) = \{(a, b) \mid (t(a, \bar{c}), t(b, \bar{c})) \in \theta \text{ for every term } t \text{ and every tuple } \bar{c}\}.$$

A variety  $\mathcal{V}$  has *finitely determined syntactic congruences* (FDSC for short) if there is a finite set  $T$  of terms such that for every  $\mathbf{A} \in \mathcal{V}$  and every equivalence relation  $\theta$  on the carrier of  $\mathbf{A}$  we have

$$\text{Syn}(\theta) = \{(a, b) \mid (t(a, \bar{c}), t(b, \bar{c})) \in \theta \text{ for every term } t \in T \text{ and every tuple } \bar{c}\}.$$

Clark, Davey, Freese and Jackson proved in [2] that the property of having FDSC indeed yields standardness.

Still, already in [6, Section VI.2.6] Johnstone speculated that it may be hard to give a simple condition for varieties which is both necessary and sufficient for standardness. And, somehow confirming this speculation, Jackson proved in [5] that there is no algorithm which decides if a given finite set of identities defines a standard variety or a variety with FDSC. Our main result is a proof of a similar fact, but for finitely generated varieties [9].

**Theorem 1.** *There is no algorithm which decides if a given finite algebra of a finite type generates a standard variety or a variety with FDSC.*

From the perspective of axiomatization of  $\mathcal{V}_{Bc}$  relative to  $\mathcal{V}_{Bt}$ , standardness describes just the best (the simplest) possible situation. In [3] a weakening to the first-order axiomatization was proposed. Also, a technique for showing the lack of such axiomatizations was presented. With the use of this technique, we obtained the following fact [9].



**Theorem 2.** *There is no algorithm which decides if a given finite algebra of a finite type generates a variety  $\mathcal{V}$  such that  $\mathcal{V}_{Bc}$  is first-order axiomatizable relative to  $\mathcal{V}_{Bt}$ .*

Let us say at this place that we would not obtain these results without earlier works of McKenzie and Moore. In [7] McKenzie presented a construction which effectively assigns to each Turing machine  $\mathbf{T}$  the algebra  $A(\mathbf{T})$  such that  $\mathbf{T}$  halts iff there is a finite bound on the cardinality of subdirectly irreducible algebras in the variety generated by  $A(\mathbf{T})$ . In [8] Moore modified this construction to  $A'(\mathbf{T})$  and proved that a Turing machine  $\mathbf{T}$  halts iff the variety generated by  $A'(\mathbf{T})$  has definable principal subcongruences. This property was invented by Baker and Wang in the context of finitely axiomatizable varieties [1]. As noted in [2], proving that the variety generated by  $A(\mathbf{T})$  or by  $A'(\mathbf{T})$  has FDSC when  $\mathbf{T}$  halts would yield Theorem 1. Moore observed that it is not true for McKenzie’s algebras  $A(\mathbf{T})$ . But the question for algebras  $A'(\mathbf{T})$  was open.

We have not worked with the algebra  $A'(\mathbf{T})$  directly. Instead, we simply proved that having definable principal subcongruences yields having FDSC for varieties. It is worth emphasizing that this connection was not expected.

Let us finish with a remark that in [2, 3] the issue of axiomatization of Boolean cores relative to the classes of Boolean topological structures was presented for universal Horn classes, not for varieties. This was more natural from the perspective of duality theory. However, it seems, today we do not have tools to attack the problem of (un)decidability of standardness for finitely generated universal Horn classes.

## References

- [1] Kirby A. Baker and Ju Wang. Definable principal subcongruences. *Algebra Universalis*, 47(2):145–151, 2002.
- [2] David M. Clark, Brian A. Davey, Ralph S. Freese, and Marcel Jackson. Standard topological algebras: syntactic and principal congruences and profiniteness. *Algebra Univers.*, 52:343–376, 2004.
- [3] David M. Clark, Brian A. Davey, Marcel G. Jackson, and Jane G. Pitkethly. The axiomatizability of topological prevarieties. *Adv. Math.*, 218(5):1604–1653, 2008.
- [4] David M. Clark and Peter H. Krauss. Topological quasivarieties. *Acta Sci. Math. (Szeged)*, 47(1-2):3–39, 1984.
- [5] Marcel Jackson. Residual bounds for compact totally disconnected algebras. *Houston J. Math*, 34(1), 2008.
- [6] Peter T. Johnstone. *Stone spaces*, volume 3 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1986. Reprint of the 1982 edition.
- [7] Ralph McKenzie. The residual bound of a finite algebra is not computable. *Internat. J. Algebra Comput.*, 6(1):29–48, 1996.
- [8] Matthew Moore. The undecidability of the definability of principal subcongruences. *J. Symb. Log.*, 80(2):384–432, 2015.
- [9] Anvar M. Nurakunov and Michał M. Stronkowski. Profiniteness in finitely generated varieties is undecidable. *J. Symb. Log.*, 83(4):1566–1578, 2018.

# Overview and perspectives of the general construction of spectra

Axel Osmond

Some of the most prominent dualities between spaces and algebras, as Stone-like dualities for ordered structures or scheme construction for commutative rings, are instantiations of a common construction, namely the spectral construction. The general template is a duality between a category of algebra-like objects, and a corresponding category of “structured spaces”. Those are space-like objects (topos, posites, proper spaces...) equipped with a distinguished sheaf of algebras, with a specific class of “local objects” as stalks. The duality consists of a contravariant adjunction between:

- a spectrum functor assigning to any algebra a certain structured space,
- and a global sections functor reconstructing algebraic objects from the geometric information attached to structured spaces.

The exact construction of such a spectrum functor entangles model theoretical, categorical, topological and geometric aspects. While the general philosophy was initiated by Hakim[7] in the specific context of rings, the first general method was suggested by Cole[2] who isolated the central notion of admissibility and proceeded with the construction in an abstract 2-categorical way. In Coste[3] the syntactical and topological aspects were first made explicit. In the more recent works of Dubuc[6] and Lurie[9] the topological and geometrical aspects were developed, while Anel[1] provided a deep topological interpretation. In parallel Diers[4] gave a lesser known yet handy categorical interpretation of this construction. Here we provide a synthesis of those different methods, identifying the bridges between them. At the end we also give an insight into ongoing developments.

The core concept is that of “structure of admissibility” as defined in the seminal work of Cole[2]. The context to start with is the data of i) a locally finitely presentable category of models of a given essentially algebraic theory ii) together with a geometric extension of this theory, coding for local objects iii) and a given factorization system; the morphisms in the left class are therein called etale transformations while those in the right class are the local transformations.

Then admissibility defines how the “global data” encoded by algebraic objects have to be related to the “local data” encoded by local objects and local transformations. This relation just states that in the ambient category, morphisms toward local objects admit an initial factorization with a local transformation on the right. Equivalently, it can be rephrased, as in Diers[4], as a situation of multireflectivity of the category made of local objects plus local maps amongst the ambient category of local objects.

Now it happens that admissibility encodes topological behaviors in the opposite category of the category of global objects, as described in Anel[1]. Maps that are dual to the etale transformations behave as open inclusions, and define a generalized specialization order between the points. Then, if we gather the etale transformations under a given global object  $B$ , this defines a site (and sometimes a proper space) associated to  $B$ . This is the spectrum of  $B$ , and an etale transformation from  $B$  to a local object just exhibits the local object as a “point” of the spectrum of  $B$ . The structural sheaf of the spectrum is then defined from the presheaf returning the codomain of etale morphisms with domain  $B$ . In particular it returns as stalks “local objects under  $B$ ”.

This spectrum will play the role of a free object for the theory of local objects in the following sense. The condition of admissibility corresponds in fact to a situation of “almost reflectivity”. Rather than having a universal free model of the theory of local objects under a global object  $B$ , one has a universal cone of “locally free” local objects that jointly behave as a free object. The spectrum of  $B$  provides then a convenient topos one can construct a free model associated to  $B$  into: this free model is just the structural sheaf that gathers the local objects under  $B$  into an object of the topos  $Sh(SpecB)$ . Hence, the spectrum deploys a geometry encoded as a solution to a defect of universality; the bigger the defect of universality, the richer the geometry.

Recall that the stalk at a point is obtained as a filtered colimit of the values of the sheaf on its neighborhoods. This means that local data can be conveniently approximated by global data. In some situations, the defined geometry enjoys a symmetric property ensuring that any global object can be reconstructed as a limit over the stalks of its structural sheaf: this is a condition of representability. As explained in Diers[5] and also in Kennison & Ledbetter[8], representability is ensured as soon as the class of local objects contains enough cogenerators.

After explaining how this general construction proceeds, we shall address some ongoing developments and perspectives:

- We will present a functorialization of the process which associates a geometry to a situation of admissibility, in order to obtain comparison functors, or construct geometries from other ones. This may be helpful in contexts of residuated lattices where plenty of interrelated varieties are still to be dualized.
- We want this construction to encompass the functor  $pt$  that takes a frame to its set of points, though the category of frames is not locally finitely presentable. However it enjoys many algebraic-like properties as a monadic category. We will give some insights into the monadic aspects of the spectral construction, in order to understand how it should be adapted to match with properly monadic situations.
- As a perspective, we shall discuss why the 2-categorification of this process is expected to capture the syntax-semantics dualities corresponding with the propositional Stone-like dualities. Models of a propositional theory  $T$  correspond with the points of the spectrum  $Spec(A_T)$  of its Lindenbaum-Tarski algebra  $A_T$ . Similarly, consider the 2-functor associating to a theory  $T$  in a certain doctrine its category of models  $Mod(T)$ . The similarities between these two functors  $Spec$  and  $Mod$  suggest that semantics functors could be constructed as 2-categorical spectra, exhibiting semantics as some kind of 2-categorical geometry.

## References

- [1] Anel, Mathieu. *Grothendieck topologies from unique factorisation systems*. arXiv:0902.1130v2. 22 Oct 2009
- [2] Cole, J.C. *The bicategory of Topoi and Spectra*. Reprints in Theory and Applications of Categories, No. 25 (2016) pp. 1-16 (TAC)
- [3] Coste, Michel. *Localisation, spectra and sheaf representation*. pp.212-238 in Fourman, Mulvey Scott (eds.), Applications of Sheaves , Springer LNM 753 (1979)
- [4] Diers, Yves. *Une construction universelle des spectres et faisceaux structuraux*. Communication in Algebra, 12(17),2141-2183 (1984).
- [5] Diers, Yves. *Un critère de représentabilité par sections continues de faisceaux*. Category Theory: Applications to Algebra, Logic and Topology. Proceedings of the International Conference Held at Gummersbach, July 6-10, 1981, pp51-61
- [6] Dubuc, Eduardo. *Axiomatic etal maps and a theory of spectrum*. Journal of pure and Applied Algebra 149 (2000) pp15-45.
- [7] Hakim, Monique. *Topos Annelés et schémas relatifs*. Springer Heidelberg 1972.
- [8] Kennison, John F. Ledbetter, Carl S. *Sheaf representations and the Dedekind reals* Application of sheaves, pp500-513
- [9] Lurie, Jacob. *Derived Algebraic Geometry V: Structured Spaces* <http://www.math.harvard.edu/~lurie/papers/DAG-V.pdf>
- [10] Taylor, Paul. *The trace factorisation of stable functors*. 1998.

# Completion of pseudo-orthomodular posets

Jan Paseka<sup>1\*</sup>

Department of Mathematics and Statistics  
Masaryk University  
Czech Republic [paseka@math.muni.cz](mailto:paseka@math.muni.cz)

It is well-known that every poset  $(P, \leq)$  can be embedded into a complete lattice  $\mathbf{L}$ . We frequently take the so-called Dedekind-MacNeille completion  $\mathbf{DM}(P, \leq)$  for this  $\mathbf{L}$ . By Schmidt [3] the Dedekind-MacNeille completion of a poset  $\mathbf{P}$  is (up to isomorphism) any complete lattice  $\mathbf{L}$  into which  $\mathbf{P}$  can be supremum-densely and infimum-densely embedded (i.e., for every element  $x \in L$  there exist  $M, Q \subseteq P$  such that  $x = \bigvee \varphi(M) = \bigwedge \varphi(Q)$ , where  $\varphi: P \rightarrow L$  is the embedding). In this paper we get some classes of pseudo-orthomodular posets for which their Dedekind-MacNeille completion is an orthomodular lattice. To obtain more information on these topic or on notions used in this paper, we direct the reader to [1] and [2].

In what follows, we will work with posets  $\mathbf{P} = (P, \leq, ', 0, 1)$  where  $'$  is an antitone involution or a complementation. The precise definition is the following.

A *poset with antitone involution* is an ordered quintuple  $\mathbf{P} = (P, \leq, ', 0, 1)$  such that  $(P, \leq, 0, 1)$  is a bounded poset and  $'$  is a unary operation on  $P$  satisfying the following conditions for all  $x, y \in P$ :

- (i)  $x \leq y$  implies  $y' \leq x'$ ,
- (ii)  $(x')' \approx x$ .

For  $M \subseteq P$  denote by  $U(M) := \{x \in P \mid y \leq x \text{ for all } y \in M\}$  the so-called *upper cone of*  $M$ , and by  $L(M) = \{x \in P \mid x \leq y \text{ for all } y \in M\}$  the so-called *lower cone of*  $M$ . If  $M = \{a, b\}$  or  $M = \{a\}$ , we will write simply  $U(a, b)$ ,  $L(a, b)$  or  $U(a)$ ,  $L(a)$ , respectively.

A *poset with complementation* is a poset with antitone involution  $\mathbf{P} = (P, \leq, ', 0, 1)$  satisfying the following LU-identities:

- (iii)  $L(x, x') \approx \{0\}$  and  $U(x, x') \approx \{1\}$ .

A subset  $S \subseteq P$  of a poset  $\mathbf{P}$  with complementation such that  $s \leq t'$  for any pair  $s, t \in S$ ,  $s \neq t$  is called *orthogonal*. A poset  $\mathbf{P}$  with complementation is called an *orthocomplete poset* if every orthogonal subset of  $\mathbf{P}$  has a supremum. A poset  $\mathbf{P}$  is said to have a *finite rank* if every orthogonal subset of  $\mathbf{P}$  is finite.

A *poset with complementation*  $\mathbf{P} = (P, \leq, ', 0, 1)$  is called *orthomodular* if for all  $x, y \in P$  with  $x \leq y'$  there exists  $x \vee y$  and then  $\mathbf{P}$  satisfies one of the following equivalent identities:

$$\begin{aligned} ((x \wedge y) \vee y') \wedge y &\approx x \wedge y, \\ ((x \vee y) \wedge y') \vee y &\approx x \vee y \end{aligned}$$

where  $x \wedge y$  stands for  $(x' \vee y')'$  (De Morgan laws).

It is known that for an orthomodular poset (in fact a lattice)  $\mathbf{P} = (P, \leq, ', 0, 1)$ , its Dedekind-MacNeille completion  $\mathbf{DM}(\mathbf{P})$  need not be an orthomodular lattice.

---

\*The author was supported by the project Group Techniques and Quantum Information, Nr. MU-NI/G/1211/2017 by Masaryk University Grant Agency (GAMU).

A poset  $\mathbf{P}$  with complementation is called a *pseudo-orthomodular poset* (see [1]) if it satisfies one of the following equivalent conditions:

$$\begin{aligned}L(U(L(x, y), y'), y) &\approx L(x, y), \\U(L(U(x, y), y'), y) &\approx U(x, y).\end{aligned}$$

Note that that an lattice with complementation is orthomodular if and only if it is pseudo-orthomodular.

It is easy to find an example of a finite pseudo-orthomodular poset  $\mathbf{P}$  such that  $\mathbf{DM}(\mathbf{P})$  is a nonmodular orthomodular lattice. Conversely, one can assume that  $\mathbf{DM}(\mathbf{P})$  is really an orthomodular lattice for a poset  $\mathbf{P}$  with complementation and ask what is  $\mathbf{P}$ . The answer is as follows.

**Theorem 1.** *Let  $\mathbf{P} = (P, \leq, ', 0, 1)$  be a complemented poset such that  $\mathbf{DM}(\mathbf{P})$  is an orthomodular lattice. Then  $\mathbf{P}$  is pseudo-orthomodular.*

An element  $a$  of a poset  $\mathbf{P}$  with least element  $0$  is an *atom* if  $0 < a$  and there is no  $x \in P$  such that  $0 < x < a$ . A poset  $\mathbf{P}$  with a least element  $0$  is *atomic* if every element  $b > 0$  has an atom  $a$  below it.

The following series of theorems and their corollaries forms the main results of our paper.

**Theorem 2.** *Let  $\mathbf{P} = (P, \leq, ', 0, 1)$  be an orthocomplete atomic orthomodular poset. The following conditions are equivalent:*

- (i)  $\mathbf{P}$  is pseudo-orthomodular.
- (ii)  $\mathbf{P}$  is a complete orthomodular lattice.
- (iii)  $\mathbf{DM}(\mathbf{P})$  is orthomodular.

From the following result we see that non-lattice finite orthomodular posets do not have orthomodular Dedekind-MacNeille completion. Hence the right generalization of orthomodularity for posets in the context of an orthomodular Dedekind-MacNeille completion is pseudo-orthomodularity.

**Corollary 3.** *Let  $\mathbf{P} = (P, \leq, ', 0, 1)$  be a finite orthomodular poset which is not a lattice. Then its Dedekind-MacNeille completion  $\mathbf{DM}(\mathbf{P})$  is not orthomodular.*

**Theorem 4.** *Let  $\mathbf{P} = (P, \leq, ', 0, 1)$  be an atomic pseudo-orthomodular poset. Then any element of  $\mathbf{P}$  is a join of an orthogonal set of atoms lying under it.*

**Theorem 5.** *Let  $\mathbf{P} = (P, \leq, ', 0, 1)$  be an atomic pseudo-orthomodular poset with finite rank. Then  $\mathbf{DM}(\mathbf{P})$  is orthomodular.*

Getting together the previous results we can formulate our closing corollary.

**Corollary 6.** *Let  $\mathbf{P} = (P, \leq, ', 0, 1)$  be a finite pseudo-orthomodular poset. Then  $\mathbf{DM}(\mathbf{P})$  is a complete orthomodular lattice.*

## References

- [1] I. Chajda and H. Länger. Residuated operators in complemented posets. *Asian-European Journal of Mathematics*, 11:(15pp), 2018, doi:10.1142/S1793557118500973.
- [2] Gudrun Kalmbach. *Orthomodular Lattices*. Academic Press, London, 1983, ISBN 0-12-394580-1.
- [3] Jürgen Schmidt. Zur Kennzeichnung der Dedekind-MacNeilleschen Hülle einer geordneten Menge. *Archiv der Mathematik*, 7:241–249, 1956.

# Axiom $T_D$ and the relation between sublocales and subspaces of a space

Jorge Picado<sup>1,\*</sup> and Aleš Pultr<sup>2</sup>

<sup>1</sup> CMUC, Department of Mathematics, University of Coimbra, PORTUGAL  
picado@mat.uc.pt

<sup>2</sup> CE-ITI, Department of Applied Mathematics, Charles University, Prague, Czech Republic  
pultr@kam.mff.cuni.cz

Let **Top** denote the category of topological spaces and continuous maps and let **Loc** denote the category of locales (=frames) and localic maps ([6]). The structure of their subobject lattices differ substantially: while the subspaces (=subsets) of a space  $X$  form a complete and atomic Boolean algebra  $\mathcal{P}(X)$ , the lattice of sublocales of a locale  $L$  is a coframe  $\mathbf{S}(L)$ . Moreover, a topological space  $X$  (more precisely its associated locale  $\Omega(X)$  of open sets) has typically more sublocales than subspaces.

This talk will be about sublocales *induced by* subspaces ([7]). More specifically, consider a space  $X$  and a subspace  $Y \subseteq X$ . The embedding  $j: Y \subseteq X$  is represented in **Loc** via the ‘open-set’ functor  $\Omega$  by the localic map  $\kappa: \Omega(Y) \rightarrow \Omega(X)$ , given by  $\kappa(V) = \text{int}((X \setminus Y) \cup V)$ . The sublocale of  $\Omega(X)$  *induced by* subspace  $Y$  is

$$\begin{aligned} S_Y = \kappa[\Omega(Y)] &= \{\text{int}((X \setminus Y) \cup V) \mid V \text{ open in } Y\} = \\ &= \{\text{int}((X \setminus Y) \cup (U \cap Y)) \mid U \in \Omega(X)\}. \end{aligned}$$

We say that the representation  $Y \mapsto S_Y$  of subspaces is *precise* if it constitutes a one-to-one correspondence between subspaces and induced sublocales. It turns out that unless the space in question satisfies a certain weak separation condition  $T_D$  ([1, 3]), representation of subspaces of  $X$  by sublocales of  $\Omega(X)$  is imperfect: distinct subspaces can induce the same sublocale. In fact, one has the following (see e.g. [6]):

*Induced sublocales* constitute a *precise* representation of *subspaces* of  $X$  if and only if  $X$  is  $T_D$ .

The first result concerning the question when every sublocale is (induced by) a subspace was presented by Simmons ([8]). Specifically, Simmons proved that

every *sublocale* of  $X$  is complemented in  $\mathbf{S}(\Omega(X))$  if and only if  $X$  is *weakly scattered*,

providing a necessary and sufficient condition for  $\mathbf{S}(\Omega(X))$  being Boolean which is slightly different: if sublocales are in a one-to-one correspondence with subspaces they do form a Boolean algebra, while the converse implication does not hold.

Later, Niefield and Rosenthal ([5]) treated more directly the question of every sublocale being spatial and gave a characterization of the respective locales.

In both cases, however, the question of the one-to-one correspondence between subspaces and sublocales is somehow circumvented. While, as we have already pointed out, typically one has more sublocales than subspaces, there are already cases when there are less sublocales than subspaces.

---

\*Presenter at TACL 2019. A part of the talk is joint work with D. Baboolal and P. Pillay ([2]).

In this talk, we will present a proof of the following result (the Simmons sublocale theorem for  $T_D$ -spaces), included in [7]:

**Theorem 1.** *For a  $T_D$ -space  $X$ , the sublocales are in a one-to-one correspondence with subspaces if and only if  $X$  is scattered.*

Consequent use of properties of  $T_D$ -spaces and the sublocale technique makes the proof simpler, and we think more transparent, than those in [8, 5]. Also, since we do not need the concept of a minimal prime (and that of an essential one) we can do it without any choice principle.

We will also present a characteristics of the subspaces that are complemented in  $S(\Omega(X))$  and as a consequence obtain the following result ([2]):

**Theorem 2.** *Every subspace of  $X$  is complemented in  $S(\Omega(X))$  if and only if  $X$  is hereditarily irresolvable.*

Thus, using results from [4], we learn that

- in a large class  $\mathcal{C}$  of spaces (containing e.g. all metrizable spaces, locally compact Hausdorff spaces, Alexandroff spaces, first countable spaces and spectral spaces), every *sublocale* is complemented (that is,  $S(\Omega(X))$  is Boolean) if and only if every *subspace* is complemented (and, indeed, if every subspace is complemented then each sublocale is a subspace),
- in other words, a space  $X$  in  $\mathcal{C}$  has a sublocale that is not a subspace if and only if it has a subspace that is not complemented,
- and, on the other hand, there exist spaces such that each of their subspaces is complemented in  $S(\Omega(X))$  while this coframe contains also non-complemented elements.

## References

- [1] C.E. Aull and W.J. Thron, *Separation axioms between  $T_0$  and  $T_1$* , Indag. Math. 24 (1963), 26-37.
- [2] D. Baboolal, J. Picado, P. Pillay and A. Pultr, *Hewitt's irresolvability and induced sublocales in spatial frames*, DMUC preprint 19-05, 2019 (submitted for publication).
- [3] B. Banaschewski and A. Pultr, *Pointfree aspects of the  $T_D$  axiom of classical topology*, Quaest. Math. 33 (2010), 369-385.
- [4] G. Bezhanishvili, R. Mines and P. Morandi, *Scattered, Hausdorff-reducible, and hereditarily irresolvable spaces*, Topology Appl. 132 (2003), 291-306.
- [5] S.B. Niefield and K.I. Rosenthal, *Spatial sublocales and essential primes*, Topology Appl. 26 (1987), 263-269.
- [6] J. Picado and A. Pultr, *Frames and locales: Topology without points*, Frontiers in Mathematics, vol. 28, Springer, Basel (2012).
- [7] J. Picado and A. Pultr, *Axiom  $T_D$  and the Simmons sublocale theorem*, DMUC preprint 18-48, 2018 (submitted for publication).
- [8] H. Simmons, *Spaces with Boolean assemblies*, Colloq. Math. 43 (1980), 23-29.

# Topological representations of congruence lattices

Miroslav Ploščica

University of P. J. Šafárik , Košice, Slovakia

For a given class of algebras  $\mathcal{K}$  let  $\text{Con } \mathcal{K}$  be the class of all lattices isomorphic to congruence lattices of algebras from  $\mathcal{K}$ . The problem of describing  $\text{Con } \mathcal{K}$  is very difficult and there are very few relevant classes  $\mathcal{K}$ , for which a satisfactory answer is known. The two most common investigation methods are based on topological representation of (distributive) algebraic lattices and on lifting of diagrams of (distributive) semilattices by the  $\text{Con}$  functor. (See [2], chapter 3, for some overview.) A considerable part of research in this direction is aimed at comparing  $\text{Con } \mathcal{K}$  and  $\text{Con } \mathcal{L}$  for different classes  $\mathcal{K}$  and  $\mathcal{L}$  using the concept of critical point. (See [1].)

We concentrate on the case when  $\mathcal{K}$  is a congruence-distributive variety (equational class) of algebras. Then the congruence lattice  $\text{Con } A$  ( $A \in \mathcal{K}$ ) can be regarded as the lattice of all open sets of a suitable topological space defined on the sets of all subdirectly irreducible quotients of  $A$ . And it turns out that the relationships between subdirectly irreducible members of  $\mathcal{K}$  correspond to some separation properties in these topological spaces.

A special attention will be given to varieties  $\mathcal{K}$  with the Compact Congruence Intersection Property, which means that the intersection of any two compact congruences is compact. In this case, the compact elements of every  $\text{Con } A$  form a distributive lattice, and it is convenient to investigate it using the Priestley duality. This enables a description of  $\text{Con } \mathcal{K}$  for some  $\mathcal{K}$ . (See [3] for some important examples.) Further, we connect topological properties of dual spaces with the liftability of certain semilattice diagrams by the  $\text{Con}$  functor. This is a systematic attempt to link the two previously mentioned investigation methods.

## References

- [1] P. Gillibert, *Critical points of pairs of varieties of algebras*, International Journal of Algebra and Computation 19 (2009), 1–40.



- [2] G. Grätzer, F. Wehrung (eds.) *Lattice Theory: Special Topics and Applications, Vol. 1* Birkhäuser Verlag, 2014, ISBN 978-3-319-06412-3.
- [3] F. Krajník, M. Ploščica, *Compact intersection property and description of congruence lattices*, *Mathematica Slovaca* **64** (2014), 643-664.

# Axiomatising categories of spaces: the case of compact Hausdorff spaces \*

Luca Reggio

Institute of Computer Science, Czech Academy of Sciences, Prague, Czech Republic  
reggio@cs.cas.cz

In this talk I will report on recent joint work (see [4]) with Vincenzo Marra on a categorical characterisation of the category **KH** of compact Hausdorff spaces and continuous maps.

The characterisation of subcategories of **Top**, the category of topological spaces and continuous maps, is an important concern of categorical topology. At the same time it can provide, in some cases, an abstract approach to dualities for categories of spaces.

The discrete case corresponds to Lawvere’s *Elementary Theory of the Category of Sets*, outlined in [1] (see also [2]). Lawvere gives eight elementary axioms (in the language of categories) such that every complete category satisfying these axioms is equivalent to **Set**, the category of sets and functions. His characterisation of **Set** was later adapted by Schlomiuk in [6] to capture the category of topological spaces. Concerning the category of compact Hausdorff spaces, a purely categorical description of it was provided by Richter in [5, Remark 4.7].

Our main result is a new categorical axiomatisation of **KH**. It relies on two main ingredients. The first one is the pretopos structure of **KH**, and the second one is a condition that we call *filtrality*. The latter notion makes sense in any coherent category, and is related to (the dual of) the one introduced by Magari in universal algebra [3]. Filtrality asserts that every object is covered by one whose lattice of subobjects is isomorphic to the lattice of filters of its Boolean center. Our main result reads as follows:

*Up to equivalence, **KH** is the unique non-trivial well-pointed pretopos which is filtral and admits all set-indexed copowers of its terminal object.*

In the talk I will argue that, compared to Richter’s, our approach is more natural from a duality theoretic standpoint. Further, I will indicate how to recover Richter’s characterisation of **KH** from ours.

In more detail, under mild hypotheses every positive coherent category **X** which is well-pointed admits a topological representation, i.e. a faithful functor  $\mathbf{X} \rightarrow \mathbf{Top}$ . Coherent categories (i.e. finitely complete categories with stable images, and stable joins of subobjects) can be thought of as a categorical abstraction of distributive lattices, and positivity captures the topological intuition that the coproduct of any two objects is disjoint. Finally, well-pointedness (or “existence of enough points”) means that two distinct morphisms  $f, g \in \text{hom}_{\mathbf{X}}(X, Y)$  can be separated by a morphism from the terminal object of **X** into  $X$ .

The notion of filtrality is used in order to co-restrict the functor  $\mathbf{X} \rightarrow \mathbf{Top}$  to the category of compact Hausdorff spaces. If we consider those positive coherent categories in which every internal equivalence relation is effective (hence ensuring a good correspondence between congruences and quotients), we arrive precisely at the notion of *pretopos*. Our main result shows

---

\*This work was carried out as part of the project DualL which has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No.670624). The author also acknowledges financial support from Sorbonne Paris Cité (PhD agreement USPC IDEX – REGGI15RDXMTSPC1GEHRKE).

that, under this extra assumption, the topological representation  $\mathbf{X} \rightarrow \mathbf{KH}$  is an equivalence of categories.

We also specialise the characterisation of  $\mathbf{KH}$  to its full subcategory  $\mathbf{BStone}$  on the Boolean (Stone) spaces, i.e. compact Hausdorff spaces admitting a basis of clopens. In this framework, our main result can be exploited to give a proof of the folklore result stating that  $\mathbf{KH}$  is the exact (equivalently, pretopos) completion of  $\mathbf{BStone}$ .

If time allows, I will discuss the possibility of adapting the construction outlined above to deal with other classes of spaces, e.g. with categories of ordered topological spaces.

## References

- [1] F. W. Lawvere, *An elementary theory of the category of sets*, Proc. Nat. Acad. Sci. U.S.A. **52** (1964), 1506–1511.
- [2] ———, *An elementary theory of the category of sets (long version) with commentary*, Repr. Theory Appl. Categ. (2005), no. 11, 1–35, With comments by the author and Colin McLarty.
- [3] R. Magari, *Varietà a quozienti filtrali*, Ann. Univ. Ferrara Sez. VII (N.S.) **14** (1969), 5–20.
- [4] V. Marra and L. Reggio, *A characterisation of the category of compact Hausdorff spaces*, preprint available at [arXiv:1808.09738](https://arxiv.org/abs/1808.09738) (2018).
- [5] G. Richter, *Axiomatizing algebraically behaved categories of Hausdorff spaces*, Category theory 1991 (Montreal, PQ, 1991) (R. A. G. Seely, ed.), CMS Conf. Proc., vol. 13, Amer. Math. Soc., Providence, RI, 1992, pp. 367–389.
- [6] D. I. Schlomiuk, *An elementary theory of the category of topological spaces*, Trans. Amer. Math. Soc. **149** (1970), 259–278.

# Duality for two-sorted lattices

Achim Jung, University of Birmingham, United Kingdom  
Umberto Rivieccio, Universidade Federal do Rio Grande do Norte, Brazil

Several algebras related to non-classical logics can be viewed or represented (in some cases, have originally been introduced) as two-sorted algebras in the sense of many-sorted universal algebra (see e.g. [4]). This is a (novel) way of stating the main content of many so-called *product* or *twist-structure* representation theorems, which have been shown to hold for Nelson algebras, N4-lattices (the algebraic counterparts of, respectively, Nelson’s constructive logic with strong negation [13] and Nelson’s paraconsistent logic [1]) and bilattices endowed with various extra operations (negation, implications, modalities); see e.g. [14, 16, 5, 11, 10].

More recently, similar representation results have been established for further classes of algebras of non-classical logics, which are even more general than the preceding ones, in that their negation operator is not a “strong negation” (is not involutive): e.g. the quasi-Nelson algebras of [12, 8] and Sankappanavar’s semi-De Morgan algebras [17, 8].

The above-mentioned algebras can all in fact be viewed as *two-sorted lattices*, i.e. tuples  $\langle \mathbf{L}_+, \mathbf{L}_-, n, p \rangle$  such that  $\mathbf{L}_+$  and  $\mathbf{L}_-$  are (usually, distributive) lattices, perhaps endowed with additional operations (e.g. implications, modalities), and  $n: L_+ \rightarrow L_-$  and  $p: L_- \rightarrow L_+$  are (meet-preserving) maps, in each case satisfying different additional requirements. From the point of view of many-sorted universal algebra, the maps  $n$  and  $p$  can be treated as (unary) many-sorted algebraic operations, whereas the lattice operations of  $\mathbf{L}_+$  and  $\mathbf{L}_-$  act within a single sort. Imposing restrictions on the structure of the two lattices and/or the maps, one obtains (tuples corresponding to) the various classes of algebras; as limit cases, requiring the two maps to be mutually inverse lattice isomorphisms, one recovers standard bilattices/Nelson/N4-lattices having an involutive negation.

The correspondence between each class of (uni-sorted) algebras and the respective tuples (viewed as two-sorted lattices) often yields a (co-variant) categorical equivalence between naturally associated categories. This suggests that a Priestley-style duality approach can be developed for general two-sorted lattices and then specialised, via the co-variant equivalences, to each of the above-mentioned classes of algebras. The preceding discussion also indicates that a suitable base category to work with is one whose objects are bounded distributive (semi)-lattices and whose morphisms are meet-preserving maps; this indeed allows us to view tuples  $\langle \mathbf{L}_+, \mathbf{L}_-, n, p \rangle$  as diagrams in the base category. Following this intuition, we have mainly built on the Priestley-style duality for meet-semilattices (and for meet-semilattices enriched with an intuitionistic implication) introduced by G. Bezhanishvili and R. Jansana [2, 3].

The strategy outlined above allowed us, in some cases, to establish the only duality results currently available for these structures (e.g. quasi-Nelson algebras); but in all cases we have obtained a “two-sorted duality” that is, we believe, neater and much easier to work with than any of the existing (uni-sorted) approaches (see

e.g. [9, 6] for semi-De Morgan algebras and [7, 15] for Nelson/N4-lattices).

## References

- [1] A. Almukdad and D. Nelson. Constructible falsity and inexact predicates. *The Journal of Symbolic Logic*, 49(1):231–233, 1984.
- [2] G. Bezhanishvili and R. Jansana. Priestley style duality for distributive meet-semilattices. *Studia Logica*, 98(1-2):83–122, 2011.
- [3] G. Bezhanishvili and R. Jansana. Esakia style duality for implicative semilattices. *Applied Categorical Structures*, 21(2):181–208, 2013.
- [4] G. Birkhoff and J. D. Lipson. Heterogeneous algebras. *J. Combinatorial Theory*, 8:115–133, 1970.
- [5] F. Bou, R. Jansana, and U. Rivieccio. Varieties of interlaced bilattices. *Algebra Universalis*, 66(1):115–141, 2011.
- [6] S. Arturo Celani. Distributive lattices with a negation operator. *Mathematical Logic Quarterly*, 45(2):207–218, 1999.
- [7] R. Cignoli. The class of Kleene algebras satisfying an interpolation property and Nelson algebras. *Algebra Universalis*, 23(3):262–292, 1986.
- [8] F. Greco, F. Liang, A. Moshier, and A. Palmigiano. Multi-type display calculus for semi de morgan logic. In J. Kennedy and R. de Queiroz, editors, *Proc. WoLLIC 2017*, 199–215, 2017.
- [9] D. Hobby. Semi-De Morgan algebras. *Studia Logica*, 56(1-2):151–183, 1996. Special issue on Priestley duality.
- [10] R. Jansana and U. Rivieccio. Dualities for modal N4-lattices. *Logic Journal of the I.G.P.L.*, 22(4):608–637, 2014.
- [11] A. Jung and U. Rivieccio. Kripke semantics for modal bilattice logic. *Proceedings of the 28th Annual ACM/IEEE Symposium on Logic in Computer Science*, IEEE Computer Society Press, 2013, pp. 438–447.
- [12] U. Rivieccio and M. Spinks. Quasi-Nelson algebras. *Proceedings of the 13th workshop on Logical and Semantic Frameworks, with Applications (LSFA 2018)*, Elsevier Science b.v., 189–202, 2018.  
<http://lia.ufc.br/~lsfa2018/wp-content/uploads/2018/09/LSFA18.pdf>
- [13] D. Nelson. Constructible falsity. *The Journal of Symbolic Logic*, 14:16–26, 1949.
- [14] S. P. Odintsov. On the representation of N4-lattices. *Studia Logica*, 76(3):385–405, 2004.
- [15] S. P. Odintsov. Priestley duality for paraconsistent nelson’s logic. *Studia Logica*, 96(1):65–93, 2010.
- [16] S. P. Odintsov and E. I. Latkin. BK-lattices. Algebraic semantics for Belnapian modal logics. *Studia Logica*, 100(1-2):319–338, 2012.
- [17] H. P. Sankappanavar. Semi-De Morgan algebras. *The Journal of symbolic logic*, 52(3):712–724, 1987.

# Quantale semantics for Lambek calculus with subexponentials \*

Daniel Rogozin<sup>12</sup>

<sup>1</sup> Lomonosov Moscow State University,  
Moscow, Russia

<sup>2</sup> Serokell OÜ

daniel.rogozin@serokell.io

Lambek calculus is a non-commutative version of linear logic [2] that was initially proposed by J. Lambek [4]. The multiplicative and additive Lambek calculus with subexponentials was introduced by M. Kanovich, A. Scedrov, S. Kuznetsov and V. Nigam [3].

The first is to define a subexponential signature as a quintuple  $\Sigma = \langle \mathcal{I}, \preceq, \mathcal{W}, \mathcal{E}, \mathcal{C} \rangle$ , where  $\langle \mathcal{I}, \preceq \rangle$ , and  $\mathcal{W}, \mathcal{E}, \mathcal{C}$  are upwardly closed subsets with respect to  $\preceq$ . Note that  $\mathcal{W} \cap \mathcal{C} \subseteq \mathcal{E}$ .

Thus, the multiplicative and additive Lambek calculus with subexponentials is defined as an extension of Lambek calculus with additives with the following inference rules for subexponentials:

$$\begin{array}{c}
 \frac{\Gamma, A, \Delta \rightarrow C}{\Gamma, !^s A, \Delta \rightarrow C} !_s \rightarrow \qquad \frac{!^{s_1} A_1, \dots, !^{s_n} A_n \rightarrow A}{!^{s_1} A_1, \dots, !^{s_n} A_n \rightarrow !^s A} \rightarrow !_s, \forall j, s_j \succeq s \\
 \\
 \frac{\Gamma, !^s A, \Delta, !^s A, \Theta \rightarrow B}{\Gamma, !^s A, \Delta, \Theta \rightarrow B} \mathbf{ncontr}_1, s \in C \qquad \frac{\Gamma, !^s A, \Delta, !^s A, \Theta \rightarrow B}{\Gamma, \Delta, !^s A, \Theta \rightarrow B} \mathbf{ncontr}_2, s \in C \\
 \\
 \frac{\Gamma, \Delta, !^s A, \Theta \rightarrow B}{\Gamma, !^s A, \Delta, \Theta \rightarrow B} \mathbf{ex}_1, s \in E \qquad \frac{\Gamma, !^s A, \Delta, \Theta \rightarrow B}{\Gamma, \Delta, !^s A, \Theta \rightarrow B} \mathbf{ex}_2, s \in E \\
 \\
 \frac{\Gamma, \Delta \rightarrow B}{\Gamma, !^s A, \Delta \rightarrow B} \mathbf{weak}_1, s \in W
 \end{array}$$

Initially, quantale semantics of linear logic was introduced by D. Yetter [6], where some connection between linear logic and models of the logic of quantum mechanics was established.

We develop these ideas proposed by Yetter, and also by Brown and Gurr [1], and consider a quantale model for this extension of Lambek calculus, where subexponential modalities will be considered as quantic conuclei (or open modalities) [5].

Now, let us define a quantale:

**Definition 1.** Quantale is a quadruple  $\mathcal{Q} = \langle A, \cdot, \bigvee, \bigwedge \rangle$ , where  $\langle A, \cdot \rangle$  is a semigroup,  $\langle A, \bigvee, \bigwedge, \cdot \rangle$  is a complete lattice with additional axioms for each indexing set  $I$ :

$$\begin{aligned}
 a \cdot \bigvee_{i \in I} b_i &= \bigvee_{i \in I} (a \cdot b_i) \\
 \bigvee_{i \in I} a_i \cdot b &= \bigvee_{i \in I} (a_i \cdot b)
 \end{aligned}$$

A quantale  $\mathcal{Q}$  is called unital, if  $\langle A, \cdot, \varepsilon \rangle$  is a monoid, where  $\varepsilon$  is a neutral element.

---

\*The research is supported by the by the Presidential Council, reseach grant MK-430.2019.1

Also, we define a quantic conucleus on a unital quantale as a map  $\Box : \mathcal{Q} \rightarrow \mathcal{Q}$  with the following data:

$$\begin{aligned} \Box a &\leq a \\ \Box a &= \Box^2 a \\ a \leq b &\Rightarrow \Box a \leq \Box b \\ \Box a \cdot \Box b &\leq \Box(a \cdot b) \\ \Box \varepsilon &= \varepsilon \end{aligned}$$

A quantic conucleus is called unital, if for each  $a \in \mathcal{Q}$ ,  $\Box a \leq \varepsilon$ ; central, if for all  $a \in \mathcal{Q}$ ,  $\Box a \in \mathcal{Z}(\mathcal{Q})$  (where  $\mathcal{Z}(\mathcal{Q})$  is a central subquantale, that is,  $\mathcal{Z}(\mathcal{Q}) = \{a \in \mathcal{Q} \mid \forall b \in \mathcal{Q}, a \cdot b = b \cdot a\}$ ); non-local weak idempotent, if for all  $b \in \mathcal{Q}$ ,  $\Box a \cdot b \leq \Box a \cdot b \cdot \Box a$  and  $b \cdot \Box a \leq \Box a \cdot b \cdot \Box a$ .

Note that, unital (central) conucleus corresponds to weakening rule (exchange) for some  $!_s$  subexponential, if  $s \in \mathcal{W}$  (or  $s \in \mathcal{E}$ ). We introduce non-local weak idempotent open modalities as counterparts for non-local contraction rules for  $!_s$ , such that  $s \in \mathcal{C}$ .

After that, we introduce subexponential interpretation as a special contravariant functor  $\sigma : \Sigma \rightarrow \Box_{\mathcal{Q}}$ , where  $\Sigma = \langle I, \preceq, W, C, E \rangle$  is a subexponential signature and  $\Box_{\mathcal{Q}}$  is a category of open modalities on a quantale  $\mathcal{Q}$ . In our report, we define this map in more detail.

Let  $\mathcal{Q}$  be a quantale,  $f : Tp \rightarrow \mathcal{Q}$  a valuation (where  $Tp$  is a set of propositional variables) and  $\sigma : I \rightarrow \Box_{\mathcal{Q}}$  a subexponential interpretation, then interpretation  $\llbracket A \rrbracket$  is defined by induction on formula  $A$ .

**Definition 2.**  $\Gamma \models A \Leftrightarrow \forall f, \forall \sigma, \llbracket \Gamma \rrbracket \leq \llbracket A \rrbracket$

There is the following theorem:

**Theorem 1.**  $\Gamma \vdash A \Leftrightarrow \Gamma \models A$

*Proof.* In completeness proof, we modify the technique used by Yetter [6], Brown, and Gurr [1].  $\square$

It is easy to see that open modality on some quantale is the special case of comonad. So, we also will consider the categorical model of this calculus that allows one to generalize quantale semantics with open modalities.

## References

- [1] Brown, C., and Gurr, D. (1995) Relations and non-commutative linear logic. Journal of Pure and Applied Algebra, vol. 105, No. 2, 117-136.
- [2] Girard, Jean-Yves. (1987). Linear logic. Theoretical Computer Science, Volume 50, Issue 1, 1-101.
- [3] Kanovich, M., Kuznetsov, S., Nigam, V., and Scedrov, A. (2018). Subexponentials in non-commutative linear logic. Mathematical Structures in Computer Science, 1-33.
- [4] Lambek, J. (1958). The mathematics of sentence structure. American mathematical monthly, vol. 65, No. 3, 154-170.
- [5] Rosenthal, Kimmo I. (1990) Quantales and their applications, Pitman Res. Notes in Math. Series 234, Longman.
- [6] Yetter, David N. (1990). Quantales and (noncommutative) linear logic. Journal of Symbolic Logic, vol. 55, No. 1, 41-64.

# Derivations on bounded pocrimms and MV-algebras with product

Jan Kühr<sup>1</sup>, Jiří Rachůnek<sup>1</sup>, and Dana Šalounová<sup>2</sup>

<sup>1</sup> Department of Algebra and Geometry,  
Faculty of Sciences, Palacký University,  
17. listopadu 12, 771 46 Olomouc, Czech Republic,  
e-mail: jan.kuhr@upol.cz; jiri.rachunek@upol.cz  
<sup>2</sup> VŠB–Technical University Ostrava,  
Sokolská 33, 701 21 Ostrava, Czech Republic,  
e-mail: dana.salounova@vsb.cz

The concept of a derivation became from the mathematical analysis. Derivations on lattices were established by Szász [5] in 1974, formally in the same manner as derivations on rings. In the last decade, derivations on some kinds of algebras have been introduced and investigated. More specifically, derivations on MV-algebras were studied e.g. in [1, 6]. Further, derivations on basic algebras [3], residuated lattices [2] and on GMV-algebras [4] followed.

Pocrims, i.e. partially ordered commutative residuated integral monoids, form a large class of algebras that contains, among others, some classes of algebras behind quantum and fuzzy logics.

In this paper we study derivations on bounded pocrimms and on PMV-algebras, such that they are generalizations of the derivations on MV-algebras. By a *derivation* on a bounded pocrim  $M = (M; \odot, \rightarrow, 0, 1)$  we mean a map  $d: M \rightarrow M$  satisfying

$$d(x \oplus y) = d(x) \oplus d(y) \quad \text{and} \quad d(x \odot y) = (d(x) \odot y) \oplus (x \odot d(y)), \quad (1)$$

for all  $x, y \in M$ , where the addition  $\oplus$  is defined by  $x \oplus y = (x^- \odot y^-)^-$ .

We first observe that every derivation  $d$  on  $M$  is completely determined by its restriction to the set  $Reg(M)$  of regular elements in  $M$ , since  $d(x) = d(x^-)$  where  $x^- \in Reg(M)$ , for every  $x \in M$ . Consequently, there is a one-one correspondence between the derivations on  $M$  and those on  $Reg(M)$ , and so, roughly speaking, it suffices to characterize derivations on involutive pocrimms.

For any derivation  $d$  on an involutive pocrim  $M$  we prove that  $d(x) = x \odot d(1)$  for all  $x \in M$ , whence it follows that  $d$  is a conucleus on  $M$  and, in fact,  $d$  is



a homomorphism of  $M$  onto  $M_d$ , the conucleus image of  $M$ , which is a Boolean algebra. Moreover, the derivations  $d$  on  $M$  with the property that  $d(1)$  is a Boolean element correspond one-one to the direct decompositions  $M \cong K \times L$  where  $K$  is a Boolean algebra; the natural derivation on  $K \times L$  is given by  $d(x, y) = (x, 0)$ .

As a result we obtain that for any derivation  $d$  on an arbitrary bounded pocrim  $M$  we have  $d(x) = (x \odot d(1))^{\circ}$  for all  $x \in M$ . This formula can be somewhat improved when  $M$  is divisible or prelinear.

The concept of a *coderivation* is defined by interchanging  $\oplus$  and  $\odot$  in (1). We investigate coderivations on the so-called normal pocrimms only. For any map  $d: M \rightarrow M$  we define  $\tilde{d}: M \rightarrow M$  by  $\tilde{d}(x) = d(x^-)^{\circ}$ . If  $d$  is a derivation, then  $\tilde{d}$  is a coderivation, and if  $d$  is a coderivation, then  $\tilde{d}$  is a derivation. In fact, when the set of derivations  $D(M)$  and the set of coderivations  $C(M)$  are equipped with pointwise order, then the assignment  $d \mapsto \tilde{d}$  is an antitone Galois connection between  $D(M)$  and  $C(M)$ . Every derivation on  $M$  is closed, but a coderivation  $d$  is closed if and only if  $d(x) = d(x^-)$  for all  $x \in M$ , if and only if  $d$  is determined by its restriction to  $Reg(M)$ .

PMV-algebras are MV-algebras equipped with product  $\cdot$  that satisfies a certain equation that holds in the standard MV-algebra  $[0, 1]_{MV}$  with the usual product of reals. We prove that  $d$  is a derivation if and only if  $d$  satisfies (1) with  $\cdot$  in place of  $\odot$ , although the two products are distinct.

## References

- [1] N. O. Alshehri, *Derivations of MV-algebras*. Inter. J. Math. and Math. Sci., 312027:7, 2010.
- [2] P. He, X. Xin and J. Zhan: *On derivations and their fixed point sets in residuated lattices*. Fuzzy Sets Syst. 303 (2016), 97–113.
- [3] J. Krňávek and J. Kühr: *A note on derivations on basic algebras*. Soft Comput. 19 (2015), 1765–1771.
- [4] J. Rachůnek, D. Šalounová: *Derivations on algebras of a non-commutative generalization of the Łukasiewicz logic*. Fuzzy Sets Syst. 333 (2018), 11–16.
- [5] G. Szász: *Derivations of lattices*. Acta Sci. Math. (Szeged) 37:1-2(1975), 149–154.
- [6] H. Yazarli, *A note on derivations in MV-algebras*. Miskolc Math. Notes 14, 345–354, 2013.

# Decidability of the equational theory of the natural join and inner union

Luigi Santocanale

LIS, UMR CNRS 7020, Aix-Marseille Université, FRANCE

This talk reports on the research available as [5].

The natural join and the inner union are operations that combine relations of a relational database. Tropashko and Spight realized that these two operations are the meet and join operations in a class of lattices, known by now as the relational lattices. They proposed then lattice theory as an algebraic approach to the theory of databases, alternative to the relational algebra. Previous works [2] proved that the quasiequational theory of these lattices—that is, the set of definite Horn sentences valid in all the relational lattices—is undecidable, even when the signature is restricted to the pure lattice signature [4]. These results add up to a long list of undecidability results in relation algebra [3, 1].

In this talk I'll show how different ideas (duality for finite lattices, generalized ultrametric spaces on a powerset algebra, injectivity, ...) combine to yield decidability of the equational theory of relational lattices.

Namely, we provide an algorithm to decide if two lattice theoretic terms  $t, s$  are made equal under all interpretations in some relational lattice. The algorithm stem from a countermodel construction of bounded size : we show that if an inclusion  $t \leq s$  fails in any of these lattices, then it fails in a relational lattice whose size is bound by a triple exponential function of the sizes of  $t$  and  $s$ .

## References

- [1] R. Hirsch and I. Hodkinson. Representability is not decidable for finite relation algebras. *Trans. Amer. Math. Soc.*, 353:1403–1425, 2001.
- [2] T. Litak, S. Mikuláš, and J. Hidders. Relational lattices: From databases to universal algebra. *J. Log. Algebr. Meth. Program.*, 85(4):540–573, 2016.
- [3] R. Maddux. The equational theory of  $CA_3$  is undecidable. *The Journal of Symbolic Logic*, 45(2):311–316, 1980.
- [4] L. Santocanale. Embeddability into relational lattices is undecidable. *J. Log. Algebr. Meth. Program.*, 97:131–148, 2018.
- [5] L. Santocanale. The equational theory of the natural join and inner union is decidable. In C. Baier and U. D. Lago, editors, *FOSSACS 2018, Proceedings*, volume 10803 of *Lecture Notes in Computer Science*, pages 494–510. Springer, 2018.

# Mix $\star$ -autonomous quantales and the continuous weak order

Luigi Santocanale and Maria João Gouveia

<sup>1</sup> LIS, UMR CNRS 7020, Aix-Marseille Université, FRANCE

<sup>2</sup> Departamento de Matemática, Universidade de Lisboa, PORTUGAL

This talk reports on the research available as [1, 2].

The set of permutations on a finite set can be given the lattice structure known as the weak Bruhat order. This lattice structure is generalized to the set of words on a fixed alphabet  $\Sigma = \{x, y, z, \dots\}$ , where each letter has a fixed number of occurrences. These lattices are known as multinomial lattices and, when  $\text{card}(\Sigma) = 2$ , as lattices of lattice paths. By interpreting the letters  $x, y, z, \dots$  as axes, these words can be interpreted as discrete increasing paths on a grid of a  $d$ -dimensional cube, with  $d = \text{card}(\Sigma)$ .

I'll explain how to extend this order to images of continuous monotone functions from the unit interval to a  $d$ -dimensional cube that preserve the end-points. The order so obtained, denoted by  $L(\mathbb{I}_d)$ , is proved to be a complete lattice.

With respect to previous knowledge on the lattices  $L(\mathbb{I}_d)$ ,  $d \geq 3$  (see our TACL 2011 talk), a main advance is the recognition of the key role in this construction of the quantale  $L_\vee(\mathbb{I})$  of join-continuous functions from the unit interval to itself. All the construction relies on a few algebraic properties of this quantale: it is cyclic  $\star$ -autonomous and it satisfies the mix rule. Many generalizations of permutohedra (the permutohedra themselves, the multinomial lattices, lattices of pseudopermutations) can be constructed from a cyclic/mix  $\star$ -autonomous quantale (or a involutive residuated lattice) in a functorial way.

We begin developing a structural theory of the lattices  $L(\mathbb{I}_d)$ : they are self-dual, they are generated under infinite joins from their join-irreducible elements, they have no completely irreducible elements, nor compact elements. The colimit of all the  $d$ -dimensional multinomial lattices embeds into  $L(\mathbb{I}_d)$ . When  $d = 2$ ,  $L(\mathbb{I}_d) = L_\vee(\mathbb{I})$  is the Dedekind-MacNeille completion of this colimit. When  $d \geq 3$ , every element of  $L(\mathbb{I}_d)$  a join of meets of elements from this colimit.

## References

- [1] M. J. Gouveia and L. Santocanale. Mix  $\star$ -autonomous quantales and the continuous weak order. In J. Desharnais, W. Guttmann, and S. Joosten, editors, *RAMiCS 2018, Proceedings*, volume 11194 of *Lecture Notes in Computer Science*, pages 184–201, 2018.
- [2] L. Santocanale and M. J. Gouveia. The continuous weak order. Submitted, available from Hal : <https://hal.archives-ouvertes.fr/hal-01944759>, Dec. 2018.

## PARTIAL FRAMES

This talk will fit squarely into the topic of pointfree topology, but whereas the usual fundamental object under consideration is a frame or locale, I will be discussing partial frames. In his influential 1991 paper on kappa-frames Jim Madden states: "It will be possible, I believe, to formulate a useful notion of a partial frame. This would be a meet-semilattice in which certain distinguished subsets would all have suprema and in which meets would distribute over joins of such subsets... My hope is that a theory of partial frames could provide substantial insight into large classes of epireflective properties and covering properties in locale theory and topology..."

It is in this spirit that we proceed, using the concept of a selection function as introduced by Zhao, Paseka and Zenk. A selection function must satisfy certain axioms to produce a tractable theory, and each of these authors uses different but overlapping collections of such axioms, as do we. Our axioms are sufficiently general to include as examples of partial frames meet-semilattices, bounded distributive lattices, sigma-frames, kappa-frames and frames. We note that this idea has been used by other authors in more general contexts as well; see, for instance, Erné below.

We have found this to be a rich context in which to do topology, both in the unstructured situation – for example, in the construction of compactifications; general, largest and one-point compactifications – and in the structured situation, for example in the construction of completions of uniform partial frames. Our particular context often brings into relief what is generic and what is intrinsic to specific examples. For instance: nuclei and right adjoints of frame maps are extremely useful tools in pointfree topology, but they are unavailable to us. The more algebraic aspects of the subject are equally amenable to this approach. For example, we have used Johnstone's idea of coverages to create partial frames freely generated by sites, so that we are able to define objects via generators and relations. However, since nuclei cannot be used here, we adapted his technique by using congruences instead. We have further considered partial spaces, and accompanying notions of soberness and spatiality, and see that a knowledge of (partial) frames is exceedingly useful, even when considering compactifications of (partial) spaces.

## References

- M. Erné, *Z-Continuous Posets and Their Topological Manifestation*, Appl. Categ. Struct. 7 (1999) 31 – 70.
- J. Frith and A. Schauerte, *Completions of uniform partial frames*, Acta Math, Hungar. 147 (2015) 116 – 134.
- J. Frith and A. Schauerte, *One-point compactifications and continuity for partial frames*, Categ. Gen. Alg. Str. Appl. 7 (2017) 57 – 88.
- J. Frith and A. Schauerte, *Coverages give free constructions for partial frames*, Appl. Categ. Struct 25 (2017) 303 – 321.
- J. Frith and A. Schauerte, *The congruence frame and the Madden quotient for partial frames*, Alg. Universalis 25 (2018) 79:73.

J.J. Madden,  *$\kappa$ -frames*, J. Pure Appl Algebra 70 (1991) 107-127.

J. Paseka, *Covers in generalized frames*, in: General Algebra and Ordered Sets (Horní Lipová 1994), Palacký Univ. Olomouc, Olomouc pp. 84-99.

E.R. Zenk, *Categories of partial frames*, Algebra Univers. 54 (2005) 213-235.

D. Zhao, *On projective Z-frames*, Canad. Math. Bull. 40(1) (1997) 39-46.

# Two Approaches to Substructural Modal Logic

Igor Sedlár<sup>1</sup>

The Czech Academy of Sciences, Institute of Computer Science,  
Prague, Czech Republic  
sedlar@cs.cas.cz

Two semantic approaches to substructural modal logics may be distinguished. Firstly, one takes Kripke frames  $\langle S, R \rangle$ , where  $R$  is a binary relation on the non-empty set  $S$ , and extends the frame to a model by adding an evaluation function from atomic formulas to a residuated lattice. More generally, one may use any lattice of truth values. The standard approach here is to define logics semantically by fixing one specific lattice of truth values. Approaches of this kind, which we may call *lattice-valued*, go back at least to [11]; see also [7, 2, 8, 6, 1], for example. Secondly, one takes Kripke frames and adds to them additional relations that yield non-classical behaviour of some propositional connectives in the language (these frames are known as Routley–Meyer frames); models are obtained by adding a two-valued valuation function. Approaches of this kind, which we may call *relevant*, go back to the studies of modal relevant logics [3, 4, 5, 10], but they can be applied to other substructural logics as well [9].

In this contribution we study the relationship between these two approaches. Using elementary dualities between frames and residuated lattices we show that logics defined in terms of classes of Routley–Meyer frames correspond to logics defined semantically by reference to a *class* of truth-value lattices, not one specific lattice. Our main result is that the logic of all Routley–Meyer frames is identical to the logic of all Kripke models evaluated in complete distributive FL-algebras. In addition, we suggest that a *generalized* Routley–Meyer frames can be used as equivalent semantics for logics originally defined in terms of Kripke models evaluated in one particular lattice of truth values. (Similarly as in classical modal logic, a generalized frame is a frame with a distinguished subalgebra of its full complex algebra, seen as the algebra of *admissible* truth sets of formulas).

## References

- [1] F. Bou, F. Esteva, L. Godo, and R. O. Rodríguez. On the minimum many-valued modal logic over a finite residuated lattice. *Journal of Logic and Computation*, 21(5):739–790, 2011.
- [2] M. Fitting. Many-valued modal logics. *Fundamenta Informaticae*, 15:235–254, 1991.
- [3] A. Fuhrmann. Models for relevant modal logics. *Studia Logica*, 49(4):501–514, 1990.
- [4] E. D. Mares. The semantic completeness of RK. *Reports on Mathematical Logic*, 26:3–10, 1992.
- [5] E. D. Mares and R. K. Meyer. The semantics of R4. *Journal of Philosophical Logic*, 22(1):95–110, Feb 1993.
- [6] S. Odintsov and H. Wansing. Modal logics with Belnapian truth values. *Journal of Applied Non-Classical Logics*, 20(3):279–301, 2010.
- [7] P. Ostermann. Many-valued modal propositional calculi. *Mathematical Logic Quarterly*, 34(4):343–354, 1988.
- [8] G. Priest. Many-valued modal logics: A simple approach. *The Review of Symbolic Logic*, 1(2):190–203, 2008.
- [9] G. Restall. *An Introduction to Substructural Logics*. Routledge, London, 2000.
- [10] R. Routley and R. K. Meyer. The semantics of entailment—ii. *Journal of Philosophical Logic*, 1(1):53–73, Feb 1972.

- [11] K. Segerberg. Some modal logics based on a three-valued logic. *Theoria*, 33(1):53–71, 1967.

# On global algebraic completeness of the Gödel-Löb provability logic

Daniyar S. Shamkanov<sup>12</sup>

<sup>1</sup> Steklov Mathematical Institute of Russian Academy of Sciences  
Gubkina str. 8, 119991, Moscow, Russia

<sup>2</sup> National Research University Higher School of Economics  
Moscow, Russia

daniyar.shamkanov@gmail.com

The Gödel-Löb provability logic **GL** is a modal logic describing all universally valid principles of the formal provability in Peano arithmetic [9]. While **GL** is complete with respect to its Kripke semantics [5], it is not strongly complete. Neighbourhood semantics is a generalization of Kripke semantics independently developed by D. Scott and R. Montague in [4] and [3]. A neighbourhood frame can be defined as a pair  $(X, \Box)$ , where  $X$  is a set and  $\Box$  is a unary operation in  $\mathcal{P}(X)$ . The logic **GL** appears to be compact with respect to its neighbourhood interpretation, which immediately implies that it is strongly neighbourhood complete (see [7, 1]). This completeness result holds for the case of the so-called local semantic consequence relation. Recall that, over neighbourhood **GL**-models, a formula  $A$  is a local semantic consequent of  $\Gamma$  if for any neighbourhood **GL**-model  $\mathcal{M}$  and any world  $x$  of  $\mathcal{M}$

$$(\forall B \in \Gamma \mathcal{M}, x \models B) \Rightarrow \mathcal{M}, x \models A.$$

A formula  $A$  is a global semantic consequent of  $\Gamma$  if for any neighbourhood **GL**-model  $\mathcal{M}$

$$(\forall B \in \Gamma \mathcal{M} \models B) \Rightarrow \mathcal{M} \models A.$$

Notice that this global semantic consequence relation coincides with the following one:  $A$  is a consequent of  $\Gamma$  if for any neighbourhood **GL**-model  $\mathcal{M}$ , any world  $x$  of  $\mathcal{M}$  and any open neighbourhood  $U$  of  $x$ , that is  $x \in U$  and  $U \subset \Box U$ ,

$$(\forall B \in \Gamma \forall y \in U \mathcal{M}, y \models B) \Rightarrow \mathcal{M}, x \models A.$$

In the paper [6], I considered Hilbert-style non-well-founded derivations in **GL** and established that **GL** with the obtained derivability relation is strongly neighbourhood complete in the case of the global semantic consequence relation. A non-well-founded derivation is defined as a (possibly infinite) tree whose nodes are marked by modal formulas and that is constructed according to the rules of modus ponens and necessitation. In addition, any infinite branch in this tree must contain infinitely many applications of the necessitation rule. The global neighbourhood completeness result from [6] means that a formula  $A$  is a global semantic consequent of  $\Gamma$  if and only if there is a non-well-founded derivation of the formula  $A$  from assumptions  $\Gamma$ . This completeness result rests on the Boolean ultrafilter theorem.

The Gödel-Löb provability logic **GL** can be additionally defined as the logic of the class of all Magari algebras [2, 8]. A *Magari algebra*  $\mathcal{A} = (Y, \wedge, \vee, \rightarrow, 0, 1, \Box)$  is a Boolean algebra  $(Y, \wedge, \vee, \rightarrow, 0, 1)$  together with a unary map  $\Box: Y \rightarrow Y$  satisfying the identities:

$$\Box 1 = 1, \quad \Box(x \wedge y) = \Box x \wedge \Box y, \quad \Box(\Box x \rightarrow x) = \Box x.$$



A Magari algebra is  $\sigma$ -complete if its every countable subset  $S$  has the least upper bound  $\bigvee S$ . Over  $\sigma$ -complete Magari algebras, a formula  $A$  is a local semantic consequent of  $\Gamma$  if for any  $\sigma$ -complete Magari algebra  $\mathcal{A}$  and any valuation  $\theta$  on  $\mathcal{A}$

$$\bigwedge\{\theta(B) \mid B \in \Gamma\} \leq \theta(A).$$

A formula  $A$  is a global semantic consequent of  $\Gamma$ , over  $\sigma$ -complete Magari algebras, if for any  $\sigma$ -complete Magari algebra  $\mathcal{A}$  and any valuation  $\theta$  on  $\mathcal{A}$

$$\bigwedge\{\theta(B) \mid B \in \Gamma\} \wedge \square \bigwedge\{\theta(B) \mid B \in \Gamma\} \leq \theta(A).$$

The logic GL enriched with non-well-founded derivations is sound for its algebraic interpretation over  $\sigma$ -complete Magari algebras in the case of the global semantic consequence relation<sup>1</sup>. Together with the global neighbourhood completeness result, it implies global algebraic completeness of GL with non-well-founded derivations. Moreover, it follows that a Magari algebra  $\mathcal{A} = (Y, \wedge, \vee, \rightarrow, 0, 1, \square)$  can be embedded into a ( $\sigma$ -)complete Magari algebra if and only if, for any  $a \in Y$ ,  $a = 1$  whenever there exists a sequence  $(a_i)_{i \in \mathbb{N}}$  in  $\mathcal{A}$  such that  $\square a_{i+1} \leq a_i$  and  $a_0 = a$ .

In the talk, I would like to discuss another algebraic proof of global completeness of GL enriched with non-well-founded derivations for its algebraic interpretation over  $\sigma$ -complete Magari algebras, which doesn't rest on the Boolean ultrafilter theorem.

## References

- [1] J.P. Aguilera and D. Fernandez-Duque. “Strong Completeness of Provability Logic for Ordinal Spaces”. In: *The Journal of Symbolic Logic* 82.1 (2 2017), pp. 608–638.
- [2] R. Magari. “The diagonalizable algebras (the algebraization of the theories which express Theor. II)”. In: *Bollettino dell'Unione Matematica Italiana*. 4th ser. 12 (1975), pp. 117–125.
- [3] R. Montague. “Universal Grammar”. In: *Theoria* 373–98 (1970).
- [4] D. Scott. “Advice in modal logic”. In: *Philosophical problems in Logic*. Ed. by K. Lambert. Reidel, 1970.
- [5] K. Segerberg. *An essay in classical modal logic*. Vol. 13. Filosofiska Studier. Uppsala University, 1971.
- [6] D. Shamkanov. “Global neighbourhood completeness of the Gödel-Löb provability logic”. In: *Logic, Language, Information, and Computation. 24th International Workshop, WoLLIC 2017 (London, UK, July 18-21, 2017)*. Ed. by Juliette Kennedy and Ruy de Queiroz. Lecture Notes in Computer Science 103888. Springer, 2017, 358–371.
- [7] V. Shehtman. “On neighbourhood semantics thirty years later”. In: *We Will Show Them! Essays in Honour of Dov Gabbay*. Ed. by S. Artemov et al. Vol. 2. London: College Publications, 2005, pp. 663–692.
- [8] C. Smoryński. *Self-Reference and Modal Logic*. New York: Springer, 1985.
- [9] R. Solovay. “Provability Interpretations of Modal Logic”. In: *Israel Journal of Mathematics* 25 (1976), pp. 287–304.

---

<sup>1</sup>This observation has been inspired by a correspondence with T. Litak.

# Glivenko's theorem, finite height, and local finiteness\*

Ilya Shapirovsky

Steklov Mathematical Institute of Russian Academy of Sciences  
Institute for Information Transmission Problems of Russian Academy of Sciences

**Glivenko's theorem for arbitrary finite height** Glivenko's theorem states that a formula is derivable in the classical propositional logic CL iff under the double negation it is derivable in the intuitionistic propositional logic IL [Gli29]. Similarly, for modal logics S5 and S4 we have  $S5 \vdash \varphi$  iff  $S4 \vdash \neg \Box \neg \Box \varphi$  [Mat55]. In Kripke semantics, IL is the logic of partial orders, and CL is the logic of partial orders of height 1. Likewise, S4 is the logic of preorders, and S5 is the logic of equivalence relations, which are preorders of height 1. For a modal or intermediate logic L, let  $L[h]$  be its extension with the formula of height  $h$ , restricting the height of a Kripke frame by the finite  $h$ . In the intermediate case, these formulas are defined as  $B_0^{(i)} = \perp$ ,  $B_h^{(i)} = p_h \vee (p_h \rightarrow B_{h-1}^{(i)})$ , and in the modal transitive case — as  $B_0 = \perp$ ,  $B_h = p_h \rightarrow \Box(\Diamond p_h \vee B_{h-1})$ . In particular,  $IL[1] = CL$ ,  $S4[1] = S5$ , and the above translations can be formulated as follows:

$$IL[1] \vdash \varphi \text{ iff } IL \vdash \neg \neg \varphi, \quad S4[1] \vdash \varphi \text{ iff } S4 \vdash \Diamond \Box \varphi. \quad (1)$$

For finite variable fragments of IL and S4, (1) can be generalized for arbitrary finite height.

A  $k$ -formula is a formula in variables  $p_0, \dots, p_{k-1}$ . Consider the  $k$ -canonical frame  $(W, R)$  of a logic S4 built from maximal S4-consistent sets of  $k$ -formulas. It follows from [She85] that there exist formulas  $\mathbf{B}_{h,k}$  (and their intuitionistic analogs  $\mathbf{B}_{h,k}^{(i)}$ ) such that for every  $x \in W$  and every finite  $h$ ,  $\mathbf{B}_{h,k} \in x$  iff the depth of  $x$  in  $W$  is less than or equal to  $h$ .

**Theorem 1.** Fix a finite  $k$ . For all  $k$ -formulas  $\varphi$  we have:

- (a)  $IL[h+1] \vdash \varphi$  iff  $IL \vdash \bigwedge_{i \leq h} ((\varphi \rightarrow \mathbf{B}_{i,k}^{(i)}) \rightarrow \mathbf{B}_{i,k}^{(i)})$ ;
- (b)  $S4[h+1] \vdash \varphi$  iff  $S4 \vdash \bigwedge_{i \leq h} (\Box(\Box \varphi \rightarrow \mathbf{B}_{i,k}) \rightarrow \mathbf{B}_{i,k})$ .

In particular, for  $h = 0$  we obtain (1), since for every  $k$ , the formulas  $\mathbf{B}_{0,k}$  and  $\mathbf{B}_{0,k}^{(i)}$  are  $\perp$ .

**Modal non-transitive case** The proof of Theorem 1 is essentially based on formulas  $\mathbf{B}_{h,k}$ . Sometimes, their analogs exist in the non-transitive case. A modal logic L is pretransitive (or weakly transitive, in another terminology), if the transitive reflexive closure modality  $\Diamond^*$  is expressible in L [Kra99]. Namely, for a language with  $n$  modalities  $\Diamond_i$  ( $i < n$ ), put  $\Diamond^0 \varphi = \varphi$ ,  $\Diamond^{m+1} \varphi = \Diamond^m \vee_{i < n} \Diamond_i \varphi$ ,  $\Diamond^{\leq m} \varphi = \bigvee_{l \leq m} \Diamond^l \varphi$ . A logic L is *pretransitive* if  $L \vdash \Diamond^{m+1} p \rightarrow \Diamond^{\leq m} p$  for some finite  $m$ . In this case  $\Diamond^{\leq m}$  plays the role of  $\Diamond^*$ . Examples of pretransitive logics are S4, K5, or the modal product  $S4 \times S4$ . The *height of a polymodal frame*  $(W, (R_i)_{i < n})$  is the height of the preorder  $(W, (\bigcup_{i < n} R_i)^*)$ . In the pretransitive case, the formulas of finite height can be defined analogously to the transitive case.

L is said to be *k-tabular* if, up to the equivalence in L, there exist only finitely many  $k$ -formulas. L is *locally tabular* (or *locally finite*) if it is  $k$ -tabular for every finite  $k$ .

**Theorem 2.** Let L be a pretransitive logic,  $h, k < \omega$ . If  $L[h]$  is  $k$ -tabular, then:

- (a) For every  $i \leq h$ , there exists a formula  $\mathbf{B}_{i,k}$  such that  $\mathbf{B}_{i,k} \in x$  iff the depth of  $x$  in the  $k$ -canonical frame of L is less than or equal to  $h$ .

---

\*This note is based on the manuscript [arxiv.org/abs/1806.06899](https://arxiv.org/abs/1806.06899). This work was supported by the RSF grant 16-11-10252 and performed at Steklov Mathematical Institute of Russian Academy of Sciences.

(b) For all  $k$ -formulas  $\varphi$ ,  $L[h+1] \vdash \varphi$  iff  $L \vdash \bigwedge_{i \leq h} (\Box^*(\Box^*\varphi \rightarrow \mathbf{B}_{i,k}) \rightarrow \mathbf{B}_{i,k})$ .

In particular, we have the following generalization of (1):  $L[1] \vdash \varphi$  iff  $L \vdash \Diamond^*\Box^*\varphi$  for every pretransitive  $L$ , since the inconsistent logic  $L[0]$  is locally tabular (for the unimodal case, a more direct proof of this equivalence is given in [KS17]).

**The one-variable fragment of a non-locally tabular modal logic can be finite** Theorem 2 generalizes Theorem 1, and has an additional condition,  $k$ -tabularity of  $L[h]$ . Indeed, a transitive logic is locally tabular iff it is of finite height iff it is 1-tabular ([Seg71], [Mak75]), but in the non-transitive case the situation is much more complicated. Every 1-tabular logic is a pretransitive logic of finite height [SS16], and there exists a pretransitive  $L$  such that none of the logics  $L[h]$  are 1-tabular [Mak81]. In general,  $k$ -tabularity of  $L[h]$  depends on  $h$  and  $k$ .

For example, let  $L$  be the least unimodal logic containing  $p \rightarrow \Diamond p$ ,  $\Diamond^3 p \rightarrow \Diamond^2 p$ , and  $\Box^2 \Diamond^2 p \rightarrow \Diamond^2 \Box^2 p$ . One can see that  $L[1] \vdash p \leftrightarrow \Box p$ . Thus,  $L[1]$  is locally tabular (and Theorem 2 describes translations from  $L[2]$  to  $L$  for all  $k < \omega$ ). One can check that  $L[2]$  is not 1-tabular.

With the parameter  $k$ , the situation is even more interesting. We know that a unimodal transitive logic is locally tabular iff it is 1-tabular [Mak75]. This equivalence also holds for other families of modal logics [SS16] (for example, it holds for unimodal logics containing  $\Diamond^{m+1} p \rightarrow \Diamond p \vee p$  for some  $m > 0$ ). However, this equivalence does not hold in general. For the counterexample, consider the frame  $(\omega+1, R)$ , where  $xRy$  iff  $x \leq y$  or  $x = \omega$ . It can be shown that its logic is 1-tabular but not locally tabular.

**Theorem 3.** There exists a unimodal 1-tabular logic which is not locally tabular.

**Questions** 1. It is unknown whether 2-tabularity of a modal logic implies its local tabularity. At least, does  $k$ -tabularity imply local tabularity, for some fixed  $k$  for all modal logics? The same questions are open in the intuitionistic case. 2. Finite height is not a necessary condition for local tabularity of intermediate logics. What can an analog of Glivenko's translation be in the case of a locally tabular intermediate logic with no finite height axioms? 3. In [Bez01], Glivenko type theorems were proved for intuitionistic modal logics above MIPC, the intuitionistic variant of S5. What can be an analog of Theorem 2 for modal intuitionistic logics?

## References

- [Bez01] G. Bezhanishvili. Glivenko type theorems for intuitionistic modal logics. *Studia Logica*, 67(1):89–109, 2001.
- [Gli29] V. Glivenko. Sur quelques points de la logique de M. Brouwer. *Bulletins de la classe des sciences*, 15:183–188, 1929.
- [Kra99] M. Kracht. *Tools and techniques in modal logic*. Elsevier, 1999.
- [KS17] A. Kudinov and I. Shapirovsky. Partitioning Kripke frames of finite height. *Izvestiya: Mathematics*, 81(3):592, 2017.
- [Mak75] L. Maksimova. Modal logics of finite slices. *Algebra and Logic*, 14(3):304–319, 1975.
- [Mak81] D. Makinson. Non-equivalent formulae in one variable in a strong omnitemporal modal logic. *Math. Log. Q.*, 27(7):111–112, 1981.
- [Mat55] K. Matsumoto. Reduction theorem in Lewis's sentential calculi. *Mathematica Japonicae*, 3:133–135, 1955.
- [Seg71] K. Segerberg. *An essay in classical modal logic*. Filosofiska Studier, vol.13. Uppsala Universitet, 1971.
- [She85] V. Shehtman. *Applying Kripke models to the investigation of superintuitionistic and modal logics*. PhD thesis, Moscow State University, 1985. In Russian.
- [SS16] I. Shapirovsky and V. Shehtman. Local tabularity without transitivity. In *Advances in Modal Logic*, volume 11, pages 520–534. College Publications, 2016.

# Simplicial semantics and one-variable fragments of modal predicate logics

Valentin Shehtman\*

We consider normal 1-modal predicate logics in the signature with predicate letters of all arities, but without equality, constants or functions letters. A logic is a set of formulas containing all classical validities and the axioms of **K** and closed under the standard rules including the predicate substitution, cf. [2].

**QA** denotes the minimal predicate extension of a propositional modal logic **Λ**, and **QΛC** := **QA** +  $\forall x \Box P(x) \rightarrow \Box \forall x P(x)$ .

*1-variable formulas* are constructed from a single variable  $x$  and monadic predicate letters. Every such formula  $A$  translates into a bimodal propositional formula  $A_*$  if every atom  $P_i(x)$  is replaced with the proposition letter  $p_i$  and every quantifier  $\forall x$  with  $\blacksquare$ . The *1-variable fragment* of a predicate logic  $L$  is the set  $L-1 := \{A_* \mid A \in L, A \text{ is 1-variable}\}$ ; this is always a bimodal propositional logic.

Our goal in this talk is to describe some logics of the form **QA** – 1.

For a monomodal logic **Λ** (in the language with  $\Box$ ), **Λ**\***S5** denotes its fusion with **S5** (in the language with  $\blacksquare$ ), and

$$\Lambda \_ \mathbf{S5} := \Lambda * \mathbf{S5} + \Box \blacksquare p \rightarrow \blacksquare \Box p, [\Lambda, \mathbf{S5}] := \Lambda * \mathbf{S5} + \Box \blacksquare p \leftrightarrow \blacksquare \Box p.$$

A propositional modal logic is *Horn axiomatizable* if it is axiomatized by modal axioms corresponding to first-order Horn formulas and (maybe) variable-free modal axioms, cf. [1]. The following theorem is well-known ([1], [3]):

**Theorem 1** *If  $\Lambda$  is Horn axiomatizable and Kripke complete, then  $\mathbf{Q}\Lambda\mathbf{C} - 1 = [\Lambda, \mathbf{S5}]$ .*

Our new result is similar:

**Theorem 2** *If  $\Lambda$  is Horn axiomatizable and Kripke complete, then  $\mathbf{Q}\Lambda - 1 = \Lambda \_ \mathbf{S5}$ .*

The proof of Theorem 1 was based on the observation that a predicate Kripke frame with a constant domain can be regarded as a product of a propositional frame with a cluster. This method does not work for Theorem 2, because now we need “expanding products”, and a logic  $\Lambda \_ \mathbf{S5}$  may not axiomatize them.

So instead of Kripke semantics, we use simplicial semantics of predicate modal logics. Let us recall related definitions. Let  $I_n = \{1, \dots, n\}$ ,  $I_0 = \emptyset$ , and let  $\Sigma_{mn}$  be the set of all maps  $I_m \rightarrow I_n$  ( $\Sigma_{0n}$  consists of a single map  $\emptyset_n$ , and  $\Sigma_{m0} = \emptyset$  for  $m > 0$ ). Also let  $\Sigma = \bigcup_{m,n} \Sigma_{mn}$ . There are specific

---

\*Institute for Information Transmission Problems, RAS; Steklov Mathematical Institute; Moscow State University; National Research University Higher School of Economics

maps:  $\delta_i^n \in \Sigma_{n-1,n}$  sends  $1, \dots, n-1$  respectively to  $1, \dots, i-1, i+1, \dots, n$ ;  $\sigma^+ \in \Sigma_{m+1,n+1}$  prolongs  $\sigma \in \Sigma_{mn}$  with  $\sigma^+(m+1) = n+1$ .

A *simplicial frame* based on a propositional frame  $F_0$  is  $\mathbb{F} = ((F_n)_{n \geq 0}, \pi)$ , where each  $F_n = (D^n, R^n)$  is a propositional frame,  $\pi = (\pi_\sigma)_{\sigma \in \Sigma}$  is a family of maps  $\pi_\sigma : D^n \rightarrow D^m$  for  $\sigma \in \Sigma_{mn}$ . A *valuation* in  $\mathbb{F}$  is a function  $\xi$  sending every predicate letter  $P_k^n$  to a subset  $\xi(P_k^n) \subseteq D^n$ . An *assignment* in  $\mathbb{F}$  is a pair  $(\mathbf{x}, \mathbf{a})$ , where  $\mathbf{a} \in D^n$ ,  $\mathbf{x} = (x_1, \dots, x_n)$  is a list of different variables. For a formula  $A$ , an assignment  $(\mathbf{x}, \mathbf{a})$  involving all its parameters and a valuation  $\xi$  the *truth relation*  $(\mathbb{F}, \xi), \mathbf{a}/\mathbf{x} \models A$  is defined by induction, in particular

- $\mathbf{a}/\mathbf{x} \models P_k^m(x_{\sigma(1)}, \dots, x_{\sigma(n)})$  iff  $\pi_\sigma \mathbf{a} \in \xi(P_k^m)$  (for  $\sigma \in \Sigma_{mn}$ );
- $\mathbf{a}/\mathbf{x} \models \Box B$  iff  $\forall \mathbf{b} \in R^n(\mathbf{a}) \mathbf{b}/\mathbf{x} \models B$ ;
- $\mathbf{a}/\mathbf{x} \models \exists y B$  iff  $\exists \mathbf{c} \in D^{n+1}(\pi_{\delta_{n+1}^n} \mathbf{c} = \mathbf{a} \ \& \ \mathbf{c}/\mathbf{xy} \models B)$ , where  $y$  does not occur in  $\mathbf{x}$ ;
- $M, \mathbf{a}/\mathbf{x} \models \exists x_i B$  iff  $M, \pi_{\delta_i^n} \mathbf{a}/(\mathbf{x} - x_i) \models \exists x_i B$  (where  $\mathbf{x} - x_i$  is obtained by crossing  $x_i$  out of  $\mathbf{x}$ ).

A formula  $A$  is *valid* in a simplicial frame if it is true under every valuation and variable assignment (for its parameters);  $A$  is *strongly valid* if all its substitution instances are valid.

**Theorem 3** [4] *Let  $\mathbb{F} = ((F_n)_{n \geq 0}, \pi)$  be a simplicial frame such that: (1)  $\pi_{\emptyset_1}$  is surjective; (2) every  $\pi_\sigma$  for  $\sigma \in \Sigma_{mn}$  is a  $p$ -morphism from  $F_n$  to  $F_m$  (perhaps, not surjective); (3)  $\pi$  reverses composition and sends identity maps to identity maps; (4) if  $\pi_{\delta_{m+1}^m}(\mathbf{b}) = \pi_\sigma(\mathbf{a})$ ,  $\sigma \in \Sigma_{mn}$ , then there exists  $\mathbf{c} \in D^{n+1}$  such that  $\pi_{\sigma^+}(\mathbf{c}) = \mathbf{b}$ ,  $\pi_{\delta_{n+1}^n}(\mathbf{c}) = \mathbf{a}$ . (Such a frame is called sound).*

*Then the set of formulas strongly valid in  $\mathbb{F}$  is a modal predicate logic.*

For the proof of Theorem 2, we assume that  $\mathbf{\Lambda} \_ \mathbf{S5} \not\models A_*$ . By Kripke completeness of this logic, we obtain a bimodal frame separating  $A_*$  from  $\mathbf{\Lambda} \_ \mathbf{S5}$ , which produces two 1-modal frames  $F_0$  and  $F_1$ . Basing on them we construct a sound simplicial frame  $\mathbb{F}$  such that  $F_n \models \mathbf{\Lambda}$  for every  $n$ . Then  $\mathbb{F}$  refutes  $A$  and strongly validates  $\mathbf{QL}$ , which implies  $\mathbf{QL} \not\models A$ .

This work was supported by the Russian Science Foundation, Project No. 16-11-10252 and was carried out at Steklov Mathematical Institute, RAS.

## References

- [1] D. GABBAY, A. KURUCZ, F. WOLTER, M. ZAKHARYASCHEV. *Many-dimensional modal logics: theory and applications*, Elsevier, 2003.
- [2] D. GABBAY, V. SHEHTMAN, D. SKVORTSOV, *Quantification in nonclassical logic*, Vol. 1. Elsevier, 2009.
- [3] D. GABBAY, V. SHEHTMAN. Products of modal logics, part 1. *Logic Journal of the IGPL*, v.6 (1998), 73–146.
- [4] D. SKVORTSOV, V. SHEHTMAN. Maximal Kripke-type semantics for modal and superintuitionistic predicate logics. *Annals of Pure and Applied Logic*, 63, 69–101, 1993.

# Order-enriched solid functors

Lurdes Sousa<sup>12\*</sup> and Walter Tholen<sup>3</sup>

<sup>1</sup> CMUC, University of Coimbra, Portugal

<sup>2</sup> Polytechnic Institute of Viseu, Portugal

sousa@estv.ipv.pt

<sup>3</sup> York University, Toronto, Canada

tholen@mathstat.yorku.ca

Many well-known categories, such as semilattices and their homomorphisms, frames and their homomorphisms,  $T_0$  topological spaces and continuous maps, has not only colimits, but weighted colimits, when regarded as order-enriched categories. We show that in all these, and many other cases, this is a consequence of a stronger property: all of them are the domain of an order-solid functor into the category of partially ordered sets and monotone maps.

Solid functors were studied extensively in [13], [15], [14] and [12] under the name of semi-topological functors, having been previously approached, independently and under different names, in [8] and [16]. They encompass functors from algebra and topology and enjoy many good properties including the detection of limits and colimits.

In the general enriched context, solid functors were studied by Anghel [3, 4, 5]. Here we are interested in the order-enriched version of these functors, continuing the study of order-enriched categories developed in the papers [6, 2, 10, 7, 1, 9]. Thus the hom-sets of our categories are equipped with a partial order preserved by the composition of morphisms on the left and on the right, and by functors between them. We consider order-solid functors, the order-enriched version of Anghel's notion, but we pay special attention to a slightly simplified notion, strongly order-solid functors, which, in some respects, proves to be more fruitful and justifies itself by a long list of examples. A functor  $P : \mathcal{A} \rightarrow \mathcal{X}$  is said to be *strongly order-solid* if, for every family  $\xi = (\xi_i : PD_i \rightarrow X)_{i \in I}$  in  $\mathcal{X}$ , there is a family  $\alpha = (\alpha_i : D_i \rightarrow A)_{i \in I}$  in  $\mathcal{A}$  and an  $\mathcal{X}$ -arrow  $q : X \rightarrow PA$  such that (i)  $(\alpha, A, q)$  is a  $P$ -extension of  $\xi$ , that is,  $P\alpha = q \cdot \xi$ ; (ii)  $(\alpha, A, q)$  is universal with respect to property (i), that is, it is initial in the obvious category of  $P$ -extensions of  $\xi$ ; and (iii)  $q : X \rightarrow PA$  is order- $P$ -epimorphic, i.e. the inequality  $Pf \cdot q \leq Pg \cdot q$  implies  $f \leq g$ . Every strongly order-solid functor is order-solid, but the question whether the converse property holds seems to be hard and is left open.

An important goal is the characterization of (strongly) order-solid functors in terms of their behaviour with respect to weighted limits and colimits. For a codomain category with inserters, we characterize strongly order solid functors as those which are solid in the ordinary sense, order-faithful, and have a domain category with inserters which are preserved by  $P$ . And we show that the existence of weighted (co)limits for diagrams of any shape is lifted by (strongly) order-solid functors.

We prove that algebraic functors between categories of general ordered algebras are always strongly order-solid provided that they admit free algebras over every ordered set. But not all examples of order-solid functors with an algebraic flavour fall in the scope of this result, as shown by the category of ordered vector spaces, considered as an order-enriched category via the positive cones of its objects: its positive-cone functor to the category of partially ordered sets is still strongly order-solid. But ordered vector spaces, when considered as a discretely ordered category, show that the condition of preservation of inserters in the above characterization is essential.

This presentation is based on the paper [11].

---

\*Presenter

## References

- [1] J. Adámek, L. Sousa, KZ-monadic categories and their logic, *Theory Appl. Categ.* 32 (2017), 338-379.
- [2] J. Adámek, L. Sousa, J. Velebil, Kan injectivity in order-enriched categories, *Math. Structures Comput. Sci.* 25 (2015), no. 1, 6-45.
- [3] C. Anghel, *Factorizations and initiality in enriched categories*, doctoral dissertation, Fernuniversität, Hagen, 1987.
- [4] C. Anghel, Semi-initial and semi-final  $\mathcal{V}$ -functors, *Comm. Algebra* 18 (1990), no. 1, 135-192.
- [5] C. Anghel, Lifting properties of  $\mathcal{V}$ -functors, *Comm. Algebra* 18 (1990), no. 1, 183-181.
- [6] M. Carvalho, L. Sousa, Order-preserving reflectors and injectivity, *Topology Appl.* 158 (2011), no. 17, 2408-2422.
- [7] M. Carvalho, L. Sousa, On Kan-injectivity of locales and spaces, *Appl. Categorical Structures* 25 (2017), no. 1, 1-22.
- [8] R.-E. Hoffmann, *Die kategorielle Auffassung der Initial- und Finaltopologie*, doctoral dissertation, Ruhr-Universität, Bochum, 1972.
- [9] D. Hofmann, L. Sousa, Aspects of algebraic algebras, *Logical Methods Computer Sci.* 13 (2017), no. 3, 25 pp.
- [10] L. Sousa, A calculus of lax fractions, *J. Pure Appl. Algebra* 221 (2017), 422-448.
- [11] L. Sousa, W. Tholen, Order-enriched solid functors, arXiv:1812.11222.
- [12] R. Street, W. Tholen, M. B. Wischnewsky, H. Wolff, Semitopological functors III. Lifting of monads and adjoint functors. *J. Pure Appl. Algebra* 16 (1980), 299314.
- [13] W. Tholen, Semitopological functors I, *J. Pure Appl. Algebra* 15 (1979), 53-73.
- [14] W. Tholen, A note on total categories, *Bull. Austr. Math. Soc.* 21 (1980), 169-173.
- [15] W. Tholen, M. B. Wischnewsky, Semitopological functors II. External characterizations. *J. Pure Appl. Algebra* 15 (1979), 7592.
- [16] V. Trnkova, Automata in categories, *Lecture Notes in Computer Science* 32, Springer-Verlag (1975), 132-152.

# Divisibility and diagonals

Isar Stubbe

Université du Littoral  
 isar.stubbe@univ-littoral.fr

In a frame  $L$ , from the Modus Ponens  $a \wedge (a \Rightarrow b) \leq b$  we can obviously derive the equality

$$a \wedge (a \Rightarrow b) = a \wedge b. \quad (1)$$

On the left hand side we can read here the interplay between the multiplication  $\wedge$  and its residuation in  $L$ , on the right hand side the  $\wedge$  plays its role as the infimum in  $L$ .

Now let  $Q$  be a quantale, i.e. a monoid in the category **Sup** of complete lattices and supremum-preserving maps. It still has residuations  $a \cdot - \dashv a \searrow -$  and  $- \cdot a \dashv - \swarrow a$ , but in general the Modus Ponens no longer implies a suitable equivalent to the equation (1). Therefore we say that  $Q$  is *divisible* if the equation

$$a \cdot (a \searrow b) = a \wedge b = (b \swarrow a) \cdot a \quad (2)$$

holds in  $Q$ . More generally, if we let  $Q$  be a quantaloid (i.e. a category enriched in **Sup**), then it still makes sense to say that  $Q$  is *divisible* if, for all pairs of parallel arrows  $f$  and  $f'$ , we have

$$f \circ (f \searrow f') = f \wedge f' = (f' \swarrow f) \circ f. \quad (3)$$

To gain a better understanding of this condition, and to make better use of it in applications, we consider the following: given any two arrows  $f$  and  $g$  in a quantaloid  $Q$ , define a *diagonal*  $d: f \rightarrow g$  to be an arrow  $d: \text{dom}(f) \rightarrow \text{cod}(g)$  for which the following diagram commutes:

$$\begin{array}{ccc}
 \cdot & \xrightarrow{d \swarrow f} & \cdot \\
 f \downarrow & \searrow d & \downarrow g \\
 \cdot & \xrightarrow{g \searrow d} & \cdot
 \end{array} \quad (4)$$

Such diagonals are the arrows of a new quantaloid  $D(Q)$  – the composition of diagonals is done by suitably “pasting” commutative squares as in (4) – which fully faithfully contains  $Q$ : indeed,  $D(Q)$  is the universal “splitting-of-everything” completion of  $Q$ . It is then a fact that the quantaloid  $Q$  is divisible if and only if

$$D(Q)(f, f') = \downarrow(f \wedge f') \quad (5)$$

for all pairs of parallel arrows  $f$  and  $f'$ .

In this talk, apart from explaining the above, I shall indicate some (easy) consequences of the divisibility of  $Q$ , sketch a useful context for these definitions (“partial enrichment”), and discuss a few examples (in part. continuous  $t$ -norms). This is a continuation of the work that I presented at TACL 2017 in Prague.



## References

- [1] Dirk Hofmann and Isar Stubbe, Topology from enrichment: the curious case of partial metrics, *Cahiers Topol. Géom. Différ. Catég.* 59 (2018) 307–353.
- [2] Ulrich Höhle, Tomasz Kubiak, A non-commutative and non-idempotent theory of quantale sets, *Fuzzy Sets and Systems* 166 (2011) 1–43.
- [3] Isar Stubbe, An introduction to quantaloid-enriched categories, *Fuzzy Sets and Systems* 256 (2014) 95–116.

# The coproduct of frames as encoding d-frame structure

Anna Laura Suarez    Achim Jung  
University of Birmingham

*D-frames* are pointfree duals of bitopological spaces; this duality was introduced and explored in [1]. The aim of the talk is to illustrate applications of both the coproduct of frames and Galois connections in this theory. A *d-frame* is a quadruple  $(L^+, L^-, \text{con}, \text{tot})$  where  $L^+$  and  $L^-$  are frames, and  $\text{con}, \text{tot} \subseteq L^+ \times L^-$  satisfy certain axioms. The frames  $L^+$  and  $L^-$  represent the two topologies, the subset  $\text{con}$  represents all the pairs in  $L^+ \times L^-$  that are disjoint, the subset  $\text{tot}$  all of those which cover the whole space. The motivation behind the axioms is precisely that  $\text{con}$  and  $\text{tot}$  should reflect the behaviour of disjointness and covering. The work we have been doing centres around trying to understand bitopological sublocales, which we identify with d-frame surjections. These come equipped with a natural order which we take as being the ordering of d-sublocales. Under this ordering, the d-sublocales form a complete lattice, as shown in [2]. There we are also given a notion of what it means for a d-frame surjection to be *generated* by updates of the form  $\text{con} \mapsto \text{con}' \supseteq \text{con}$  or  $\text{tot} \mapsto \text{tot}' \supseteq \text{tot}$ . These are not all of the sublocales, but the whole lattice of d-sublocales is generated by them in the subbasis sense. For d-frames of the form  $(L^+, L^-, \text{con}_{\min}, \text{tot}_{\min})$  the d-sublocales obtained by updating  $\text{con}$  are anti-isomorphic to the suitable  $\text{con}$  subsets of  $L^+ \times L^-$ , ordered under set inclusion. These form a frame, which we call  $\text{con}(L^+ \times L^-)$ . The d-sublocales obtained via  $\text{tot}$  updates are anti-isomorphic to a simple sublocale of the analogous poset of  $\text{tot}$  subsets  $\text{tot}(L^+ \times L^-)$  (this poset of  $\text{tots}$  is a frame too). So, we study the frames  $\text{con}(L^+ \times L^-)$  and  $\text{tot}(L^+ \times L^-)$  to gain insight about these subbasic d-sublocales.

Subsets of  $L^+ \times L^-$  such that they satisfy the  $\text{con}$  axioms are completely determined by their *d-pseudocomplementation maps*  $\sim : L^+ \rightleftharpoons L^- : \sim$  mapping each element of a frame to its *d-pseudocomplement*, that is the maximal element in the other frame such that it is in  $\text{con}$  with it. That this indeed exists is ensured by the defining axioms of a d-frame. Because a pair of monotone maps  $f : L^+ \rightleftharpoons L^- : g$  is a d-pseudocomplementation map if and only if it is an antitone Galois connection, we have an isomorphism  $\text{con}(L^+ \times L^-) \cong \text{Gal}(L^+, L^-)$ , where the second poset is that of Galois connections between  $L^+$  and  $L^-$ , ordered pointwise. The frame  $\text{Gal}(L^+, L^-)$  is also isomorphic to the coproduct  $L^+ \oplus L^-$ , as shown in [3].

Let us now look at the main diagram of my talk, shown below. The diagram is in the category **Frm**. The frame  $\text{totFilt}(L^+ \oplus L^-)$  is a subframe of  $\text{Filt}(L^+ \oplus L^-)$ , consisting of all filters generated by a collection of elements of the form  $e_+(x^+) \vee e_-(x^-)$ , where  $e_{+,-} : L^{+,-} \hookrightarrow L^+ \oplus L^-$  are the canonical embeddings into the coproduct. The maps  $i$  and  $j$  are both order isomorphisms. We have an antitone Galois connection  $\uparrow' : L^+ \oplus L^- \rightleftharpoons \text{totFilt}(L^+ \oplus L^-) : \bigwedge^1$ , which we lift along the two isomorphisms to obtain the  $\overline{(-)}$  maps.

$$\begin{array}{ccc}
 \text{con}(L^+ \times L^-) & \xleftarrow{\overline{(-)}} & \text{tot}(L^+ \times L^-) \\
 j(\cong) \downarrow & & \downarrow i(\cong) \\
 L^+ \oplus L^- & \xleftarrow[\bigwedge]{\uparrow'} & \text{totFilt}(L^+ \oplus L^-)
 \end{array}$$

The first relevant fact this tells us is that the coproduct of two frames  $L^+$  and  $L^-$  encodes all of  $\text{con}(L^+ \times L^-)$  and  $\text{tot}(L^+ \times L^-)$ . An intuitively appealing way of seeing this is the following: both  $\text{con}(L^+ \times L^-)$  and  $\text{tot}(L^+ \times L^-)$  represent subcollections of  $\mathbf{S}(L^+ \oplus L^-)$ , the coframe of all

<sup>1</sup>Since not all principal filters are  $\text{tot}$ -filters, we adapt the usual Galois connection and define  $\uparrow'(c) = \bigvee\{F \in \mathbf{tFilt}(L^+ \oplus L^-) : F \subseteq \uparrow c\}$ , where this join is taken in  $\text{Filt}(L^+ \oplus L^-)$ .

sublocales of  $L^+ \oplus L^-$  (although this representation is only unique in the case of  $\text{con}$ ). The significance of  $L^+ \oplus L^-$  is that it is the pointfree representation of the join topology<sup>2</sup> from  $L^+$  and  $L^-$ .

- The  $\text{con}$  are unique representations of *closed sublocales* of the coproduct. The main idea is that each  $c \in \text{con}(L^+ \times L^-)$  represents  $\mathfrak{c}(\bigvee\{a^+ \wedge a^- : a^+ a^- \in c\})$ . The correspondence is an order anti-isomorphism.

Consider that a certain  $\text{con}$  subset  $c$  imposes that some pairs of abstract patch opens  $a^+ a^-$  are disjoint. But in the pointfree patch topology  $L^+ \oplus L^-$  these abstract opens appear too, and imposing those pairs to be disjoint in this setting means identifying each  $a^+ \wedge a^-$  (for  $a^+ a^- \in c$ ) with the bottom element, which is exactly what our closed sublocale above does. Notice that this means that the d-sublocales of  $(L^+, L^-, \text{con}_{\min}, \text{tot}_{\min})$  coming from  $\text{con}$  updates do not depend on the frames  $L^+$  and  $L^-$ , but only on their coproduct. These d-sublocales can be completely described monotopologically, by identifying them when the closed sublocales of  $L^+ \oplus L^-$ .

- The  $\text{tot}$  subsets are representations of *certain fitted sublocales* of the coproduct. The main idea is that each  $t \in \text{tot}(L^+ \times L^-)$  represents  $\bigwedge\{\mathfrak{o}(a^+ \vee a^-) : a^+ a^- \in t\}$ .

Any  $\text{tot}$  subset  $t$  imposes a certain collection of pairs of patch opens  $a^+ a^-$  to be covering. In the coproduct this means identifying all the  $a^+ \vee a^-$ 's with  $a^+ a^- \in t$  with the top element, which gives the fitted sublocale above. Each of these sublocales is generated by a  $\text{tot}$ -filter. In contrast with the  $\text{con}$  case,  $\text{tot}(L^+ \times L^-)$  does depend on the particular frames  $L^+$  and  $L^-$ , and in this sense the d-sublocales of  $(L^+, L^-, \text{con}_{\min}, \text{tot}_{\min})$  obtained by  $\text{tot}$  updates are genuinely bitopological.

The second main result is that we can get a d-frame canonically whose two frame components are  $\text{con}(L^+ \times L^-)$  and  $\text{tot}(L^+ \times L^-)$ . We can equip this pair of frames with a suitable  $\text{con}$  relation, obtained by taking (the lifting of) the Galois connection  $(\bigwedge, \uparrow')$  to be a d-pseudocomplementation map. One can also define a totality relation  $\text{Tot}$ , satisfying the balance axiom with  $\text{Con}$ . So the structure  $(\text{con}(L^+ \times L^-), \text{tot}(L^+ \times L^-), \text{Con}, \text{Tot})$  is a d-frame. In the “monotopological” case – that is in the case where the pair of frames is of the form  $(L, 2)$  – this d-frame is  $(L, \text{Filt}(L), \text{Con}, \text{Tot})$  and it was already known to have an interesting property. In fact its patch frame<sup>3</sup> is isomorphic to the frame of nuclei  $\mathbf{N}(L)$  on  $L$ , as explained in [4]. Motivated by this, in our future work we would like to see whether in general the d-frame  $(\text{con}(L^+ \times L^-), \text{tot}(L^+ \times L^-), \text{Con}, \text{Tot})$  satisfies a similar universal property, or even whether there is a certain sense in which it acts as the d-frame containing information about all of the d-sublocales of  $(L^+, L^-, \text{con}_{\min}, \text{tot}_{\min})$ .

## References

- [1] A. Jung and M. A. Moshier. *On the bitopological nature of Stone duality*. Technical Report CSR-06-13, School of Computer Science, The University of Birmingham, 2006. 110 pages.
- [2] T. Jakl, A. Jung, and A. Pultr. Quotients of d-frames.
- [3] D. Wigner. Two notes on frames. *J. Austral. Math. Soc.*, Series A(28):257–268, 1979.
- [4] O. K. Klinke. *A presentation of the assembly of a frame by generators and relations exhibits its bitopological structure*. *Algebra universalis*, 71(1):55–64, Feb 2014.

---

<sup>2</sup>We refer to this alternatively as the join topology or the patch topology. This is because in the bitopological setting we see  $L^+$  and  $L^-$  as being topologies on the same set. When  $\text{con}$  and  $\text{tot}$  are minimal, their join topology can be seen as the patch topology generated by taking  $L^+ \cup L^-$  as a subbasis.

<sup>3</sup>The *patch frame* of a d-frame  $(L^+, L^-, \text{con}, \text{tot})$  is a frame  $(L^+ \oplus L^-)/R$  where  $R$  forces the sets of identifications  $\{e_+(a^+) \wedge e_-(a^-) = \perp : a^+ a^- \in \text{con}\}$  and  $\{e_+(a^+) \vee e_-(a^-) = \top : a^+ a^- \in \text{tot}\}$ .

# Gluing residuated lattices

Nikolaos Galatos<sup>1</sup> and Sara Ugolini<sup>1</sup>

University of Denver, Colorado, USA  
 nikolaos.galatos@du.edu, sara.ugolini@du.edu

Residuated lattices constitute the equivalent algebraic semantics, in the sense of Blok-Pigozzi [2], of substructural logics, which encompass most of the interesting nonclassical logics: intuitionistic logic, fuzzy logics, relevance logics, linear logic, besides including classical logic as a limit case. Thus, the investigation of the variety of residuated lattices is a powerful tool for analyzing such logics comparatively, as deeply explored in [4]. The multitude of different structures makes the study fairly complicated, and at the present moment large classes of residuated lattices lack a structural understanding. Thus, the study of constructions that allow to obtain new structures starting from known ones, is extremely important to improve our grasp of residuated lattices, and as a result of substructural logics.

The construction we introduce *glues* together two integral residuated lattices that share a principal filter. Let us call *conical* an element that is comparable to every other element of a residuated lattice, while an element  $x$  is said *idempotent* if  $x \cdot x = x$ . Let  $\mathbf{B} = (B, \cdot, \backslash, /, \wedge, \vee, 1)$ ,  $\mathbf{C} = (C, \cdot, \backslash, /, \wedge, \vee, 1)$  be integral residuated lattices, with  $\mathbf{C}$  having a lower bound  $0_C$ , that have an isomorphic principal filter generated by a conical idempotent element, that we will call  $a$  in both  $\mathbf{B}$  and  $\mathbf{C}$ . We assume that the two algebras intersect in the filter generated by  $a$ ,  $B_a \cap C_a = \langle a \rangle$  (where  $\langle a \rangle$  coincides with the upset of  $a$ , since  $a$  is idempotent). Let us call  $C_a = C \setminus \langle a \rangle$ , and  $B_a = B \setminus \langle a \rangle$ .

We define the *gluing* of  $\mathbf{B}$  and  $\mathbf{C}$  with respect to  $a$  as the structure

$$\mathbf{B} \oplus_a \mathbf{C} = (B \cup C, \cdot_a, \backslash_a, /_a, \wedge_a, \vee_a, 1)$$

Where the operations are defined as follows:

$$\begin{aligned} x \cdot_a y &= \begin{cases} x \cdot y & \text{if } x, y \in B, \text{ or } x, y \in C \\ a \cdot y & \text{if } x \in C_a, y \in B_a \\ x \cdot a & \text{if } x \in B_a, y \in C_a \end{cases} \\ x \backslash_a y &= \begin{cases} x \backslash y & \text{if } x, y \in B, \text{ or } x, y \in C \\ a \backslash y & \text{if } x \in C_a, y \in B_a \\ 1 & \text{if } x \in B_a, y \in C_a \end{cases} \\ x /_a y &= \begin{cases} x / y & \text{if } x, y \in B, \text{ or } x, y \in C \\ x / a & \text{if } x \in B_a, y \in C_a \\ 1 & \text{if } x \in C_a, y \in B_a \end{cases} \\ x \wedge_a y &= \begin{cases} x \wedge y & \text{if } x, y \in B, \text{ or } x, y \in C \\ x & \text{if } x \in B_a, y \in C_a \\ y & \text{if } y \in B_a, x \in C_a \end{cases} \\ x \vee_a y &= \begin{cases} x \vee y & \text{if } x, y \in C, \text{ or } x, y \in B \text{ with } x \vee y \neq a \\ 0_C & \text{if } x, y \in B_a, x \vee y = a \\ y & \text{if } x \in B_a, y \in C_a \\ x & \text{if } y \in B_a, x \in C_a \end{cases} \end{aligned}$$

Notice that the lattice structure is given by copying  $B_a$  below  $C$ , and the product and the residuals between elements in  $C_a$  and elements in  $B_a$  derive from products and residuals between

$a$  and the elements in  $B_a$  and  $C_a$ . We can prove that  $\mathbf{B} \oplus_a \mathbf{C}$  is an integral residuated lattice, that has a lower bound if and only if  $\mathbf{B}$  does. Notice that in general  $\mathbf{C}$  is a subalgebra of the gluing  $\mathbf{B} \oplus_a \mathbf{C}$ , while  $\mathbf{B}$  is a subalgebra with respect to all operations except for  $\vee_a$ . More precisely,  $\mathbf{B}$  is a subalgebra iff  $a$  is join-irreducible.

The gluing construction generalizes the *ordinal sum* construction introduced in [3], that has played an important role in the study of residuated structures, in particular of commutative integral residuated lattices (or CIRLs) satisfying prelinearity (the equational property that characterizes CIRLs that are a subdirect product of chains), and the divisibility condition  $x \wedge y = x \cdot (x \rightarrow y)$  (see [1]). The ordinal sum construction is a special case of the gluing operation, where the conical idempotent element  $a$  is 1. This also means that given any pair of integral residuated lattices  $\mathbf{B}$  and  $\mathbf{C}$ , with  $\mathbf{B}$  having a lower bound, we can always *glue* them.

It is worth noticing that the gluing preserves commutativity, prelinearity and divisibility. Thus, for instance, gluing two GMTL-algebras (prelinear CIRL) gives a GMTL-algebra, and the gluing of two basic hoops (divisible GMTL-algebras) is a basic hoop. Moreover, it can be easily shown that in general, the gluing operation preserves all unary equations without join (e.g. it preserves  $n$ -potency,  $x^n = x^{n+1}$ , for every  $n \geq 1$ ). Furthermore, whenever  $a$  is join irreducible in  $\mathbf{B}$ , any gluing of the kind  $\mathbf{B} \oplus_a \mathbf{C}$  will preserve all unary equations satisfied by both  $\mathbf{B}$  and  $\mathbf{C}$ .

The gluing construction can be extended to *bounded* integral residuated lattices. Then we get for instance that a gluing of MTL-algebras (prelinear bounded CIRLs) is an MTL-algebra, and gluings of BL-algebras (divisible MTL-algebras) are BL-algebras. It can further be proved that a gluing of bounded CIRLs,  $\mathbf{B} \oplus_a \mathbf{C}$ , is bipartite iff  $\mathbf{B}$  is bipartite, where we call a bounded CIRL  $\mathbf{R}$  bipartite if it is the disjoint union of the intersection of its maximal filters  $\mathcal{R}(\mathbf{R})$  and  $\mathcal{C}(\mathbf{R}) = \{x \in R : x \rightarrow 0_R \in \mathcal{R}(\mathbf{R})\}$ . More interestingly, we can use the gluing construction to generate and characterize new varieties of bounded CIRLs generated by classes of bipartite residuated lattices. As a result, interesting subvarieties of  $n$ -potent CIRLs can be characterized using this construction.

## References

- [1] P. Aglianò, F. Montagna, *Varieties of BL-Algebras I: General Properties*, Journal of Pure and Applied Algebra, 181: 105-129, 2003.
- [2] W. Blok, D. Pigozzi, *Algebraizable Logics*, Mem. Amer. Math. Soc, 396(77), Amer. Math. Soc. Providence, 1989.
- [3] I.M.A. Ferreira, *On varieties and quasi varieties of hoops and their reducts*, Ph.D. Thesis, University of Illinois at Chicago, 1992.
- [4] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono, *Residuated Lattices: an algebraic glimpse at substructural logics*, Studies in Logics and the Foundations of Mathematics, Elsevier, 2007.

# Point-free theories of space and time: a short history, models and representation theory

Dimitar Vakarelov

Faculty of Math. and Informatics, Sofia University, 5 J. Bourchier Blvd., 1164 Sofia, Bulgaria  
dvak@fmi.uni-sofia.bg

One of the main primitive notions of the standard Euclidean approach to the theory of space is the notion of point (as well as line and plane). In the beginning of 20 century some philosophers including mainly Alfred North Whitehead [8] formulated certain criticism to this point-based approach, because points (lines, planes) are some abstract things which have no separate existence in reality and consequently should not be put on the base of the theory. Accordingly the theory of space should be point-free and should be based on some simple spatial relations between things. The same criticism is given by Whitehead to the theory of time - time points (moments of time) also do not have separate existence in reality. Influenced by the Relativity theory Whitehead stated that space and time should be considered as one integrated point-free axiomatic theory. This means that neither space points, nor time moments should be taken as primitive notions at the beginning of the theory but should be introduced later on by appropriate formal definitions. In the resulting theory the structure of space-time should be extracted from some simple spatiotemporal relations between things. Let us note that till now various point-free formulations of the theory of space have been given (see, for instance, the survey [4] for some historical information, results and references). One of the main formulation of the point-free theory of space is based on the abstract notion of **contact algebra** [2]. A contact algebra  $\mathbf{B} = (B, C)$  is a Boolean algebra  $B$  with one binary relation  $C$  called **contact**. The elements of the Boolean algebra are considered as abstractions of spatial regions and its operations are considered as constructions of new regions by given ones. The relation of Boolean ordering  $a \leq b$  is interpreted as "the region  $a$  is part of the region  $b$ " and  $a \neq 0$  is interpreted as " $a$  exists". The intuitive meaning of the contact relation  $aCb$  is that  $a$  and  $b$  has a common point, but as point is not a primitive notion,  $C$  is characterized by several simple axioms. Standard point-based models of contact algebras are Boolean algebras of regular closed sets of a given topological space with contact non-empty intersection. Detailed topological representation theory for some classes of contact algebras is given in [2] and duality results for some categories of contact algebras and some generalizations are presented in [3] with references to other papers on the same topic.

While the point-free theory of space is developing quite well with some applications in *Knowledge representation*, the same cannot be said for the integrated point-free theory of space and time. According to the author's information only the initial steps have been done in the papers [5,6,7,1]. As a result several notions of **dynamic contact algebra** (DCA) have been obtained. These are Boolean algebras whose elements are considered as changing or moving regions, called **dynamic regions** equipped with some spatio-temporal relations between them. In [5] these relations are:  $aC^{\forall}b$  - **stable contact** (intuitively  $a$  and  $b$  are in a contact in all moments of time) and  $aC^{\exists}b$  - **unstable contact** ( $a$  and  $b$  are in contact in some moments of time). In [6] we consider a different set of spatio-temporal relations:  $aC^s b$  - **spatial contact** (the same as  $aC^{\exists}b$ ),  $aC^t b$  - **time contact** - ( $a$  and  $b$  exist simultaneously at some moment of time, *simultaneity* or *contemporaneity* relation in Whitehead's terminology), and  $aBb$  - **precedence** - (there is a moment of time in which  $a$  is existing and a later moment of time in which  $b$  is existing). In [7] we add to these relations some special regions called **representatives of time** one of them called **NOW**. Intuitively each representative of time exists at exactly one moment of time ("epoch" in the Whitehead's terminology) and **NOW** is existing only at the **present epoch**. In the ordinary language we use as time representatives some things identifying the time epoch in which they are existing. For instance: "the epoch of Leonardo", "the epoch of dinosaurs". Axioms for all these versions of DCA are chosen as true sentences from a concrete point-based model, called

---

This research was supported by Bulgarian National Science Fund, contract no. DN02/15/19.12.2016.

**snapshot model** (or **cinematographic model**). The informal idea of the snapshot model is the following. If we want to describe an area of changing regions, then for each moment of time we make a picture (snapshot) of the corresponding spatial configurations of regions at that time assuming that each picture is a given contact algebra. We identify a given dynamic region  $a$  with its time history  $\langle \dots a_t \dots \rangle_{t \in T}$ , where  $a_t$  is the region  $a$  at the moment  $t$ . Formally we start with a given temporal structure  $(T, \prec, \mathbf{now})$  where  $T$  is the set of time points,  $\prec$  is a time order relation **before-after**, and **now** is the present epoch. Then we associate to each  $t \in T$  a contact algebra  $\mathbf{B}_t = (B_t, C_t)$ , called the **coordinate contact algebra** corresponding to  $t$ . Then for each dynamic region  $a = \langle \dots a_t \dots \rangle_{t \in T}$  we have that  $a_t \in B_t$ . All dynamic regions belong to a (subset)  $\mathbf{B}$  of the Cartesian product  $\prod_{t \in T} B_t$  of Boolean algebras  $\mathbf{B}_t$ ,  $t \in T$ , and  $\mathbf{B}$  is considered as a Boolean subalgebra of  $\prod_{t \in T} B_t$ . Then all mentioned above spatio-temporal relations have exact definition in  $\mathbf{B}$ :

$$aC^s b \text{ iff } (\exists t \in T)(a_t C_t b_t), \quad aC^t b \text{ iff } (\exists t \in T)(a_t \neq 0_t \text{ and } b_t \neq 0_t),$$

$$aBb \text{ iff } (\exists s, t \in T)(s \prec t \text{ and } a_s \neq 0_s \text{ and } b_t \neq 0_t),$$

and similarly for the time representatives. A special number of conditions is chosen using only the spatio-temporal relations and regions called **time axioms** validity of which in the snapshot model is equivalent to some good properties of the time-order relation  $\prec$  in the time structure  $(T, \prec, \mathbf{now})$ . For instance the relation  $\prec$  satisfies the density axiom

$$(\mathbf{Dens} \ s \prec t \Rightarrow (\exists k)(s \prec k \prec t) \text{ iff the axiom } (\mathbf{dens} \ aBb \rightarrow aBp \text{ or } -pBb) \text{ is true in the model.}$$

For all versions of DCA mentioned in [5,6,7] a special representation theorem is proved showing that each (DCA) is isomorphic with a special DCA obtained by the snapshot construction, showing in this way that the information of the concrete model is coded by the axioms of the abstract point-free formulation. This means that a (canonical) time structure  $(T, \prec, \mathbf{now})$  and (canonical) coordinate contact algebras has to be extracted from the abstract definition of DCA to build a (canonical) snapshot model and to show that the obtained model is isomorphic to the given DCA.

In [1] some variations of DCA-s from [7] has been introduced with relational characteristic models which are used for a complete Kripke style semantics of some spatio-temporal logics based on DCA-s. The aim of the present talk is to give a short survey of the obtained so far results for DCA-s and to present a third kind of models of DCA-s - topological models and the expected representation theory. We will formulate also various open problems for further investigations.

## REFERENCES

1. P. Dimitrov and Dimiter Vakarelov. Dynamic contact algebras and quantifier-free logics for space and time. *Siberian Electronic Mathematical Reports* <http://semr.math.nsc.ru>, DOI 10.17377/semi.2018.15.092 MSC 03B60, 03G25
2. G. Dimov and D. Vakarelov. Contact Algebras and Region-based Theory of Space. A proximity approach. I and II. *Fundamenta Informaticae*, 74(2-3):209-249, 251-282, 2006.
3. G. Dimov, E. Ivanova-Dimova and D. Vakarelov. A generalization of the Stone Duality Theorem. *Topology and its Applications*, 221(2017)237-261.
4. D. Vakarelov, Region-Based Theory of Space: Algebras of regions, Representation Theory and Logics. In: Dov Gabbay, Sergey Goncharov and Michael Zakharyashev (Eds.) *Mathematical problems from Applied Logics. New Logics for XXIst Century. II. Springer, 2007, 267-348.*
5. D. Vakarelov, Dynamic Mereotopology: A point-free Theory of Changing Regions. I. Stable and unstable mereotopological relations. *Fundamenta Informaticae*, vol 100, (1-4) (2010), 159-180.
6. D. Vakarelov, Dynamic Mereotopology II: Axiomatizing some Whiteheadean Type Space-time logics. In: Th. Bolander, T. Braüner, S. Ghilardi and L. Moss Eds., *Advances in Modal logic*, vol. 9, 538-558, 2012.
7. D. Vakarelov, Dynamic mereotopology III. Whiteheadean type of integrated point-free theories of space and tyme. Part I, *Algebra and Logic*, vol. 53, No 3, 2014, 191-205. Part II, *Algebra and Logic*, vol. 55, No 1, 2016, 9-23. Part III, *Algebra and Logic*, vol. 55, No 3, 2016, 181-197.
8. A. N. Whitehead, *Process and Reality*, New York, MacMillan, 1929.

# Axiomatizing the crisp Gödel modal logic

Ricardo O. Rodriguez<sup>1,2</sup> and Amanda Vidal<sup>3</sup>

<sup>1</sup> UBA. FCEyN. Departamento de Computación.

<sup>2</sup> CONICET-UBA. Inst. de Invest. en Cs. de la Computación.  
Buenos Aires, Argentina. [ricardo@dc.uba.ar](mailto:ricardo@dc.uba.ar)

<sup>3</sup> Institute of Computer Sciences of the Czech Academy of Sciences, Czech Republic  
[amanda@cs.cas.cz](mailto:amanda@cs.cas.cz)

Several recent publications in the literature address the study of modal expansions of many-valued logics (see eg. [4, 3], [6], [1], [5], [7]), and it is the aim of the current work to contribute to the better understanding of this topic. In particular, we close the problem left open in [6] and [3] of finding an axiomatization for the Gödel Modal Logic with crisp accessibility and the two usual modal operators  $\Box$  and  $\Diamond$ .

While in [4] the Gödel modal logics with one modal operator are studied, only the  $\Box$  fragment of the logic<sup>1</sup> arising from crisp-accessibility models is axiomatized there (turning out it coincides with the logic with arbitrary accessibility models). The modal logic with  $\Diamond$  and crisp accessible relation is axiomatized in [6]. Further, in [3], the authors study the Gödel modal logics with both  $\Box$  and  $\Diamond$  usual operators, axiomatizing the logic arising from the class of models with valued accessibility relation. However, the completeness proof is highly dependant in the possibility of assigning values strictly in  $(0, 1)$  to the accessibility relation, and thus it was not clear how to face the axiomatization of the restriction to the the model with  $\{0, 1\}$  (crisp) valued accessibility relation. Further, the addition of the axioms and rules from [6] to the system in [4] was not clear to be enough to axiomatize the bi-modal (crisp) logic, leaving the axiomatization of the Bi-modal Gödel logic with crisp accessibility as a non-trivial open problem.

Let us use  $Fm$  to denote the set of formulas in the language  $\{\wedge/2, \vee/2, \neg/1, \rightarrow/2, 0/0, 1/0\} \cup \{\Box/1, \Diamond/1\}$  (the formulas in the first set of operations are the propositional ones). A (**crisp accessibility**) **Gödel-Kripke model**  $\mathfrak{M}$  is a tripla  $\langle W, R, e \rangle$  with  $W \neq \emptyset$ ,  $R \subseteq W \times W^2$  and a mapping  $e: W \times \mathcal{V}ars \rightarrow [0, 1]$ . The evaluation  $e$  is uniquely extended to a mapping  $W \times Fm \rightarrow [0, 1]$  by evaluating the propositional connectives by their corresponding operations in the standard Gödel algebra  $[0, 1]_G$ , and letting  $e(v, \Box\varphi) := \bigwedge_{Rvw} e(w, \varphi)$  and  $e(v, \Diamond\varphi) := \bigvee_{Rvw} e(w, \varphi)$ .

We write  $\Gamma \models \varphi$ , and say that  $\varphi$  **follows from**  $\Gamma$  (**in the crisp Gödel modal logic**) whenever, for any Gödel Kripke model  $\mathfrak{M}$  and any  $v \in W$ ,  $e(v, [\Gamma]) \subseteq \{1\}$  implies  $e(v, \varphi) = 1$ .

In their work [3], Caicedo and Rodriguez axiomatize  $K(G)$ , the minimum Gödel modal logic, which is the logic arising from Kripke models similar to the above ones but where  $R: W^2 \rightarrow [0, 2]$   $K(\mathbf{G})$  and then  $e(v, \Box\varphi) := \bigwedge_{w \in W} R(v, w) \rightarrow e(w, \varphi)$  and  $e(v, \Diamond\varphi) := \bigvee_{w \in W} R(v, w) \wedge e(w, \varphi)$ .  $K(G)$  is axiomatized extending Gödel-Dummett propositional calculus (i.e, Heyting calculus plus the prelinearity law) with the following additional axioms and rules:

$$\begin{array}{ll} (K_{\Box}) & \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi) & (K_{\Diamond}) & \Diamond(\varphi \vee \psi) \rightarrow (\Diamond\varphi \vee \Diamond\psi) \\ (F_{\Box}) & \Box\top & (P) & \Box(\varphi \rightarrow \psi) \rightarrow (\Diamond\varphi \rightarrow \Diamond\psi) \\ (FS2) & (\Diamond\varphi \rightarrow \Box\psi) \rightarrow \Box(\varphi \rightarrow \psi) & (Nec) & \text{from } \vdash \varphi \text{ infer } \vdash \Box\varphi \end{array}$$

The logic we propose now,  $K^c(\mathbf{G})$ , complete wrt. crisp accessibility Gödel-Kripke models is defined by adding to  $K(\mathbf{G})$  the following axiom schemata:

$$\begin{array}{l} (\text{Cr}) \quad \Box(\varphi \vee \psi) \rightarrow (\Box\varphi \vee \Box\psi) \\ (A_{\Box\Diamond}^{\leq}) \quad ((\Box\varphi \rightarrow \Box\psi) \rightarrow \Box\psi) \rightarrow \Diamond((\varphi \rightarrow \psi) \rightarrow \psi) \vee \Box \perp \end{array}$$

<sup>1</sup>It is the case that in Gödel modal logic  $\Box$  and  $\Diamond$  are not interdefinable.

<sup>2</sup>A usual accessibility relation, for which we will write  $Rab \Leftrightarrow \langle a, b \rangle \in R$ .



$$\begin{aligned} (A_{\Box}^{\leq}) \quad & ((\Box\varphi \rightarrow \Box\psi) \rightarrow \Box\psi) \rightarrow ((\Box((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow \Box\psi) \rightarrow \Box\psi) \\ (A_{\Diamond}^{\leq}) \quad & ((\Diamond\varphi \rightarrow \Diamond\psi) \rightarrow \Diamond\psi) \rightarrow \Diamond((\varphi \rightarrow \psi) \rightarrow \psi) \end{aligned}$$

In what follows, for any formula  $\varphi$  we denote by  $SFm(\varphi) \subseteq Fm$  the set of subformulas of  $\varphi$  and containing the formulas  $\perp$  and  $\top$ . Further, as usual, let us denote by  $Fm^{\sharp}$  the set of sentences built only with propositional connectives over an extended set of variables which denote the formulas in  $Fm$  starting by a modality. For checking the previous system is complete w.r.t  $\models$ , it suffices to do that for theorems (since the logic enjoys the DT). A relevant observation is that the canonical model arising in the proof depends on the formula being studied; namely, if  $\not\models_{K^c(\mathbf{G})} \varphi$ , we let the canonical model  $\mathfrak{M}^{\varphi}$  be

- $W^{\varphi} := \{h \in \text{hom}(Fm^{\sharp}, [0, 1]_G) : h(\text{Th}(K^c(\mathbf{G}))) \subseteq \{1\}\}$ ,
- $R^{\varphi}hg \iff h(\Box\chi) \leq g(\chi)$  and  $h(\Diamond\chi) \geq g(\chi)$  for all  $\chi \in SFm(\varphi)$ ,
- $e^{\varphi}(h, p) = h(p)$  for any  $p$  propositional variable.

The Truth Lemma, which easily implies completeness, is formulated as  $h(\Box\psi) = \bigwedge_{R^{\varphi}hg} g(\psi)$  for all  $\Box\psi \in SFm(\varphi)$ , and the analogous result for  $\Diamond$  formulas. The following can be proven by relying in the proposed axiomatic system and dividing the proof in two cases depending on the behaviour of  $v$ . It is the key result to check the Truth Lemma over  $\Box$ -formulas:

**Lemma.** *Let  $v \in W^{\varphi}$ , and  $\psi \in Fm(\varphi)$  such that  $v(\Box\psi) < 1$ . Then there exists a  $G$ -homomorphism  $u$  from  $Fm^{\sharp}$  into  $[0, 1]_G$  such that*

1.  $u(\text{Th}(K^c(\mathbf{G}))) = 1$ ,
2.  $u(\theta) = 1$  for all  $\theta \in SFm(\varphi)$  such that  $v(\Box\theta) = 1$ ,
3.  $u(\chi) < 1$  for all  $\chi \in SFm(\varphi)$  such that  $v(\Diamond\chi) < 1$ ,
4.  $u(\psi) < u(\chi)$  for all  $\chi \in SFm(\varphi)$  such that  $v(\Box\psi) < v(\Box\chi)$  (in particular,  $u(\Box\psi) < 1$ ).

Now, for an arbitrary  $\epsilon > 0$ , it can be defined, in a similar fashion to [3], a standard Gödel endomorphism  $\sigma$  such that  $R^{\varphi}v(\sigma \circ u)$ , and  $\sigma \circ u(\varphi) \in [v(\Box\varphi), v(\Box\varphi) + \epsilon]$ , proving the Truth Lemma. The proof for the  $\Diamond$  formulas can be done in a dual way.

**Theorem.**  *$K^c(\mathbf{G})$  is sound and complete with respect to  $\models$ .*

Interestingly enough, this approach also provides a new axiomatization of the  $\Diamond$  fragment of the (crisp) Gödel modal logic, different from the one in [6]. For the interested reader, the logic  $G^c(G)$  is proven complete (via its semantics) in [2].

## References

- [1] F. Bou, F. Esteva, L. Godo, and R. Rodríguez. On the minimum many-valued modal logic over a finite residuated lattice. *Journal of Logic and Computation*, 21(5):739–790, 2011.
- [2] X. Caicedo, G. Metcalfe, R. Rodríguez, and J. Rogger. A finite model property for Gödel modal logics. In L. Libkin et. al. eds, *Logic, Language, Information, and Computation*, vol. 8071 of LNCS.2013.
- [3] X. Caicedo and R. Rodríguez. Bi-modal Gödel logic over  $[0, 1]$ -valued kripke frames. *Journal of Logic and Computation*, 25(1):37–55, 2015.
- [4] X. Caicedo and R. Rodríguez. Standard Gödel modal logics. *Studia Logica*, 94(2):189–214, 2010.
- [5] G. Hansoul and B. Teheux. Extending Łukasiewicz logics with a modality: Algebraic approach to relational semantics. *Studia Logica*, 101(3):505–545, 2013.
- [6] G. Metcalfe and N. Olivetti. Towards a Proof Theory of Gödel Modal Logics. *Logical Methods in Computer Science*, 7(2), 2011.
- [7] A. Vidal, F. Esteva, and L. Godo. On modal extensions of product fuzzy logic. *Journal of Logic and Computation*, 27 (1): 299–336, 2017.

# Correspondence, Canonicity, and Model Theory for Monotonic Modal Logics

Kentarô Yamamoto

Group in Logic and the Methodology of Science  
 University of California, Berkeley  
 Berkeley, California USA  
 ykentar@math.berkeley.edu

Monotonic modal logics generalize normal modal logics by dropping the K axiom  $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$  and instead requiring only that  $\vdash \phi \rightarrow \psi$  imply  $\vdash \Box \phi \rightarrow \Box \psi$ . There are a number of reasons for relaxing the axioms of normal modal logics and considering monotonic modal logics. For instance, monotonic modal logics are considered more appropriate to describe the ability of agents or systems to make certain propositions true in the context of games and open systems [4, 5, 1]. The standard semantics for monotonic modal logics is provided by monotonic neighborhood frames (see, e.g., [2]).

Just as the first-order language with a relation symbol is a useful correspondence language for Kripke frames, it is natural to consider what would be a useful correspondence language for monotonic neighborhood frames. Litak et al. [3] studied *coalgebraic predicate logic* (CPL) as a logic that plays that role and proved a characterization theorem in the style of van Benthem and Rosen [6]. In this article, we continue that path for monotonic neighborhood frames and prove variants of the Goldblatt-Thomason theorem and the Fine canonicity theorem in the setting of coalgebraic predicate logic.

Subclass	Closed under ...
monotonic	supersets
quasi-filter	supersets, intersections of nonempty finite families of neighborhoods
augmented quasi-filter	supersets, intersections of nonempty families of neighborhoods
filter	supersets, intersections of finite families of neighborhoods
augmented filter	supersets, intersections of families of neighborhoods

Table 1: Classes of monotonic neighborhood frames and their definitions

The analogue of the Goldblatt-Thomason theorem in this article is that a class of monotonic neighborhood frames closed under CPL-elementarity relative to any of the classes of neighborhood frames in Table 1 is modally definable if and only if it is closed under disjoint unions, bounded morphic images, and generated subframes, and it reflects ultrafilter extensions; and the analogue of Fine's theorem we will prove states that a sufficient condition for the canonicity of a monotonic modal logic is that it is complete with respect to the class of monotonic neighborhood frames it defines and that that class is closed under CPL-elementarity relative to any of the classes of neighborhood frames in Table 1.

**Definition 1.** Let  $L_0$  be a language of first-order logic. The *language of coalgebraic predicate logic  $L$  based on  $L_0$*  is the least set of formulas containing  $L_0$  and closed under Boolean combinations, existential quantification, and formation of formulas of the form  $x \Box_y \phi$  where  $\phi \in L$ ,  $x$  is a term, and  $y$  is a variable. An  *$L$ -structure*  $F = (F, N^F)$  is an  $L_0$ -structure  $F$  with an additional datum  $N^F : F \rightarrow \mathcal{P}(\mathcal{P}(F))$ , where  $\mathcal{P}$  is the powerset operation.

**Definition 2.** Let  $L$  be a language of coalgebraic predicate logic and  $F$  an  $L$ -structure. We define the satisfaction predicate  $F \models \phi$  for a sentence  $\phi \in L$ . It is convenient to define the predicate for the expanded language  $L(F)$  of coalgebraic predicate logic. In general, for  $A \subseteq F$ , we define  $L(A)$  to be the language of coalgebraic predicate logic that has all symbols of  $L$  and for each  $w \in A$  a constant symbol  $w$  that is intended to be interpreted as  $w$  itself. Now,  $F$  is an  $L(F)$ -structure in the obvious way. We define the satisfaction predicate  $F \models \phi$  for  $\phi \in L(F)$ . The predicate is defined by recursion on  $\phi$ . For symbols of first-order logic in  $L$ , the predicate is defined in the ordinary way. For  $\phi = w \Box_y \phi_0$ , we define  $F \models w \Box_y \phi_0(y) \iff \phi_0(F) \in N^F(w)$ , where  $\phi_0(F) = \{v \in F \mid F \models \phi_0(v)\}$  and  $\phi_0(v)$  stands for the substitution instance of  $\phi_0(y)$  with  $v$  substituted for  $y$ .

**Definition 3.** Let  $\mathcal{K}_0$  be a class of monotonic neighborhood frames. A class  $\mathcal{K}$  of monotonic neighborhood frames is *CPL-elementary relative to  $\mathcal{K}_0$*  if there is a set of  $T$  of sentences of  $L_{=}$  with  $\mathcal{K} = \{F \in \mathcal{K}_0 \mid F \models T\}$ . Two monotonic neighborhood frames  $F$  and  $F'$  are *CPL-elementarily equivalent relative to  $\mathcal{K}_0$*  if  $F, F' \in \mathcal{K}_0$  and  $\text{Th}_{L_{=}}(F) = \text{Th}_{L_{=}}(F')$ .

**Theorem 4.** Let  $\mathcal{K}$  be a class of monotonic neighborhood frames that is closed under CPL-elementary equivalence relative to any of the classes in Table 1.  $\mathcal{K}$  is modally definable if and only if it is closed under bounded morphic images, generated subframes, and disjoint unions, and it reflects ultrafilter extensions. Moreover, if such a class is modally definable, then it is definable by a canonical set of formulas.

**Theorem 5.** A set  $\Sigma$  of modal formulas is canonical if it is complete with respect to the class  $\mathcal{K}$  of monotonic neighborhood frames that it defines, and  $\mathcal{K}$  is closed under CPL-elementary equivalence relative to any of the classes in Table 1.

## References

- [1] Rajeev Alur, Thomas Henzinger, and Orna Kupferman. Alternating-time temporal logic. In *Journal of the ACM*, pages 100–109. IEEE Computer Society Press, 1997.
- [2] Helle Hvid Hansen. Monotonic modal logics. Master’s thesis, University of Amsterdam, 2003.
- [3] T. Litak, D. Pattinson, K. Sano, and L. Schröder. Model Theory and Proof Theory of CPL. *ArXiv e-prints*, January 2017.
- [4] Rohit Parikh. The logic of games and its applications. In *Annals of Discrete Mathematics*, pages 111–140. Elsevier, 1985.
- [5] Marc Pauly. *Logic for Social Software*. PhD thesis, University of Amsterdam, 2001.
- [6] Lutz Schröder, Dirk Pattinson, and Tadeusz Litak. A van benthem/rosen theorem for coalgebraic predicate logic. *Journal of Logic and Computation*, 27(3):749–773, 2017.

**THE BOHR COMPACTIFICATION OF AN ABELIAN GROUP  
AS A QUOTIENT OF ITS STONE-ČECH COMPACTIFICATION**

PAVOL ZLATOŠ

COMENIUS UNIVERSITY  
BRATISLAVA

For any (discrete) semigroup  $S$ , its Stone-Čech compactification

$$\beta S = \{u \mid u \text{ is an ultrafilter on the set } S\},$$

endowed with the topology given by the base of clopen sets

$$\{u \in \beta S \mid X \in u\}$$

where  $X \subseteq S$ , admits a semigroup operation  $(u, v) \mapsto uv$  given by

$$A \in uv \Leftrightarrow \{s \in S \mid s^{-1}A \in v\} \in u,$$

for  $u, v \in \beta S$ ,  $A \subseteq S$ , where

$$s^{-1}A = L_s^{-1}[A] = \{x \in S \mid sx \in A\},$$

for  $s \in S$ . Then  $\beta S$  becomes a compact right topological semigroup, i.e., all the right shifts  $R_v: \beta S \rightarrow \beta S$  where  $R_v(u) = uv$  are continuous, densely extending  $S$ . Moreover,  $\beta S$  can be characterized by the following universal property:

*Every homomorphism  $h: S \rightarrow K$  from  $S$  to a compact hausdorff right topological semigroup  $K$  extends to a unique continuous homomorphism  $h: \beta S \rightarrow K$ ;  $h$  is onto if and only if  $h[S]$  is dense in  $K$ .*

For an abelian group  $G$ , its dual  $\widehat{G} = \text{Hom}(G, \mathbb{T})$ , where  $\mathbb{T} \subseteq \mathbb{C}$  is the unit circle, is again an abelian group under the pointwise multiplication of characters  $\gamma, \chi \in \widehat{G}$  given by

$$(\gamma\chi)(x) = \gamma(x)\chi(x) \quad (x \in G).$$

Being a closed subgroup of  $\mathbb{T}^G$ ,  $\widehat{G}$  is a compact hausdorff topological group.

Let  $\widehat{G}_d$  denote the dual of  $G$  endowed with the discrete topology. Using the Pontryagin-van Kampen duality theory for locally compact abelian groups, the Bohr compactification of  $G$  can be defined as the dual  $\mathfrak{b}G = \widehat{G}_d$  of  $\widehat{G}_d$ .

Then  $\mathfrak{b}G$  is a compact topological group and  $G$  can be canonically embedded into  $\mathfrak{b}G$  as a dense subset, identifying any  $x \in G$  with the character  $x: \widehat{G}_d \rightarrow \mathbb{T}$  of  $\widehat{G}_d$ , given by  $x(\gamma) = \gamma(x)$  for  $\gamma \in \widehat{G}_d$ . This is justified by the following universal property of  $\mathfrak{b}G$ , characterizing the Bohr compactification in category-theoretical terms:

*Every homomorphism  $h: G \rightarrow K$  from  $G$  to a compact hausdorff topological group  $K$  extends to a unique continuous homomorphism  $h^\sharp: \mathfrak{b}G \rightarrow K$ ;  $h^\sharp$  is onto if and only if  $h[G]$  is dense in  $K$ .*

Since the Stone-Čech compactification is “more universal” than the Bohr one, there is a canonical continuous map  $\xi_G: \beta G \rightarrow \mathfrak{b}G$  from the Stone-Čech compactification

$\beta G$  of  $G$  to its Bohr compactification  $\mathfrak{b}G$ , such that  $\xi_G(x) = x$  for  $x \in G$ . Obviously,  $\xi_G$  is surjective and it can be easily shown that it is a homomorphism with respect to the semigroup operation on  $\beta G$ , extending the multiplication on  $G$ , and the group operation on  $\mathfrak{b}G$ . Hence the description of  $\mathfrak{b}G$  as a quotient of  $\beta G$  reduces to the description of the equivalence relation

$$\text{Eq}(\xi_G) = \{(u, v) \in \beta G \times \beta G \mid \xi_G(u) = \xi_G(v)\}$$

on  $\beta G$ .

An ultrafilter  $u \in \beta G$  is called a *Schur ultrafilter* if, for any set  $A \in u$ , there exist elements  $a, b \in A$  such that  $ab \in A$ , as well. Let  $\Xi(G)$  denote the least closed congruence relation on  $\beta G$  merging all the Schur ultrafilters on  $G$  to the unit of  $G$ .

**Theorem.** *For any (discrete) abelian group  $G$  we have  $\text{Eq}(\xi_G) = \Xi(G)$ , hence the Bohr compactification  $\mathfrak{b}G$  of  $G$  is isomorphic (both algebraically and topologically) to the quotient  $\beta G/\Xi(G)$  of the Stone-Čech compactification  $\beta G$ .*

The proof relies on the fact that every idempotent ultrafilter on  $G$  is Schur and on the two Ellis' theorems, the first one of which states that every compact right topological semigroup contains an idempotent, while, according to the second one, every locally compact right and left topological group is already a topological group.

FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS, COMENIUS UNIVERSITY,  
 MLYNSKÁ DOLINA, 842 48 BRATISLAVA, SLOVAKIA  
*E-mail address:* `zlatos@fmph.uniba.sk`

# Pierce stalks in preprimal varieties

D.J. Vaggione and W.J. Zuluaga Botero

<sup>1</sup> CONICET and Universidad Nacional de Córdoba  
Córdoba, Argentina

<sup>2</sup> Universidad Nacional del Centro de la Provincia de Buenos Aires  
Tandil, Argentina

A variety with  $\vec{0}$  and  $\vec{1}$  is a variety  $\mathcal{V}$  in which there are 0-ary terms  $0_1, \dots, 0_N, 1_1, \dots, 1_N$  such that  $\mathcal{V} \models \vec{0} = \vec{1} \rightarrow x = y$ , where  $\vec{0} = (0_1, \dots, 0_N)$  and  $\vec{1} = (1_1, \dots, 1_N)$ . The terms  $\vec{0}$  and  $\vec{1}$  are analogous, in a rather general manner, to identity (top) and null (bottom) elements in rings (lattices), and their existence in a variety, when the language has at least a constant symbol, is equivalent to the fact that no non-trivial algebra in the variety has a trivial subalgebra [9]. In order to simplify and clarify our treatment we will assume that  $N = 1$  (i.e.  $\mathcal{V} \models 0 = 1 \rightarrow x = y$ , for some 0-ary terms 0 and 1). The results exposed remain valid in the general case.

If  $A \in \mathcal{V}$ , then we say that  $e \in A$  is a *central element* of  $A$  if there exists an isomorphism  $A \rightarrow A_1 \times A_2$  such that  $e \mapsto (0, 1)$ . We use  $Z(A)$  to denote the set of central elements of  $A$ . Central elements are a generalization of both central idempotent elements in rings with identity and neutral complemented elements in a bounded lattice. In these classical cases it is well known that the central elements concentrate the information concerning the direct product representations. This happens when  $\mathcal{V}$  has the Fraser-Horn property (FHP) [8]. Let  $A$  be an algebra. By  $Con(A)$  we denote the congruence lattice of  $A$ . It is well known that the set of factor congruences of an algebra  $A$  in a variety with the Fraser-Horn property forms a Boolean algebra  $FC(A)$  which is a sublattice of  $Con(A)$  (see [1]). In [8] it is proved that if  $\mathcal{V}$  has the Fraser-Horn property, then for  $A \in \mathcal{V}$ , the map

$$\begin{aligned} \lambda : FC(A) &\rightarrow Z(A) \\ \theta &\rightarrow \text{unique } e \text{ satisfying } 0 \theta e \theta^* 1 \end{aligned}$$

(where  $\theta^*$  is the complement of  $\theta$  in  $FC(A)$ ) is bijective. Thus via the above bijection we can give to  $Z(A)$  a Boolean algebra structure.

Many of the usual properties of central elements in rings with identity or bounded lattices hold when  $\mathcal{V}$  has the Fraser-Horn property. For example, there is a set  $\{\zeta_r(z) : r \in R\}$  of  $(\forall \exists \wedge p = q)$ -formulas such that for any  $A \in \mathcal{V}$ , we have that  $e \in Z(A)$  iff  $A \models \zeta_r(z)$ , for every  $r \in R$ . Also in [8] it is proved that there is a  $(\exists \wedge p = q)$ -formula  $\epsilon(x, y, z)$  such that for all  $A, B \in \mathcal{V}$ ,  $A \times B \models \epsilon((a, b), (c, d), (0, 1))$  if and only if  $a = c$ . The formula  $\epsilon(-, -, e)$  defines the factor congruence associated (via the map  $\lambda^{-1}$ ) with the central element  $e$ . Also we note that the existence of  $\epsilon(x, y, z)$  and  $\{\zeta_r(z) : r \in R\}$  implies that the central elements (and its Boolean algebra structure) are preserved by surjective homomorphisms and products.

Let  $A \subseteq \prod\{A_i : i \in I\}$  be a subdirect product. Given  $x, y \in \prod\{A_i : i \in I\}$ , the *equalizer* of  $x$  and  $y$  is the set  $E(x, y) = \{i \in I : x(i) = y(i)\}$ . We say that the subdirect product  $A \subseteq \prod\{A_i : i \in I\}$  is *global* if there is a topology  $\tau$  on  $I$  such that  $E(x, y) \in \tau$  for every  $x, y \in A$  and the following property holds:

**PP** (*Patchwork Property*) For every  $\{F_r : r \in R\} \subseteq \tau$  such that  $\bigcup\{F_r : r \in R\} = I$ , and  $\{x_r : r \in R\} \subseteq A$  such that for every  $r, s \in R$ ,  $x_r$  and  $x_s$  match in  $F_r \cap F_s$ , there exists  $x \in A$  such that  $x(i) = x_r(i)$ , provided that  $i \in F_r$  and  $r \in R$ .

Let  $\mathcal{M}$  be a class of algebras and let us assume that  $A$  is a global subdirect product of  $\{A_i : i \in I\}$ . We say that  $A$  is a global subdirect product with factors in  $\mathcal{M}$  if  $A_i \in \mathcal{M}$ , for every  $i \in I$ .

An algebra  $A$  is *directly indecomposable* if  $FC(A) = \{\Delta^A, \nabla^A\}$ . Given a variety  $\mathcal{V}$ , we write  $\mathcal{V}_{DI}$  for the class of all directly indecomposable members of  $\mathcal{V}$ .

**Theorem 1.** Let  $\mathcal{L}$  be a language of algebras with at least a constant symbol. Let  $\mathcal{V}$  be a variety of  $\mathcal{L}$ -algebras with the FHP. Suppose that there is a universal class  $\mathcal{F} \subseteq \mathcal{V}_{DI}$  such that every member of  $\mathcal{V}$  is isomorphic to a global subdirect product with factors in  $\mathcal{F}$ . Then there exists a  $(n+2)$ -ary term  $u(x, y, \vec{z})$  and 0-ary terms  $0_1, \dots, 0_n, 1_1, \dots, 1_n$  such that

$$\mathcal{V} \models u(x, y, \vec{0}) = x \wedge u(x, y, \vec{1}) = y.$$

An algebra  $P$  is called preprimal if  $P$  is finite and  $Clo(P)$  is a maximal clone. For each preprimal algebra  $P$ , Rosenberg [6] described an  $m$ -ary relation  $\sigma$  on  $P$  such that the  $n$ -ary term-operations of  $P$  are precisely the  $n$ -ary functions  $f$  on  $P$  that preserve  $\sigma$ . He gives seven types of relations which are sufficient to describe all preprimal algebras. A preprimal variety is a variety generated by a preprimal algebra.

In [5] Knoebel studies the Pierce sheaf ([4], [2], [3]) of the different preprimal varieties and he asks for a description of the Pierce stalks. He solves this problem for some of the Rosenberg types and left open some other types such as those of a central relation and of a non-trivial proper equivalence relation. In this paper we use the above Theorem and some results of [7] to give important information on the Pierce stalks for these two unsolved types.

## References

- [1] D. Bigelow and S. Burris, *Boolean algebras of factor congruences*, Acta Sci. Math. (Szeged) 54:1-2(1990).
- [2] S. Comer, *Representations by algebras of sections over Boolean spaces*, Pacific Journal of Mathematics 38 (1971), no. 1, 29–38.
- [3] B. A. Davey, *m-Stone lattices*, Can. J. Math., Vol. XXIV, No. 6, (1972), 1027-1032.
- [4] K. Keimel, *Darstellung von Halbgruppen und universellen Algebren durch Schnitte in Garben; bireguläre Halbgruppen*, Math. Nachrichten 45 (1970), 81-96.
- [5] A. Knoebel, *Sheaves of algebras over Boolean spaces*, Birkhauser, (2012).
- [6] I. Rosenberg, *Über die funktionale Vollständigkeit in den mehrwertigen Logiken*, Rozpr. CSAV Rada Mat. Pfir. Ved, 80 (1970), 3-93.
- [7] D. Vaggione, *Varieties in which the Pierce stalks are directly indecomposable*, Journal of Algebra 184 (1996), 424-434.
- [8] D. Vaggione, *Central elements in varieties with the Fraser-Horn property*, Advances in Mathematics 148, 193-202, 1999.
- [9] D. Vaggione, *Varieties of shells*, Algebra Universalis, **36** (1996) 483-487.

