Notes on minimization the area of self-intersection of a folded rectangle.
Mikhail Gaichenkov, October,2023

The problem of minimization the area of self-intersection of a folded rectangle: rectangle with sides a, b (a<b) is bent along the line that passes through the center of the rectangle in order to get the minimum area of crossing intersections: a unique rectangle exists for two solutions with equal area but different shapes - triangle and pentagon. The unique ratio of sides a/b=T=0.81502370129163... is derived based on the real root of the quintic. If a/b<T ('long' rectangle) the angle to bent is pi/4. If a/b=1 (square) the angle is Pi/8.
In more details:

$$\frac{a}{b} = \frac{1+4t_0^2-t_0^4}{4t_0+(1+t_0)(1-t_0)\sqrt{6t_0^2-1-t_0^4}},$$

t_0 – the real root of eq. $t^5+3t^4+4t^3+t-1=0$

$t_0 \approx 0{,}459133723310207539 47\ldots$

$\frac{a}{b} \approx 0{,}815023701291631086 87409\ldots$

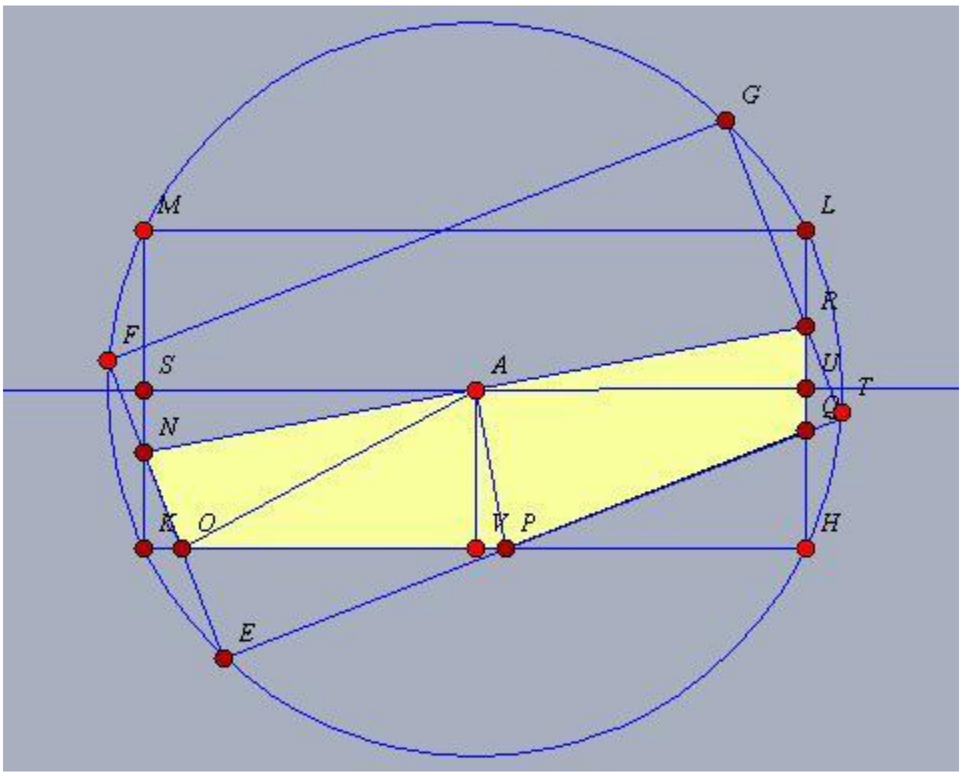
Overview.

Let the sides of the rectangle: a, b and a<b (the case a=b will be reviewed separately), also let the angle β: tg β = a/b. Also α is the angle to band the rectangle.

There are a few cases:

1. The angle α changes from 0 to β
2. α changes from β to (π/2-β)
3. α changes from (π/2-β) to π/2

Case 1.



Let tg α=t, then the area:

$$S=\frac{2(1+t^2)}{4(1-t^2)}*(ab-ta^2+ab-tb^2)=\frac{(1+t^2)(2ab-(a^2+b^2)t)}{4(1-t^2)}=\frac{2ab-(a^2+b^2)t+2abt^2-(a^2+b^2)t^3}{4(1-t^2)}$$

$$S'=\frac{(-(a^2+b^2)+4abt-3(a^2+b^2)t^2)(4-4t^2)-(2ab-(a^2+b^2)t+2abt^2-(a^2+b^2)t^3)(-8t)}{(4-4t^2)^2}=$$

$$=\frac{-4(a^2+b^2)+32abt-16(a^2+b^2)t^2+4(a^2+b^2)t^4}{(4-4t^2)^2}$$

So, we get the eq:

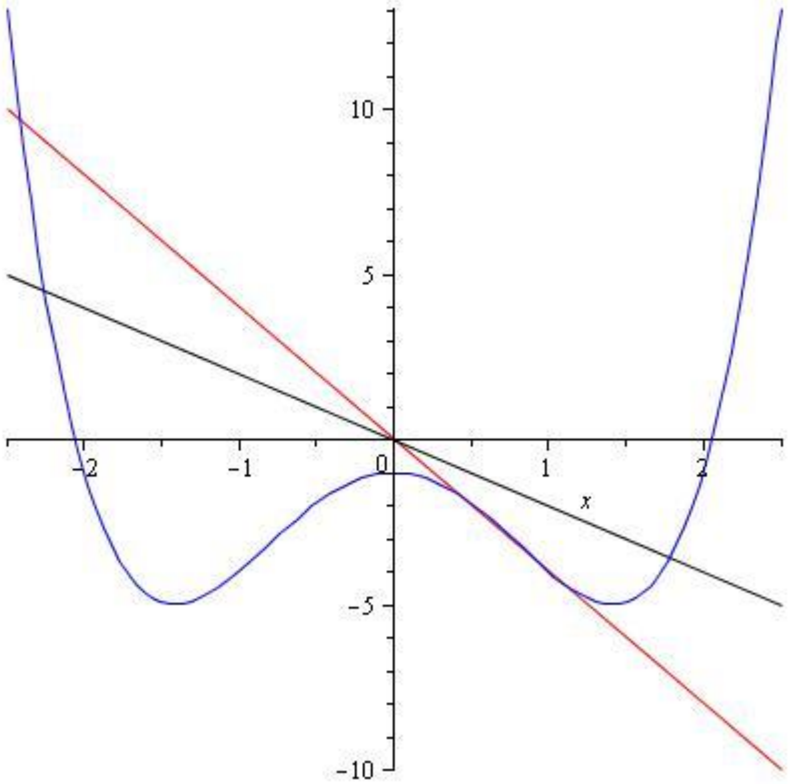
$$-(a^2+b^2)+8abt-4(a^2+b^2)t^2+(a^2+b^2)t^4=0$$

Due to a/b=tg β, then k=sin 2β = 2tg β /(1+tg^2β)= $\frac{\frac{2a}{b}}{1+(\frac{a}{b})^2}=\frac{2ab}{a^2+b^2}$

This results in the eq: $t^4-4t^2+4kt-1=0$

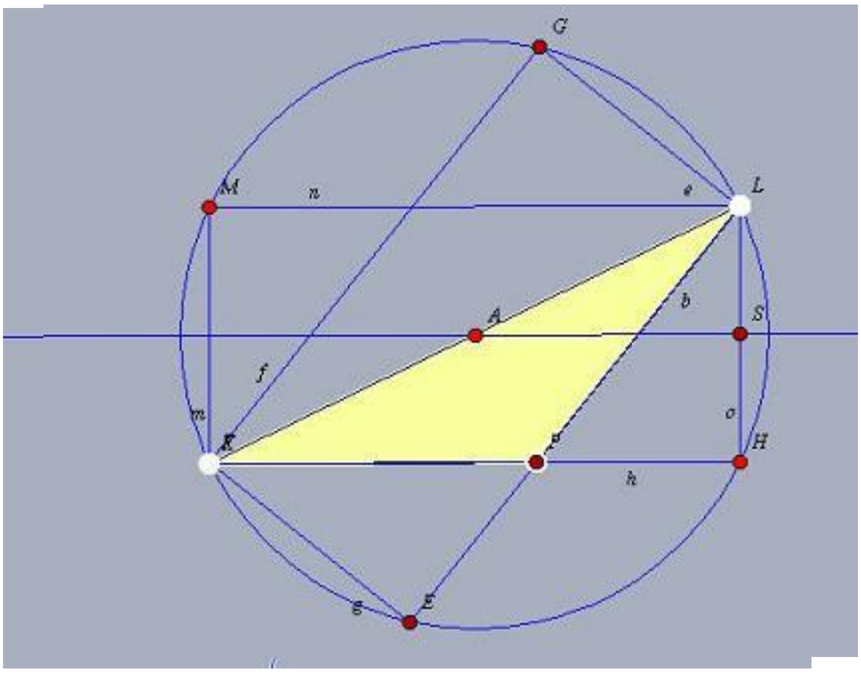
Here k changes from 0 to 1 (β changes from 0 to π/4).

Consider the functions: $f=t^4-4t^2-1$ and $g=-4kt$ for further analysis.



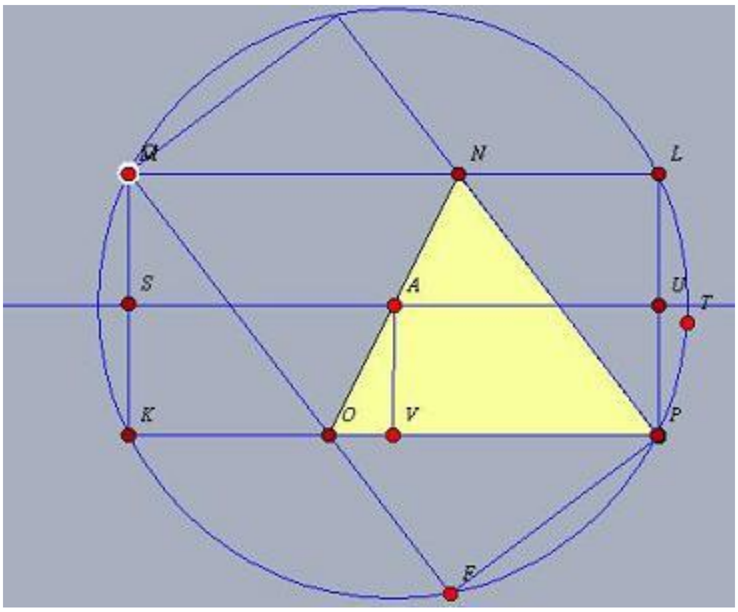
Case 2.

Pentagon converts into the triangle.



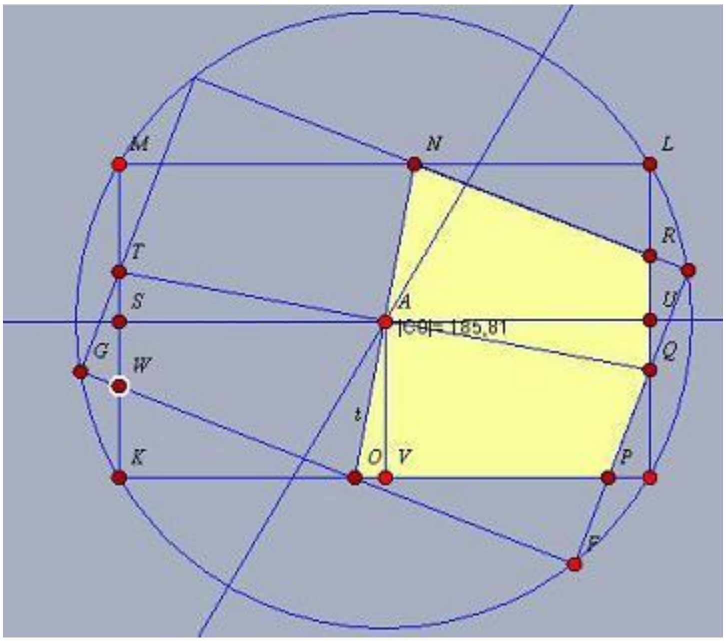
The minimum area of self-intersection appears at sin 2α=1, i.e. α=π/4, and the area is $\frac{a^2}{2}$

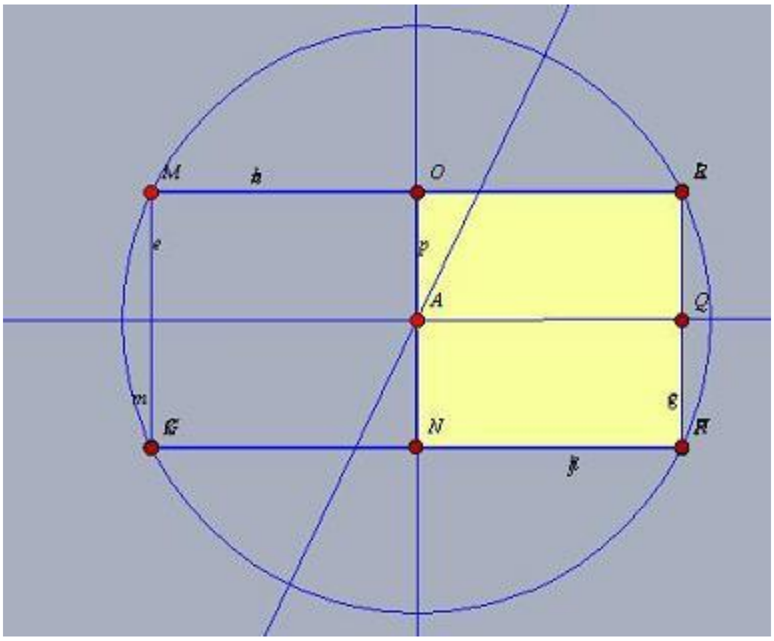
If α gets to (π/2-β), the intersections is like below.



Case 3.

α changes from (π/2-β) to π/2, we get the pentagon or a rectangle like below





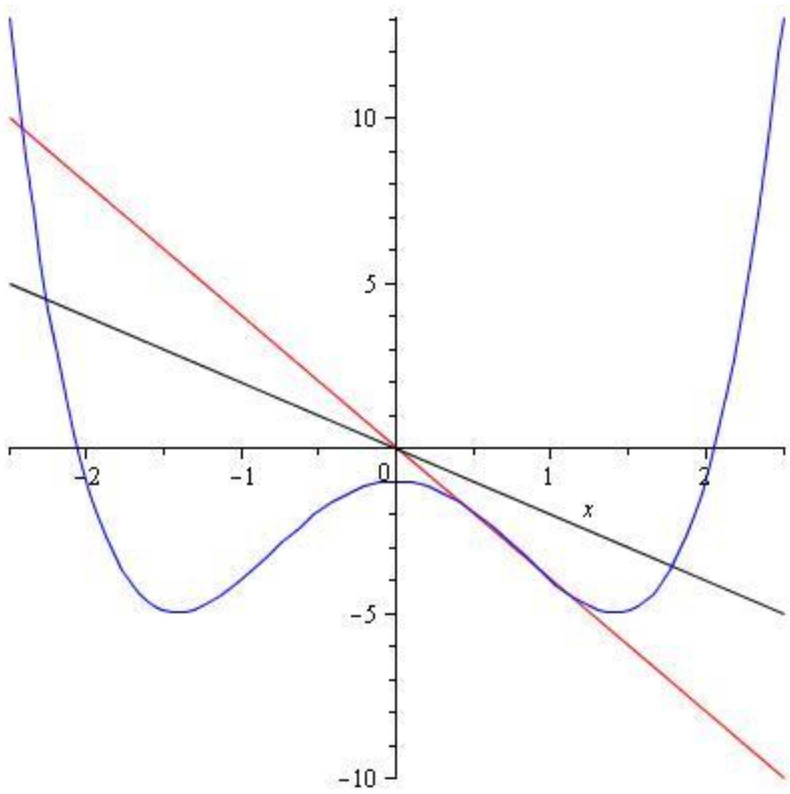
The case of square.

tg β=1, the root of the eq. $t^4 - 4t^2 + 4t - 1 = 0$ is t1=$\sqrt{2} - 1$

This means α=π/8 and S=$(\sqrt{2} - 1)a^2$.

Now, consider

$f = t^4 - 4t^2 - 1$ and $g = -4kt$. If k<C=$\frac{5}{3\sqrt{3}}$ then the line g does not cross the f for t ∈ (0,1). If k=C then g touches f at t0=$\frac{1}{\sqrt{3}}$. If k changes from k1=1 to k2=$\frac{5}{3\sqrt{3}}$ then the appropriate root t1 also changes: $(\sqrt{2} - 1, \frac{1}{\sqrt{3}})$.



So, we can consider the inverse function $k=\frac{1+4t^2-t^4}{4t}$

Let s=tg β=$\frac{a}{b}$ and u=$\frac{b}{a}$, this results in s=$\frac{1-\sqrt{1-k^2}}{k}$, u=$\frac{1+\sqrt{1-k^2}}{k}$.

$$S=\frac{2ab-(a^2+b^2)t+2abt^2-(a^2+b^2)t^3}{4(1-t^2)}=\frac{(1+t^2)}{4(1-t^2)}*(2ab-(a^2+b^2)t)=$$

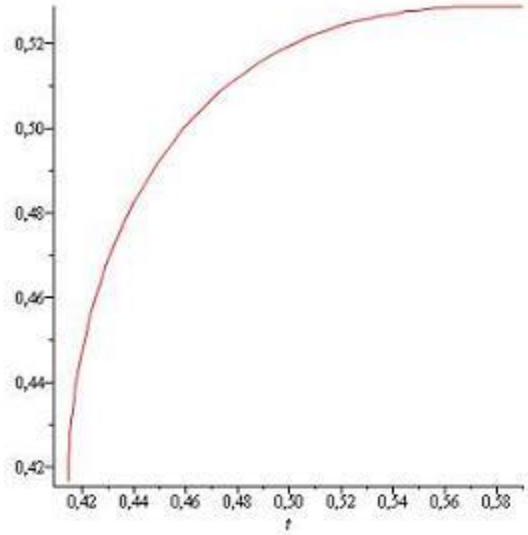
$$=\frac{(1+t^2)}{4(1-t^2)}*a^2*(2u-(1+u^2)t)$$

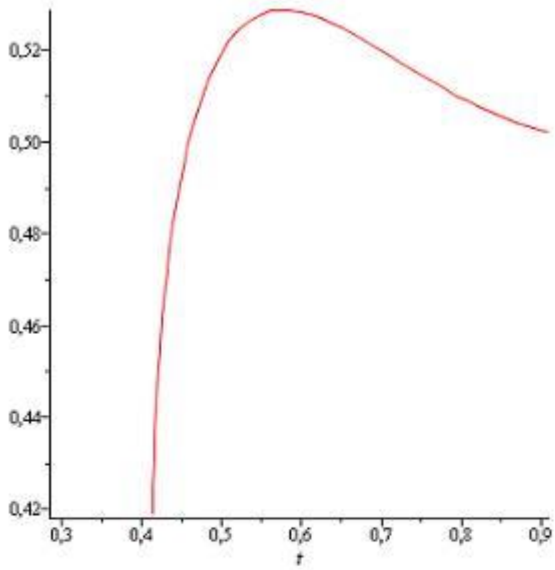
$$u=\frac{1+\sqrt{1-k^2}}{k}=\frac{1+\sqrt{(1-k)(1+k)}}{k}=\frac{1+\sqrt{(1-\frac{1+4t^2-t^4}{4t})(1+\frac{1+4t^2-t^4}{4t})}}{\frac{1+4t^2-t^4}{4t}}=$$

$$=\frac{4t(1+\sqrt{\frac{(t+1)^2(t-1)^2(1+2t-t^2)(t^2+2t-1)}{16t^2}})}{1+4t^2-t^4}=\frac{4t+(t+1)(1-t)\sqrt{6t^2-1-t^4}}{1+4t^2-t^4}$$

$$S=\frac{(1+t^2)}{4(1-t^2)}*a^2*(2\frac{4t+(t+1)(1-t)\sqrt{6t^2-1-t^4}}{1+4t^2-t^4}-(1+(\frac{4t+(t+1)(1-t)\sqrt{6t^2-1-t^4}}{1+4t^2-t^4})^2)t)$$

The graph of S(t) at ($\sqrt{2}-1,\frac{1}{\sqrt{3}}$) and (0.3, 0.9):





There is such a special t0: if t<t0 then S<$\frac{a^2}{2}$, and t>t0, S>$\frac{a^2}{2}$. t0=0.459133..

$u=\frac{4t0+(t0+1)(1-t0)\sqrt{6t0^2-1-t0^4}}{1+4t0^2-t0^4}$=1.226958..

tg β=$\frac{a}{b}=\frac{1}{u}=0.8150237..$ Finally, if $\frac{a}{b}$<0.8150237…, then the optimal figure is the triangle with the area $\frac{a^2}{2}$ and α=π/4

If $\frac{a}{b}$>0.8150237…, the optimal figure is the pentagon with the area S and α=arctg (t1), and the pentagon with the same area and α=π/2-arctg (t1).

The tables for different $\frac{a}{b}$ and the angles α:

$\frac{a}{b}$	1	0.9	0.8150237	0.7	0.6
α	π/8, 3π/8, -π/8, -3π/8	0.40117, π/2-0.40117, -0.40117, -π/2+0.40117	π/4, 0.430423, π/2-0.430423, -π/4, -0.430423, -π/2+0.430423	π/4, -π/4	π/4, -π/4
S	$(\sqrt{2}-1)a^2$	$0.458392\,a^2$	$\frac{a^2}{2}$	$\frac{a^2}{2}$	$\frac{a^2}{2}$

The special case for $\frac{a}{b}$(approx. 0.8150237…) results in the optimal figures are pentagon and triangle with S=$\frac{a^2}{2}$.

SUMMARY

Theorem

Rectangle with sides a, b (a<b) is bent along the line that passes through the center of the rectangle in order to get the minimum area of crossing intersections: a unique rectangle exists for two solutions with equal area but different shapes - triangle and pentagon. The unique ratio of sides a/b=T=0.81502370129163... is derived based on the real root of the quintic. If a/b<T ('long' rectangle) the angle to bent is pi/4. If a/b=1 (square) the angle is Pi/8.

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