To John Riordan

CONTENTS

PREFACE

In spite of the large number of existing mathematical tables, until now there has been no table of sequences of integers. Thus someone coming across the sequence 1, 2, 5, 15, 52, 203, 877, 4140, . . . would have had difficulty in finding out that these are the Bell numbers, and that they have been extensively studied. This handbook remedies this situation. The main table contains a list of some 2300 sequences of integers, collected from all branches of mathematics and science. The sequences are arranged in numerical order, and for each one a brief description and a reference is given.

The first part of the book describes how to use the table, gives methods for analyzing unknown sequences, and contains an illustrated description of the most important sequences.

Who will use this handbook? Anyone who has ever been confronted with a strange sequence, whether in an intelligence test in high school, e.g.,

$$1, 8, 11, 69, 88, 96, 101, 111, 181, 609, . . .$$

(guess![1]), or in solving a mathematical problem, e.g.,

$$1, 2, 5, 14, 42, 132, 429, 1430, . . .$$

(the Catalan numbers), or from a counting problem, e.g.,

$$1, 1, 2, 4, 9, 20, 48, 115, 286, 719, . . .$$

(the number of rooted trees with n points), or in physics, e.g.,

$$1, 0, 3, 22, 192, 2046, 24853, . . .$$

[1]For many more terms and the explanation, see the main table.

(coefficients of the partition function for a cubic lattice), or in chemistry, e.g.,

$$1, 1, 1, 2, 3, 5, 9, 18, 35, 75, 159, \ldots$$

(the number of distinct hydrocarbons of the methane series), or in electrical engineering, e.g.,

$$3, 7, 46, 4336, 134281216, \ldots$$

(the number of Boolean functions of n variables), will find this handbook useful.

Besides identifying sequences, the handbook will serve as an index to the literature for locating references on a particular problem, and for quickly finding numbers like 7^{12}, the number of partitions of 30, the 18th Catalan number, or the expansion of π to 60 decimal places. It might also be useful to have around when the first signals arrive from Betelgeuse (sequence **2311** for example would be a friendly beginning).

ACKNOWLEDGMENTS

ABBREVIATIONS

Abbreviations of the references are listed in the bibliography

$[a]$	the largest integer $\leqslant a$
$a ** b$	a^b
$C(i, j)$	the binomial coefficient $\binom{i}{j}$
$\exp(a)$	e^a
gf	generating function
LCM	least common multiple
REF	reference(s)
seq.	sequence

CHAPTER
1

DESCRIPTION OF THE BOOK

It is the fate of those who toil at the lower employments of life, to be driven rather by the fear of evil, than attracted by the prospect of good; to be exposed to censure, without hope of praise; to be disgraced by miscarriage, or punished for neglect, where success would have been without applause, and diligence without reward.

Among these unhappy mortals is the writer of dictionaries; whom mankind have considered, not as the pupil, but the slave of science, the pionier of literature, doomed only to remove rubbish and clear obstructions from the paths of Learning and Genius, who press forward to conquest and glory, without bestowing a smile on the humble drudge that facilitates their progress. Every other authour may aspire to praise; the lexicographer can only hope to escape reproach, and even this negative recompense has yet been granted to very few.

Samuel Johnson, Preface to the "Dictionary," 1755

1.1 DESCRIPTION OF A TYPICAL ENTRY

The main table is a list of about 2300 sequences of integers. A typical entry is:

256 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040, 1346269
FIBONACCI NUMBERS A(N) = A(N − 1) + A(N − 2). REF HW1 148. REC 11 20 62. HO1.

and consists of the following items:

256	the sequence identification number
1, 2, 3, 5, 8, 13, 21, . . .	the sequence itself
FIBONACCI NUMBERS	a name or descriptive phrase (in this case a recurrence)
A(N) = A(N − 1) + A(N − 2)	for the sequence
REF	references

HW1 148	G. H. Hardy and E. M. Wright, "An Introduction to the Theory of Numbers," Oxford Univ. Press, 3rd ed., page 148, 1954
REC 11 20 62	*Recreational Mathematics Magazine*, Volume 11, page 20, 1962.
HO1	V. E. Hoggatt, Jr., "Fibonacci and Lucas Numbers," Houghton Mifflin, Boston, 1969

1.2   ARRANGEMENT

The entries are arranged in lexicographic order, so that sequences beginning 1, 2, 1 come before those beginning 1, 2, 2, and so on.

1.3   NUMBER OF TERMS GIVEN

Whenever possible enough terms are given to fill two lines. If fewer terms are given, it is because they have never been calculated so far as the author knows. (He would be very pleased to be corrected.) Finding the next term in the following sequences is known to be difficult (others of a similar type can be located via the index):

11, 48, 66–68, 124, 125, 129, 142, 143, 149, 150, 181, 189, 195, 226, 246, 248, 271, 304, 309, 317, 321–325, 329, 330, 358, 373, 380, 393, 435, 450, 465, 477, 516, 559–561, 580, 581, 595, 596, 614, 615, 621, 648, 650, 730, 731, 745, 757, 782, 788, 809, 812, 911, 954, 972, 994, 998, 1052, 1099, 1115, 1133, 1167, 1210, 1244, 1245, 1339, 1340, 1403, 1404, 1467, 1518, 1537, 1803, 2248, 2342.

These sequences all represent unsolved problems.

1.4   REFERENCES

To conserve space, journal references are extremely abbreviated. They usually give the exact page on which the sequence may be found, but neither the author nor the title of the article. To find out more the reader must go to a library; this book is meant to used in conjunction with a library. Quite a small one will do. A considerable fraction of the sequences will be found in the following nine great works:

Indexes	Dickson [D12] Lehmer [LE1] Fletcher, Miller, Rosenhead, and Comrie [FMR]
Tables	Davis [DA2] Abramowitz and Stegun [AS1] David, Kendall, and Barton [DKB]
Combinatorics	Riordan [R1] David and Barton [DB1] Comtet [CO1]

and in four journals:

> *American Mathematical Monthly* [AMM]
> *Fibonacci Quarterly* [FQ]
> *Journal of Combinatorial Theory* [JCT]
> *Mathematics of Computation* [MTAC]

Unusual sequences may send the reader to more exotic sources, but in any case he should first check Chapter III where additional information about some of the commoner sequences is given, and the index to see if other sequences (and hence references) of a similar type are listed.

Journal references usually give volume, page, and year, in that order. (See the example at beginning of this chapter.) Years after 1899 are abbreviated, by dropping the 19. Earlier years are not abbreviated. Sometimes to avoid ambiguity we use the more expanded form of: journal name (series number), volume number (issue number), page number, year.

References to books give volume (if any) and page. (See the example at the beginning of this chapter.)

The references do not attempt to give the discoverer of a sequence, but rather the most extensive table of the sequence that has been published.

1.5 WHAT SEQUENCES ARE INCLUDED?

Rule 1 The sequence must consist of nonnegative integers. (Sequences alternating in sign have been replaced by their absolute values. Interesting sequences of fractions have been entered by numerators and denominators separately. Some sequences of real numbers have been replaced by their integer parts, others by the nearest integers.)

Rule 2 The sequence must be infinite.

A few, like the Mersenne primes, have been given the benefit of the doubt.

Rule 3 The first two terms must be 1, *n*, where *n* is between 2 and 999.

An initial 1 has been silently inserted before the first term if this is greater than 1, and extra 1's and 0's at the beginning have been silently deleted. (See the beginning of Chapter II for examples.)

Rule 3 Enough terms must be known to separate the sequence from its neighbors in the table.

Rule 4 The sequence should have appeared in the scientific literature, and must be well-defined and interesting.

The selection has inevitably been subjective, but the goal has been to

include a broad variety of sequences and as many as possible.

1.6 HOW ARE ARRAYS OF NUMBERS TREATED?

Arrays of numbers (binomial coefficients, Stirling numbers of the first kind, etc.) have been entered by rows, columns, or diagonals, whichever seemed appropriate.

1.7 SUPPLEMENTS

It is planned to issue supplements to the Handbook from time to time, containing new sequences and corrections and extensions to the original sequences. Readers wishing to receive these supplements should notify the author.

HOW TO HANDLE A STRANGE SEQUENCE

2.1 HOW TO SEE IF A SEQUENCE IS IN THE TABLE

Obtain as many terms of the sequence as possible. The initial terms are handled as follows: Recall that the sequence must begin 1, n, where n is between 2 and 999. Find the first term in the sequence that is greater than 1, and replace all the terms that come before it by a single 1. Then look it up in the table. The initial 1 is just a marker, and need not be in the original sequence. For example, if the sequence begins

1, 2, 3, 5, 8, 13, . . .	see under	1, 2, 3, 5, 8, 13, . . .
2, 3, 5, 8, 13, . . .	see under	1, 2, 3, 5, 8, 13, . . .
−1, 1, 0, 1, 1, 2, 3, 5, 8, . . .	see under	1, 2, 3, 5, 8, . . .
1, 0, 0, 2, 24, 552, 21280, . . .	see under	1, 2, 24, 522, 21280, . . .

2.2 IF THE SEQUENCE IS NOT IN THE TABLE

(i) Try changing or redefining the sequence. Some typical changes are inserting or deleting an initial term (e.g., seq. *46* occurs as both 1, 2, 1, 2, 3, 6, 9, 18, . . . and 1, 2, 3, 6, 9, 18, . . .); adding or subtracting 1 or 2 from all the terms (e.g., seq. *309* occurs as both 1, 2, 3, 6, 20, 168, . . . and 1, 4, 18, 166, . . .); and multiplying all the terms by 2 or dividing by any common factor.

(ii) If all these methods fail, and it seems certain that the sequence is not in this handbook, please send the sequence and anything that is known about it, including appropriate references, to the author for possible inclusion in later editions.[1]

[1]Address: Mathematics Research Center, Bell Telephone Laboratories, Inc., Murray Hill, New Jersey   07974.

2.3 FINDING THE NEXT TERM

Suppose the beginning of a sequence is given as

$$
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7
\end{array}
$$

and a rule or explanation for it is desired. If nothing is known about the history of the sequence or if it is an arbitrary sequence, nothing can be said and any continuation is possible. (Any $n + 1$ points can be fitted by an nth degree polynomial.)

But the sequences normally encountered, and those in this handbook, are distinguished in that they have been produced in some intelligent and systematic way. Occasionally such sequences have a simple explanation, and if so, the methods given below may help to find it. These methods can be divided roughly into two classes: those which look for a systematic way of generating the nth term a_n from the terms $a_0, \ldots, a_{n-1}$ before it, e.g., $a_n = a_{n-1} + a_{n-2}$, i.e., methods which seek an internal explanation; and those which look for a systematic way of going from n to a_n, e.g., a_n is the number of divisors of n, or the number of trees with n nodes, or the nth prime number, i.e., methods which seek an external explanation. The former methods are described in the rest of this chapter, the latter in Chapter III.

In practice it is usually clear for one reason or another when a correct explanation for a sequence has been found.

(For the related problems of defining the complexity of a sequence, and extrapolating a sequence of real numbers, see the interesting work of Martin-Lof [IC 9 602 66] and Fine [IC 16 331 70 and F11].)

2.4 LOOK FOR A RECURRENCE

Let the sequence be $a_0, a_1, a_2, a_3, \ldots$. Is there a systematic way of getting the nth term a_n from the preceding terms $a_{n-1}, a_{n-2}, \ldots$? A rule for doing this, such as $a_n = a_{n-1}^2 - a_{n-2}$, is called a *recurrence*, and of course provides a method for getting as many terms of the sequence as desired.

In studying sequences and recurrences it is useful to define a *generating function* (gf) associated with the sequence, usually an ordinary gf:

$$
A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots,
$$

but sometimes an exponential gf:

$$
E(x) = a_0 + a_1 \frac{x}{1!} + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \cdots.
$$

(These are formal power series having the sequence as coefficients; questions of convergence do not arise.)

Once a recurrence has been found for the sequence, techniques for solving it will be found in the works by Riordan [R1 19], Batchelder [BAT], and Levy and Lessman [LE2].

For example, consider seq. **256,** the Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, 34, ... These are generated by the recurrence $a_n = a_{n-1} + a_{n-2}$, and from this it is not difficult to obtain the generating function

$$1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots = \frac{1}{1 - x - x^2},$$

and the explicit formula for the nth term

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right].$$

2.4.1 METHOD OF DIFFERENCES

This is the standard method for finding recurrences. In simple cases, it will even find an explicit formula for the nth term of a sequence, e.g., if this is a polynomial (such as $a_n = n^2 + 1$) or a simple exponential (such as $a_n = 2^n + n + 1$).

If the sequence is

$$a_0, a_1, a_2, a_3, a_4, \ldots,$$

its first differences are the numbers

$$\Delta a_0 = a_1 - a_0, \qquad \Delta a_1 = a_2 - a_1, \qquad \Delta a_2 = a_3 - a_2, \qquad \ldots,$$

its second differences are

$$\Delta^2 a_0 = \Delta a_1 - \Delta a_0, \qquad \Delta^2 a_1 = \Delta a_2 - \Delta a_1, \qquad \Delta^2 a_2 = \Delta a_3 - \Delta a_2, \qquad \ldots,$$

and so on. The 0th differences are the original sequence: $\Delta^0 a_0 = a_0$, $\Delta^0 a_1 = a_1$, $\Delta^0 a_2 = a_2, \ldots$; and the mth differences are

$$\Delta^m a_n = \Delta^{m-1} a_{n+1} - \Delta^{m-1} a_n$$

or, in terms of the original sequence,

$$\Delta^m a_n = \sum_{i=0}^{m} (-1)^i \binom{m}{i} a_{m+n-i}. \tag{1}$$

Therefore if the differences of some order can be identified, Eq. (1) gives a recurrence for the sequence.

Furthermore, if the differences a_k, Δa_k, $\Delta^2 a_k$, $\Delta^3 a_k$, . . . are known for some fixed value of k, then a formula for the nth term is given by

$$a_{n+k} = \sum_{m=0}^{n} \binom{n}{m} \Delta^m a_k. \qquad (2)$$

Example (i) Seq. *1562*

n	1	2	3	4	5	6	7	8
a_n	1	5	12	22	35	51	70	92
Δa_n		4	7	10	13	16	19	22
$\Delta^2 a_n$			3	3	3	3	3	3
$\Delta^3 a_n$				0	0	0	0	0

Since $\Delta^2 a_n = 3$, $\Delta a_{n+1} - \Delta a_n = 3$, or $a_{n+2} - 2a_{n+1} + a_n = 3$, a recurrence for the sequence. An explicit formula is obtained from Eq. (2) with $k = 1$:

$$a_{n+1} = 1 + 4\binom{n}{1} + 3\binom{n}{2} = 1 + 4n + \frac{3}{2}n(n-1) = \frac{1}{2}(n+1)(3n+2).$$

In general, if the mth differences are zero, a_n is a polynomial in n of degree $m - 1$.

Example (ii) Seq. *1382*

n	1	2	3	4	5	6	7
a_n	1	4	11	26	57	120	247
Δa_n		3	7	15	31	63	127
$\Delta^2 a_n$			4	8	16	32	64

Here $\Delta^2 a_n = 2^{n+1}$, $\Delta a_n = 2^{n+1} - 1$, and $a_n = 2^{n+1} - n - 2$. Equation (2) gives the same answer.

Example (iii) Seq. *552* (the Pell numbers)

n	1	2	3	4	5	6	7
a_n	1	2	5	12	29	70	169
Δa_n		1	3	7	17	41	99
$\Delta^2 a_n$			2	4	10	24	58
$\frac{1}{2}\Delta^2 a_n$			1	2	5	12	29

Since $\frac{1}{2}\Delta^2 a_n = a_n$, Eq. (1) gives the recurrence $a_{n+2} - 2a_{n+1} - a_n = 0$. Calculating further differences shows that $\Delta^m a_1 = 2^{[m/2]}$ and so Eq. (2) gives the formula

$$a_{n+1} = \sum_{m=0}^{n} \binom{n}{m} 2^{[m/2]}.$$

Example (iv) Seq. *469*

n	1	2	3	4	5	6	7	8
a_n	1	2	4	10	26	76	232	764
Δa_n		1	2	6	16	50	156	532
$n^{-1} \Delta a_n$		1	1	2	4	10	26	76

Notice that Δa_n is divisible by n, and in fact $n^{-1} \Delta a_n = a_{n-1}$, so that $a_{n+1} = a_n + na_{n-1}$. Again Eq. (2) gives a formula for a_n.

2.4.2 OTHER METHODS OF ATTACK

Is the sequence close to a known sequence, such as the powers of 2? If so, try subtracting off the known sequence. For example, seq. *1382* (again): 1, 4, 11, 26, 57, 120, 247, 502, 1013, 2036, 4083, ... The last four numbers are close to powers of 2: 512, 1024, 2048, 4096; and then it is easy to find $a_n = 2^n - n - 1$.

Is a simple recurrence such as $a_n = \alpha a_{n-1} + \beta a_{n-2}$ likely? For this to happen, the ratio $\rho_n = a_{n+1}/a_n$ of successive terms must approach a constant as n increases. Use the values a_2 to a_5 to determine α and β and then see if a_6, a_7, . . . are generated correctly.

If the ratio ρ_n has first differences which are approximately constant, this suggests a recurrence of the type $a_n = \alpha n a_{n-1} + \cdots$. For example, seq. *704*: 1, 2, 7, 30, 157, 972, 6961, 56660, 516901, . . . has successive ratios 2, 3.5, 4.29, 5.23, 6.19, 7.16, 8.14, 9.12, . . . with differences approaching 1, suggesting $a_n = na_{n-1} + ?$. Subtracting na_{n-1} from a_n, we obtain the original sequence 0, 1, 2, 7, 30, 157, 972, . . . again, so $a_n = na_{n-1} + a_{n-2}$.

This example illustrates the principle that whenever $\rho_n = a_{n+1}/a_n$ seems to be close to a recognizable sequence r_n, one should try to analyze the sequence $b_n = a_{n+1} - r_n a_n$.

A recurrence of the form $a_n = na_{n-1} + $ (small term) can be identified by the fact that the 10th term is approximately 10 times the 9th. For example, seq. *766*: 0, 1, 2, 9, 44, 265, 1854, 14833, 133496, 1334961, . . . , $a_n = na_{n-1} + (-1)^n$.

The recurrence $a_n = a_{n-1}^2 + \cdots$ is characterized by the fact that each term is about twice as long as the one before. For example, seq. *331*: 1, 2, 3, 7, 43, 1807, 3263443, 10650056950807, . . . , and $a_n = a_{n-1}^2 - a_{n-1} + 1$.

2.4.3 FACTORIZING

Does the sequence, or one obtained from it by some simple operation, have many factors?

Example (i) Seq. *1614*: 1, 5, 23, 119, 719, 5039, 40319, ... As it stands, the sequence cannot be factored, since 719 is prime, but the addi-

tion of 1 to all the terms gives the highly composite sequence $2, 6 = 2 \cdot 3$, $24 = 2 \cdot 3 \cdot 4$, $120 = 2 \cdot 3 \cdot 4 \cdot 5, \ldots$, which are the factorial numbers (see Section 3.13).

The presence of only small primes may also suggest binomial coefficients:

Example (ii)   Seq. *577* (the Catalan numbers): 1, 2, 5, $14 = 2 \cdot 7$, $42 = 2 \cdot 3 \cdot 7$, $132 = 4 \cdot 3 \cdot 11$, $429 = 3 \cdot 11 \cdot 13$, $1430 = 2 \cdot 5 \cdot 11 \cdot 13$, $4862 = 2 \cdot 11 \cdot 13 \cdot 17, \ldots$ and

$$a_n = \frac{1}{n+1} \binom{2n}{n}$$

(see Section 3.5).

Sequences arising in number theory are sometimes *multiplicative*, i.e., have the property that $a_{mn} = a_m a_n$ whenever m and n have no common factor. For example, seq. *86*:   1, 2, 2, 3, 2, 4, 2, 4. . . . , the number of divisors of n.

2.4.4   SELF-GENERATING SEQUENCES

This section describes some recurrences of a simple yet unusual type. They have been called (rather arbitrarily) *self-generating*.

In the first two examples let $A = \{a_0 = 1, a_1, a_2, \ldots\}$ be a sequence of 1's and 2's.

(i)   If every 1 in A is replaced by 1, 2 and every 2 by 2, 1 a new sequence A' is obtained. Imposing the condition that $A = A'$ forces A to be seq. *71*: 1, 2, 2, 1, 2, 1, 1, ... Sequences *21* and *36* are of the same type.

(ii)   Let $A'' = \{b_0, b_1, b_2, \ldots\}$, where b_n is the length of the nth run in A. (A *run* is a maximal string of identical symbols.) The condition $A = A''$ forces A to be seq. *70*: 1, 2, 2, 1, 1, 2, 1, 2, 2, 1, ...

In the remaining examples, $A = \{a_0 = 1, a_1, a_2, \ldots\}$ is a nondecreasing sequence of integers.

(iii)   Let c_n be the number of times n occurs in A, for $n = 1, 2, \ldots$. If $c_n = n$, A is seq. *89*: 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, ... If $c_n = a_{n-1}$, A is seq. *91*: 1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 6, ... (Seq. *965* is related to the latter sequence.)

(iv)   The condition that $a_{n+1} - a_n$ be the smallest positive integer not equal to $a_i - a_j$ for any $i, j \leq n$ forces a to be seq. *416*: 1, 2, 4, 8, 13, 21, ... The conditions $a_0 = 1$, $a_2 = 2$, and that a_n be the smallest integer which can be written uniquely as the sum of two distinct preceding terms force A to be seq. *201*: 1, 2, 3, 4, 6, 8, 11, 13, ... Sequences *231, 254, 425*, and *909* have similar explanations.

CHAPTER
III
___

ILLUSTRATED DESCRIPTION OF SOME
IMPORTANT SEQUENCES

While Chapter II studied ways of getting the nth term of a sequence from the preceding terms, this chapter considers externally generated sequences, such as the sequences in which the nth term is the number of graphs with n nodes or the nth triangular number. An informal and illustrated description is given of some of the most important such sequences.

3.1 GRAPHS AND TREES

Stated informally, a *graph* consists of a finite set of points (or nodes) some of which are joined by lines (or edges). Figure 1 illustrates seq. *479*, the number of graphs with n nodes.
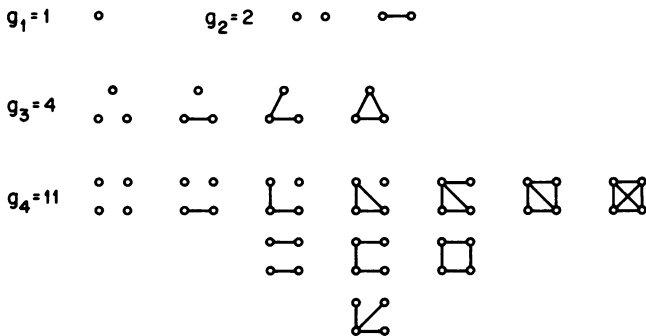

Fig. 1. Seq. *479*, graphs or reflexive symmetric relations.

A *digraph*, or directed graph, is a graph with arrows on the edges (Fig. 2, seq. *1229*). Figure 3 shows seq. *1069*, digraphs of functions, i.e., digraphs with exactly one arrow directed out of each node.
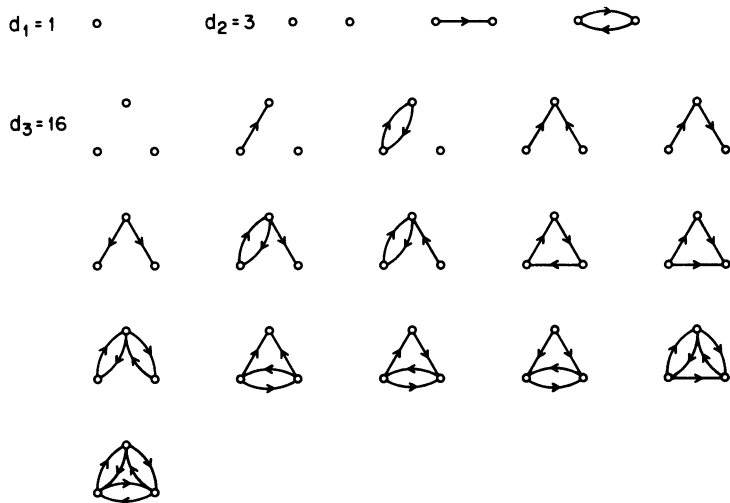
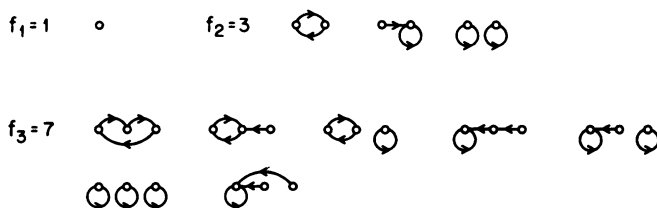Fig. 2.   Seq. *1229*, digraphs or reflexive relations.

Fig. 3.   Seq. *1069*, functional digraphs.

A *tree* is a connected graph containing not closed paths (Fig. 4, seq. *299*). A *rooted* tree is a tree with a distinguished node called Eve, or the root. Figure 5 illustrates seq. *454*, the number of rooted trees with n nodes. The generating function (gf) of this sequence is

$$r(x) = x + x^2 + 2x^3 + 4x^4 + 9x^5 + \cdots$$

and satisfies

$$r(x) = x \exp[r(x) + \tfrac{1}{2}r(x^2) + \tfrac{1}{3}r(x^3) + \cdots].$$

The generating function for trees,

$$t(x) = x + x^2 + x^3 + 2x^4 + 3x^5 + 6x^6 + \cdots.$$

is then given by

$$t(x) = r(x) - \tfrac{1}{2}r^2(x) + \tfrac{1}{2}r(x^2).$$
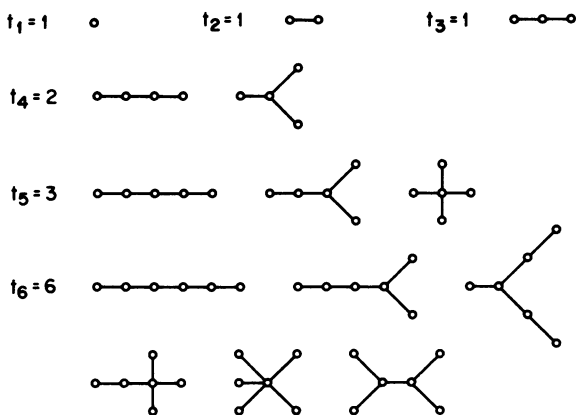
Fig. 4. Seq. *299*, trees.
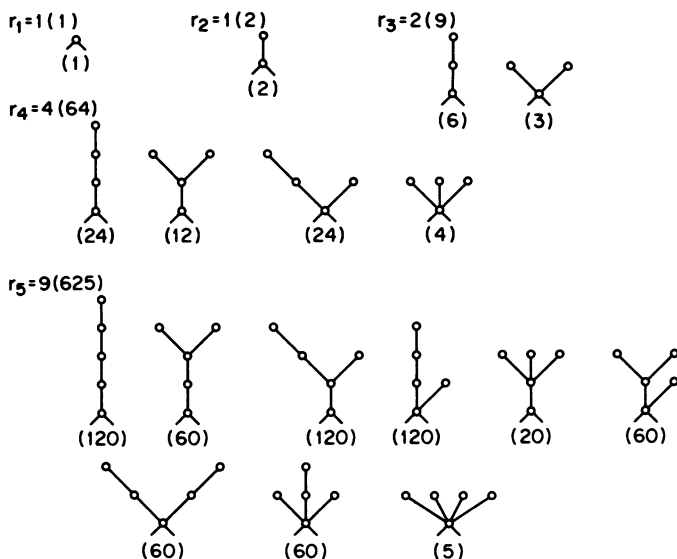
Fig. 5. Seq. *454*, rooted trees. (The numbers in parentheses give seq. *771*, labeled rooted trees.)

Any of these graphs may be *labeled* by (if there are n nodes) attaching the numbers from 1 to n to the nodes. For example in Fig. 5, the numbers in parentheses give the number of ways of labeling each tree, and then the total number of labeled rooted trees with n nodes is n^{n-1}, seq. *771*. Usually when graphs are mentioned in the main table they are unlabeled unless stated otherwise.

The degree of a node is the number of edges meeting it. Figure 6 shows seq. *118*, series-reduced trees, or trees without nodes of degree 2.

For further information about the preceding sequences and for the enumeration of other kinds of graphs, see Riordan [R1] and Harary [HA5].
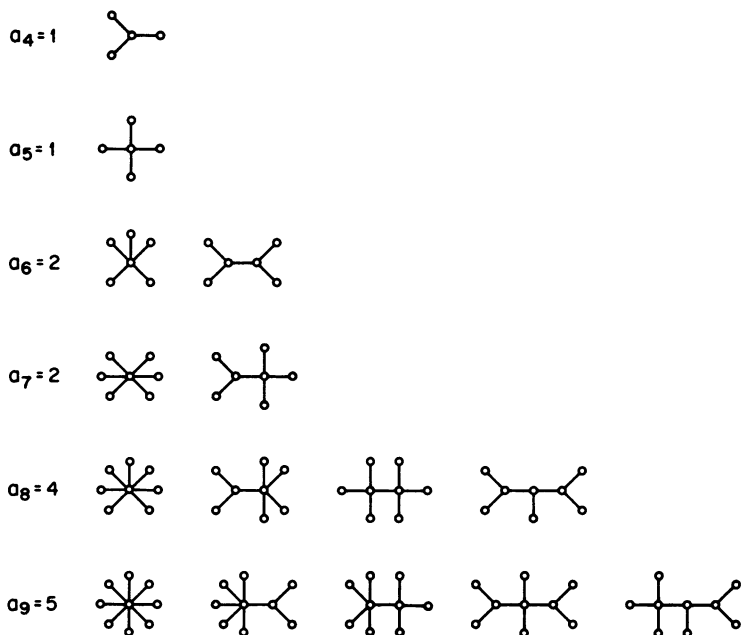
Fig. 6. Seq. *118*, series-reduced trees.

3.2 RELATIONS

A *relation R* on a set *S* is any subset of $S \times S$, and xRy means (x, y) $\in R$ or "*x* is related to *y*." A relation is *reflexive* if xRx for all *x* in *S, symmetric* if $xRy \Rightarrow yRx$, *antisymmetric* if xRy and $yRx \Rightarrow x = y$, and *transitive* if xRy and $yRz \Rightarrow xRz$.

The most important types of relations are:

(1) unrestricted, or digraphs with loops of length 1 allowed (seq. *784*: 2, 10, 104, 3044, 291968, . . .);

(2) symmetric, or graphs with loops of length 1 allowed (seq. *646*: 2, 6, 20, 90, 544, 5096, 79264, . . .);

(3) reflexive, or digraphs (Fig. 2, seq. *1229* again);

(4) reflexive symmetric, or graphs (Fig. 1, seq. *479* again);

(5) reflexive transitive, or topologies (Fig. 7, seq. *1133*: 1, 3, 9, 33, 139, 718, 4535, ?. For the connection between digraphs and topologies, see Birkhoff [B11 117]);

(6) reflexive symmetric transitive, or partitions (Fig. 20, p. 24, seq. *244*);

(7) reflexive antisymmetric transitive, or partially ordered sets (Fig. 8, seq. *588*: 1, 2, 5, 16, 63, 318, 2045, ?).
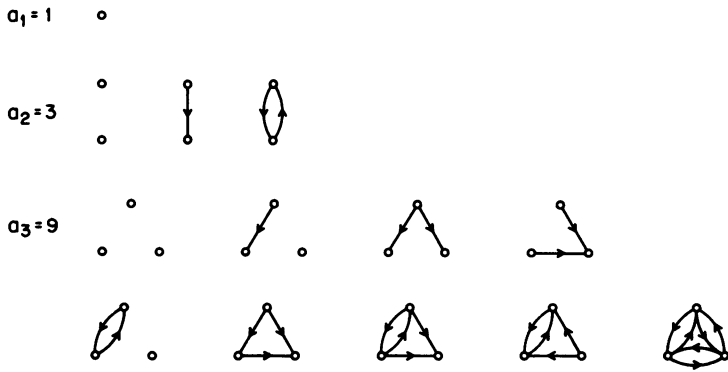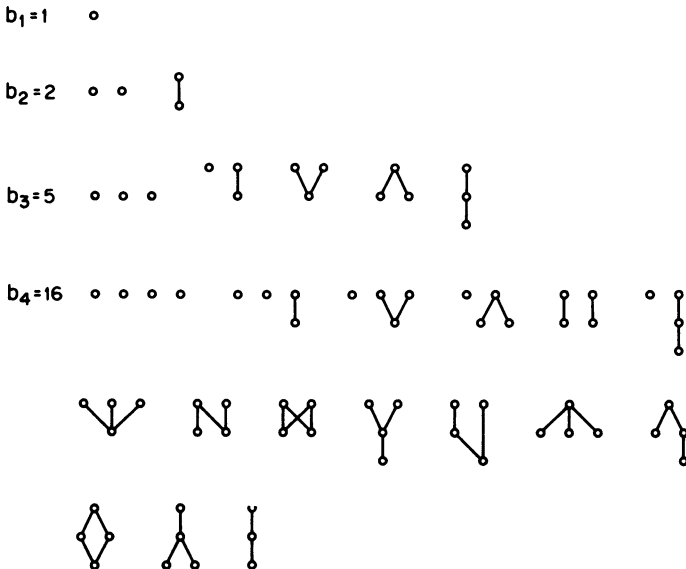

Fig. 7. Seq. *1133*, topologies.

Fig. 8. Seq. *588*, partially ordered sets.

This assumes that the graphs are unlabeled, i.e., that the elements of the set S are indistinguishable. If the elements of S are labeled 1 through n, the corresponding numbers are:

(1) 2^{n^2};
(2) $2^{n(n+1)/2}$;
(3) $2^{n(n-1)}$;
(4) $2^{\binom{n}{2}}$

(these four [(1)–(4)] are not in the table, but the sequences of their exponents are);

(5) seq. *1476*: 1, 4, 29, 355, 6942, 209527, 9535241, ?;
(6) seq. *585*: the Bell numbers or the number of equivalence relations on a set of n objects (see Fig. 22, p. 25);
(7) seq. *1244*: 1, 3, 19, 219, 4231, 130023, 6129859, ?.

3.3 GEOMETRIES

The numbers of topologies were shown in Fig. 7; the following are also basic geometrical sequences:

A *linear space* is a system of (abstract) points and lines such that every two points lie on a unique line, and every line contains at least two points. A *geometry* is a system of points, lines, planes, . . . with an analogous definition. Figure 9 shows seq. *462*: 1, 1, 2, 4, 9, 26, 101, 950, ?, the number of geometries with n points. (See Crapo and Rota [JM2 49 127 70]. The * denotes 5 points in general position in 4-dimensional space.) The planar figures in Fig. 9 form seq. *271*: 1, 1, 2, 3, 5, 10, 24, 69, 384, ?, the number of linear spaces (Doyen [BSM 19 424 67]).
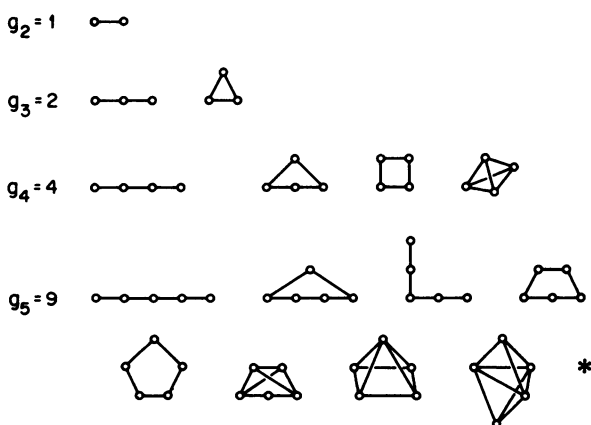
Fig. 9. Seq. *462*, geometries (for * see text).

3.4 COMBINATIONS AND FIGURATE NUMBERS

The most basic combinatorial number is the *binomial coefficient*

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots r} = \frac{n!}{r!(n-r)!},$$

which is the number of selections, or combinations, of *n* unlike things taken *r* at a time, has gf

$$(1+x)^n = \sum_{r=0}^{n}\binom{n}{r}x^r,$$

and is the $(r+1)$th term in the $(n+1)$th row of Pascal's triangle

```
                    1
                1       1
            1       2       1
        1       3       3       1
    1       4       6       4       1
  1     5      10      10       5      1
1     6      15      20      15       6      1
                    .
                    .
                    .
```
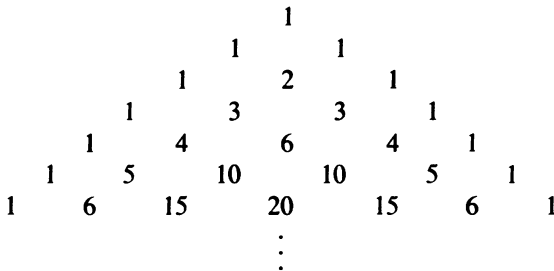
These are also called *figurate* numbers since they are the numbers of points in certain figures. For example, $\binom{n}{2}$ and $\binom{n}{3}$ are the *triangular* and *tetrahedral* numbers (Fig. 10, seqs. *1002*, *1363*).

Fig. 10. Seqs. *1002* and *1363*, the triangular and tetrahedral numbers.

Other examples of figurate numbers are the *polygonal* numbers $P(r, s) = \frac{1}{2}r(rs - s + 2)$. Figure 11 shows seq. *1350*, the *square* numbers $P(r, 2) = r^2$; and seq. *1562*, the *pentagonal* numbers $P(r, 3) = \frac{1}{2}r(3r - 1)$.

Many other figurate numbers, including cubes, fourth powers, etc., will be found in the table. For further pictures see Hogben [HO3].
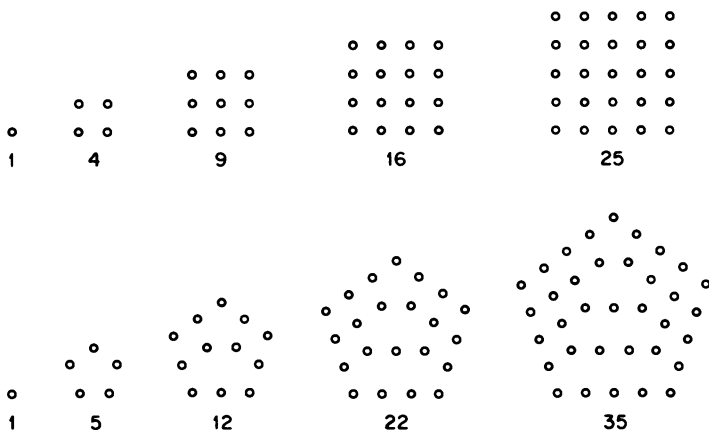
Fig. 11.   Seqs. *1350* and *1562*, the square and pentagonal numbers.

3.5   CATALAN NUMBERS AND DISSECTIONS

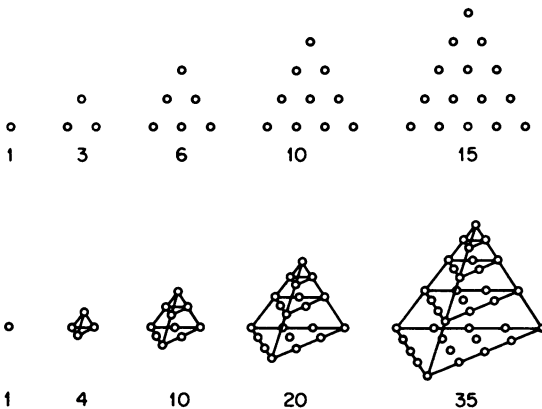Next to the figurate numbers, the Catalan numbers are the most frequently occurring combinatorial numbers. (Gould [GO4] lists over 240 references.) They are defined by

$$c_n = \frac{1}{n + 1} \binom{2n}{n},$$

and form seq. *557*: 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, ... A gf is

$$1 + x + 2x^2 + 5x^3 + \cdots = (2x)^{-1}[1 - (1 - 4x)^{1/2}].$$

Some of the interpretations of c_n are:

(1)   The number of ways of dissecting a convex polygon of $n + 2$ sides into n triangles by drawing nonintersecting diagonals (Fig. 12a).

(2)   The number of ways of completely parenthesizing a product of $n + 1$ letters (so that there are two factors inside each set of parentheses). The examples for $n = 1, 2, 3$ (arranged to show the correspondence with the dissections of Fig. 12a) are:

$n = 1$ $(ab);$ $n = 2$ $a(bc),$ $(ab)c;$
$n = 3$ $(ab)(cd),$ $a((bc)d),$ $((ab)c)d,$ $a(b(cd)),$ $(a(bc))d.$

(3) The number of bifurcated rooted planar trees with $n + 1$ endpoints. (A planar tree is one which has been drawn on a plane, and bifurcated means that each edge splits in two at each node. See Fig. 12b. The trees are drawn to show the correspondence with the dissections and the parentheses.)

(4) In an election with two candidates A and B, each receiving n votes, c_n is the number of ways the votes can come in so that A is never behind B (Feller [FE1 1 71] and Comtet [CO1 1 94]).
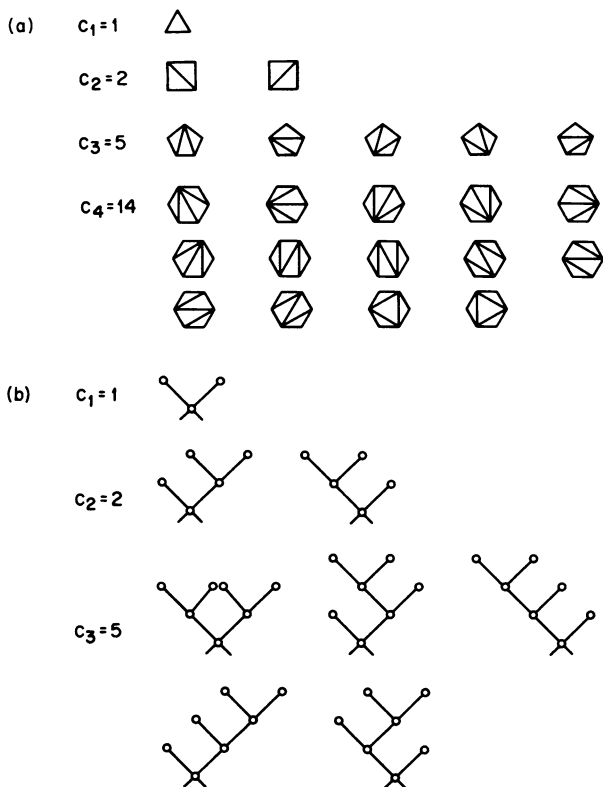
Fig. 12. Seq. *577*, the Catalan numbers.

Figure 13 illustrates seq. *942*, the number of different dissections of a polygon when two dissections are considered to be the same if a rotation or reflection sends one into the other.

Figure 14 illustrates seq. *391*, giving $\frac{1}{2}n(n + 1) + 1$, the maximum number of pieces obtained by slicing a pancake with n slices. The numbers

of n-sided polygons in the nth diagram of Fig. 14 form seq. *1181*: 0, 0, 1, 3, 12, 70, 465, 3507, 30016, . . . (Robinson [AMM 58 462 51]). Seq. *491*: 2, 4, 8, 15, 26, 42, 64, 93, 130, 176, . . . gives $(n + 2)(n + 3)/6$, the maximum number of pieces obtained with n slices of a cake.
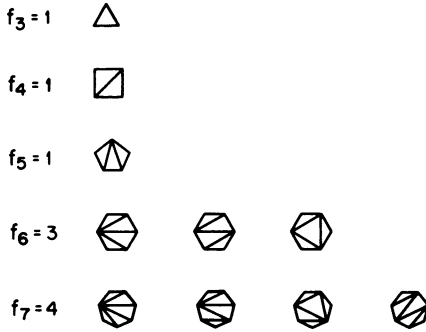
Fig. 13. Seq. *942*, dissections of a polygon.
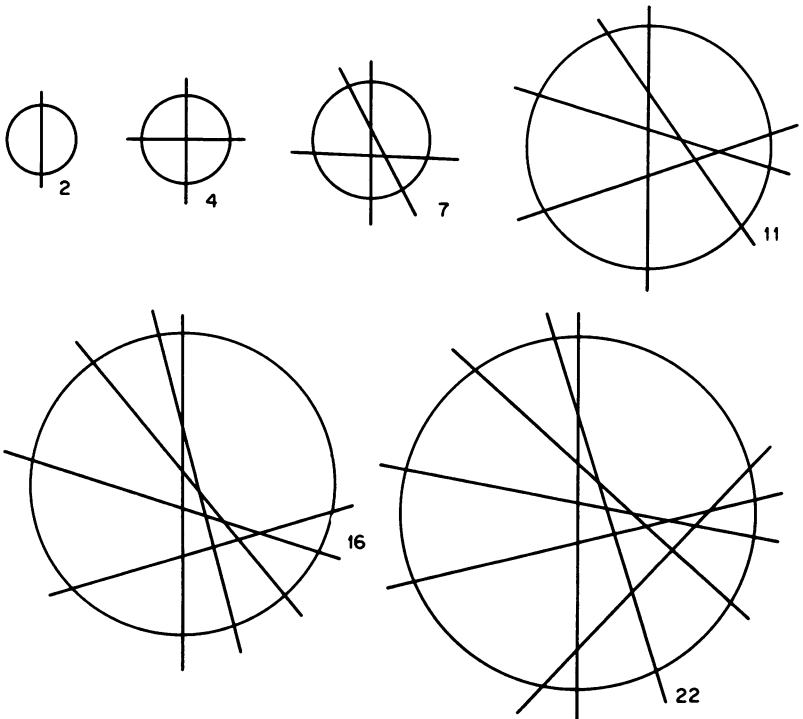
Fig. 14. Seq. *391*, slicing a pancake.

3.6 NECKLACES AND IRREDUCIBLE POLYNOMIALS

Figure 15 illustrates seq. *203*, T_n, the number of different necklaces that can be made from beads of two colors, when the necklaces can be rotated but not turned over. This is also the number of irreducible binary polynomials whose degree divides n, an important sequence in digital circuitry; and has the formula $T_n = \Sigma\ \phi(d)2^{n/d}$, where $\phi(d)$ is the Euler totient function (seq. *111*, Section 3.14) and the sum is over all divisors d of n. (See Berlekamp [BE2 70] and Golomb [CMA 1 358 69].) If turning over is allowed, the number of different necklaces is given by seq. *202*: 2, 3, 4, 6, 8, 13, 18, 30, 46, 78, ... (See Gilbert and Riordan [IJM 5 657 61].)
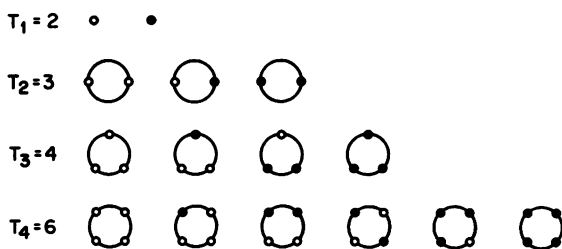
Fig. 15.   Seq. *203*, necklaces.

3.7 KNOTS

Figure 16 shows seq. *322*: 0, 0, 1, 1, 2, 3, 7, 18, 41, 123, 367, ?, the number of knots with n crossings, in which the crossings alternate. (See Tait [TA1 1 334] and Conway [JL2 343].)
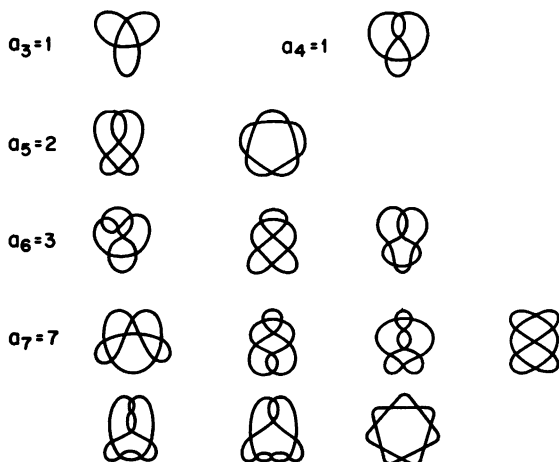
Fig. 16.   Seq. *322*, knots.

3.8   STAMPS

Figure 17 shows seq. *576*: 1, 1, 2, 5, 14, 39, 120, 358, 1176, 3527, . . .
(six more terms are known), the number of ways of folding a strip of
stamps.

$a_1 = 1$ $a_2 = 1$ $a_3 = 2$

$a_4 = 5$

$a_5 = 14$
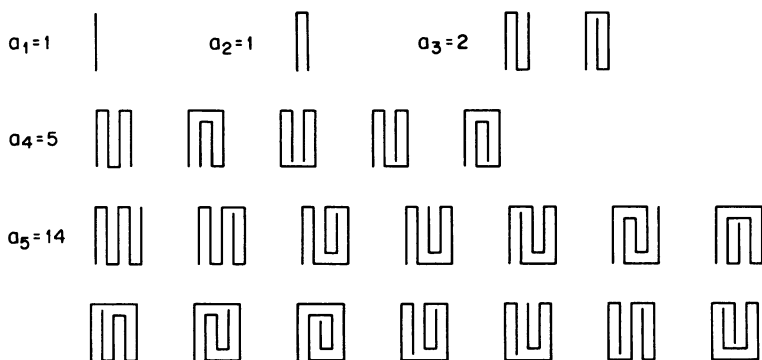
Fig. 17.   Seq. *576*, folding a strip of stamps.

3.9   POLYOMINOES

A *polyomino* with p squares is a connected set of p squares from a
chessboard pattern. Polyominoes are *free* if they can be rotated and turned
over (Fig. 18), and *fixed* otherwise. Unless otherwise stated, all polyomi-
noes are free. Polyominoes may also be formed from triangles, rectangles,
cubes (Fig. 19), etc. In no case is a formula known for the general term.
(See Golomb [GO2].)

$a_1 = 1$ $a_2 = 1$

$a_3 = 2$

$a_4 = 5$

$a_5 = 12$
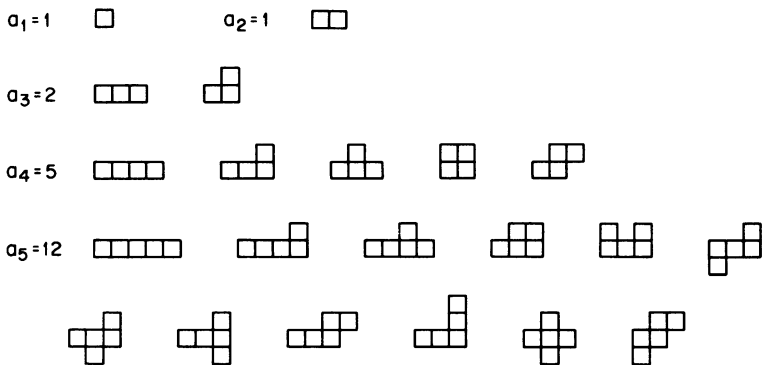
Fig. 18.   Seq. *561*, square polyominoes.

$d_1 = 1$

$d_2 = 1$

$d_3 = 2$

$d_4 = 8$
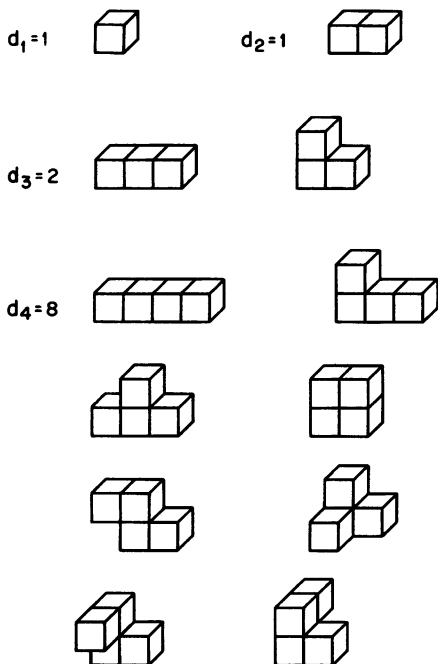
Fig. 19. Seq. *731*, polyominoes made from cubes.

3.10 BOOLEAN FUNCTIONS

A Boolean (or switching) function is a function $f(x_1, \ldots, x_n)$, where each variable x_i is 0 or 1, and f takes on the values 0 or 1.

These arise in the design of logical circuits, when the names of the variables do not matter. So it is natural to say that two such functions are equivalent if they differ only in the names of the variables (so that $x_1 + x_2 x_3$ is equivalent to $x_2 + x_1 x_3$), and to ask for the number of inequivalent functions. The answers to this (which is seq. *1405*: 4, 12, 80, 3984, . . .) and to many similar questions (allowing complementation of the variables, etc.) are given by the Pólya counting theory (Section 3.11).

Two generalizations that will be found in the table are (i) Post functions, which are functions $f(x_1, \ldots, x_n)$, where each x_i and f can take any value from 0 to $m - 1$; and (ii) switching networks, which are n-input, k-output networks such that each of the outputs is a Boolean function of the n inputs. For details see Harrison [HA2, MU3 85].

3.11 PÓLYA COUNTING THEORY

A large number of counting problems involving graphs, necklaces, Boolean functions, and patterns of various kinds have been solved by the

theorems of Redfield, Pólya, and De Bruijn. (See Riordan [R1 131], De Bruijn [BE6 144], Harrison [HA2 127, MU3 85], and Harary [HA5 178].)

3.12 PARTITIONS

The following are the most important sequences of partitions.

The main such sequence is number *244*: 1, 2, 3, 5, 7, 11, . . . , giving the number of partitions of n into integer parts (Fig. 20). A gf is

$$1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots = \prod_{i=1}^{\infty} (1 - x^i)^{-1}.$$

(See Gupta [RS2] and David *et al.* [DKB 273].)

Those partitions of n in which all parts are distinct form seq. *100*: 1, 1, 2, 2, 3, 4, 5, . . . with gf

$$1 + x + x^2 + 2x^3 + 2x^4 + \cdots = \prod_{i=1}^{\infty} (1 + x^i).$$

The partitions of the even numbers into parts which are powers of two form the binary partition function $b(n)$, seq. *378*: 1, 2, 4, 6, 10, 14, 20, 26, 36, 46, . . . , with recurrence $b(n) = b(n-1) + b([\frac{1}{2}n])$.

$p(1) = 1$ 1

$p(2) = 2$ 2, 1^2

$p(3) = 3$ 3, 21, 1^3

$p(4) = 5$ 4, 31, 2^2, 21^2, 1^4

$p(5) = 7$ 5, 41, 32, 31^2, $2^2 1$, 21^3, 1^5

$p(6) = 11$ 6, 51, 42, 41^2, 3^2, 321, 31^3, 2^3, $2^2 1^2$, 21^4, 1^6

$p(7) = 15$ 7, 61, 52, 51^2, 43, 421, 41^3, $3^2 1$, 32^2, 321^2, 31^4, $2^3 1$, $2^2 1^3$, 21^5, 1^7

Fig. 20. Seq. *244*, the number of partitions of n.

Figure 21 illustrates the number of planar partitions of n, seq. *1016*, with gf

$$1 + x + 3x^2 + 6x^3 + \cdots = \prod_{i=1}^{\infty} (1 - x^i)^{-i}.$$

Figure 22 shows $S(n, k)$, the *Stirling numbers of the second kind,* or the number of partitions of a set of n labeled objects into k parts.

$r(1) = 1$ 1

$r(2) = 3$ 2   1 1   1
                                       1

$r(3) = 6$ 3   2 1   2   1 1 1   1 1   1
                             1   1   1
                                                         1

$r(4) = 13$ 4   3 1   3   2   2 1 1   2 1   2
                              1   2   1   1
                                                                                              1

1 1 1 1   1 1 1   1 1   1 1   1
                                1   1 1   1   1
                                                    1   1
                                                                 1

Fig. 21.   Seq. *1016*, planar partitions.

n \\ k	1	2	3	4	Total
1	1				1
2	12	1, 2			2
3	123	1, 23 2, 13 3, 12	1, 2, 3		5
4	1234	1, 234 2, 134 3, 124 4, 123 12, 34 13, 24 14, 23	1, 2, 34 1, 3, 24 1, 4, 23 2, 3, 14 2, 4, 13 3, 4, 12	1, 2, 3, 4	15

Fig. 22. $S(n, k)$, the Stirling numbers of the second kind, and seq. *585*, the Bell numbers.

The numbers continue:

                                     row sums
                                     $B(n)$

1							1
1	1						2
1	3	1					5
1	7	6	1				15
1	15	25	10	1			52
1	31	90	65	15	1		203
1	63	301	350	140	21	1	877

⋮   ⋮

A gf for $S(n, k)$ is

$$x^n = \sum_{k=0}^{n} S(n, k)\, x(x - 1) \cdots (x - k + 1).$$

Both the columns and diagonals of this array will be found in the main table.

The row sums are the *Bell numbers* $B(n)$, seq. **585.** $B(n)$ is also the number of equivalence relations on a set of n objects (Section 3.2) and has gf

$$1 + x + 2\frac{x^2}{2!} + 5\frac{x^3}{3!} + \cdots = e^{e^x - 1}.$$

(See Abramowitz and Stegun [AS1 835], David *et al.* [DKB 223], and Comtet [CO1 2 38].)

3.13 PERMUTATIONS

A *permutation* of n objects is any rearrangement of them, and is specified either by a table:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{array}$$

or by a product of cycles: (134)(25), both of which mean replace 1 by 3, 3 by 4, 4 by 1, 2 by 5, and 5 by 2.

Figure 23 shows $s(n, k)$, the *Stirling numbers of the first kind,* or the numbers of permutations of n objects containing k cycles. The numbers continue:

							row sums $n!$
1							1
1	1						2
2	3	1					6
6	11	6	1				24
24	50	35	10	1			120
120	274	225	85	15	1		720
720	1764	1624	735	175	21	1	5040

A gf for $s(n, k)$ is

$$x(x - 1) \cdots (x - n + 1) = \sum_{k=0}^{n} (-1)^{n-k} s(n, k) x^k.$$

Both the columns and diagonals of this array will be found in the main table. The row sums are the *factorial* numbers $n!$, seq. **659**, the total num-

ber of permutations of n objects. References are as given above for the Stirling numbers of the second kind.

Factorial n is the product $1 \cdot 2 \cdot 3 \cdots n$ of the first n numbers. The products of the first n even numbers, $(2n)!! = 2 \cdot 4 \cdot 6 \cdots (2n) = 2^n \cdot n!$, seq. *742*: 2, 8, 48, 384, 3840, 46080, . . . , and of the first n odd numbers, $(2n - 1)!! = 1 \cdot 3 \cdot 5 \cdots (2n - 1) = (2n)!/(2^n \cdot n!)$, seq. *1217*: 1, 3, 15, 105, 945, 10395, . . . , are called *double factorials*.

n \ k	1	2	3	4	Total
1	(1)				1
2	(12)	(1)(2)			2
3	(123) (132)	(1)(23) (2)(13) (3)(12)	(1)(2)(3)		6
4	(1234) (1243) (1324) (1342) (1423) (1432)	(1)(234) (1)(243) (2)(134) (2)(143) (3)(124) (3)(142) (4)(123) (4)(132) (12)(34) (13)(24) (14)(23)	(1)(2)(34) (1)(3)(24) (1)(4)(23) (2)(3)(14) (2)(4)(13) (3)(4)(12)	(1)(2)(3)(4)	24

Fig. 23. $s(n, k)$, the Stirling numbers of the first kind; and seq. *659*, the factorial numbers.

$D_2 = 1$ 　 1 2
　　　　　 2 1

$D_3 = 2$ 　 1 2 3 　　 1 2 3
　　　　　 2 3 1 　　 3 1 2

$D_4 = 9$ 　 1 2 3 4 　 1 2 3 4 　 1 2 3 4
　　　　　 2 1 4 3 　 2 3 4 1 　 2 4 1 3

　　　　　 1 2 3 4 　 1 2 3 4 　 1 2 3 4
　　　　　 3 1 4 2 　 3 4 1 2 　 3 4 2 1

　　　　　 1 2 3 4 　 1 2 3 4 　 1 2 3 4
　　　　　 4 1 2 3 　 4 3 1 2 　 4 3 2 1

Fig. 24.　Seq. *766*, derangements.

Figure 24 shows D_n, the number of *derangements* of n objects, or the permutations in which every object is moved from its original position (seq. *766*). These are also called subfactorial or *rencontres* numbers, and have the recurrence $D_n = nD_{n-1} + (-1)^n$. (See Riordan [R1 57].)

Figure 25 illustrates seq. *587*, the *Euler numbers E_n*, or the number of permutations of *n* objects which first rise and then alternately fall and rise. (Only the second rows of the permutations are shown.)

The even numbered Euler numbers form seq. *1667*: 1, 5, 61, 1385, 50521, . . . , and have gf

$$1 + 1\frac{x^2}{2!} + 5\frac{x^4}{4!} + 61\frac{x^6}{6!} + \cdots = \sec x.$$

(Often these are called the Euler numbers instead of seq. *587*.)

The odd numbered Euler numbers form seq. *829*: 1, 2, 16, 272, 7936, 353792, . . . , and are called the *tangent numbers $T_n = E_{2n-1}$*. They have gf

$$x + 2\frac{x^3}{3!} + 16\frac{x^5}{5!} + \cdots = \tan x.$$

$E_1 = 1$	1				
$E_2 = 1$	1 2				
$E_3 = 2$	1 3 2	2 3 1			
$E_4 = 5$	1 3 2 4	1 4 2 3	2 3 1 4	2 4 1 3	3 4 1 2
$E_5 = 16$	1 3 2 5 4	1 4 2 5 3	1 4 3 5 2	1 5 2 4 3	
	1 5 3 4 2	2 3 1 5 4	2 4 1 5 3	2 4 3 5 1	
	2 5 1 4 3	2 5 3 4 1	3 4 1 5 2	3 4 2 5 1	
	3 5 1 4 2	3 5 2 4 1	4 5 1 3 2	4 5 2 3 1	

Fig. 25.   Seq. *587*, the Euler numbers.

The *Bernoulli numbers B_n* are defined by

$$B_n = \frac{2nE_{2n-1}}{2^{2n}(2^{2n}-1)},$$

and form the sequence

$$\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \frac{5}{66}, \frac{691}{2730}, \frac{7}{6}, \frac{3617}{510}, \cdots$$

with gf

$$1 - \frac{x}{2} + \frac{1}{6}\frac{x^2}{2!} - \frac{1}{30}\frac{x^4}{4!} + \frac{1}{42}\frac{x^6}{6!} - \cdots = \frac{x}{e^x - 1}.$$

The numerators and denominators form seqs. *1677* and *1746*.

Finally the *Genocchi numbers* are defined by $G_n = 2^{2-2n} n E_{2n-1}$, and form seq. *1233*: 1, 1, 3, 17, 155, 2073, 38227, . . . , with gf

$$1 \frac{x}{2!} + 1 \frac{x^3}{4!} + 3 \frac{x^5}{6!} + 17 \frac{x^7}{8!} + \cdots = \tan \frac{1}{2} x.$$

The Euler, tangent, Bernoulli, and Genocchi numbers arise in all branches of mathematics. For applications and properties see Jordan [JO2], David and Barton [DB1], Comtet [CO1] and Gould [AMM 79 44 72]; for tables see Fletcher *et al.* [FMR 1 65] and Knuth and Buckholtz [MTAC 21 663 67].

3.14   SEQUENCES FROM NUMBER THEORY

The table contains many number-theoretic sequences, of which the following are typical:

(1)   The prime numbers, lucky numbers, and other sequences generated by sieves (seqs. *241, 377, 1035, 1048*);

(2)   the Euler totient function $\phi(n)$: the number of integers not exceeding and relatively prime to n (seq. *111*);

(3)   from the Goldbach conjecture: the number of ways of writing $2n$ as a sum of two primes (various sequences—see index);

(4)   quadratic partitions of primes: a prime of the form $4n + 1$ has a unique representation as $a^2 + b^2$ with $a \geqslant b$. Sequences *169* and *33* give a and b;

(5)   the number of integers less than or equal to 2^n expressible in the form $u^2 + v^2$, where u and v are integers (seq. *265*);

(6)   Mersenne primes: the numbers n such that $2^n - 1$ is prime (seq. *248*);

(7)   from Euler's proof that there are an infinity of primes: let $p_1 = 2$, $p_2, \ldots, p_n$ be primes, and define p_{n+1} to be the smallest (largest) prime factor of $p_1 p_2 \cdots p_n + 1$ (seqs. *329, 330*);

(8)   Beatty sequences: if α, β are positive irrational numbers such that $(1/\alpha) + (1/\beta) = 1$, then the *Beatty sequences*

$$[\alpha], [2\alpha], [3\alpha], \ldots \quad \text{and} \quad [\beta], [2\beta], [3\beta], \ldots$$

together contain all the positive integers without repetition, where $[x]$ denotes the greatest integer less than or equal to x. (See Honsberger [HO2].) For example, $\alpha = \frac{1}{2}(1 + \sqrt{5}) = 1.61803\ldots$ gives seqs. *917*: 1, 3, 4, 6, 8, 9, . . . and *509*: 2, 5, 7, 10, 13, 15, ...

The following test for Beatty sequences is due to R. L. Graham. If $a_1, a_2, \ldots$ is a Beatty sequence, then the values of $a_1, \ldots, a_{n-1}$ determine

a_n to within 1. Look at the sums $a_1 + a_{n-1}$, $a_2 + a_{n-2}, \ldots, a_{n-1} + a_1$. If all these sums have the same value, V say, then a_n must equal V or $V + 1$; but if they take on the two values V and $V + 1$, and no others, then a_n must equal $V + 1$. If anything else happens, it is not a Beatty sequence. For example, in seq. *917*, $a_1 + a_1 = 2$ so a_2 must be 2 or 3 (it is 3); $a_1 + a_2$ = 4 so a_3 must be 4 or 5 (it is 4); $a_1 + a_3 = 5$ and $a_2 + a_2 = 6$, so a_4 must be 6 (it is); and so on.

For further information about number-theoretic sequences see the comprehensive works of Dickson [DI2] and Lehmer [LE1].

3.15 PUZZLE SEQUENCES

This section describes some sequences with simple yet unexpected generating principles. They have all been given as puzzles at one time or another. Of course all of the sequences given in Chapters II and III make good puzzles.

(1) Sequences related to well-known constants (e.g., seq. *1291*: 1, 4, 1, 4, 2, 1, 3, 5, 6, 2, 3, . . . , the decimal expansion of $\sqrt{2}$) or to other common sequences (seq. *2127*: 1, 15, 29, 12, 26, 12, 26, 9, . . . is related to the calendar—guess!). See also seqs. *684*, *880*, *1679*, *1812*, etc.

(2) Sequences depending on the binary expansions of numbers (e.g., seq. *41*: 1, 2, 1, 2, 2, 3, 1, 2, 2, . . . gives the number of 1's in the binary expansion of $n + 1$; see also seqs. *360*, *388*).

(3) Sequences depending on the English words or Arabic numerals used to describe them (e.g., seq. *2218*: 1, 21, 21000, 101, 121, 1101, . . . , the smallest number requiring n words in English; see also seqs. *1818*, *1897*).

(4) The terms not in some well-known sequence (e.g., seq. *1319*: 4, 6, 7, 9, 10, 11, 12, 14, 15, . . . , the non-Fibonacci numbers).

(5) Sequences obtained by *bisecting* (i.e., taking every other term of) well-known sequences (e.g., seq. *1067*: 1, 3, 7, 18, 47, 123, 322, . . . , a bisection of seq. *924*, the Lucas numbers; see also seqs. *569*, *1101*).

(6) Sequences obtained by alternating the terms of two sequences (seq. *889*: 3, 2, 1, 7, 4, 1, 1, 8, 5, 2, 9, . . . , mixing π and e, is the only example given).

The following pleasing puzzles are not in the table because they are finite or are not integers.

(7) $\frac{1}{4}, \frac{1}{2}$, 1, 3, 6, 12, 24, 30, 120, 240, 1200, 2400, English money in 1950.

(8) 3, 8, 8, 4, 89, 75, 30, 28, ?, planetary diameters in thousands of statute miles.

(9)   8, 5, 4, 9, 1, 7, 6, 3, 2, 0; or 8, 8000000000, . . . , 18, 18000000000, . . . , 18000000, . . . , 18000, . . . , 80, . . . , 88, . . . , 85, . . . , 84, . . . , 11, . . . , 15, . . . , 5, . . . , 4, . . . , the numbers arranged in alphabetical order (in English).

(10)   12, 13, 14, 15, 20, 22, 30, 110, 1100, the number 12 written to the bases 10, 9, 8, . . . , 2.

(11)   14, 18, 23, 28, 34, 42, 50, 59, 66, 72, 79, 86, 96, 103, 110, 116, 125, 137, 145, 157, 168, 181, 191, 207, 215, 225, 231, 238, 242, the local stops on the New York IRT subway.

(12)   1714, 1727, 1760, 1820, 1910, 1936, dates of the accessions of the Georges to the English throne.

(13)   1732, 1735, 1743, 1751, 1758, 1767, 1767, 1782, 1773, 1790, 1795, 1784, 1800, 1804, 1791, 1809, 1808, 1822, 1822, 1831, 1830, 1837, 1833, 1837, 1843, 1858, 1857, 1856, 1865, 1872, 1874, 1882, 1884, 1890, 1917, 1908, 1913, dates of birth of presidents of the U.S.A.

(14)   The integers 1, 2, 3, . . . drawn next to a mirror. (See Fig. 26.)

(15)   O, T, T, F, F, S, S, E, N, T, E, T, T, F, F, S, S, E, N, T, T, T, T, . . . , the initial letters of the English names for the numbers.
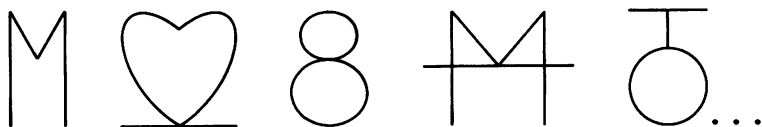

Fig. 26.   A puzzle.

3.16   SEQUENCES FROM LATTICE STUDIES IN PHYSICS

In the last twenty years physicists have studied a number of basic combinatorial problems related to crystal lattices. Typical problems are to find the number of self-avoiding paths of length n on a given lattice, or the number of ways a particular graph can be drawn on the lattice. A number of such sequences will be found in the main table. For further information see Montroll [BE6 96], Sykes *et al.* [JMP 7 1557 66], Kasteleyn [HA1 43], Percus [PE3], and Domb [ACP 15 229 69].